PHYSICAL REVIEW D VOLUME 46, NUMBER 6 15 SEPTEMBER 1992

Stochastic inflation: Quantum phase-space approach

Salman Habib*

Department of Physics, The University of British Columbia, 6224 Agriculture Road, Vancouver, British Columbia, Canada V6T 1Z1 and T-6, Theoretical Division,† Los Alamos National Laboratory, Los Alamos, New Mexico 87545

(Received 13 April 1992)

In this paper a quantum-mechanical phase-space picture is constructed for coarse-grained free quantum fields in an inflationary universe. The appropriate stochastic quantum Liouville equation is derived. Explicit solutions for the phase-space quantum distribution function are found for the cases of power-law and exponential expansions. The expectation values of dynamical variables with respect to these solutions are compared to the corresponding cutoff regularized field-theoretic results (we do not restrict ourselves only to $\langle \Phi^2 \rangle$). Fair agreement is found provided the coarse-graining scale is kept within certain limits. By focusing on the full phase-space distribution function rather than a reduced distribution it is shown that the thermodynamic interpretation of the stochastic formalism faces several difficulties (e.g., there is no fluctuation-dissipation theorem). The coarse graining does not guarantee an automatic classical limit as quantum correlations turn out to be crucial in order to get results consistent with standard quantum field theory. Therefore, the method does *not* by itself constitute an explanation of the quantum to classical transition in the early Universe. In particular, we argue that the stochastic equations do not lead to decoherence.

PACS number(s): 98.80.Dr, 03.70.+k, 05.40.+j

I. INTRODUCTION

The paradigm of stochastic inflation, first introduced explicitly by Starobinsky [1], has recently become popular as a means of investigating various features of inflation. Some studies using this formalism are those of density perturbations from inflation [2], the very large scale structure of the Universe [3], "eternal inflation" [4], power-law inflation [5,6], and speculations regarding the relationship of this formalism to quantum cosmology [7] (this list is by no means exhaustive).

It must be admitted, however, that there is still no iron-clad justification for the systematics of the method nor, for that matter, a clear-cut interpretational scheme. The claim at issue is that the infrared behavior of massless or small mass quantized scalar fields in an inflationary universe can be described in terms of a real time classical random process. The source of the noise is taken to be large scale quantum fluctuations which are continuously generated in an inflationary universe by redshifting of the ultraviolet sector. A key question here is: can these quantum fluctuations be treated as being classical?

These fundamental issues have been considered previously for free fields [8]; however, the situation for interacting fields is not clear, and it is not obvious how far, if it all, any of the present "derivations" are correct [9]. In this paper we leave aside for the moment the problem of interacting fields and attempt a further clarification of the issues addressed in Ref. [8]. To do so we will derive a phase-space quantum master equation for a quantum distribution function (the Wigner function) and study its solutions. In our stochastic approach averages with respect to this distribution function are supposed to reproduce quantum field theoretic expectation values. A study of the solution itself is supposed to enable one to judge the "classicality" of each physical situation. The new method is distinct from the conventional approach (where one takes as given a *classical* Langevin equation), and enables the inclusion of crucial quantum correlations that have been unjustifiably neglected in the past.

A subtle and important aspect of the Wigner distribution function is the fact that essential quantum features are hidden in quantum correlation "cross terms" that disappear when one integrates over any one of the phase-space variables to produce a one-variable (necessarily positive definite) distribution function. Such a reduced distribution is *essentially useless* as a diagnostic tool for studying quantum correlations in phase space (as will be seen forcefully in this paper). Unfortunately it is on precisely such objects that attention has been focused till now. Here, with the full Wigner function at hand we will be able to go much further with regard to clarifying the physics behind the stochastic approach. It has been noticed previously [8] that the reduced distribution for a massive scalar field in de Sitter space has, at late times, an intriguing thermodynamic interpretation: it corresponds to a Boltzmann distribution at the Gibbons-Hawking temperature. However, the full distribution found in this paper does not have a thermal form even though the reduced distribution is the same. This can be traced directly to the fact that quantum correlations have not been neglected; indeed they are every bit as important as the remaining contributions. We will go more deeply into this question in Sec. V.

It has long been appreciated that the stochastic approach probes the infrared sector of the relevant field theory. The length scale is set by a certain parameter ϵ

*Electronic address: habib@eagle.lanl.gov

†Present address.

which is usually taken to be small (i.e., attention is confined to length scales much larger than the time-dependent horizon length). Assuming inflation began at a finite time in the past, one cannot take ϵ to be arbitrarily small *independent* of the time scale of interest: a small ϵ is consistent only with the "late times." It is also well known that the quantum theory of free fields in an inflationary spacetime has a nontrivial infrared sector and that a simple (though certainly not rigorous) way to calculate expectation values of field variables is to set an upper momentum cutoff at the Hubble scale (corresponding to $\epsilon \simeq 1$). The stochastic picture conflicts with these field-theoretic results unless ϵ is small; this is due to the fact that in the stochastic approach one focuses essentially on the zero mode and attempts to include inhomogeneities only through a noise term. Though this approximation appears to be quite drastic, we will show that the stochastic calculations even for $\epsilon \sim 1$ are never too far from the naive field-theoretic results. (Unlike some previous work, our formalism does not restrict the value of ϵ.)

Previous work in stochastic inflation has concentrated mainly on the quantity $\langle \Phi^2 \rangle$. In this paper we extend the method to compute $\langle \Phi\pi + \pi\Phi \rangle$ and $\langle \pi^2 \rangle$. Such quantum averages would be needed if one wished to compute the expectation value of the stress tensor. (While for a small mass field, and an exponential inflation, the dominant contribution to to $\langle T_{\mu\nu} \rangle$ is from terms $\propto \langle \Phi^2 \rangle$, it is important to check if the calculation of the other terms is trustworthy.) Earlier approaches to stochastic inflation implement approximations which lead to incorrect values for these quantities. Indeed, precisely these approximations formed the basis of some arguments about stochastic inflation leading to an automatic classical limit for the quantum field theory. We will argue against any such result in Sec. V.

An interesting (and somewhat uncomfortable) feature of the quantum phase-space distribution found in this paper is that, in some cases, it depends quite strongly on ϵ and is indeed singular in the limit $\epsilon \to 0$. Therefore, while it is true that $\langle \Phi^2 \rangle$ (as well as the reduced distribution for the field alone) may be independent of ϵ for small values of ϵ, this parameter does not drop out of the physics. A finite value of ϵ is necessary for the distribution function to exist; this is true even for a massive field in an exponentially expanding Universe where to leading order all quadratic phase-space expectation values are independent of ϵ.

An important issue that seems to have received insufficient attention in the stochastic inflation literature is a discussion of the role of initial conditions. Massless theories in inflationary spacetimes suffer from infrared divergences. Typically these divergences are "fixed" by assuming that inflation began a finite time in the past and thereby modifying the infrared structure of the quantum state of the field. Expectation values then have two contributions: one each from the preinflationary and inflationary sectors. One can show that the preinflationary contribution falls rapidly with time and can always be neglected compared to the inflationary one (see the Appendix). In the stochastic paradigm there are also two contributions to expectation values: a systematic piece arising from the dynamical evolution of an initial condition and a stochastic piece due entirely to the noise source. We will show that, while at late times, and for arbitrary initial conditions, the second piece always dominates the first, this is *not* true at early times. While the matching of the quantum states in the preinflationary and inflationary regimes at the onset of inflation can also provide an initial condition for the stochastic method, the time dependences of the systematic contribution in the stochastic method do not always match the time dependences of the preinflationary contribution in the field-theoretic calculation. This fact coupled with the small ϵ restriction might limit the application of stochastic techniques in accurately studying the onset of inflation. The use of the method for studying phase transitions in the early Universe should also be approached with some caution [10]. (Of course all this is not a serious problem if one is only interested in late-time results.)

The attempt in this paper is to push the formalism of stochastic inflation as hard as possible in simple examples: we find that some of the appealing original results no longer appear as compelling as at first sight. However, it is still a remarkable fact that a simple stochastic model suffices to (almost) correctly calculate field-theoretic expectation values and further that the essentially nonstationary phase-space distribution nevertheless yields a thermal (or "random walk") distribution for the reduced distribution function. Whether this has a deep significance is unfortunately not clear.

The organization of the paper is as follows: In Sec. II we derive the appropriate stochastic quantum Liouville equation for the coarse-grained field using the phase-space formulation of quantum mechanics and obtain the general solution. In Sec. III we apply these results to the case of an inflationary expansion; power-law expansions are dealt with in Sec. IV. The existence of classical stochastic interpretations is discussed in Sec. V via a study of the solutions of the quantum Liouville equation. We conclude with Sec. VI where the results are reviewed and future directions for research are suggested. The quantum field theoretic derivations of the results obtained via the stochastic approach are given in an Appendix.

II. THE STOCHASTIC QUANTUM LIOUVILLE EQUATION

In this section we will set up a formalism to study the evolution of coarse-grained free scalar fields in a spatially flat inflationary Friedmann-Robertson-Walker (FRW) universe. We will work under the "test field" assumption; i.e., the contribution to the stress tensor from the field is taken to be small compared to that of the matter driving the expansion. All of our results will therefore not be applicable to an inflaton field but some may indeed be extended to that case.

The line element for the spacetime is

$$ds^2 = -dt^2 + a(t)^2 d\mathbf{x}\cdot d\mathbf{x} \tag{1}$$

$$= S(\eta)^2(-d\eta^2 + d\mathbf{x}\cdot d\mathbf{x}) \ , \tag{2}$$

where, in terms of the cosmic time t, the conformal time

$$\eta = \int^{t} \frac{dt'}{a(t')} . \tag{3}$$

A massive minimally coupled scalar field has the Lagrangian

$$\mathcal{L}(\Phi,\Phi_{,\mu}) = -\tfrac{1}{2}\sqrt{-g}\,(g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu} + m^2\Phi^2) , \tag{4}$$

which, with the metric choice (2), reduces to

$$\mathcal{L}(\Phi,\Phi_{,\mu}) = -\tfrac{1}{2}(-S^2\dot{\Phi}^2 + S^2\Phi_{,i}\Phi_{,i} + S^4 m^2\Phi^2) . \tag{5}$$

The overdot represents differentiation with respect to the conformal time. In terms of the "conformal field," defined via the time-dependent canonical transformation,

$$\chi \equiv S\Phi , \tag{6}$$

and modulo an integration by parts, the Lagrangian (5) becomes

$$\mathcal{L}(\chi,\chi_{,\mu}) = -\frac{1}{2}\left[-\dot{\chi}^2 + \chi_{,i}\chi_{,i} + \left[S^2 m^2 - \frac{\ddot{S}}{S} \right] \chi^2 \right] . \tag{7}$$

One advantage of working with the conformal field and the Lagrangian (7) is that the equation of motion for the field

$$\ddot{\chi} - \nabla^2\chi + \left[S^2 m^2 - \frac{\ddot{S}}{S} \right] \chi = 0 \tag{8}$$

does not have the first derivative in time "Hubble damping" term found in the equation of motion for the original field

$$\ddot{\Phi} + 2\frac{\dot{S}}{S}\dot{\Phi} - \nabla^2\Phi + S^2 m^2\Phi = 0 , \tag{9}$$

and that the canonical momentum

$$\pi_\chi \equiv \frac{\partial\mathcal{L}}{\partial\dot{\chi}} = \dot{\chi} \tag{10}$$

is of the usual flat-space form.

The Hamiltonian corresponding to (7) is

$$H(\chi,\pi_\chi) = \frac{1}{2}\int d\mathbf{x} \left[\pi_\chi^2 + \chi_{,i}\chi_{,i} + \left[S^2 m^2 - \frac{\ddot{S}}{S} \right] \chi^2 \right] \tag{11}$$

which is that for a free field in flat spacetime with a time-dependent mass. The Hamiltonian equations of motion are

$$\dot{\chi} = \frac{\delta H}{\delta \pi_\chi} = \pi_\chi , \tag{12}$$

$$\dot{\pi}_\chi = -\frac{\delta H}{\delta\chi} = \nabla^2\chi - \left[S^2 m^2 - \frac{\ddot{S}}{S} \right] . \tag{13}$$

The form of the Hamiltonian (11), though "canonical," is hardly unique; if we had worked with some other choice of time and field it would have been "natural" to consider a different description in terms of a different Hamiltonian. At the level of a classical treatment, and even at the level of quantum dynamics, this difference is largely irrelevant. However, there is an aspect of the quantum treatment where such a difference does indeed matter: The two Hamiltonian descriptions will, upon canonical quantization, lead to inequivalent descriptions in terms of different vacua. Furthermore, there are well known difficulties if one chooses to select the time-dependent ground state of the Hamiltonian as the "instantaneous diagonalization" vacuum [11]. These problems will be of no concern to us as in our case the choice of quantum state will be an independently defined adiabatic vacuum (details will be given later). (We note in passing that for spatially flat FRW models Weiss has shown [12] that with the specific Hamiltonian (11) the instantaneous diagonalization approach can be made consistent with a "mode quantization" for certain special choices of the latter.)

