Rotation halts cylindrical, relativistic gravitational collapse

Theocharis A. Apostolatos and Kip S. Thorne
Theoretical Astrophysics, California Institute of Technology, Pasadena, California 91125
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It is shown, in a simple analytic example, that an infinitesimal amount of rotation can halt the general relativistic gravitational collapse of a pressure-free cylindrical body. The example is a thin cylindrical shell (a shell with translation symmetry and rotation symmetry), made of counter-rotating dust particles. Half of the particles rotate about the symmetry axis in one direction with (conserved) angular momentum per unit rest mass α, and the other half rotate in the opposite direction with the same α. It is shown, using C-energy arguments, that the shell can never collapse to a circumference smaller than $C = 8\pi\alpha\Lambda$, where Λ is the shell's nonconserved mass per unit proper length. Equivalently, if $R \equiv |\partial/\partial\phi||\partial/\partial z|$ is the product of the lengths of the rotational and translational Killing vectors at the shell's location and λ is the shell's conserved rest mass per unit Killing length z, then the shell can never collapse smaller than $R = 4\alpha\lambda$. It is also shown that after its centrifugally induced bounce, the shell will oscillate radially and will radiate gravitational waves as it oscillates, the waves will carry away C energy, and this loss of C energy will force the shell to settle down to a static, equilibrium radius.

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I. INTRODUCTION AND SUMMARY

A. Motivation

A recent numerical solution of the Einstein field equations by Shapiro and Teukolsky [1] suggests that it may be possible for a naked singularity to form in the gravitational collapse of a highly nonspherical body, in violation of Penrose's [2] cosmic censorship conjecture. The Shapiro-Teukolsky solution describes the collapse of a prolate spheroid of dust particles, all of which initially are at rest. Near the end point of their collapse, the dust particles form a thin spindle that is imploding radially. If the spindle is so long that its poloidal circumference exceeds $4\pi M$ (where M is its mass and we set Newton's gravitation constant and the speed of light to unity), then in accord with the hoop conjecture [3, 4] no apparent horizon forms around the spindle at least up to the termination of the numerical solution, and in apparent violation of cosmic censorship, naked singularities appear to be forming in the vacuum just beyond the spindle's two pointed ends. The growth of these singularities forces the numerical integration to terminate.

It is not surprising that the collapse forms a singularity, since the dust spindle is more or less a finite-length version of an infinitely long dust cylinder, and it has long been known that collapsing infinite dust cylinders form naked singularities [3, 5, 6]. What *is* a bit surprising is that the Shapiro-Teukolsky singularity appears to be forming most rapidly in the vacuum just beyond the spindle's end rather than inside the spindle, where the dust resides. We shall discuss the significance of this below.

The cosmic censorship conjecture (the impossibility of naked singularities) is generally thought to be correct only for *realistic* gravitational collapse—collapse with rotation and realistic amounts of pressure. It therefore is of interest to ask whether the Shapiro-Teukolsky collapse would still produce a naked singularity if the collapsing body were endowed with rotation and/or a realistically stiff equation of state.

In this paper we investigate the effects of rotation in the idealized limit of an infinitely long spindle, i.e., an infinite cylinder. We show analytically that the centrifugal forces associated with an arbitrarily small amount of rotation, by themselves, without the aid of any pressure, can halt the collapse and prevent a singularity from forming [7]. Elsewhere Piran [8] has shown, in specific numerical examples, that realistic pressure (pressure p such that $\gamma \equiv d\ln p/d\ln n > 1$ where n is the number density of conserved baryons) can also halt cylindrical collapse.

These results make it seem likely that also in the Shapiro-Teukolsky case of a finite-length spindle, an arbitrarily small amount of rotation and/or a realistic pressure will halt the spindle's radial collapse. If so, however, this by no means would *guarantee* an absence of naked singularities. The reason is as follows.

The fact that the singularity appears to be forming most rapidly in the vacuum just beyond the spindle's end suggests that the vacuum part of the singularity might be spacelike or timelike with respect to the singularity in the dust interior, or might even precede it causally. If so, then a realistic but tiny rotation or pressure in the spindle's imploding matter would make itself felt too late to influence the vacuum singularity. The imploding matter might bounce, but the vacuum singularity, causally oblivious of the bounce, would still form in precisely the same manner as if there were no rotation or pressure. The singularity presumably would be created by nonlinear gravity that is triggered by the sharp spacetime curvature that occurs near the bouncing spindle's pointed ends.

We must emphasize that this scenario is pure speculation. The Shapiro-Teukolsky numerical solution is by no means accurate enough nor carried to late enough times to reveal (i) whether the vacuum singularity is spacelike

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or timelike with respect to the interior dust singularity, or (ii) whether a horizon forms around the vacuum singularity at late times.

B. Collapsing shell of counter-rotating dust

The system whose collapse is analyzed in this paper is a thin cylindrical shell made of pressure-free counter-rotating dust. Half of the dust particles orbit around the symmetry axis in a right-handed direction with angular momentum per unit rest mass α, and the other half orbit in the opposite, left-handed direction with angular momentum per unit rest mass $-\alpha$, so there is vanishing total angular momentum. This counter-rotation guarantees that there will be no dragging of inertial frames and thereby simplifies the analysis. (It seems likely on intuitive grounds that, as for our counter-rotating shell, so also for shells with net angular momentum, an arbitrarily small amount of rotation will cause the collapsing shell to bounce. However, we have not attempted to analyze shells with net angular momentum.)

In our analysis we describe the counter-rotating shell mathematically by Israel's thin-shell junction conditions [9] (Sec. II B). In the vacuum interior and exterior of the shell, the Einstein-Rosen canonical cylindrical coordinates [10, 11] (t, r, z, ϕ) are introduced and the line element takes the form

$$ds^2 = e^{2(\gamma-\psi)}(-dt^2 + dr^2) + e^{2\psi}dz^2 + r^2e^{-2\psi}d\phi^2 \,. \quad (1)$$

Here $\psi = \psi(t,r)$ is a gravitational field whose static part is an analogue of the Newtonian potential and whose ripples represent gravitational waves; $\gamma = \gamma(t,r)$ is a metric function that will play an important role in the details of our analysis (Secs. II–VI) but is irrelevant for our discussion of the results (Sec. I); t is the coordinate time; r is the coordinate radius; $\partial/\partial z$ and $\partial/\partial\phi$ are the Killing vector generators of translational symmetry along the cylinder and rotational symmetry around the symmetry axis; z is the Killing coordinate length with $-\infty < z < +\infty$; and ϕ is angle around the axis with $0 \leq \phi \leq 2\pi$. Here and throughout we set Newton's gravitation constant and the speed of light to unity. Notice that r has the geometric meaning $r = |\partial/\partial z||\partial/\partial\phi| =$ (product of lengths of the two Killing vectors), and that the circumference around the symmetry axis is $2\pi r e^{-\psi}$.

We restrict attention to shells whose mass per unit length is small enough that, at some initial moment of time, they do *not* close space up around themselves radially (subsection D). This implies (Appendix A) that they will never close up space radially, and correspondingly r always increases monotonically as one travels radially outward from the symmetry axis ($r = 0$) to the shell and then onward; i.e., r varies over the range $0 \leq r < \infty$.