In our case there is a conceptually important consequence of (11) being the chosen Hamiltonian. In the stochastic-inflation literature there appears to be a tendency of interpreting the Hubble damping term in (9) as being of a truly dissipative nature. This runs the risk of repeating an old error in quantum mechanics: the confusion of a time-dependent mass with true damping [13]. Working with (11) and the associated equation of motion (8) manifestly eliminates the possibility of such misinterpretations.

The scalar field is now quantized in the standard manner [14]; first we introduce the modes

$$\chi_{\mathbf{k}}(\mathbf{x},\eta) = S(\eta)\Phi_{\mathbf{k}}(\mathbf{x},\eta) \tag{14}$$

$$= \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}}\chi_k(\eta) , \tag{15}$$

where $\chi_k(\eta)$ is a solution of

$$\ddot{\chi}_k + \omega_k^2\chi_k = 0 \tag{16}$$

with ω_k the oscillator "frequency," defined via

$$\omega_k^2 \equiv k^2 + \left[S^2 m^2 - \frac{\ddot{S}}{S} \right] . \tag{17}$$

The annihilation and creation operators with respect to these modes, which satisfy the commutation relations

$$[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = \delta(\mathbf{k}-\mathbf{k}') , \tag{18}$$

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = 0 \tag{19}$$

are then used to build the field operator

$$\hat{\chi}(\mathbf{x},\eta) = S(\eta)\hat{\Phi}(\mathbf{x},\eta) \tag{20}$$

$$= \int d\mathbf{k}[\hat{a}_{\mathbf{k}}\chi_{\mathbf{k}}(\mathbf{x},\eta) + \hat{a}^\dagger_{\mathbf{k}}\chi^*_{\mathbf{k}}(\mathbf{x},\eta)] . \tag{21}$$

As is well known the annihilation and creation operators are not uniquely specified by (18) and (19). Further restrictions are needed to fix these operators and thereby to uniquely specify the "vacuum" state annihilated by $\hat{a}_{\mathbf{k}}$. We will turn to these issues shortly.

A seemingly generic feature of inflationary spacetimes is the "destabilization" of massless scalar fields [15,16] due in part to infrared divergences [17]. To render the quantum state infrared finite one assumes a benign Robertson-Walker expansion in which there is no infrared divergent vacuum prior to inflation (a radiation-dominated universe, for example). The quantum state in the inflationary phase is matched to an infrared finite quantum state (e.g., the conformal vacuum) at the time when inflation takes over from the previous epoch. It is then possible to show that the new state is always free from infrared divergences [17]. The key result, however, is that the expectation value $\langle \Phi^2 \rangle$ in the infrared finite (and ultraviolet regulated) state starts to grow at the onset of inflation; for power-law inflation $\langle \Phi^2 \rangle$ grows to an asymptotic constant value, whereas in the case of an exponential expansion it grows linearly with cosmic time without any upper limit. These otherwise puzzling results have a natural interpretation within the framework of stochastic inflation [8].

Crudely speaking, "destabilization" occurs when, with $m=0$, the ω_k^2 term in (16) goes negative with the passage of time, at ever higher values of k. Modes at long wavelengths behave as amplitudes for upside-down harmonic oscillators (with time-dependent "frequencies") and due to the inflationary expansion there is a continuous flow of short-wavelength modes into this unstable infrared sector. Therefore we focus attention on the long-wavelength modes, i.e., those with $k^2 < \ddot{S}/S$, by defining the coarse-grained quantum field

$$\hat{\chi}_L(\mathbf{x},\eta) \equiv \int d\mathbf{k}\,\theta(k_S - k)[\hat{a}_{\mathbf{k}}\chi_{\mathbf{k}}(\mathbf{x},\eta) + \hat{a}^{\dagger}_{\mathbf{k}}\chi^*_{\mathbf{k}}(\mathbf{x},\eta)] \quad (22)$$

and the corresponding coarse-grained momentum

$$\hat{\pi}_L(\mathbf{x},\eta) \equiv \int d\mathbf{k}\,\theta(k_S - k)[\hat{a}_{\mathbf{k}}\dot{\chi}_{\mathbf{k}}(\mathbf{x},\eta) + \hat{a}^{\dagger}_{\mathbf{k}}\dot{\chi}^*_{\mathbf{k}}(\mathbf{x},\eta)] \; . \quad (23)$$

For the moment we will leave the upper cutoff k_S unspecified beyond the fact that it is set by $\ddot{S}/S$ (however, it is important to remember that this cutoff is time dependent). The corresponding short-wavelength fields $\hat{\chi}_S$ and $\hat{\pi}_S$ are defined by

$$\hat{\chi} = \hat{\chi}_L + \hat{\chi}_S \; , \quad (24)$$

$$\hat{\pi}_\chi = \hat{\pi}_L + \hat{\pi}_S \; . \quad (25)$$

The Heisenberg operators $\hat{\chi}$ and $\hat{\pi}_\chi$ satisfy the classical Hamiltonian equations of motion. Substituting (24) and (25) in (12) and (13) we find

$$\dot{\hat{\chi}}_L = \hat{\pi}_L + \hat{F}^c_1 \; , \quad (26)$$

$$\dot{\hat{\pi}}_L = \nabla^2 \hat{\chi}_L - \left[S^2 m^2 - \frac{\ddot{S}}{S} \right] \hat{\chi}_L + \hat{F}^c_2 \; , \quad (27)$$

where

$$\hat{F}^c_1(\mathbf{x},\eta) \equiv \dot{k}_S \int d\mathbf{k}\,\delta(k - k_S)[\hat{a}_{\mathbf{k}}\chi_{\mathbf{k}}(\mathbf{x},\eta) + \hat{a}^{\dagger}_{\mathbf{k}}\chi^*_{\mathbf{k}}(\mathbf{x},\eta)] \; , \quad (28)$$

$$\hat{F}^c_2(\mathbf{x},\eta) \equiv \dot{k}_S \int d\mathbf{k}\,\delta(k - k_S)[\hat{a}_{\mathbf{k}}\dot{\chi}_{\mathbf{k}}(\mathbf{x},\eta) + \hat{a}^{\dagger}_{\mathbf{k}}\dot{\chi}^*_{\mathbf{k}}(\mathbf{x},\eta)] \; . \quad (29)$$

The new terms $\hat{F}^c_1$ and $\hat{F}^c_2$ arise simply because k_S is time dependent. These terms represent the inflow of short-wavelength modes into the infrared "condensate." (It is important to note that this contribution exists even for free fields.)

In order to proceed further we have to decide which quantum state the field is in during the inflationary phase. It is known that the adiabatic vacuum suffers from infrared divergences; to produce states free of such divergences one usually modifies the long-wavelength mode structure of the quantum state (i.e., long compared to the time dependent horizon length at the onset of inflation) but leaves the short-wavelength structure the same. This implies that the quantum state for computing expectation values of $\hat{F}^c_1$ and $\hat{F}^c_2$, and of various powers of these operators, is the adiabatic vacuum. Other choices are possible when describing different physical situations, for example, thermal states have been considered in Ref. [6] and more general vacuum states in Ref. [8].

Eventually we will deal specifically with an exponential expansion and with power-law inflation (i.e., where the radius of the Universe goes as a power, greater than 1, of the cosmic time). For such a Robertson-Walker universe we will assume the quantum state for the short-wavelength modes to be the adiabatic vacuum (which reduces to the Bunch-Davies vacuum [18] for de Sitter space). For the moment, though, all that is relevant is that the chosen state by annihilated by the operator $\hat{a}_{\mathbf{k}}$ of (21), so that

$$\langle \hat{F}^c_1(\mathbf{x},\eta) \rangle = 0 \; , \quad (30)$$

$$\langle \hat{F}^c_2(\mathbf{x},\eta) \rangle = 0 \; . \quad (31)$$

It is also straightforward to compute that

$$\langle \hat{F}^c_i(\mathbf{x}_1,\eta_1)\hat{F}^c_j(\mathbf{x}_2,\eta_2) \rangle = 2B_{ij}(\mathbf{x}_1,\mathbf{x}_2,\eta_1)\delta(\eta_1 - \eta_2) \; , \quad (32)$$

where, with $R \equiv |\mathbf{x}_1 - \mathbf{x}_2|$,

$$B_{11}(\mathbf{x}_1,\mathbf{x}_2,\eta_1) = \frac{1}{4\pi^2} k_S^2 |\dot{k}_S| \frac{\sin k_S R}{k_S R} |\chi_{k_S}(\eta_1)|^2 \; , \quad (33)$$

$$B_{12}(\mathbf{x}_1,\mathbf{x}_2,\eta_1) = \frac{1}{4\pi^2} k_S^2 |\dot{k}_S| \frac{\sin k_S R}{k_S R} \chi_{k_S}(\eta_1)\dot{\chi}^*_{k_S}(\eta_1) \; , \quad (34)$$

$$B_{21}(\mathbf{x}_1,\mathbf{x}_2,\eta_1) = \frac{1}{4\pi^2} k_S^2 |\dot{k}_S| \frac{\sin k_S R}{k_S R} \dot{\chi}_{k_S}(\eta_1)\chi^*_{k_S}(\eta_1) \; , \quad (35)$$

$$B_{22}(\mathbf{x}_1,\mathbf{x}_2,\eta_1) = \frac{1}{4\pi^2} k_S^2 |\dot{k}_S| \frac{\sin k_S R}{k_S R} |\dot{\chi}_{k_S}(\eta_1)|^2 \; . \quad (36)$$

It is at this point that a stochastic interpretation suggests itself. The quantum expectation value may be regarded as an averaging bracket for the white (albeit nonstationary) "noise" operators $\hat{F}^c_1$ and $\hat{F}^c_2$. Since we are dealing with a free theory it is trivial to verify that the higher moments of these operators are those appropriate for Gaussian noise. The fact that the noise is white stems from the theta function cutoff in momentum space. Other cutoffs are certainly acceptable; however, they will lead to the noises being colored and unnecessarily complicate the derivation of the phase-space picture. We emphasize that physical results do not depend strongly on this choice.

The "diffusion matrix" B_{ij} has two curious features: it is complex (albeit Hermitian) and singular. We will show later that as far as the stochastic quantum Liouville equation is concerned what is really relevant is $B_{ij}+B_{ji}$, which not only is necessarily real but also has a nonzero determinant. The complex nature of B_{ij} is essential for $B_{ij}+B_{ji}$ to be nonsingular. It is important to be cautious when implementing approximations for the diffusion matrix and not to prematurely throw out the essential imaginary pieces. Finally, the fact that B_{12} and B_{21} are complex implies that these cross correlations cannot be understood on a purely classical basis.

According to the conventional stochastic interpretation we should view (26) and (27) as Langevin equations for the *classical* stochastic variables χ_L and π_L with F_1^c and F_2^c viewed as *classical* noises [19]. However, it is not at all obvious why this should be true. The Langevin equations (26) and (27) are *operator* equations and we must have further information about the quantum state of the system before any classical interpretation can be accepted. Furthermore, the noise operators do not commute:

$$[\hat{F}_1^c(\mathbf{x}_1,\eta_1),\hat{F}_2^c(\mathbf{x}_2,\eta_2)]=\frac{i}{2\pi^2}k_S^2|\dot{k}_S|\frac{\sin k_S R}{k_S R}\delta(\eta_1-\eta_2) . \tag{37}$$

In principle the noises are certainly not classical: we will go on to show that neglecting the quantum correlations buried in the noises produces results conflicting with standard field theory. (The question of the quantum state and the quantum nature of the coarse-grained fields has been taken up in more detail in Ref. [8] where it has been pointed out that the coarse-graining by itself does not lead to a set of classical equations.) Here we follow a different path by deriving a *quantum* stochastic Liouville equation that incorporates, at least to some extent, the correlations between the noises.

The fact that $\hat{F}_1^c$ and $\hat{F}_2^c$ do not commute implies that (26) and (27) should be treated as two separate Langevin equations. Strictly speaking it is not valid to substitute (26) in (27) and treat the resulting equation as a stochastic equation second order in time. This will lead to wrong answers for averages involving $\hat{\pi}_L$. The approach we will follow avoids this pitfall.

The Hamiltonian equations of motion (26) and (27) are exact, as no approximations have been made so far. The first approximation we make is to drop the spatial-derivative term in (27); this is because we will be interested only in the behavior of the quantum field at "large" scales. The coarse-graining will be implemented in the sense of a temporal ensemble, i.e., we focus attention on one spatially fixed coarse-grained domain and consider the evolution of the coarse-grained quantum field defined on that domain. The spatial coarse-graining and this interpretation imply that all two-point objects be evaluated with the spatial separation between the points being much less than the coarse-graining scale, i.e., $R \ll 2\pi k_S^{-1}$. With this limit in place and with the neglect of spatial derivatives, the quantum Langevin equations are

$$\dot{\hat{\chi}}_L=\hat{\pi}_L+\hat{F}_1^c , \tag{38}$$

$$\dot{\hat{\pi}}_L=-\left[S^2m^2-\frac{\ddot{S}}{S}\right]\hat{\chi}_L+\hat{F}_2^c , \tag{39}$$

where

$$\langle \hat{F}_i^c(\eta_1)\hat{F}_j^c(\eta_2)\rangle \simeq 2B_{ij}(\eta_1)\delta(\eta_1-\eta_2) , \tag{40}$$

and

$$B_{11}(\eta_1)\simeq\frac{1}{4\pi^2}k_S^2|\dot{k}_S|\,|\chi_{k_S}(\eta_1)|^2 , \tag{41}$$

$$B_{12}(\eta_1)\simeq\frac{1}{4\pi^2}k_S^2|\dot{k}_S|\chi_{k_S}(\eta_1)\dot{\chi}_{k_S}^*(\eta_1) , \tag{42}$$

$$B_{21}(\eta_1)\simeq\frac{1}{4\pi^2}k_S^2|\dot{k}_S|\dot{\chi}_{k_S}(\eta_1)\chi_{k_S}^*(\eta_1) , \tag{43}$$

$$B_{22}(\eta_1)\simeq\frac{1}{4\pi^2}k_S^2|\dot{k}_S|\,|\dot{\chi}_{k_S}(\eta_1)|^2 . \tag{44}$$

Spatial variations within one coarse-grained domain cannot be sampled by the coarse-grained field; this accounts for the fact that there are no terms reflecting such a dependence in (38)–(44).

The dynamical equations (38) and (39) can just as well be obtained from the stochastic Hamiltonian

$$H(\chi_L,\pi_L)=\tfrac{1}{2}\pi_L^2+\tfrac{1}{2}\omega^2(\eta)\chi_L^2+F_1^c\pi_L-F_2^c\chi_L , \tag{45}$$

where the time-dependent "frequency"

$$\omega^2\equiv S^2m^2-\frac{\ddot{S}}{S} . \tag{46}$$

The coarse-grained field is now viewed as the coordinate variable in the one-dimensional quantum-mechanical problem specified by (45). The terms containing F_1^c and F_2^c are taken to represent stochastic external perturbations with correlations specified by (40)–(44). (The Hamiltonian (45) is a time-dependent generalization of the randomly forced oscillator considered previously in a different context by Merzbacher [20].) The idea now is to study the one-dimensional quantum mechanical problem instead of the original field theory. It is important to note that for a quantum analysis we cannot just use the equations of motion (38) and (39); a Hamiltonian is *necessary*. On the other hand, were we only interested in a classical analysis, the equations of motion would suffice. (A discussion of this point is given in Ref. [21].)

Before proceeding further some cautionary remarks are in order. First, while the above assumption is an improvement on previous work to the extent that we are not assuming the system to be classical, it still does not constitute a well controlled approximation scheme. In particular, the Hamiltonian (45) has been written down simply by fiat. Nevertheless, to see whether the results and insights obtained using this approach are persuasive, our attitude will be to take the formalism as it stands and proceed as far as possible without any further assumptions. Second, there is a coordinate dependence inherent in the phase-space formalism we will be employing shortly: the distribution function is not invariant under canonical transformations (this feature is generic to quantum mechanics and is not specific to our problem). We will return to these problems in more detail later on.

The quantum system is completely described by its density matrix, which, written in the coordinate representation

$$\rho(\chi_L,\chi_L')=\sum_j W_j\psi_j(\chi_L)\psi_j^*(\chi_L') , \tag{47}$$

obeys the quantum Liouville equation

$$i\dot\rho(\chi_L,\chi_L')=[H(\chi_L)-H^*(\chi_L')]\rho(\chi_L,\chi_L') . \tag{48}$$

The passage to a quantum phase space is now made via the Wigner transform [22] of the density matrix:

$$f_W(X_L,p_L)=\int\frac{dx_L}{2\pi}e^{ip_Lx_L}\rho(X_L+x_L/2,X_L-x_L/2) , \tag{49}$$

where the new variables

$$X_L=(\chi_L+\chi_L')/2 , \tag{50}$$

$$x_L=\chi_L-\chi_L' . \tag{51}$$

The Wigner function $f_W(X_L,p_L)$ is always real and properly normalized over phase space (for bounded systems); moreover, it is square integrable (a property not shared in general by classical distribution functions):

$$\int dX_Ldp_Lf_W(X_L,p_L)=1 , \tag{52}$$

$$\int dX_Ldp_Lf_W^2(X_L,p_L)\le\frac{1}{2\pi} , \tag{53}$$

where in the second expression the equality holds for pure states. Quantum expectation values for functions of $\hat\chi_L$ and $\hat\pi_L$ alone are given correctly as phase space averages with respect to the Wigner function, as, for example,

$$\langle h(\hat\pi_L)\rangle=\int dX_Ldp_Lh(p_L)f_W(X_L,p_L) \tag{54}$$

but not for mixed operators such as $\hat\chi_L^2\hat\pi_L^2$. This is related to the ordering problem in quantum mechanics; the Wigner formalism is associated with Weyl's rule for the ordering of operators [23]. A further obstacle to the literal interpretation of a Wigner function as a true distribution function over a classical phase space is the fact that in general it is not positive definite. (Fortunately we will not encounter the ordering problem nor the lack of positivity in our example.) More on the Wigner function can be found in the reviews of Hillery *et al.* [24] and Narcowich [25].

The Wigner transform of the quantum Liouville equation (48) yields

$$\frac{\partial}{\partial\eta}f_W(X_L,p_L;\eta)=-L_0f_W(X_L,p_L;\eta)-L_Sf_W(X_L,p_L,\eta) , \tag{55}$$

where the Liouville operator has been written as the sum of a systematic piece

$$L_0=p_L\frac{\partial}{\partial X_L}-\omega^2X_L\frac{\partial}{\partial p_L} \tag{56}$$

and a stochastic piece

$$L_S=F_1^c\frac{\partial}{\partial X_L}+F_2^c\frac{\partial}{\partial p_L} \tag{57}$$

$$\equiv F_i^c\frac{\partial}{\partial z_i}\quad(z_1\equiv X_L,\ \ z_2\equiv p_L) . \tag{58}$$

We now implement the strategy of Kubo [26] in order to obtain a simple derivation of the stochastic quantum Liouville equation (cases more complicated than the one considered here are treated elsewhere [21]). To begin, we focus attention on the dynamical effect of L_S by shifting to the interaction picture

$$f_W(X_L,p_L;\eta)=e^{-L_0\eta}\sigma(X_L,p_L;\eta) . \tag{59}$$

In terms of $\sigma(X_L,p_L;\eta)$ the Liouville equation (55) becomes

$$\frac{\partial}{\partial\eta}\sigma(X_L,p_L;\eta)=-e^{L_0\eta}L_Se^{-L_0\eta}\sigma(X_L,p_L;\eta) \tag{60}$$

$$\equiv\Omega(\eta)\sigma(X_L,p_L;\eta) . \tag{61}$$

This equation has the formal time-ordered exponential solution

$$\begin{aligned}\sigma(X_L,p_L;\eta)&=\left[1+\int_{\eta_0}^{\eta}d\eta_1\Omega(\eta_1)+\int_{\eta_0}^{\eta}d\eta_1\int_{\eta_0}^{\eta_1}d\eta_2\Omega(\eta_1)\Omega(\eta_2)+\cdots\right]\sigma(X_L,p_L;\eta_0)\\&=\left[\exp\left[\int_{\eta_0}^{\eta}d\eta'\Omega(\eta')\right]\right]_T\sigma(X_L,p_L;\eta_0) ,\end{aligned} \tag{62}$$

where the initial value $\sigma(X_L,p_L;\eta_0)$ is specified at some initial time η_0. All the noise terms come multiplied together in each term of the series. If we take the average over noise of (62), these terms will either be zero, or will produce δ functions. It is easy to see that only the quadratic product of noise terms needs to be computed, this following from the Gaussian nature of the noises. With $\langle\ \rangle_N$ denoting an average over noise, we find

$$\langle\Omega(\eta_1)\Omega(\eta_2)\rangle_N=2B_{ij}(\eta_1)\delta(\eta_1-\eta_2)\left[e^{L_0\eta_1}\frac{\partial^2}{\partial z_i\partial z_j}e^{-L_0\eta_1}\right] . \tag{63}$$

The noise averaged version of the time-ordered exponential solution (62) then turns out to be

$$\langle \sigma(X_L,p_L;\eta)\rangle_N=\sigma(X_L,p_L;\eta_0)+\int_{\eta_0}^{\eta}d\eta_1 B_{ij}(\eta_1)\left[e^{L_0\eta_1}\frac{\partial^2}{\partial z_i \partial z_j}e^{-L_0\eta_1}\right]\langle \sigma(X_L,p_L;\eta_1)\rangle_N \ , \tag{64}$$

which may be immediately differentiated to yield

$$\frac{\partial}{\partial \eta}\langle \sigma(X_L,p_L;\eta)\rangle_N=B_{ij}(\eta)\left[e^{L_0\eta}\frac{\partial^2}{\partial z_i \partial z_j}e^{-L_0\eta}\right]\langle \sigma(X_L,p_L;\eta)\rangle_N \ . \tag{65}$$

We recall that the transformation to the interaction picture involved only L_0, which is of course unaffected by averages over the noise. Therefore, there is no difficulty in writing (65) in terms of the original distribution function:

$$\frac{\partial}{\partial \eta}\langle f_W(X_L,p_L;\eta)\rangle_N=-L_0\langle f_W(X_L,p_L;\eta)\rangle_N+B_{ij}(\eta)\frac{\partial^2}{\partial z_i \partial z_j}\langle f_W(X_L,p_L;\eta)\rangle_N \ . \tag{66}$$

This is the required stochastic quantum Liouville equation and, as is obvious, it has the standard Fokker-Planck form. Alternatively, (66) may be written in a more convenient form in terms of the explicitly symmetrized diffusion matrix $D_{ij}=(B_{ij}+B_{ji})$ as

$$\frac{\partial}{\partial \eta}\langle f_W(X_L,p_L;\eta)\rangle_N=-L_0\langle f_W(X_L,p_L;\eta)\rangle_N+\frac{1}{2}D_{ij}(\eta)\frac{\partial^2}{\partial z_i \partial z_j}\langle f_W(X_L,p_L;\eta)\rangle_N \ . \tag{67}$$

The stochastic equation (67) and the nature of its derivation merit a few clarifying remarks. First, it is not necessary to being with the Wigner formalism; we can just as well employ the density matrix (either by following the procedure used here or the influence functional approach [27]). The stochastic equation for the density matrix can then be converted to one for the Wigner function by implementing the "twisted product" [25]. Second, as only the case of free fields is treated here, the Hamiltonian is at most quadratic in the dynamical variables. This is why (67) is of the standard classical Fokker-Planck form; such a simplification does not obtain in general [21]. Of course, even if the form of (67) is classical, this does *not* imply that all solutions be classical distribution functions. It should be emphasized though that X_L and p_L are not operators and can be treated as ordinary classical objects.

We point out that there is no need to invoke any *ad hoc* thermodynamic analogy (e.g., fluctuation-dissipation relations) in our derivation of the stochastic quantum Liouville equation as was done by Graziani [28] in a first attempt to apply the Wigner-function formalism to stochastic inflation. As we will show in the following sections, such relations do not hold in general and any analogy with conventional Brownian motion must be treated with extreme caution. A related remark is that since (67) is formally a master equation one might expect to define a suitable entropy satisfying some variant of the H theorem [29]. For example, it is easy to see that because of the diffusion term, the "linear entropy" or "mixing parameter" $\mathrm{Tr}\rho^2=\int dX_L dp_L f_W^2$ will always decrease with time (implying that the quantum state is getting more and more mixed). This must not be interpreted in the sense of "quantum decoherence" [30] as we are dealing with a *free* theory and there is no coupling to some external environment. (One way to understand this result may be that this decrease simply mirrors the loss of information inherent in our time-dependent coarse graining.)