We shall use the following parameters to characterize the shell: (i) the angular momentum per unit rest mass of its particles, $\pm\alpha$; (ii) its coordinate radius $R =$ (value of r on shell); (iii) the value ψ_s of ψ on the shell; (iv) $\mathcal{R} = Re^{-\psi_s} =$ (circumference of shell)$/2\pi$; (v) $u \equiv \alpha/\mathcal{R} \equiv v/\sqrt{1-v^2}$ where $\pm v$ is the velocity of orbital motion of the dust particles as measured by an observer who rides on the shell and orbits neither rightward nor leftward, and where $\pm u$ is the dust particles' corresponding linear momentum per unit rest mass as measured by these observers; (vi) $\lambda \equiv dm/dz \equiv$ (shell's total rest mass per unit Killing length); and (vii) $\Lambda \equiv \lambda e^{-\psi_s} =$ (shell's total rest mass per unit proper length).

Of these parameters, α, $\mathcal{R}$, u, v, and Λ are unaffected by a rescaling of the Killing coordinate length z [i.e., by $z \to \text{const} \times z$; Eq. (15a) below], and in fact they are locally measurable with no ambiguity. By contrast, R and λ are scale dependent and thus are not locally measureable; however, their dynamical changes (doublings, halvings, ...) are readily measureable locally. The parameters α and λ are conserved as the shell evolves and emits gravitational waves, but the other parameters change.

C. Nearly Newtonian shell

If the shell's rest mass per unit proper length Λ and linear momentum per unit rest mass u are very small compared to one, $\Lambda \ll 1$ and $u \ll 1$ (in geometrized units where the speed of light and Newton's gravitation constant are equal to unity), and if the gravitational waves initially are very weak, so that (with an appropriate choice of z scaling) $|\psi| \ll 1$ everywhere except at extremely large radii, then the shell and its evolution will be very nearly Newtonian. More specifically, $\Lambda \simeq \lambda$ will be conserved as will be α, and the shell's radius $\mathcal{R} \simeq R$ will obey the rather obvious equation of motion

$$\mathcal{C} = \frac{\Lambda}{2}\left(\frac{d\mathcal{R}}{dt}\right)^2 + \mathcal{C}_{\rm MS}(\mathcal{R}) \,. \quad (2)$$

Here $\mathcal{C}$ is the shell's conserved energy per unit length and

$$\mathcal{C}_{\rm MS}(\mathcal{R}) = \Lambda\frac{\alpha^2}{2\mathcal{R}^2} + \Lambda^2 \ln\mathcal{R} + \text{const} \quad (3)$$

is the energy the shell would have if it were radially momentarily static. (We use the symbol $\mathcal{C}$ because these energies are the Newtonian limits of the shell's relativistic "C energy".) Note that $\mathcal{C}_{\rm MS}(\mathcal{R})$ plays the role of an effective potential for the shell's radial motion (Fig. 1).

From the shape of the effective potential, it should be clear that the shell oscillates back and forth between a maximum radius $\mathcal{R}_{\rm max}$ and a minimum radius $\mathcal{R}_{\rm min}$, whose values depend on its initial conditions. At $\mathcal{R}_{\rm max}$, gravity overwhelms the centrifugal force and pulls the shell inward; at $\mathcal{R}_{\rm min}$, the centrifugal force overwhelms gravity and pushes it outward.

General relativity insists that these nearly Newtonian oscillations produce very weak gravitational waves which carry off energy. As a result, $\mathcal{R}_{\rm max}$ decreases a bit from one oscillation to the next and $\mathcal{R}_{\rm min}$ increases a bit, until finally the shell settles down into equilibrium at the minimum of its effective potential $\mathcal{C}_{\rm MS}(\mathcal{R})$. The equilibrium radius is clearly

$$\mathcal{R}_{\rm eq} = \alpha/\sqrt{\Lambda} \,. \quad (4)$$

From this simple analysis it is obvious that (i) if $\alpha = 0$ (no counter-rotation), then the shell collapses to a New-

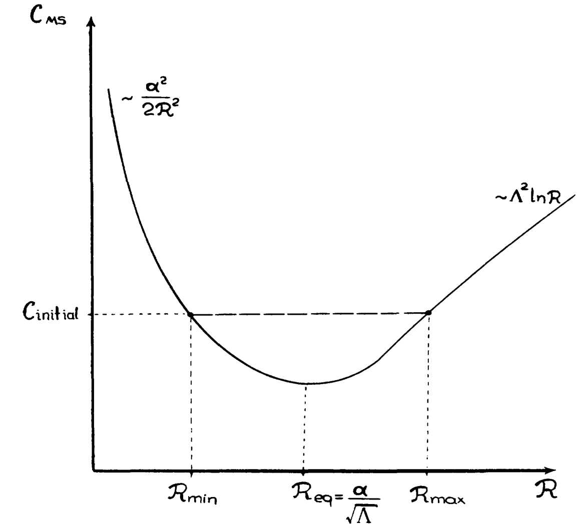

FIG. 1. The energy per unit length $\mathcal{C}$ for a cylindrical, Newtonian shell of counter-rotating dust, plotted as a function of the shell's radius $\mathcal{R}$. For a dynamical shell, $\mathcal{C}$ is conserved (dashed line) as the shell bounces back and forth between the radii $\mathcal{R}_{\rm min}$ and $\mathcal{R}_{\rm max}$. The effective potential $\mathcal{C}_{\rm MS}(\mathcal{R})$ in which it moves is given by Eq. (3).

tonian singularity, $\mathcal{R} = 0$, but (ii) an arbitrarily small amount of angular momentum per unit rest mass, α, causes any collapsing shell to bounce and thereby prevents a singularity from forming.

As we shall see, in full general relativity the shell behaves qualitatively the same as this, though the quantitative details are different.

In our relativistic analysis we shall find it convenient to think about the equilibrium configuration from a different viewpoint than Eq. (4). We shall ask the following question: "If the shell at some moment of time has a rotational linear momentum per unit rest mass $u = \alpha/\mathcal{R}$, then how large must its rest mass per unit length Λ be in order for its inward gravitational force to precisely counterbalance its outward centrifugal force?" The answer is easily seen from Eq. (4) to be

$$\Lambda_{\rm eq} = u^2 \; . \tag{5}$$

D. Static and momentarily static, relativistic shell

Gravitational radiation causes severe complications in the theory of a fully relativistic cylindrical shell. A useful tool for cutting through those complications is the concept of a *momentarily static and radiation-free* (MSRF) shell, i.e., a configuration that, at some chosen moment of coordinate time t, (i) has no gravitational radiation ($\partial\psi/\partial t = \partial^2\psi/\partial t^2 = 0$ everywhere), and (ii) has its shell momentarily radially at rest ($dR/dt = 0$ and thus also $d\mathcal{R}/dt = 0$ since $\mathcal{R} = Re^{-\psi_s}$).

As we shall see in Sec. III, a MSRF shell is characterized fully (at the chosen time t) by the scale-invariant parameters α = (angular momentum per unit rest mass), Λ = (rest mass per unit proper length), and $\mathcal{R}$ = (circumference)/2π. From these we can construct two dimensionless parameters, e.g., Λ and $u = \alpha/\mathcal{R}$ = (linear momentum per unit rest mass). In Fig. 2 we plot Λ upward and u rightward. There are two special lines in this Λ-u plane. The upper line $\Lambda_{\rm max}(u)$ is given by

$$\Lambda_{\rm max} = \frac{1}{4\sqrt{1+u^2}} \tag{6}$$

or equivalently

$$\frac{\Lambda_{\rm max}}{\sqrt{1-v^2}} = \frac{1}{4} \; , \tag{7}$$

where $\pm v$ is the speed of orbital motion of the shell's particles as measured by an observer at rest on the shell (cf. subsection B), so $\Lambda/\sqrt{1-v^2}$ is their total mass (rest mass plus kinetic energy) per unit proper length, as measured by that observer. Any MSRF shell above this upper line, $\Lambda > \Lambda_{\rm max}(u)$, i.e., any MSRF shell with total mass per unit proper length greater than 1/4, is so massive that it closes space up around itself radially (see Appendix B for a proof). (In the language of Appendix A, the spacetime has character $D^{(-)}$ outside the shell.) In this paper, we are seeking insight into the collapse of bodies around which spacetime is asymptotically flat, not closed, so we constrain our analysis to MSRF shells below the upper line of Fig. 2, $\Lambda < \Lambda_{\rm max}$.

The lower line $\Lambda_{\rm eq}$ (u) in Fig. 2 represents MSRF shells that are in equilibrium—i.e., shells that, when evolved to the future of the chosen initial time t, never change their radii R or $\mathcal{R}$ and never develop any gravitational radiation and thus always remain static. At small u this $\Lambda_{\rm eq}(u)$ has the Newtonian form (5), $\Lambda_{\rm eq} \simeq u^2$. The precise formula for $\Lambda_{\rm eq}(u)$ is (Sec. III)

$$\Lambda_{\rm eq} = \frac{u^2\sqrt{1+u^2}}{(1+2u^2)^2} \; . \tag{8}$$

At $u < 0.8836$, $\Lambda_{\rm eq}(u)$ increases with increasing u (cf. Fig. 2) because larger u means larger centrifugal forces

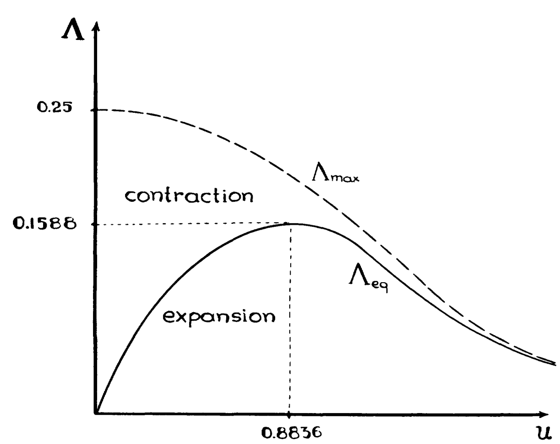

FIG. 2. The dimensionless parameter space for momentarily static and radiation-free (MSRF) shells.

and requires larger mass to produce enough gravity to hold the shell together. At $u > 0.8836$, $\Lambda_{\rm eq}(u)$ decreases with increasing u because larger u means larger total mass per unit rest mass and thus less rest mass is needed to produce enough total mass to hold the shell together.

It turns out (Sec. III) that any MSRF shell with $\Lambda < \Lambda_{\rm eq}(u)$ begins to expand when released from its initial state, because its gravity is too weak to counterbalance its centrifugal forces; similarly, any MSRF shell with $\Lambda > \Lambda_{\rm eq}(u)$ [but $\Lambda < \Lambda_{\rm max}(u)$] begins to contract when released.

E. Dynamical relativistic shell

In Secs. III–VII, we use the concept of C energy to prove that a fully relativistic, cylindrical shell evolves dynamically in the same qualitative manner as a nearly Newtonian one. More specifically, we place the shell in an initial MSRF configuration that is arbitrary (arbitrary values of $\mathcal{R}$, u, Λ) except that $\Lambda < \Lambda_{\rm max}(u)$ so it does not close up space radially around itself. We then select a radius r_0 that is arbitrarily large and evaluate the total amount of C energy $\mathcal{C}_0$ inside r_0. As the shell evolves dynamically, emitting gravitational waves, this total C energy $\mathcal{C}_0$ will be conserved until the waves reach r_0 (which means for an arbitrarily long time), and then as the waves carry C energy outwards through r_0 (Sec. VI), $\mathcal{C}_0$ will begin to decrease.

During the shell's arbitrarily long evolution with fixed $\mathcal{C}_0$, it conserves its values of α, λ, and $\psi_{0^\cdot} =$ (value of ψ at r_0), while $\psi(r,t)$ and $R(t)$ evolve dynamically.

In Sec. V we prove that $\mathcal{C}_0$, at any moment of time t, consists of a *positive* contribution associated with the shell's radial velocity dR/dt and with the gravitational waves (GW's) it has emitted, plus the contribution $\mathcal{C}_{\rm MSRF}(R)$ that the configuration would have if it were MSRF and had the same shell radius R as it actually has

$$\mathcal{C}_0 = \mathcal{C}_{\rm MSRF}(R) + (\text{positive kinetic and GW energy}) . \tag{9}$$

In Secs. III and IV we derive the somewhat complicated functional form of $\mathcal{C}_{\rm MSRF}(R)$ and show that it has the following properties. (i) $\mathcal{C}_{\rm MSRF}(R)$ has the same qualitative form as the Newtonian $\mathcal{C}_{\rm MS}(\mathcal{R})$ (Fig. 1): It has a single minimum at a radius $R_{\rm eq}$, and it increases monotonically as one moves away from that minimum toward either decreasing or increasing shell radii R. (ii) In the Newtonian limit $\mathcal{C}_{\rm MSRF}(R)$ becomes $\mathcal{C}_{\rm MS}(R)$ [Eq. (3)]. (iii) $\mathcal{C}_{\rm MSRF}(R)$ reaches the largest value $\frac{1}{8}$, that any shell's C energy can have without closing the universe up radially, as R approaches r_0 from below and as it approaches

$$R_{\rm abs\ min} = 4\alpha\lambda \tag{10}$$

from above.

These properties of $\mathcal{C}_{\rm MSRF}(R)$, together with relation (9) and the conservation of $\mathcal{C}_0$, imply the following dynamics for the shell. (i) Once released from its initial MSRF radius, the shell must oscillate back and forth in R, emitting gravitational waves, and finally settle down into an equilibrium state. (ii) If the shell's initial radius R_i is larger than the minimum point $R_{\rm eq}$ of $\mathcal{C}_{\rm MSRF}(R)$, then the shell can never oscillate out to a radius R larger than R_i; furthermore, if $R_i < R_{\rm eq}$, then it can never oscillate to a smaller R than R_i. (iii) No matter what the initial state may be, so long as the shell initially does not close space up radially, the centrifugal force always keeps its radius R larger than $R_{\rm abs\ min} = 4\alpha\lambda$, and correspondingly keeps $\mathcal{R} =$ circumference$/2\pi$ larger than

$$\mathcal{R}_{\rm abs\ min} = 4\alpha\Lambda . \tag{11}$$

(Recall that λ and α are conserved during the evolution; $\Lambda = \lambda e^{-\psi_s}$, however, will generally vary.) From Eqs. (10) and (11), it is clear that an arbitrarily small angular momentum α prevents the shell from collapsing to a singularity [7].