Finally we draw attention to some technical issues. Note that no assumption is needed as to the symmetry properties of $B_{ij}(\eta)$; this allows for the fact that the noises do not commute. Note also that while separately B_{12} and B_{21} need not be real, they appear in (66) only in the symmetrized combination $B_{12}+B_{21}$ (since partial derivatives commute), which, as is clear from (42) and (43) is always real. The derivation of (67) is also free from any kind of "slow-roll" assumption although this merit is mainly technical as physical results at late times remain unaffected when such conditions are imposed (see Ref. [8], Appendix A).

We now turn to the problem of solving the stochastic quantum Liouville equation. Formally, the solutions are not difficult to obtain as (67) is just a Kramers equation describing a time-dependent Ornstein-Uhlenbeck process [31]. The average values satisfy

$$\frac{d}{d\eta}\langle z_i\rangle_N=A_{ij}(\eta)\langle z_j\rangle_N \tag{68}$$

given the initial condition $\langle z_i(\eta_0)\rangle_N=z_{i0}$. The matrix A_{ij} is defined by

$$L_0 f_W=A_{ij}\frac{\partial}{\partial z_i}(z_j f_W) \tag{69}$$

and in our case, $A_{11}=A_{22}=0$, $A_{12}=1$, $A_{21}=-\omega^2(\eta)$. The propagator for the average values $\langle z_i\rangle$, G_{ij}, satisfies

$$\frac{d}{d\eta}G_{ij}=A_{ik}G_{kj} \ , \quad G_{ij}(\eta_0)=\delta_{ij} \ . \tag{70}$$

The second moments follow from

$$\frac{d}{d\eta}\langle z_i z_j\rangle_N=A_{ik}\langle z_k z_j\rangle_N+A_{jk}\langle z_i z_k\rangle_N+D_{ij} \ . \tag{71}$$

This equation is also obeyed by the covariances $\Xi_{ij}=\langle z_i z_j\rangle_N-\langle z_i\rangle_N\langle z_j\rangle_N$, which can themselves be expressed in terms of the propagator as

$$\Xi(\eta)=G(\eta)\Xi(\eta_0)\tilde{G}(\eta)+\int_{\eta_0}^{\eta}d\eta' G(\eta)G^{-1}(\eta')D(\eta')\tilde{G}^{-1}(\eta')\tilde{G}(\eta) \ . \tag{72}$$

The first term is the systematic contribution arising from a reversible dynamical evolution from the given initial condition. The second term represents the irreversible stochastic contribution due to the diffusion matrix.

It can be shown [31] that the general solution of (67) with δ-function initial conditions $W(z,\eta_0)=\prod_i \delta(z_i - z_{i0})$ is

$$W(z,\eta;z_0,\eta_0)=\frac{1}{2\pi}(\mathrm{Det}\Xi)^{-1/2} \times \exp\left[-\frac{1}{2}(\tilde{z}-\langle \tilde{z}\rangle_N)\Xi^{-1}(z-\langle z\rangle_N)\right] . \tag{73}$$

The function $W(z,\eta;z_0,\eta_0)$ serves as the propagator for the Fokker-Planck equation (67). Solutions for arbitrary initial conditions $f_W(z_0,\eta_0)$ can be generated from it by

$$f_W(z,\eta)=\int dz_0 W(z,\eta;z_0,\eta_0) f_W(z_0,\eta_0) . \tag{74}$$

These general results will be applied to exponential and power-law expansions in the following sections.

III. EXPONENTIAL INFLATION

The case of a de Sitter expansion furnishes a particularly simple example in which to implement the procedures of Sec. II. Our stochastic approach not only reproduces some previous field theoretic results but also introduces a new interpretive framework. In this section we aim mainly to obtain quantitative results.

The scale factor for de Sitter space is

$$S(\eta)=-\frac{1}{H_0\eta} , \tag{75}$$

and the conformal time

$$\eta=-\frac{1}{H_0}e^{-H_0 t} . \tag{76}$$

We note that here we are not really interested in the case of an eternal de Sitter expansion. Initial conditions for stochastic inflation will be assigned in the finite past, at the beginning of the inflationary phase.

The scalar field modes now satisfy

$$\ddot{\chi}_k+\left[k^2+\frac{1}{\eta^2}\left[\frac{m^2}{H_0^2}-2\right]\right]\chi_k=0 . \tag{77}$$

If the mass is zero or at least small compared to H_0^2, the "unstable" sector is characterized by $k^2<2/\eta^2$. Therefore, following Starobinsky [1] it makes sense to set

$$k_S(\eta)=\frac{\epsilon}{\eta} \tag{78}$$

where ϵ is a constant that serves to parametrize the cutoff. If we assume that inflation began at time η_0, implying a natural infrared cutoff η_0^{-1} (more details may be found in the Appendix), it is clear that ϵ cannot be arbitrarily small, as we must have $\epsilon\eta^{-1}>\eta_0^{-1}$. If one is interested only in late-time results, i.e., when $\eta \ll \eta_0$, then ϵ may be taken to be small. In this paper we will not restrict ϵ to be arbitrarily small but will allow it to be as large as unity. In principle, it is desirable that physical answers not depend on ϵ; this will turn out *not* to be the case. As will be shown later all infrared divergent quantities are only weakly dependent on ϵ, but this does not hold in general (the situation is more complicated for power-law inflation). Our stochastic approach will correctly reproduce the cutoff dependences for infrared finite quantities calculated from conventional quantum field theory (see the Appendix).

A curious special property of de Sitter space is that even when the mass is nonzero (no infrared divergence), there is still an initial growth of $\langle \Phi^2\rangle$ to an asymptotic limit $\langle \Phi^2\rangle_{\mathrm{BD}}$, which is the value in the Bunch-Davies vacuum [15]. The mathematical reason for this behavior is simply that the mass and curvature contributions in (77) scale identically with conformal time and that for m^2 small compared to H_0^2 there is still an "unstable" infrared regime despite there being no infrared divergence. As will be made clear in the next section, this feature is not shared by power-law inflation.

The mode equation (77) admits the general solution

$$\chi_k(\eta)=C_1\eta^{1/2}e^{i\nu\pi/2}H_\nu^{(1)}(k\eta)+C_2\eta^{1/2}e^{-i\nu\pi/2}H_\nu^{(2)}(k\eta) , \tag{79}$$

where

$$\nu^2=\frac{9}{4}-\frac{m^2}{H_0^2} . \tag{80}$$

The arbitrariness of the de Sitter vacuum is reflected in the various possible choices for C_1 and C_2. In this paper the quantum state we will use is the Bunch-Davies vacuum [18], characterized by $C_1=0$ and $C_2=\sqrt{\pi}/2$.

The symmetrized diffusion matrix D_{ij} now follows from (41)–(44):

$$D_{11}(\eta_1)=2B_{11}(\eta_1)\simeq\frac{\epsilon^3}{8\pi}\eta_1^{-3}|H_\nu^{(2)}(\epsilon)|^2 , \tag{81}$$

$$D_{12}(\eta_1)=B_{12}(\eta_1)+B_{21}(\eta_2)\simeq\frac{\epsilon^3}{8\pi}\eta_1^{-4}\mathrm{Re}\left\{H_\nu^{(2)}(\epsilon)\left[\left[\frac{1}{2}-\nu\right]H_\nu^{(2)*}(\epsilon)+\epsilon H_{\nu-1}^{(2)*}(\epsilon)\right]\right\} \simeq D_{21}(\eta_1) , \tag{82}$$

$$D_{22}(\eta_1)=2B_{22}(\eta_1)\simeq\frac{\epsilon^3}{8\pi}\eta_1^{-5}\left|\left[\frac{1}{2}-\nu\right]H_\nu^{(2)}(\epsilon)+\epsilon H_{\nu-1}^{(2)}(\epsilon)\right|^2 . \tag{83}$$

Our eventual goal is to solve the Fokker-Planck equation (67) given the diffusion coefficients (81)–(83). The potential term in the systematic component (56) of the stochastic Liouville operator is characterized by

$$\omega^2(\eta)=\left[\frac{1}{4}-\nu^2\right]\frac{1}{\eta^2} . \tag{84}$$

For all the cases we consider in this paper $\omega^2(\eta)$ will be negative (as $\nu^2>\frac{1}{4}$). We are dealing therefore with a time-dependent upside-down harmonic oscillator. The equation for the propagator (70) can now be easily solved, and in terms of the initial time η_0, we find

$$G_{11}(\eta)=\frac{1}{2\nu}\left[(\nu-\tfrac{1}{2})\left[\frac{\eta}{\eta_0}\right]^{\nu+1/2}+(\nu+\tfrac{1}{2})\left[\frac{\eta}{\eta_0}\right]^{-\nu+1/2}\right] , \tag{85}$$

$$G_{12}(\eta)=\frac{(\eta\eta_0)^{1/2}}{2\nu}\left[\left[\frac{\eta}{\eta_0}\right]^{\nu}-\left[\frac{\eta}{\eta_0}\right]^{-\nu}\right] , \tag{86}$$

$$G_{21}(\eta)=\frac{\nu^2-\frac{1}{4}}{2\nu}(\eta\eta_0)^{-1/2}\left[\left[\frac{\eta}{\eta_0}\right]^{\nu}-\left[\frac{\eta}{\eta_0}\right]^{-\nu}\right] , \tag{87}$$

$$G_{22}(\eta)=\frac{1}{2\nu}\left[(\nu+\tfrac{1}{2})\left[\frac{\eta}{\eta_0}\right]^{\nu-1/2}+(\nu-\tfrac{1}{2})\left[\frac{\eta}{\eta_0}\right]^{-\nu-1/2}\right] . \tag{88}$$

This is the general solution, valid for all the special cases considered in this paper.

We now confine attention to the massless case where the parameter $\nu=\frac{3}{2}$, and

$$H^{(2)}_{1/2}(k\eta)=i\left[\frac{2k\eta}{\pi}\right]^{1/2}\frac{e^{-ik\eta}}{k\eta} , \tag{89}$$

$$H^{(2)}_{3/2}(k\eta)=\left[\frac{2k\eta}{\pi}\right]^{1/2}\frac{e^{-ik\eta}}{(k\eta)^2}(i-k\eta) . \tag{90}$$

The diffusion matrix then takes the simple form

$$D_{11}(\eta_1)\simeq\frac{1}{4\pi^2}\eta_1^{-3}(1+\epsilon^2) , \tag{91}$$

$$D_{12}(\eta_1)=D_{21}(\eta_1)\simeq-\frac{1}{4\pi^2}\eta_1^{-4} , \tag{92}$$

$$D_{22}(\eta_1)\simeq\frac{1}{4\pi^2}\eta_1^{-5}(1-\epsilon^2+\epsilon^4) . \tag{93}$$

Note that to leading order (for $\epsilon\ll1$) the diffusion matrix is independent of ϵ. It is misleading, however, to conclude that the actual value of ϵ is not important, as at this order $\mathrm{Det}\,\Xi_{ij}=0$ (notice that this is a direct consequence of retaining noise cross correlations). In order to eventually obtain a nonsingular covariance matrix we must go beyond this level of approximation; the final solution for the Wigner function is in fact strongly dependent on ϵ.

It is a tedious but straightforward exercise to obtain the covariance matrix using (72). We assume an initial distribution such that $z_{i0}=0$ but impose no conditions on $\Xi_{ij}(\eta_0)$. Ignoring for the moment the systematic component, the stochastic contribution turns out to be

$$\Xi_{11}=\frac{\ln(\eta_0/\eta)}{4\pi^2\eta^2}\left[1+\frac{\epsilon^2}{3}+\frac{\epsilon^4}{9}\right]+\frac{\epsilon^2}{18\pi^2\eta^2}\left[1-\frac{\epsilon^2}{4}\right]-\frac{\epsilon^2\eta}{18\pi^2\eta_0^3}\left[1-\frac{\epsilon^2}{3}+\frac{\eta^3}{12\eta_0^3}\right] , \tag{94}$$

$$\Xi_{12}=\frac{\ln(\eta_0/\eta)}{4\pi^2\eta^3}\left[1+\frac{\epsilon^2}{3}+\frac{\epsilon^4}{9}\right]+\frac{\epsilon^2}{36\pi^2\eta^3}-\frac{\epsilon^2}{36\pi^2\eta_0^3}\left[1-\frac{\epsilon^2}{3}+\frac{\epsilon^2\eta^3}{3\eta_0^3}\right]$$
$$=\Xi_{21} , \tag{95}$$

$$\Xi_{22}=\frac{\ln(\eta_0/\eta)}{4\pi^2\eta^4}\left[1+\frac{\epsilon^2}{3}+\frac{\epsilon^4}{9}\right]-\frac{\epsilon^2}{9\pi^2\eta^4}\left[1-\frac{\epsilon^2}{2}\right]+\frac{\epsilon^2}{9\pi^2\eta_0^3\eta}\left[1-\frac{\epsilon^2}{3}-\frac{\epsilon^2\eta^3}{6\eta_0^3}\right] . \tag{96}$$

One can easily check using (72) and (85)–(88) that at late times the systematic contribution to the covariance matrix is negligible compared to the stochastic piece. (Initial conditions are discussed further below.)