F. Organization of this paper

In the body of this paper, we derive the results described above.

We lay foundations for our derivation in Sec. II. In Sec. II A and Appendix A, we discuss the requirement that spacetime not be closed up radially, and then relying on that requirement, we introduce our spacetime coordinates t, r, z, ϕ, and corresponding metric coefficients ψ and γ, we write down the vacuum Einstein field equations for the shell's interior and exterior, and we introduce the concept of C energy. In Sec. II B we introduce the parameters that describe the shell and write down, in the form of thin-shell junction conditions, the Einstein field equations that govern the shell's coupling to the spacetime metric and its dynamical evolution.

In Sec. III, we analyze the structures of momentarily static and radiation-free (MSRF) configurations, and prove that among all MSRF configurations, the ones in equilibrium are those that minimize the C energy with respect to variations of the shell's radius R.

In Sec. IV, we show that the C energy of MSRF configurations, as a function of shell radius R, has the same qualitative form as in Newtonian theory (Fig. 2).

In Sec. V, we show that the C energy of a dynamical configuration is always greater than that of a MSRF configuration that has the same shell radius R, i.e., that the C energy can be written in the form (9) used above in our discussion of dynamical evolution.

In Sec. VI, we show that gravitational waves always carry C energy away from the shell, toward $r = \infty$.

In Sec. VII, we recapitulate: The properties of the C energy, as derived in Secs. III–VI, are precisely the underpinnings needed to validate the discussion of shell dynamics given in Sec. I E. Therefore, it must be that the shell can neither collapse to a line singularity nor explode to infinity, but instead must undergo damped oscillations and end up in an equilibrium configuration of finite radius.

II. FOUNDATIONS FOR THE ANALYSIS

A. Radial nonclosure, coordinates, metric, and vacuum field equations

It is well known [10, 11] that, in any cylindrically symmetric spacetime with vanishing net momentum density along the Killing directions $\partial/\partial z$ and $\partial/\partial\phi$, one can introduce coordinates $(\bar{t}, \bar{r}, z, \phi)$ in which the line element takes the form

$$ds^2 = e^{2(\bar{\gamma}-\psi)}(-d\bar{t}^2 + d\bar{r}^2) + e^{2\psi}dz^2 + \beta^2 e^{-2\psi}d\phi^2 \; . \tag{12}$$

Here $\bar{\gamma}$, ψ, and β are functions of $\bar{t}$ and $\bar{r}$. In vacuum, but not generally inside matter, the quantity $\beta = |\partial/\partial z||\partial/\partial\phi|$ obeys the Einstein field equation

$$\beta_{,\bar{t}\,\bar{t}} - \beta_{,\bar{r}\bar{r}} = 0 \; , \tag{13}$$

where commas denote partial derivatives. (This equation is the content of $R^z_z + R^\phi_\phi = 0$, where R^ν_μ is the Ricci tensor.) In Appendix A it is shown that, if (as we shall assume) space *initially* is *not* closed up radially by the shell's mass, then *everywhere* in the spacetime $\nabla\beta$ is spacelike and is directed away from the symmetry axis. This together with Eq. (13) permits us, throughout the spacetime, to perform a conformal transformation in the $(\bar{t}, \bar{r})$ plane to new (t, r) coordinates such that $\beta = r$:

$$ds^2 = e^{2(\gamma-\psi)}(-dt^2 + dr^2) + e^{2\psi}dz^2 + r^2 e^{-2\psi}d\phi^2 \; . \tag{14}$$

These are the coordinates discussed in Sec. I B.

Because the Einstein equation takes the form (13) only in vacuum and not on the shell itself, the conformal transformation turns out to be discontinuous across the shell. More specifically, although z, ϕ, $r = |\partial/\partial z||\partial/\partial\phi|$, and $\psi = \frac{1}{2}\ln|\partial/\partial z|$ (being Killing-defined quantities) are continuous across the shell, the time coordinate t and the metric function γ are discontinuous.

The Killing coordinate length z is defined only up to an arbitrary multiplicative factor. It should be obvious that a rescaling of z produces the following changes in other quantities:

$$z \to e^\mu z, \quad \psi \to \psi - \mu, \quad r \to e^{-\mu} r, \quad t \to e^{-\mu} t, \tag{15a}$$

where μ is an arbitrary constant; correspondingly,

$$\lambda \to e^{-\mu}\lambda, \quad R \to e^{-\mu} R \, . \tag{15b}$$

In the vacuum inside and outside the shell, the metric coefficients $\psi(t, r)$ and $\gamma(t, r)$ satisfy the Einstein field equations [10, 11]

$$\psi_{,tt} - \frac{1}{r}(r\psi_{,r})_{,r} = 0 \; , \tag{16a}$$

$$\gamma_{,r} = r[(\psi_{,t})^2 + (\psi_{,r})^2] \; , \tag{16b}$$

$$\gamma_{,t} = 2\psi_{,t}\psi_{,r} \; . \tag{16c}$$

Smoothness of the spacetime geometry on the symmetry axis $r = 0$ requires that

$$\gamma = 0 \text{ and } \psi \text{ finite at } r = 0. \tag{17}$$

We shall sometimes refer to ψ as the *gravitational-wave field* since it satisfies the wave equation (16a) and it governs distortions of the geometry along the polarization axes (z and ϕ directions) in the usual "transverse-traceless" manner [13] (a weak ripple $\delta\psi$ in ψ produces fractional metric perturbations $\delta g_{zz}/g_{zz} = \delta\psi$, $\delta g_{\phi\phi}/g_{\phi\phi} = -\delta\psi$ that are confined to the transverse plane and are equal and opposite along the two transverse directions).

The quantity γ and the C energy $\mathcal{C}$ are monotonic functions of each other [11, 12]:

$$\mathcal{C} = \frac{1}{8}(1 - e^{-2\gamma}) \; . \tag{18}$$

B. Description of the shell

The evolution of the shell will be characterized by $R(\tau)$, where R is the value of the radial coordinate r at the shell's location and τ is the proper time of an observer riding with the shell, but not rotating with the shell's particles. We sometimes will use $\mathcal{R} = Re^{-\psi_s} =$ circumference$/2\pi$ to describe the shell's location, instead of R; here ψ_s is the value of ψ at the shell.

As an aid in analyzing the shell's properties and motion, it will be helpful to introduce the proper reference frame of an observer riding on the shell. This frame's orthonormal tetrad is

$$\mathbf{e}_\tau \equiv \frac{d}{d\tau} \equiv \mathbf{u} = \text{four-velocity of the shell}$$

$$= X_\pm \frac{\partial}{\partial t_\pm} + V\frac{\partial}{\partial r} \; , \tag{19a}$$

$$\mathbf{e}_n \equiv \frac{d}{dn} \equiv \mathbf{n} = \text{outward unit vector normal to shell}$$

$$= X_\pm \frac{\partial}{\partial r} + V\frac{\partial}{\partial t_\pm} \; , \tag{19b}$$

$$e_{\hat{z}} \equiv \frac{1}{e^{\psi_s}}\frac{\partial}{\partial z} \; , \tag{19c}$$

$$e_{\hat{\phi}} \equiv \frac{1}{re^{-\psi_s}}\frac{\partial}{\partial\phi} \; . \tag{19d}$$

Here

$$V \equiv \frac{dR}{d\tau} \tag{20a}$$

and

$$X_\pm \equiv \frac{dt_\pm}{d\tau} = \sqrt{e^{-2(\gamma_\pm - \psi_s)} + V^2} \; . \tag{20b}$$

The subscripts + and − are used to denote quantities evaluated on the outer and inner faces of the shell. (Recall that t and γ are discontinuous across the shell.)

As was discussed in Sec. I B, the shell is made of counter-rotating particles with conserved angular momenta per unit rest mass $\pm\alpha$, and with linear velocities $\pm v$ as measured in the shell's rest frame (19) and linear momenta per unit rest mass:

$$\pm u = \frac{\pm v}{\sqrt{1-v^2}} = \pm\frac{\alpha}{\mathcal{R}} = \pm\frac{\alpha}{Re^{-\psi_s}} \; . \tag{21}$$

The shell's conserved rest mass per unit Killing length z is λ, and correspondingly its total mass per unit proper area as measured in its own rest frame (19) is

$$\sigma = \frac{\lambda}{2\pi R}\frac{1}{\sqrt{1-v^2}} = \frac{\lambda\sqrt{1+u^2}}{2\pi R} \; . \tag{22a}$$

By their orbital motion, the particles create a surface stress $S^{\hat\phi\hat\phi} = T$ whose ratio to their surface energy density $S^{\tau\tau} = \sigma$ is

$$\frac{T}{\sigma} = \frac{S^{\hat\phi\hat\phi}}{S^{\tau\tau}} = \left(\frac{p^{\hat\phi}}{p^\tau}\right)^2 = \frac{u^2}{1+u^2} \tag{22b}$$

(where $p^{\hat\phi} = \pm u$ is a particle's linear momentum per unit rest mass and $p^\tau = 1/\sqrt{1-v^2} = \sqrt{1+u^2}$ is its total mass per unit rest mass as measured in the shell's rest frame). By combining Eqs. (22a) and (22b) we see that

$$T = \frac{\lambda u^2}{2\pi R\sqrt{1+u^2}} \; . \tag{22c}$$

The shell's full surface stress-energy tensor is

$$\mathbf{S} = \sigma \mathbf{u}\otimes\mathbf{u} + T\mathbf{e}_{\hat\phi}\otimes\mathbf{e}_{\hat\phi} \; . \tag{22d}$$

Israel [9] has shown that the Einstein field equations for a thin shell reduce to

$$K^+_{\alpha\beta} - K^-_{\alpha\beta} = 8\pi(S_{\alpha\beta} - \tfrac{1}{2}S^\mu_\mu\gamma_{\alpha\beta}) \; , \tag{23}$$

where $K^\pm_{\alpha\beta}$ is the extrinsic curvature of the shell's outer (inner) face and $\gamma_{\alpha\beta}$ is the metric of its world sheet. For our thin shell, the zz component of these junction conditions reduces to a jump condition on the normal derivative of the gravitational wave field:

$$\psi_{+,n} - \psi_{-,n} = -\frac{2\lambda}{R\sqrt{1+u^2}} \; ; \tag{24a}$$

the $\phi\phi$ component, after use of Eq. (24a), reduces to a jump condition for $X = dt/d\tau$ and therefore [cf. Eq. (20b)] for the time coordinate t and the metric function γ:

$$X_+ - X_- = -4\lambda\sqrt{1+u^2} \; ; \tag{24b}$$

the $\tau\tau$ component reduces to an equation of motion for the shell:

$$A \equiv \frac{d^2R}{d\tau^2} = V\psi_{s,\tau} - R[(\psi_{s,\tau})^2 + (\psi_{-,n})^2] + X_-\frac{\psi_{-,n}}{1+u^2} - \frac{X_-\lambda}{R(1+u^2)^{3/2}} + \frac{X_-X_+u^2}{R(1+u^2)} \; . \tag{24c}$$

[In deriving Eq. (24c), the vacuum field equations (16) and the junction conditions (24a), (24b), have been used.]

In summary, Eq. (24c) governs the motion of the shell, Eqs. (16) and (17) govern the evolution of the metric functions $\gamma(r,t), \psi(r,t)$, and the junction conditions (24a) and (24b) match the metric functions across the shell. Among the various functions that we use, r, z, ϕ, ψ, $\psi_{,\tau}$ are continuous and t, γ, $\psi_{,n}$, X are discontinuous across the shell.

III. MOMENTARILY STATIC AND RADIATION-FREE CONFIGURATIONS

If the configuration, at some moment of time t, is momentarily static ($V = dR/d\tau = 0$) and radiation-free ($\psi_{,t} = \psi_{,tt} = 0$), i.e., MSRF, then it will have the following properties. (i) In the vacuum outside the shell the vacuum field equations (16) imply

$$\psi = \psi_s - \kappa\ln(r/R) \text{ at } r > R \; , \tag{25a}$$

$$\gamma = \gamma_+ + \kappa^2\ln(r/R) \text{ at } r > R \; , \tag{25b}$$

where κ and ψ_s are constants. This is the *Levi-Civita line-mass solution* to the Einstein equations [14]. (ii) In the vacuum interior, the field equations imply a similar logarithmic form for ψ and γ, and the boundary conditions (17) at $r = 0$ imply a vanishing value of κ and, correspondingly,

$$\psi = \psi_s \; , \quad \gamma = 0 \text{ at } r < R \; , \tag{25c}$$

which means that spacetime is flat and Minkowskian inside the shell. (iii) The jump condition (24a) on the normal derivative of ψ, together with Eqs. (19b), (24a), (24b), and (25c), implies that the value of the parameter κ is

$$\kappa = \frac{2\Lambda}{(1-4\Lambda\sqrt{1+u^2})\sqrt{1+u^2}} \tag{25d}$$

(where $\Lambda = \lambda e^{-\psi_s}$ is the rest mass per unit proper length; cf. Sec. I B). (iv) The jump condition (24b) on $X_\pm$, together with expressions (20b) and (25c), implies that

$$\gamma_+ = -\ln(1-4\Lambda\sqrt{1+u^2}) \; . \tag{25e}$$

In order that space not be closed radially by the shell's mass, it must be that $\Lambda\sqrt{1+u^2} < 1/4$ (see Appendix B for a proof and Sec. I D for discussion); correspondingly, γ_+ is real and positive, and κ is positive. (v) The equation of motion (24c) for the shell, when combined with $V = 0$, $\psi_{-,n} = 0$, $\Lambda = \lambda e^{-\psi_s}$ and Eqs. (20b) and (25e), takes the form

$$A = \frac{d^2R}{d\tau^2} = \begin{pmatrix}\text{positive}\\ \text{quantity}\end{pmatrix} \times [\Lambda_{\rm eq}(u) - \Lambda] \; , \tag{25f}$$

where $\Lambda_{\rm eq}(u) \equiv u^2\sqrt{1+u^2}/(1+2u^2)^2$; cf. Eq. (8).

Therefore, if the rest mass per unit proper length Λ is greater than $\Lambda_{\rm eq}$, then the MSRF shell starts contracting, and if Λ is lower than $\Lambda_{\rm eq}$, it starts expanding.