The full solution for the noise averaged Wigner distribution function follows trivially from (73) and (74). In this section we will concentrate only on the covariance matrix itself, as all average values of interest can be computed directly from it. Detailed study of the distribution function will be postponed to Sec. V.

The covariance matrix (94)–(96) refers to the "conformal" variables X_L and p_L. Reverting to the original field Φ, we introduce new c-number variables ϕ_c and p_c via the canonical transformation

$$\phi_c=\frac{X_L}{S} , \tag{97}$$

$$p_c=Sp_L-\dot{S}X_L . \tag{98}$$

The corresponding covariance matrix may be written as

$$\Xi_{11}^{(\phi)}=\langle\phi_c^2\rangle_N=S^{-2}\Xi_{11}\ , \tag{99}$$

$$\Xi_{12}^{(\phi)}=\Xi_{21}^{(\phi)}=\langle\phi_c p_c\rangle_N=\Xi_{12}-\frac{\dot S}{S}\Xi_{11}\ , \tag{100}$$

$$\Xi_{22}^{(\phi)}=\langle p_c^2\rangle_N=S^2\Xi_{22}-2\dot S S\Xi_{12}+\dot S^2\Xi_{11}\ , \tag{101}$$

where all averages of the type $\langle z_i(\eta)\rangle$ vanish as a consequence of our choice $z_{i0}=0$ for the initial condition. Comparison with the field theoretic results is simpler if we introduce the "velocity" ϕ_c' (the prime denotes differentiation with respect to cosmic time) in place of the canonical momentum p_c. Noting that for de Sitter space, $\ln\eta_0/\eta=H_0(t-t_0)$, where t_0 denotes the beginning of the exponential expansion, and using (94)–(96), the new covariance matrix (for a massless field) turns out to be

$$\langle\phi_c^2\rangle_N=\Xi_{11}^{(\phi)}=\frac{H_0^3}{4\pi^2}(t-t_0)\left[1+\frac{\epsilon^2}{3}+\frac{\epsilon^4}{9}\right]+\frac{\epsilon^2H_0^2}{18\pi^2}\left[1-\frac{\epsilon^2}{4}\right]-\frac{\epsilon^2H_0^2}{18\pi^2}e^{-3H_0(t-t_0)}\left[1-\frac{\epsilon^2}{3}+\frac{\epsilon^2}{12}e^{-3H_0(t-t_0)}\right]\ , \tag{102}$$

$$\langle\phi_c\phi_c'\rangle_N=\frac{\Xi_{12}^{(\phi)}}{S^3}=\frac{\Xi_{21}^{(\phi)}}{S^3}=\frac{\epsilon^2H_0^3}{12\pi^2}\left[1-\frac{\epsilon^2}{6}\right]-\frac{\epsilon^2H_0^3}{12\pi^2}e^{-3H_0(t-t_0)}\left[1-\frac{\epsilon^2}{3}+\frac{\epsilon^2}{6}e^{-3H_0(t-t_0)}\right]\ , \tag{103}$$

$$\langle\phi_c'^2\rangle_N=\frac{\Xi_{22}^{(\phi)}}{S^6}=\frac{\epsilon^4H_0^4}{24\pi^2}(1-e^{-6H_0(t-t_0)})\ . \tag{104}$$

The above results record only the stochastic contribution to the covariance matrix. It is easy to compute the systematic contribution for an arbitrary initial choice of Ξ_{ij} from (72) and (85)–(88) (since the Wigner distribution function must be square integrable we cannot take the initial distribution to be a δ function over phase space). The contribution to Ξ_{11} consists of a constant piece and terms that fall off exponentially with cosmic time. Contributions to Ξ_{12} and Ξ_{22} also display a similar exponential falloff. It is important to note that these contributions, though insignificant at late times, *can* dominate similar terms that already exist in the stochastic piece (especially for small values of ϵ). Therefore, *only* the late time limit is independent of initial conditions. This is in contrast with the field theoretic case where the contribution from initial conditions is usually irrelevant even at early times (see the Appendix). We also draw attention to the fact that the exponential falloffs in the stochastic calculation are not the same as the field theoretic ones; again, this is of no consequence at late times.

As long as the initial distribution is such that $\langle\phi_c\rangle_N$ and $\langle p_c\rangle_N$ are zero (i.e., $z_{i0}=0$), at late times (η small) and with $\epsilon\ll 1$, the leading-order contributions are

$$\Xi_{11}^{(\phi)}=\langle\phi_c^2\rangle_N\simeq\frac{H_0^3}{4\pi^2}(t-t_0)\ , \tag{105}$$

$$\frac{\Xi_{12}^{(\phi)}}{S^3}=\langle\phi_c\phi_c'\rangle_N\simeq\frac{\epsilon^2H_0^3}{12\pi^2}\ , \tag{106}$$

$$\frac{\Xi_{22}^{(\phi)}}{S^6}=\langle\phi_c'^2\rangle_N\simeq\frac{\epsilon^4H_0^4}{24\pi^2}\ . \tag{107}$$

The $\Xi_{11}^{(\phi)}$ term reproduces the standard quantum field-theoretic result [15] [and (A5) in the Appendix] for the expectation value $\langle\Phi^2\rangle$, here viewed as a noise average for the c-number variable ϕ_c *provided* that ϵ is small. However, if we set $\epsilon\sim 1$ the answer does not agree with the field-theoretic result (A5) found in the Appendix (which unlike the stochastic calculation is essentially cutoff independent).

For the sake of comparison, if we set $\epsilon=1$ in (102)–(104) we find, at late times,

$$\Xi_{11}^{(\phi)}=\langle\phi_c^2\rangle_N\simeq\frac{13H_0^3}{36\pi^2}(t-t_0)\ , \tag{108}$$

$$\frac{\Xi_{12}^{(\phi)}}{S^3}=\langle\phi_c\phi_c'\rangle_N\simeq\frac{5H_0^3}{72\pi^2}\ , \tag{109}$$

$$\frac{\Xi_{22}^{(\phi)}}{S^6}=\langle\phi_c'^2\rangle_N\simeq\frac{H_0^4}{24\pi^2}\ , \tag{110}$$

whereas the corresponding field theoretic results (A5), (A9), and (A12) of the Appendix give, at late times:

$$\langle\Phi^2\rangle\simeq\frac{H_0^3}{4\pi^2}(t-t_0)\ , \tag{111}$$

$$\frac{1}{2}\langle\Phi\Phi'+\Phi'\Phi\rangle\simeq\frac{\epsilon^2H_0^3}{8\pi^2}\ , \tag{112}$$

$$\langle\Phi'^2\rangle\simeq\frac{\epsilon^4H_0^4}{16\pi^2}\ . \tag{113}$$

We see that while the stochastic results for $\Xi_{12}^{(\phi)}$ and $\Xi_{22}^{(\phi)}$ correctly reproduce the the cutoff dependence found in the field-theoretic case, the numerical values of the coefficients do not match. This is not a serious problem, as these quantities need to be renormalized anyway (something that is beyond the scope of this paper).

The technical reason for the disagreement between the field-theoretic and stochastic calculations is the neglect of spatial derivatives in (39). One is attempting to approximate a time-dependent quantum sum over modes by a modified dynamics (via the noise term) for the zero mode and neglecting all the other modes (apart from their contribution to the noise). When computing $\langle\Phi^2\rangle$ via the standard field-theoretic method, the infrared sector provides the dominant contribution. On the other hand, when computing $\langle\Phi\Phi'+\Phi'\Phi\rangle/2$ and $\langle\Phi'^2\rangle$, extra multiplicative factors of k and k^2 (see the Appendix) weaken this infrared dependence. One expects therefore that the stochastic method should work better for $\langle \Phi^2 \rangle$ and, as we have seen, this is indeed the case. The fact that stochastic results are more accurate for small values of ϵ is also easy to appreciate: a small ϵ means that only long-wavelength modes are contributing to the noise so that k is indeed small and can be neglected. However, if we let ϵ be of order unity, then the neglect of spatial derivatives will lead to errors. A possible remedy is to work with a Wigner functional derived directly from the field theory but this may well be at the expense of the calculational ease that characterizes the present approach.

We now consider the massive field but confine attention to the case $m^2 \ll H_0^2$. The propagator $G(\eta)$ is still given by (85)–(88) except that the parameter ν is now given by

$$\nu \simeq \frac{3}{2} - \frac{m^2}{3{H_0}^2} . \tag{114}$$

In the limit of a small mass the diffusion matrix is essentially the same as for the massless case (since $\nu \sim \frac{3}{2}$). Keeping the diffusion matrix given by (91)–(93) but using (114) for the propagator, it is a simple matter to solve for the stochastic contribution to the covariance matrix. The final expressions are very long and not very illuminating. Here we present only the leading-order stochastic terms at late times:

$$\langle \phi_c^2 \rangle_N \simeq \frac{3H_0^4}{8m^2\pi^2}\left[1-\exp\left[-\frac{2m^2}{3H_0}(t-t_0)\right]\right]\left[1+\frac{\epsilon^2}{3}+\frac{\epsilon^4}{9}+\frac{m^2}{9H_0^2}\right] , \tag{115}$$

$$\langle \phi_c \phi_c' \rangle_N \simeq \frac{H_0^3}{8\pi^2}\left[1+\epsilon^2-\left[1+\frac{\epsilon^2}{3}+\frac{\epsilon^4}{9}\right]\exp\left[-\frac{2m^2}{3H_0}(t-t_0)\right]\right] , \tag{116}$$

$$\langle \phi_c'^2 \rangle_N \simeq \frac{H_0^4}{24\pi^2}\left\{\epsilon^4+\frac{m^2}{H_0^2}\left[1-\exp\left[-\frac{2m^2}{3H_0}(t-t_0)\right]\right]+\frac{\epsilon^2 m^2}{H_0^2}\left[1-\exp\left[-\frac{2m^2}{3H_0}(t-t_0)\right]\right]\right\} , \tag{117}$$

where we have dropped all terms that vanish faster at late times and also neglected terms that are of higher order in m^2/H_0^2. An interesting feature of the massive case is that all contributions arising from $\Xi_{ij}^{(\phi)}(\eta_0)$ are negligible even at early times (unlike the massless case). The agreement with field theory is remarkably good. With ϵ small, or more precisely, in the range

$$\exp[-3H_0^2/(2m^2)] \ll \epsilon^2 \ll m^2/H_0^2 , \tag{118}$$

all asymptotic late-time values are exactly reproduced (similar inequalities are derived somewhat differently in Refs. [1] and [19]). Unlike the massless case, this time the approach to these late-time values is also in agreement with the field-theoretical results (i.e., no mismatch in the exponentially falling-off terms). At late times the above expressions reduce to

$$\Xi_{11}^{(\phi)} = \langle \phi_c^2 \rangle_N \simeq \frac{3H_0^4}{8m^2\pi^2}\left[1+\frac{m^2}{9H_0^2} - \exp\left[-\frac{2m^2}{3H_0}(t-t_0)\right]\right] , \tag{119}$$

$$\frac{\Xi_{12}^{(\phi)}}{S^3} = \langle \phi_c \phi_c' \rangle_N \simeq \frac{H_0^3}{8\pi^2}\left[1-\exp\left[-\frac{2m^2}{3H_0}(t-t_0)\right]\right] , \tag{120}$$

$$\frac{\Xi_{22}^{(\phi)}}{S^6} = \langle \phi_c'^2 \rangle_N \simeq \frac{m^2H_0^2}{24\pi^2}\left[1-\exp\left[-\frac{2m^2}{3H_0}(t-t_0)\right]\right] , \tag{121}$$

in agreement with the field theoretic results (A17), (A21), and (A23) of the Appendix (with ϵ in the previously indicated range). Unlike the massless case there is no leading-order ϵ dependence in Ξ_{12} and Ξ_{22}. To avoid $\mathrm{Det}\,\Xi_{ij}^{(\phi)}=0$, it is important to keep the subdominant mass squared term in (119). It is again easy to verify that agreement with field-theoretic results does not extend to the case $\epsilon \sim 1$; the two calculations now differ by multiplicative factors of order unity.