In our analysis of dynamical shells (Sec. I E above) a central role is played by C energy. For a MSRF configuration with shell radius R and with the Killing coordinate z so normalized that

$$\psi_0 \equiv \psi(r_0) = 0 \tag{26a}$$

[cf. Eq. (15a)], the total C energy inside some fixed radius $r_0 > R$ is given by

$$C_{\rm MSRF}(R) = \frac{1}{8}(1 - e^{-2\gamma_0}) \tag{26b}$$

[Eq. (18)], where

$$e^{\gamma_0} = \frac{y^{-\kappa^2}}{1 - 4\lambda y^\kappa \sqrt{1 + (\alpha/r_0)^2 y^{-2-2\kappa}}} , \tag{26c}$$

$$y = R/r_0 < 1 , \tag{26d}$$

$$\kappa = \frac{2\lambda y^\kappa}{[1 - 4\lambda y^\kappa \sqrt{1 + (\alpha/r_0)^2 y^{-2-2\kappa}}]\sqrt{1 + (\alpha/r_0)^2 y^{-2-2\kappa}}}, \tag{26e}$$

cf. Eqs. (25) and (21).

A MSRF configuration will be nearly Newtonian if its mass per unit proper length Λ is small and its dust particles orbit around the axis with a small velocity:

$$\Lambda = \lambda e^{-\psi_s} = \lambda y^\kappa \ll 1 , \tag{27a}$$

$$u = \frac{\alpha}{Re^{-\psi_s}} = \frac{\alpha}{r_0} y^{-1-\kappa} \ll 1 . \tag{27b}$$

If, in addition, the Newtonian potential difference between $r = r_0$ and $r = R$ is small,

$$\Lambda \ln(r_0/r) \ll 1 , \tag{27c}$$

then throughout the region $r \lesssim r_0$ the configuration's gravity can be approximated by Newtonian theory, and the relativistic equations (26) reduce to

$$\kappa = 2\lambda , \tag{27d}$$

$$\mathcal{C}_{\rm MSRF} = \lambda + 2\lambda^2 + \frac{\lambda\alpha^2}{2R^2} + \lambda^2 \ln\left(\frac{R}{r_0}\right) . \tag{27e}$$

Since, in this Newtonian situation, $\lambda \simeq \Lambda$ and $R \simeq \mathcal{R}$, expression (27e) for the C energy $\mathcal{C}_{\rm MSRF}(R)$ is the same as Eq. (3) for the Newtonian energy per unit length $\mathcal{C}_{\rm MS}(\mathcal{R})$.

A relativistic MSRF configuration will be in permanent, static equilibrium if and only if $A = d^2R/d\tau^2 = 0$, i.e., if and only if $\Lambda = \Lambda_{\rm eq}(u)$; cf. Eqs. (25f) and (8), Fig. 2, and the discussion in Sec. I D.

An alternative, equivalent criterion for equilibrium involves C energy: Choose an arbitrary radius r_0 and for concreteness adjust the scale of z [Eq. (15a)] so that proper length and Killing length coincide at r_0, i.e., so Eq. (26a) is satisfied. Then *among all MSRF configurations with fixed α and λ and with $R < r_0$, the ones that are in equilibrium are those that extremize $\mathcal{C}_{\rm MSRF}$ with respect to variations of R. Moreover, every one of these extrema is a minimum of $\mathcal{C}_{\rm MSRF}$*. (In Sec. IV we shall prove that for the situations of interest in this paper, there is precisely one minimum.)

These properties of equilibria can be proved as follows. Since $\mathcal{C}_{\rm MSRF} = \frac{1}{8}(1 - e^{-2\gamma_0})$ is a monotonically increasing function of γ_0, it suffices to prove these properties for γ_0 instead of $\mathcal{C}_{\rm MSRF}$. Consider the first-order change $d\gamma_0$ of γ_0 caused by a change dR of R. Equation (25b) implies that

$$d\gamma_0 = d\gamma_+ + 2\kappa \ln(r_0/R) d\kappa - \kappa^2 dR/R , \tag{28a}$$

where, by (25a) and (26a),

$$\ln(r_0/R) d\kappa = \kappa\, dR/R + d\psi_s , \tag{28b}$$

and, by (25d) and (25e), with $\Lambda = \lambda e^{-\psi_s}$ and $u = \alpha/Re^{-\psi_s}$,

$$d\gamma_+ = -2\kappa\, d\psi_s - 2\kappa u^2 dR/R . \tag{28c}$$

By combining Eqs. (28a)–(28c) we obtain

$$d\gamma_0 = \kappa(\kappa - 2u^2) dR/R , \tag{28d}$$

so γ_0 is extremized if and only if $\kappa = 2u^2$. By virtue of Eq. (25d) for κ, this is equivalent to $\Lambda = \Lambda_{\rm eq}(u)$. Thus, the equilibria are the MSRF configurations that extremize γ_0, as claimed.

To show that these equilibria actually minimize γ_0, we compute the second order change $d^2\gamma_0$ produced by dR when $\Lambda = \Lambda_{\rm eq}(u)$, i.e., when $\kappa = 2u^2$. Equation (28d) implies that

$$d^2\gamma_0 = \kappa \frac{dR}{R}(d\kappa - 4u\, du) . \tag{29a}$$

By combining with Eqs. (28b), (25d), (8), $\Lambda = \lambda e^{-\psi_s}$, and $u = \alpha/Re^{-\psi_s}$ and performing a series of manipulations, we bring this into the form

$$d^2\gamma_0 = 2u^2 \left(\frac{dR}{R}\right)^2 \frac{1 + 2u^2}{\ln(r_0/R)} \left[1 - [1 - 4u^2 \ln(r_0/R)] \frac{1 + 2u^2 \ln(r_0/R)(1 + 2u^2)}{1 + 2u^2 \ln(r_0/R)[1 + 4u^2 + u^2/(1 + u^2)]}\right] . \tag{29b}$$

The last fraction is obviously less than unity, and this implies that $d^2\gamma_0$ is positive and thus γ_0 and hence $\mathcal{C}_{\rm MSRF}$ is minimized by the equilibrium configurations. Q.E.D.

IV. QUALITATIVE FORM OF C ENERGY FOR MSRF CONFIGURATIONS

Of special interest for analyzing the dynamical evolution of a shell (Sec. I E) is the form of $\gamma_0(R)$ and thence $\mathcal{C}_{\rm MSRF}(R)$, when the outer radius r_0 is chosen arbitrarily large and the shell's initial configuration is MSRF with some specific initial values $\mathcal{R}_i$ of $\mathcal{R}$ and Λ_i of Λ. In this case the initial configuration has a mass per unit Killing length λ and a value κ_i of κ given by [cf. Eqs. (26a), (25a), (25d), (21), and $\Lambda = \lambda e^{-\psi_s}$, $\mathcal{R} = Re^{-\psi_s}$]

$$\lambda = \Lambda_i (r_0/\mathcal{R}_i)^{\kappa_i/(1+\kappa_i)} , \tag{30a}$$

$$\kappa_i = \frac{2\Lambda_i}{(1-4\Lambda_i\sqrt{1+\alpha^2/\mathcal{R}_i^2})\sqrt{1+\alpha^2/\mathcal{R}_i^2}} \,. \qquad (30b)$$

Since $r_0/\mathcal{R}_i$ is arbitrarily large and κ_i is positive, λ is also arbitrarily large. With λ being arbitrarily large and $\propto r_0^{\kappa_i/(1+\kappa_i)}$, it becomes fairly straightforward to deduce the qualitative form of $\gamma_0(R)$ and thence $\mathcal{C}_{\rm MSRF}(R)$:

These two functions of R are given explicitly by Eqs. (26). By examining these equations one can show [15] that the only places where $\gamma_0 \to \infty$ (for r_0 arbitrarily large) are at $R \to r_0$ and $R \to 4\lambda\alpha$. Since γ_0 is always positive and the only extrema of $\gamma_0(R)$ are minima (cf. Sec. III), this implies that as R varies from $R_{\rm abs\ min} = 4\lambda\alpha$ to r_0, $\gamma_0(R)$ decreases from ∞ to a single, unique minimum at some $R = R_{\rm eq}$, and then increases to ∞ at $R = r_0$. Correspondingly, $\mathcal{C}_{\rm MSRF}(R)$ [Eq. (26b)] decreases monotonically from its maximum allowed value of 1/8 at $R_{\rm abs\ min} = 4\lambda\alpha$ to a minimum at $R_{\rm eq}$ and then increases monotonically back to 1/8 at $R = r_0$. This is the qualitative behavior that we stated and used in discussing the dynamics of a shell in Sec. I E.

V. C ENERGY OF DYNAMICAL CONFIGURATIONS

We are now ready to prove that *the C energy of a dynamical configuration is always greater than that of a MSRF configuration that has the same α, λ, R, and $\psi_0 \equiv \psi(r_0) = 0$, but different V, $\psi(r)$, and $\psi_{,t}(r)$* [Eq. (9)]. Since $\mathcal{C}_0$ is a monotonically increasing function of γ_0, it suffices to prove that γ_0 has this property, or equally well that $e^{-\gamma_0}$ is always smaller for a dynamical configuration than for the corresponding MSRF one. In our proof we shall denote $\dot\psi \equiv \psi_{,t}$ and $\psi' \equiv \psi_{,r}$.

Proof. We proceed in two steps. *First*, we hold the dynamical configuration's $\psi(r)$ and $\dot\psi(r)$ fixed and vary only its shell velocity V. From the junction condition (24b), Eq. (20b), and $\Lambda = \lambda e^{-\psi_s}$, it follows that

$$e^{-\gamma_+} \le e^{-\gamma_-} - 4\Lambda\sqrt{1+u^2}\,, \qquad (31a)$$

with the equality holding if and only if $V = 0$. Combining this with the field equation (16b) and boundary condition (17), we learn that, when V is varied, $e^{-\gamma_0}$ takes on an absolute maximum value at $V = 0$. The value of that maximum is

$$e^{-\gamma_0}|_{V=0} = I \equiv J(K - 4\Lambda\sqrt{1+u^2})\,, \qquad (31b)$$

where

$$K \equiv e^{-\gamma_-} = \exp\left[-\int_0^R r(\dot\psi^2 + \psi'^2)\,dr\right]\,, \qquad (31c)$$

$$J \equiv \exp\left[-\int_R^{r_0} r(\dot\psi^2 + \psi'^2)\,dr\right]\,. \qquad (31d)$$

In our *second* step, we hold $V = 0$ and ask how $e^{-\gamma_0} = I$ changes as we vary $\psi(r)$ and $\dot\psi(r)$. It is straightforward to compute the first order change δI of I around any configuration [any $\psi(r)$ and $\dot\psi(r)$], with α, λ, R, $V = 0$, r_0, $\psi_0 \equiv \psi(r_0) = 0$, and $\dot\psi_0 \equiv \dot\psi(r_0) = 0$ held fixed, and with $\Lambda = \lambda e^{-\psi_s}$, $u = \alpha e^{\psi_s}/R$, and the junction condition (24a) imposed. The result is

$$\delta I = -2I\int_R^{r_0} r\dot\psi\delta\dot\psi\,dr - 2JK\int_0^R r\dot\psi\delta\dot\psi\,dr + 2I\int_R^{r_0}(r\psi')'\delta\psi\,dr + 2JK\int_0^R (r\psi')'\delta\psi\,dr\,, \qquad (31e)$$

which implies that I is extremal ($\delta I = 0$) if and only if $\dot\psi = 0$ and $(r\psi')' = 0$, i.e., if and only if the configuration is MSRF. In fact, the extremum of I is a maximum, as one can show by computing its second variation [using in the computation the fact that the junction conditions (24a) and (24b) must be satisfied by the perturbed configuration as well as the MSRF one]:

$$\delta^2 I = -2I\int_R^{r_0} r(\delta\dot\psi^2 + \delta\psi'^2)\,dr - 2JK\int_0^R r(\delta\dot\psi^2 + \delta\psi'^2)\,dr - 4J(\delta\psi_s)^2\frac{\Lambda}{(1+u^2)^{3/2}}\left[1 + 2u^2 + \frac{4\Lambda\sqrt{1+u^2}}{1-4\Lambda\sqrt{1+u^2}}\right] < 0\,. \qquad (31f)$$

To recapitulate, in our first step we found that, when V is varied with $\dot\psi(r)$ and $\psi(r)$ held fixed at any values one wishes, then $e^{-\gamma_0}$ reaches an absolute maximum, $e^{-\gamma_0} = I$, at $V = 0$. Then in the second step we found that when V is held equal to zero and $\dot\psi(r)$ and $\psi(r)$ are varied, $e^{-\gamma_0} = I$ reaches an absolute maximum when $\dot\psi(r)$ and $\psi(r)$ assume their MSRF values. Therefore, among all configurations with fixed α, λ, R, and $\psi_0 = 0$, the MSRF has the absolute maximum value of $e^{-\gamma_0}$ and the absolute minimum C energy $\mathcal{C}_0$. Q.E.D.

This extremal property of the C energy, together with the properties of $\mathcal{C}_{\rm MSRF}(R)$ derived in Secs. III and IV, are all that we needed in Sec. I E to infer the qualitative, dynamical evolution of the shell—with one exception: We also needed the fact that the gravitational waves emitted by the shell's oscillations carry away C energy. This we prove in the next section.

VI. C ENERGY OUTFLOW

The rate of change of the C energy $\mathcal{C} = \frac{1}{8}(1 - e^{-\gamma})$ inside a radius r is given by

$$\mathcal{C}_{,t} = \frac{1}{8}e^{-\gamma}\gamma_{,t} = \frac{1}{4}e^{-\gamma}\psi_{,t}\psi_{,r} \;, \tag{32}$$

cf. Eq. (16c). We shall now show that at any radius $r \gg R$, $\psi_{,t}\psi_{,r}$ is negative and thus $\mathcal{C}_{,t} < 0$, which means that the waves carry away C energy.