IV. POWER-LAW INFLATION

In this section we treat a power-law expansion $a(t) \sim t^p$ with $p>1$ and consider only the case of a massless scalar field. Assuming that inflation set in at the initial time η_0 (with the scale factor set to unity at this time), we find, from (3),

$$S(\eta) = \left[\frac{\eta}{\eta_0}\right]^{(1-2\nu)/2} \tag{122}$$

where

$$\nu = \frac{1-3p}{2(1-p)} . \tag{123}$$

It then follows that

$$\frac{\ddot{S}}{S} = -\left[\frac{1}{4}-\nu^2\right]\frac{1}{\eta^2} . \tag{124}$$

It is useful to note that for $p>1$, $\nu > \frac{3}{2}$ and also that as $t \to \infty$, $\eta \to 0$. In the formal limit $p \to \infty$, $\nu = \frac{3}{2}$ which is the value for de Sitter space.

The scalar field modes now satisfy

$$\ddot{\chi}_k+\left[k^2+m^2\left[\frac{\eta}{\eta_0}\right]^{1-2\nu}+\left[\frac{1}{4}-\nu^2\right]\frac{1}{\eta^2}\right]\chi_k=0\ . \quad (125)$$

This time the mass and curvature contributions scale differently with conformal time. It is easy to see that at late times the mass term dominates the curvature contribution since $\nu>\frac{3}{2}$. Therefore, as cosmic time increases (and the conformal time decreases) there is a continuous flow *from* the "unstable" to the "stable" sector. However, destabilization will still occur for the massless case where the "unstable" sector is characterized by $k^2<(\nu^2-\frac{1}{4})/\eta^2$. We see also that even in the case of power-law inflation it is only natural to implement the same choice that we made in the last section, i.e., to set $k_S(\eta)=\epsilon/\eta$.

The mode equation (125) for a massless field admits the general solution

$$\chi_k(\eta)=C_1\eta^{1/2}H_\nu^{(1)}(k\eta)+C_2\eta^{1/2}H_\nu^{(2)}(k\eta) \quad (126)$$

or, in terms of the original field,

$$\phi_k(\eta)=C_1\eta_0^{1/2}\left[\frac{\eta}{\eta_0}\right]^\nu H_\nu^{(1)}(k\eta)+C_2\eta_0^{1/2}\left[\frac{\eta}{\eta_0}\right]^\nu H_\nu^{(2)}(k\eta)\ . \quad (127)$$

The adiabatic vacuum is specified by $C_1=0$, $C_2=\sqrt{\pi}/2$, i.e.,

$$\chi_k(\eta)=\left[\frac{\pi\eta}{4}\right]^{1/2}H_\nu^{(2)}(k\eta)\ . \quad (128)$$

Assuming that all "high-frequency" modes were in the adiabatic vacuum at the onset of inflation, (128) enables us to compute the diffusion coefficients (41)–(44), which are the same as (81)–(83) except that now ν is specified by (123).

Power-law inflation with $p\gg 1$ can be treated in a simple and direct manner by following the same approach as that for the massive field in exponential inflation. Note that when p is large,

$$\nu\simeq\frac{3}{2}+\frac{1}{p}+\frac{1}{p^2}+\frac{1}{p^3}+\cdots\ . \quad (129)$$

This allows us to approximate the diffusion matrix by the one for $\nu=\frac{3}{2}$, (91)–(93). The propagator $G(\eta)$ is given by (85)–(88) with ν specified by (129). The stochastic piece of the covariance matrix can now be found by a straightforward computation. The result is too long to write out in entirety and we content ourselves by just displaying the leading-order terms:

$$\Xi_{11}^{(\phi)}=\langle\phi_c^2\rangle_N\simeq\frac{p}{8\pi^2}\eta_0^{-2}\left[1-\left[\frac{\eta}{\eta_0}\right]^{2/p}\right]\left[1+\frac{1}{3}\epsilon^2+\frac{1}{9}\epsilon^4\right] \quad (130)$$

$$\simeq\frac{p}{8\pi^2}\eta_0^{-2}\left[1-\left[\frac{t_0}{t}\right]^2\right]\left[1+\frac{1}{3}\epsilon^2+\frac{1}{9}\epsilon^4\right]\ , \quad (131)$$

$$\frac{\Xi_{12}^{(\phi)}}{S^3}=\langle\phi_c\phi_c'\rangle_N\simeq\frac{1}{12\pi^2}\eta_0^{-3}\left[\frac{\eta}{\eta_0}\right]^{3/p}\left[\epsilon^2+\frac{1}{p}\right] \quad (132)$$

$$\simeq\frac{1}{12\pi^2}\eta_0^{-3}\left[\frac{t_0}{t}\right]^3\left[\epsilon^2+\frac{1}{p}\right]\ , \quad (133)$$

$$\frac{\Xi_{22}}{S^6}=\langle\phi_c'^2\rangle\simeq\frac{1}{12\pi^2}\eta_0^{-4}\left[\frac{\eta}{\eta_0}\right]^{4/p}\left[\frac{1}{2p^2}+\epsilon^2\left[\frac{1}{p}+\frac{\epsilon^2}{2}\right]\right] \quad (134)$$

$$\simeq\frac{1}{12\pi^2}\eta_0^{-4}\left[\frac{t_0}{t}\right]^4\left[\frac{1}{2p^2}+\epsilon^2\left[\frac{1}{p}+\frac{\epsilon^2}{2}\right]\right]\ , \quad (135)$$

where terms lower order in $1/p$ and vanishing faster at late times have been dropped. The variables ϕ_c and p_c are still defined by (97) and (98) except that $S(\eta)$ is now given by (122). For small ϵ, (131) is in agreement with the field theoretic calculation of Ref. [16] [also compare with (A29) of the Appendix]. However, just as for the massive field in de Sitter space, ϵ *cannot* be arbitrarily small. Consistency with the field-theoretic results (A36) and (A42) of the Appendix requires that

$$p^{-1}\ll\epsilon^2\ll 1 \quad (136)$$

in (133) and (135). The above expectation values were also computed by Kandrup using a different method [6]. While our result for $\langle\phi_c^2\rangle$ is in agreement with his, this is not true for the other two cases. The inconsistency can be traced to an approximation for the noise that does not take the commutator properly into account (see the discussion of this point in Sec. II).

In this case, while at late times $\langle\phi_c^2\rangle_N$ goes to a constant (as for the massive field in de Sitter space), $\langle\phi_c\phi_c'\rangle_N$ and $\langle\phi_c'^2\rangle_N$ vanish. This is in contrast with the case of a massive field in de Sitter space where these quantities, instead of vanishing, also go to constant values. The role of initial conditions is similar to that for the massive field in de Sitter space rather than the massless one: the systematic contribution to the covariance matrix is always negligible as long as $p\gg 1$ (but not otherwise). Consequently, the late-time results follow from (131)–(135):

$$\langle\phi_c^2\rangle_N\simeq\frac{p}{8\pi^2}\eta_0^{-2}\ , \quad (137)$$

$$\langle\phi_c\phi_c'\rangle_N\simeq\frac{\epsilon^2}{12\pi^2}\eta_0^{-3}\left[\frac{t_0}{t}\right]^3\ , \quad (138)$$

$$\langle\phi_c'^2\rangle_N\simeq\frac{\epsilon^4}{24\pi^2}\eta_0^{-4}\left[\frac{t_0}{t}\right]^4\ , \quad (139)$$

where we have taken $\epsilon\ll 1$. For $\epsilon=1$, we have

$$\langle \phi_c^2 \rangle_N \simeq \frac{13p}{72\pi^2}\eta_0^{-2} , \tag{140}$$

$$\langle \phi_c \phi_c' \rangle_N \simeq \frac{5}{72\pi^2}\eta_0^{-3}\left[\frac{t_0}{t}\right]^3 , \tag{141}$$

$$\langle \phi_c'^2 \rangle_N \simeq \frac{1}{24\pi^2}\eta_0^{-4}\left[\frac{t_0}{t}\right]^4 . \tag{142}$$

The field-theoretic results (A29), (A37), and (A43) of the Appendix yield the corresponding late-time limits:

$$\langle \Phi^2 \rangle \simeq \frac{p}{8\pi^2}\eta_0^{-2} , \tag{143}$$

$$\frac{1}{2}\langle \Phi\Phi' + \Phi'\Phi \rangle \simeq \frac{\epsilon^2}{8\pi^2}\eta_0^{-3}\left[\frac{t_0}{t}\right]^3 , \tag{144}$$

$$\langle \Phi'^2 \rangle \simeq \frac{\epsilon^4}{16\pi^2}\eta_0^{-4}\left[\frac{t_0}{t}\right]^4 . \tag{145}$$

There is reasonable agreement with the stochastic results when ϵ is small but as is expected the results diverge from each other when $\epsilon \sim 1$.

The limit $p \to \infty$ may be applied to (131) using

$$\left[\frac{\eta}{\eta_0}\right]^{2/p} = 1 + \frac{2}{p}\ln\left[\frac{\eta}{\eta_0}\right] + \cdots , \tag{146}$$

with the result

$$\langle \phi_c^2 \rangle_N \simeq \frac{H_0(t-t_0)}{4\pi^2\eta_0^2}\left[1+\frac{1}{3}\epsilon^2+\frac{1}{9}\epsilon^4\right] . \tag{147}$$

Since $H_0 = \eta_0^{-1}$, this agrees with the result (102) for a massless field in de Sitter space. In a similar manner one can check that $\langle \phi_c \phi_c' \rangle_N$ and $\langle \phi_c'^2 \rangle_N$ also reduce to the appropriate expressions for a massless field in de Sitter space as calculated from the stochastic approach.

V. SOLUTIONS AND INTERPRETATIONS

In this section we study the full phase-space distribution function. Given that we have already computed the relevant covariance matrices it is now a simple matter to write out the corresponding Wigner functions. In the examples studied here these distributions will be positive definite and as such may be interpreted as true probability distributions, at least formally.

A knowledge of the distribution function is important as it will enable us to critically address issues such as the existence of fluctuation-dissipation relations and whether or not there exist late-time thermal solutions. These are the problems we will tackle first.

The stochastic Liouville equation (67) is written in terms of the conformal variables X_L and p_L. In all the examples we studied, $z_{i0}=0$, in which case the general solution (73) becomes, at late times,

$$f_W(z,\eta) = \frac{1}{2\pi}(\mathrm{Det}\Xi)^{-1/2}\exp\left[-\frac{1}{2}\tilde{z}\Xi^{-1}z\right] , \tag{148}$$

all contributions from nontrivial initial conditions having washed out in this limit. Converting to the variables ϕ_c and p_c appropriate to the original frame, the distribution function (148) goes over to

$$f_{\mathrm{cl}}(\phi_c,p_c) = \frac{1}{2\pi}(\mathrm{Det}\Xi^{(\phi)})^{-1/2} \times \exp\left[-\frac{1}{2}\left[\Xi_{11}^{(\phi)^{-1}}\phi_c^2 + 2\Xi_{12}^{(\phi)^{-1}}\phi_c p_c + \Xi_{22}^{(\phi)^{-1}}p_c^2\right]\right] . \tag{149}$$

It is important to appreciate that while f_{cl} gives the correct expectation values (Secs. III and IV) and is a perfectly respectable classical distribution, it is not a Wigner function defined from the beginning for the variables ϕ_c and π_c. This is because, as we noted earlier, these distributions are not invariant under canonical transformations. (We are treating the conformal variables χ_L and π_L as the preferred variables to quantize.) However, the key point is that in our case the linear entropy remains invariant under this transformation.

A key observation regarding (149) is that a knowledge of the reduced distribution

$$f_r(\phi_c) \equiv \int_{-\infty}^{+\infty} dp_c f_W(\phi_c,p_c) = \frac{1}{\sqrt{2\pi}}[\Xi_{11}^{(\phi)}]^{-1/2}\exp\left[-\frac{1}{2}\frac{\phi_c^2}{\Xi_{11}^{(\phi)}}\right] \tag{150}$$

is of no use in reconstructing the original distribution. This trivial fact has important consequences if one attempts thermodynamic interpretations of the results from our stochastic analysis using only the reduced distribution. Other points to keep in mind are that, in some cases, to leading order in ϵ, $\mathrm{Det}\Xi^{(\phi)}=0$ (therefore the distribution function is not independent of the cutoff), and that the cross term proportional to $\phi_c p_c$ represents a nontrivial contribution from quantum correlations. In order to discuss these issues more concretely we now return to the specific cases studied earlier.