The general outgoing-wave solution of the wave equation (16a) is

$$\psi = \mathrm{Re}\int_0^\infty A(\omega)e^{-i\omega t}H_0^{(1)}(\omega r)\,d\omega \;, \tag{33a}$$

where Re denotes the real part and $H_m^{(1)}(x)$ is the Hankel function of the first kind. Therefore,

$$\psi_{,t} = \mathrm{Re}\int_0^\infty -i\omega A(\omega)e^{-i\omega t}H_0^{(1)}(\omega r)\,d\omega \tag{33b}$$

and

$$\psi_{,r} = \mathrm{Re}\int_0^\infty -\omega A(\omega)e^{-i\omega t}H_1^{(1)}(\omega r)\,d\omega \;. \tag{33c}$$

Using the limit $H_m^{(1)}(x) \underset{x\gg 1}{\sim} \sqrt{2/\pi x}\exp[i(x - m\pi/2 - \pi/4)]$, we see that

$$H_1^{(1)}(\omega r) \underset{\omega r \gg 1}{\sim} -iH_0^{(1)}(\omega r) \;, \tag{33d}$$

which implies that the contributions from all frequencies $\omega \gg 1/r$ satisfy $\psi_{,t} = -\psi_{,r}$ and thence $\psi_{,t}\psi_{,r} < 0$ as was to be proved. But what about contributions from $\omega \lesssim 1/r$? Because ψ is always finite and $\int_0^{x_0} x^n H_0^{(1)}(x)dx$ converges only for $n > -1$, it is always the case that as $\omega \to 0$, $A(\omega) \sim \omega^n$ with $n > -1$. This implies that the low-frequency, $\omega \lesssim 1/r$, contributions to $\psi_{,t}$ and $\psi_{,r}$ are

$$\psi_{,t}(\omega \lesssim 1/r) \sim \psi_{,r}(\omega \lesssim 1/r) \sim \frac{1}{r^{2+n}} \;, \tag{34}$$

which are negligible compared to the $O(1/\sqrt{r})$ contributions from $\omega \gg 1/r$ when r is sufficiently large. Thus, for large r the waves necessarily carry C energy outward through radius r ($\mathcal{C}_{,t} \equiv \frac{1}{4}e^{-\gamma}\psi_{,t}\psi_{,r} < 0$). Q.E.D.

VII. CONCLUSIONS

In Secs. III–VI we have derived all the properties of the C energy that were needed, in Sec. I, to infer the dynamical evolution of a thin, cylindrical shell of counter-rotating dust: By giving the dust particles arbitrarily small amounts of angular momentum per unit mass, we guarantee that centrifugal forces will convert the shell's collapse into a bounce, thereby preventing formation of a singularity. After its bounce, the shell will oscillate radially, and then as gravitational waves carry away C energy, it will settle down into a static, equilibrium state.

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APPENDIX A: RADIAL NONCLOSURE OF SPACE

In this appendix we show that, if the space around an initial, MSRF configuration is not closed radially, then throughout the shell's spacetime to the future of the initial configuration we can introduce coordinates (t, r, z, ϕ) in which the line element takes the canonical form (1) with $0 \le r < \infty$.

The proof begins by introducing coordinates $(\bar{t}, \bar{r}, z, \phi)$ in which the line element takes the form (12). Such coordinates are permitted throughout any cylindrically symmetric spacetime with vanishing net momentum density along the Killing directions [10, 11].

Following Thorne [11], we define the *character* of spacetime at any event to be $D^{(+)}$ if $\nabla\beta$ is spacelike and points away from the symmetry axis, $D^{(-)}$ if $\nabla\beta$ is spacelike and points toward the symmetry axis, $D^{(0\uparrow)}$ if $\nabla\beta$ is timelike and points toward the future, and $D^{(0\downarrow)}$ if $\nabla\beta$ is timelike and points toward the past. [Recall that $\beta = |\partial/\partial z||\partial/\partial\phi|$; cf. Eq. (12).] The vacuum field equation $\beta_{,\bar{t}\bar{t}} - \beta_{,\bar{r}\bar{r}} = 0$ [Eq. (13)] implies that spacetime can change character only on radial null surfaces, which Thorne [11] calls *critical surfaces*, or across the nonvacuum dust shell.

The nature of the character change across any critical surface is constrained by the geometric optics focusing equation for radial null geodesics (Eq. (22.37) of MTW [4]). This equation implies that β can never have a minimum along any ingoing or outgoing radial null geodesic; this, in turn, implies that the only kinds of (vacuum) critical surfaces that can ever exist are these: An outgoing null surface with spacetime character $D^{(-)}$ in the past and $D^{(0\downarrow)}$ in the future or with $D^{(0\uparrow)}$ in the past and $D^{(+)}$ in the future, and an ingoing null surface with character $D^{(+)}$ in the past and $D^{(0\downarrow)}$ in the future or with $D^{(0\uparrow)}$ in the past and $D^{(-)}$ in the future.

The nature of any character change across the dust shell is constrained by the junction condition for the $\phi\phi$ component of the extrinsic curvature [generalization of Eq. (24b) to the case where spacetime is not necessarily $D^{(+)}$ on both sides of the shell]. This junction condition says

$$\beta_{+,n} - \beta_{-,n} = -4\lambda\sqrt{1+u^2} < 0 \;. \tag{A1}$$

Since $\beta_{,\tau}$ is continuous across the shell and $\nabla\beta = \beta_{,\tau}\mathbf{e}_\tau + \beta_{,n}\mathbf{e}_n$ (where the notation is that of Sec. II B, generalized to the case where the spacetime character is not necessarily $D^{(+)}$ everywhere), Eq. (A1) implies that, as one moves from the shell's interior to its exterior (its "−" side to its "+" side), the only allowed character changes are $D^{(+)}$ to any other character, and $D^{(0\uparrow)}$ or $D^{(0\downarrow)}$ to $D^{(-)}$.

By hypothesis, there is an initial MSRF configuration in which space is not radially closed. The fact that this configuration is momentarily static implies that nowhere on its spacelike hypersurface can spacetime have character $D^{(0\uparrow)}$ or $D^{(0\downarrow)}$; radial nonclosure means the character must be $D^{(+)}$ far outside the shell; and smoothness of spacetime near the symmetry axis implies character $D^{(+)}$ there. These constraints on character, together with the constraints on character change listed in the preceding two paragraphs, imply that on the initial hypersurface the character is everywhere $D^{(+)}$.

As the spacetime evolves forward off the initial hypersurface, the only way any change of character could occur would be if a future-directed, ingoing or outgoing critical surface were to be created at some moment at the shell's location. By examining various hypothetical character changes across such a critical surface and across the shell to its future, one discovers that there are no patterns of character change that satisfy the above constraints. Therefore, the spacetime character must remain $D^{(+)}$ throughout the future of the initial hypersurface. Q.E.D.

APPENDIX B: RADIAL NONCLOSURE FOR MSRF CONFIGURATIONS

Inside any MSRF configuration, spacetime is flat and, therefore, in the notation of Sec. II B and Appendix A, $\beta_{-,n} = e^{\psi_s}$. This, together with $\Lambda = \lambda e^{-\psi_s}$ and the junction condition (A1), implies that

$$e^{-\psi_s}\beta_{+,n} = 1 - 4\Lambda\sqrt{1+u^2}\,. \tag{B1}$$

In order for space to be radially nonclosed outside the shell, the character must be $D^{(+)}$ there rather than $D^{(-)}$ (these are the only possibilities for a MSRF configuration, cf. Appendix A); this corresponds to the requirement that $\beta_{+,n}$ must be positive and not negative; this, by virtue of Eq. (B1), corresponds to $\Lambda\sqrt{1+u^2} < 1/4$. Thus, for a MSRF configuration space will be radially nonclosed (character $D^{(+)}$ everywhere) if $\Lambda\sqrt{1+u^2} < 1/4$, and radially closed (character $D^{(-)}$ outside the shell) if $\Lambda\sqrt{1+u^2} > 1/4$.

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