We consider first the massive free scalar field in an exponentially expanding universe. At late times, with ϵ satisfying the condition (118), we have

$$f_{\mathrm{cl}}(\phi_c,p_c) = \frac{12\pi}{mH_0^2S^3}\exp\left\{-\frac{12\pi^2}{H_0^2}\left[\phi_c^2 - \frac{6}{m^2S^3}\phi_c p_c + \frac{1}{m^2S^6}\left[\frac{9H_0^2}{m^2}+1\right]p_c^2\right]\right\} . \tag{151}$$

Clearly this distribution is not stationary because of the dependence on $S(\eta)$. On the other hand, the corresponding reduced distribution has the "equilibrium" form

$$f_r(\phi_c)=\frac{2m}{H_0^2}\left[\frac{\pi}{3}\right]^{1/2}\exp\left[-\frac{4m^2\pi^2}{3H_0^4}\phi_c^2\right] \tag{152}$$

$$=m\left[\frac{V_H}{2\pi T_{\rm GH}}\right]^{1/2}\exp[-\beta_{\rm GH}E_H(\phi_c)] \tag{153}$$

where $\beta_{\rm GH}^{-1}\equiv T_{\rm GH}\equiv H_0/2\pi$ is the Gibbons-Hawking temperature of de Sitter space [32], $V_H=4\pi H_0^{-3}/3$ is the three-volume within the Hubble radius, and $E_H(\phi_c)\equiv V(\phi_c)V_H$ is the energy of the scalar field within that volume (the kinetic energy is not important if the field is in the "slow-roll" regime). The thermodynamic interpretation of the stochastic formalism [8] was suggested by the striking Boltzmann-like nature of (153). However, the full distribution (151) does not seem to encourage such speculation: it is not stationary nor of the form $e^{-\beta H}$ [nor is the reduced distribution $f_r(p_c)$ of the form $e^{-\beta E_{\rm kin}}$].

We observe that transforming to a new variable $v_c=p_c/S^3$ makes the late-time distribution (153) time independent. However this transformation is not canonical and does not preserve the linear entropy. Therefore the distribution $f(\phi_c,v_c)$ is not physical. In any case such a trick fails for the case of power-law inflation: there the phase space distribution cannot be made time independent.

Turning now to the massless case, at late times:

$$f_{\rm cl}(\phi_c,p_c)=\frac{1}{\epsilon^2S^3}\left[\frac{24\pi^2}{H_0^7(t-t_0)}\right]^{1/2}\exp\left[-\frac{2\pi^2}{H_0^2}\left[\frac{\phi_c^2}{H_0(t-t_0)}-\frac{4\phi_cp_c}{\epsilon^2S^3H_0^2(t-t_0)}+\frac{6p_c^2}{\epsilon^4S^6H_0^2}\right]\right]. \tag{154}$$

The singular nature of this solution as $\epsilon\to0$ is apparent (as is the fact that it is explicitly time dependent). Note, however, that, in this case,

$$f_r(\phi_c)=\left[\frac{\sqrt{2\pi}}{H_0^3(t-t_0)}\right]^{1/2}\exp\left[-\frac{2\pi^2}{H_0^3(t-t_0)}\phi_c^2\right] \tag{155}$$

which is independent of ϵ. The late-time reduced distribution (155) is a solution of the diffusion equation

$$\frac{\partial}{\partial t}f_r(\phi_c)=\frac{1}{2}D\frac{\partial^2}{\partial\phi_c^2}f_r(\phi_c) \tag{156}$$

with $D=H_0^3/4\pi^2$. This is suggestive of a (usual, time-independent) random-walk interpretation. However, the time dependences of the terms $\propto p_c^2$ and $\propto\phi_cp_c$ in the full distribution are hard to reconcile with this view.

The case of a massless field in a power-law spacetime is treated next. Here the late-time full and reduced distributions are, respectively,

$$f_{\rm cl}(\phi_c,p_c)=\frac{4\pi}{\epsilon^2}\frac{\eta_0^3}{S^3}\left[\frac{t}{t_0}\right]^2\left[\frac{3}{p}\right]^{1/2}\exp\left\{-4\pi^2\eta_0^2\left[\frac{\phi_c^2}{p}+\frac{4\eta_0}{\epsilon^2S^3p}\left[\frac{t}{t_0}\right]\phi_cp_c+\frac{3\eta_0^2}{\epsilon^4S^6}\left[\frac{t}{t_0}\right]^4p_c^2\right]\right\}, \tag{157}$$

$$f_r(\phi_c)=2\eta_0\left[\frac{\pi}{p}\right]^{1/2}\exp\left[-\frac{4\pi^2\eta_0^2}{p}\phi_c^2\right]. \tag{158}$$

As with the massless field result (154), the full distribution is again singular in the limit $\epsilon\to0$. Also, the reduced distribution (158) is independent of ϵ, as in the other cases. Unlike the other two cases, however, it does not seem to have any "natural" interpretation.

The late-time linear entropies $\sigma=\int d\phi_c dp_c f_{\rm cl}^2$ for the three cases studied above are, respectively,

$$\sigma_{\rm dSM}=\frac{6\pi}{mH_0^2S^3}=\frac{6\pi}{mH_0^2}e^{-3H_0t}, \tag{159}$$

$$\sigma_{\rm dS}=\frac{\sqrt{6}\pi}{\epsilon^2S^3}\frac{1}{\sqrt{H_0^7(t-t_0)}}=\frac{\sqrt{6}\pi}{\epsilon^2}\frac{e^{-3H_0(t-t_0)}}{\sqrt{H_0^7(t-t_0)}}, \tag{160}$$

$$\sigma_{\rm Pl}=\frac{6\pi\eta_0^3}{\epsilon^2\sqrt{3p}S^3}\left[\frac{t}{t_0}\right]^2=\frac{6\pi\eta_0^3}{\epsilon^2\sqrt{3p}}\left[\frac{t}{t_0}\right]^{3p-2}. \tag{161}$$

In all cases σ is approximately proportional to S^{-3}. The possible significance of this result will be discussed later below.

It is by now clear that the late-time phase-space distributions obtained here are very difficult to fit into a conventional Brownian-motion picture. In fact, this is a very obvious point and manifest in our stochastic Hamiltonian (45). In standard Brownian motion the environment with which the system interacts produces both dissipative and diffusive effects. The dissipative effects arise from the back reaction of the environment. Such an effect is absent in stochastic Hamiltonians of the type (45). In principle, then, there simply cannot be a fluctuation-dissipation theorem of the usual sort: this conclusion is manifest in the fact that in no case are our late-time solutions for the distribution function stationary. (However, this does not mean that there cannot be asymptotically constant values for some average quantities.)

We recall that the origin of the stochastic noise is simply because the "system size" is changing with time and not because of some external interaction. It appears that a mistreatment of this key has led some authors to claim that quantum decoherence occurs in this model. That, in fact, it does not can be explained by the following direct argument for which the author is indebted to Juan Pablo Paz.

The transition from quantum to classical was studied in the context of stochastic inflation by Morikawa [33] and Nambu [34] who analyzed the properties of the evolution operator for the reduced density matrix of the long-wavelength modes. When this propagator is written in path integral form the effect of the short-wavelength modes appears to be contained in a term that is rather similar to the Feynman-Vernon influence functional [27], $F=\exp(i\Gamma)$. In ordinary open systems, the presence of an imaginary part in the influence action Γ produces a tendency towards diagonalization of the reduced density matrix in a fixed basis. This is known as decoherence. For stochastic inflation, the imaginary part of Γ was calculated and related to decoherence. However, it is possible to show that this interpretation is not correct and that there is no decoherence produced by the coarse-graining of stochastic inflation. The basic reason is that the time-dependent nature of the coarse-graining prevents us from interpretating the influence functional in the usual way. In fact the reduced density matrix at a given time can be written as the product $\prod_{k<k_S(t)}\rho_k$. As the number of modes present in the system varies with time, the evolution operator $J(t,t_0)$ has some peculiar properties. It can be written as a product of an evolution operator for each mode $J_k(t,t_0)$ where for $k<k_S(t_0)$ (modes that were already present in the system at $t=t_0$), the $J_k(t,t_0)$ are ordinary unitary operators while for $k_S(t_0)<k<k_S(t)$ (modes that enter the system between t_0 and t) the evolution operator is simply $J_k(t,t_0)=\rho_k(t)$. If one writes these operators in path-integral form one realizes that there are real exponential terms simply due to the fact that, if the state of the field is the vacuum, the reduced density matrix $\rho_r(\phi_k,\phi_k')$ is a Gaussian. The only effect that the "influence functional" has in this case is to generate the above Gaussian factors. It is clear that this is not related to decoherence but to the fact that new modes are entering into the system and that the evolution operator fully contains the reduced density matrix of the incoming modes.

Another argument put forward for a late-time classical limit was that since the commutator (37) is $\propto\epsilon^3$, it is "small" and can be ignored. This of course cannot be correct. The reason is that it is not just a single mode commutator one has to look at but the total integrated contribution from the initial time to the final time of interest. This is not a negligible fraction.

Is there a classical limit or not, intrinsic to the formalism? The linear entropy does decrease exceedingly rapidly as shown by (159)–(161) but it is not clear what this means: we have just argued against interpreting this sort of decrease as being due to quantum decoherence. An intuitive basis for this result may be that it reflects the loss of information inherent in our time-dependent coarse graining. With the passage of cosmic time, two-point functions are averaged over ever smaller comoving volumes. The "smearing" scale is set by $k_S^{-1}=\eta/\epsilon$ and $\eta\to 0$ as $t\to\infty$. Since in our formalism we are tracking only one coarse-grained domain throughout its history this represents a loss of information with cosmic time. We may speculate plausibly that the decrease of σ as S^{-3} supports this viewpoint. However, just because our knowledge is incomplete is no reason to suppose that the Universe is becoming more classical.

At the present stage of analysis and understanding it appears unlikely that the quantum to classical transition in the early Universe can be explained by the stochastic paradigm. In particular, the treatment of density perturbations by modeling quantum fluctuations as classical noise appears to unjustified.

VI. CONCLUSION

This paper's main concern was to model a free field theory in an inflationary universe by way of a stochastic quantum Hamiltonian. It was shown that this approach produced results that agreed well with those from straightforward quantum field theory. Furthermore, the role of the length scale parameter ϵ and of initial conditions was considered more fully than in previous work.

The quantum phase-space distribution used in this paper enables a consideration of quantum correlations that would otherwise be missed. As a result we find that the $\epsilon\to 0$ limit is singular as far as the distribution function is concerned. This means that a finite value of ϵ is essential for the formalism to make sense and that contrary to previous belief this parameter does not drop out of the problem. The full phase-space distribution also enables a critical assessment of such issues as the existence of fluctuation-dissipation relations. We showed that fluctuation-dissipation relations do not hold (as indicated by the fact that the late time solutions are not thermal or even stationary). However, at least in de Sitter space, the reduced distributions for the field variable alone have very suggestive forms corresponding as they do to a Boltzmann distribution at the Gibbons-Hawking temperature for the massive field, and to a "random-walk" distribution for the massless field. No such simple distribution appears in the case of a power-law inflation. The significance of these results remains unclear at present.

We found that, in order to obtain results more or less consistent with conventional field-theoretic calculations, quantum correlations could not be neglected. It was also pointed out that quantum decoherence does not occur in the stochastic approach. As a consequence of these two results, the quantum-to-classical transition in the early Universe does not seem to be intrinsic to the stochastic approach. Directly modeling quantum fluctuations by classical noises as a way to study density perturbations from inflation is therefore a questionable enterprise.

There are of course many unanswered questions, chief among them is what happens when interacting fields are considered and back reaction is included. This we leave to future work. Furthermore, while it is true that the stochastic model "works," at least to some extent, we have stressed that it is not free from interpretational problems. One can only speculate whether insights gained from this approach will actually turn out to be valuable when the full quantum field-theoretic computations are eventually done.

ACKNOWLEDGMENTS

It is a pleasure to thank Larry Ford, Bei-lok Hu, Henry Kandrup, Milan Mijić, Juan Pablo Paz, Bill Unruh, Wojciech Zurek, and Robert Zwanzig for helpful discussions. This research was supported by the Natural Sciences and Engineering Research Council of Canada, the Canadian Institute for Advanced Research, the Canadian Institute for Theoretical Astrophysics, and the United States Department of Energy.

APPENDIX

A brief review of conventional field-theoretic computations of the various expectation values of interest will now be given. This will enable us to check results from the stochastic analysis. To obtain finite results we will impose an infrared cutoff in momentum space at $k=\eta_0^{-1}$ and an ultraviolet cutoff at $k=\epsilon\eta^{-1}$. A discussion of the reasons for picking these cutoffs will be given at the end of the Appendix.

We begin with exponential inflation and treat the two cases of a massless field and of a massive field with a small mass ($m^2 \ll H_0^2$). Considering first the case of a massless field, the parameter $\nu=\frac{3}{2}$ [from (80)], and for the Bunch-Davies vacuum the mode functions $\phi_k \equiv \chi_k/S$ are

$$\phi_k(\eta)=\left[\frac{\pi}{4}\right]^{1/2} H_0\eta^{3/2}H_{3/2}^{(2)}(k\eta) \ . \tag{A1}$$

Consider, first, the equal time expectation value

$$\langle \Phi(\mathbf{x})\Phi(\mathbf{y})\rangle=\frac{1}{2\pi^2}\int_{\eta_0^{-1}}^{\epsilon\eta^{-1}} dk\, k^2\frac{\sin kR}{kR}|\phi_k(\eta)|^2 \ , \tag{A2}$$

where $R=|\mathbf{x}-\mathbf{y}|$. Using the exact form of the Hankel function (90) it is easy to see that

$$|\phi_k(\eta)|^2=\frac{H_0^2}{2k^3}[1+k(\eta)^2] \ ; \tag{A3}$$

hence the integral in (A2) is infrared divergent and an infrared cutoff is necessary. With $kR \ll 1$, we find

$$\langle \Phi^2\rangle=\frac{H_0^2}{4\pi^2}\left[\ln\left[\frac{\epsilon\eta_0}{\eta}\right]+\frac{1}{2}\epsilon^2-\frac{1}{2}\left[\frac{\eta}{\eta_0}\right]^2\right] . \tag{A4}$$

Notice that since $\epsilon\eta^{-1}>\eta_0^{-1}$, $\langle \Phi^2\rangle$ as computed above is strictly positive. To write the result in terms of the cosmic time, we note that $\eta_0\eta^{-1}=\exp H_0(t-t_0)$, in which case

$$\langle \Phi^2\rangle=\frac{H_0^3}{4\pi^2}(t-t_0)+\frac{H_0^2}{4\pi^2}\left[\ln\epsilon+\frac{1}{2}\epsilon^2\right] -\frac{H_0^2}{8\pi^2}e^{-2H_0(t-t_0)} \ . \tag{A5}$$

The last term vanishes at late times and the second term is an irrelevant constant absorbed in the infrared cutoff. The first term gives the usual result [15]. Notice that this term is independent of ϵ; any potential dependence is lost in the infrared cutoff.

Now we turn to the quantity

$$\frac{1}{2}\langle \Phi\dot{\Phi}+\dot{\Phi}\Phi\rangle=\frac{1}{2\pi^2}\int_{\eta_0^{-1}}^{\epsilon\eta^{-1}} dk\, k^2\mathrm{Re}[\phi_k(\eta)\dot{\phi}_k^*(\eta)] \tag{A6}$$

where we have already set $kR \ll 1$. From (A1),

$$\dot{\phi}_k=\left[\frac{\pi}{4}\right]^{1/2} H_0 k\eta^{3/2}H_{1/2}^{(2)}(k\eta) \ , \tag{A7}$$

and it is easy to compute that

$$\frac{1}{2}\langle \Phi\dot{\Phi}+\dot{\Phi}\Phi\rangle=\frac{H_0^2}{8\pi^2}\eta^{-1}\left[\epsilon^2-\left[\frac{\eta}{\eta_0}\right]^2\right] . \tag{A8}$$

If we return to the cosmic time, then with a prime denoting differentiation with respect to t,

$$\frac{1}{2}\langle \Phi\Phi'+\Phi'\Phi\rangle=\frac{H_0^3}{8\pi^2}(\epsilon^2-e^{-2H_0(t-t_0)}) \ . \tag{A9}$$

The last term vanishes at late times and the first term is a constant that is strongly dependent on the upper cutoff. [Even if $\epsilon \ll 1$, consistency requires $\epsilon^2 \gg (\eta/\eta_0)^2$ and the first term always dominates.] The late-time answer being a strong function of the cutoff is simply a consequence of the extra multiplicate factor of k in $\dot{\phi}_k$, which not only renders the integral in (A6) infrared finite but also shifts the dominant contribution from the integrand towards the upper cutoff.

To compute $\langle \dot{\Phi}^2\rangle$ we first use (A7) to show that

$$|\dot{\phi}_k(\eta)|^2=\tfrac{1}{2}H_0^2\eta^2 k \ . \tag{A10}$$

With $kR \ll 1$, it is now easy to find

$$\begin{aligned}\langle \dot{\Phi}^2\rangle&=\frac{1}{2\pi^2}\int_{\eta_0^{-1}}^{\epsilon\eta^{-1}} dk\, k^2|\dot{\phi}_k(\eta)|^2\\ &=\frac{H_0^2}{16\pi^2}\eta^{-2}\left[\epsilon^4-\left[\frac{\eta}{\eta_0}\right]^4\right] ,\end{aligned} \tag{A11}$$

or, in terms of the cosmic time,

$$\langle \Phi'^2\rangle=\frac{H_0^4}{16\pi^2}[\epsilon^4-e^{-4H_0(t-t_0)}] \ . \tag{A12}$$

The late-time value is again a strongly cutoff-dependent constant.

Similar calculations will now be performed for the case of a small mass, i.e., for $m^2 \ll H_0^2$. Disregarding an irrelevant phase term for real ν, the mode functions are now

$$\phi_k(\eta)=\left[\frac{\pi}{4}\right]^{1/2} H_0\eta^{3/2}H_\nu^{(2)}(k\eta) \tag{A13}$$

where, for a small mass,

$$\nu\simeq\frac{3}{2}-\frac{m^2}{3H_0^2} . \tag{A14}$$

Since $k\eta<1$ over the range of integration $\eta_0^{-1}<k<\epsilon\eta^{-1}$, we may approximate the Hankel functions by

$$H_\nu^{(2)}(z)\overset{z\to 0}{\longrightarrow}-\frac{i}{\pi}\Gamma(\nu)\left[\frac{z}{2}\right]^{-\nu} , \tag{A15}$$

and using (A2) compute

$$\langle\Phi^2\rangle\simeq\frac{3H_0^4}{8\pi^2m^2}\left[\epsilon^{2m^2/3H_0^2}-\left[\frac{\eta}{\eta_0}\right]^{2m^2/3H_0^2}\right] \tag{A16}$$

$$=\frac{3H_0^4}{8\pi^2m^2}\left[\epsilon^{2m^2/3H_0^2}-\exp\left[-\frac{2m^2}{3H_0}(t-t_0)\right]\right] . \tag{A17}$$

As long as $\epsilon\gg(-3H_0^2/2m^2)$ (i.e., ϵ cannot be arbitrarily small),

$$\epsilon^{2m^2/3H_0^2}\simeq 1+\frac{2m^2}{3H_0^2}\ln\epsilon+\cdots , \tag{A18}$$

and, at late times,

$$\langle\Phi^2\rangle=\frac{3H_0^4}{8\pi^2m^2} , \tag{A19}$$

a well known result [15].

The calculational strategy used above can also be implemented to find that

$$\frac{1}{2}\langle\Phi\dot{\Phi}+\dot{\Phi}\Phi\rangle\simeq\frac{H_0^2}{8\pi^2}\eta^{-1}\left\{\epsilon^{2m^2/3H_0^2}+\epsilon^2-\left[\frac{\eta}{\eta_0}\right]^{2m^2/3H_0^2}\left[1+\left[\frac{\eta}{\eta_0}\right]^2\right]\right\} , \tag{A20}$$

and that

$$\frac{1}{2}\langle\Phi\Phi'+\Phi'\Phi\rangle\simeq\frac{H_0^3}{8\pi^2}\left[\epsilon^{2m^2/3H_0^2}+\epsilon^2-\exp\left[-\frac{2m^2}{3H_0}(t-t_0)\right](1+e^{-2H_0(t-t_0)})\right] . \tag{A21}$$

Setting $m^2=0$ in (A21) we recover the previous result (A9) for a massless scalar field. The late-time limit for $\epsilon\gg\exp(-3H_0^2/2m^2)$ is

$$\frac{1}{2}\langle\Phi\Phi'+\Phi'\Phi\rangle=\frac{H_0^3}{8\pi^2}(1+\epsilon^2) . \tag{A22}$$

Unlike the massless case, here the result is essentially ϵ independent provided ϵ is small compared to 1. However, unlike the situation for $\langle\Phi^2\rangle$, the late-time value does depend on whether ϵ is small or of order unity.

The expectation value $\langle\Phi'^2\rangle$ can be found in exactly the same way:

$$\langle\Phi'^2\rangle\simeq\frac{H_0^4}{4\pi^2}\left[\frac{m^2}{6H_0^2}\left\{\epsilon^{2m^2/3H_0^2}-\exp\left[-\frac{2m^2}{3H_0}(t-t_0)\right]\right\}+\frac{1}{4}\{\epsilon^4-e^{-4H_0(t-t_0)}\}+\frac{2m^2}{3H_0^2}\{\epsilon^2-e^{-2H_0(t-t_0)}\}\right] . \tag{A23}$$

If we assume ϵ to be much smaller than unity, then at late times we obtain, to leading order, an ϵ-independent result,

$$\langle\Phi'^2\rangle\simeq\frac{m^2H_0^2}{24\pi^2} . \tag{A24}$$

On the other hand if we set ϵ to be one, then the late-time limit becomes

$$\langle\Phi'^2\rangle\simeq\frac{H_0^4}{16\pi^2} \tag{A25}$$

which is completely different from (A24). It is easy to verify that setting $m^2=0$ in (A23) reproduces the answer (A12) for the massless case. Therefore for $\epsilon\sim 1$, the massless and massive cases give the same result (but not when ϵ is small).

We turn now to power-law inflation and consider the case of a massless field. Recall that the parameter ν is now given by

$$\nu=\frac{1-3p}{2(1-p)} \tag{A26}$$

and that the adiabatic vacuum modes are

$$\phi_k(\eta)=\left[\frac{\pi\eta_0}{4}\right]^{1/2}\left[\frac{\eta}{\eta_0}\right]^\nu H_\nu^{(2)}(k\eta) . \tag{A27}$$

The expectation value $\langle\Phi^2\rangle$ is still given by (A2) and we can still use the approximation (A15) for the Hankel function. If we set $\nu=\frac{3}{2}$ then the de Sitter results for a massless field are recovered. For $\nu\neq\frac{3}{2}$, we have

$$\langle\Phi^2\rangle=\frac{2^{2\nu-3}}{\pi^3}\Gamma(\nu)^2\eta_0^{1-2\nu}\int_{\eta_0^{-1}}^{\epsilon\eta^{-1}}dk\,k^{2(1-\nu)}$$
$$=\frac{2^{2\nu-3}\Gamma(\nu)^2}{\pi^3(2\nu-3)}\eta_0^{-2}\left[1-\left[\frac{\epsilon\eta_0}{\eta}\right]^{3-2\nu}\right]$$
$$=\frac{2^{2\nu-3}\Gamma(\nu)^2}{\pi^3(2\nu-3)}\eta_0^{-2}\left[1-\epsilon^{2/(1-p)}\left[\frac{t_0}{t}\right]^2\right] \tag{A28}$$

reproducing the result of [16]. [As long as the power law $p>1$, the parameter $\nu>\frac{3}{2}$, and it follows that the integral in (A28) is infrared divergent. The lower cutoff is necessary to get a finite answer.] We see that $\langle\Phi^2\rangle$ starting from some initial value rises to a constant. When the power law $p\gg 1$, (A26) implies that $\nu\simeq\frac{3}{2}+1/p$, and

$$\langle\Phi^2\rangle\simeq\frac{p}{8\pi^2}\eta_0^{-2}\left[1-\epsilon^{-2/p}\left[\frac{t_0}{t}\right]^2\right]. \tag{A29}$$

If $\epsilon\gg e^{-p/2}$, then

$$\epsilon^{-2/p}\simeq 1-\frac{2}{p}\ln\epsilon+\cdots \tag{A30}$$

and (A29) is essentially cutoff independent not only as to the late-time constant value but also as to how this value is approached. The situation is different when p is not large. For example, if we consider $p=2$ corresponding to $\nu=\frac{5}{2}$,

$$\langle\Phi^2\rangle=\frac{9}{8\pi^2}\eta_0^{-2}\left[1-\epsilon^{-2}\left[\frac{t_0}{t}\right]^2\right]. \tag{A31}$$

This time while the asymptotic value of $\langle\Phi^2\rangle$ is indeed cutoff independent, the approach to it is a strong function of ϵ.

We proceed now to evaluate $\langle\Phi\dot\Phi+\dot\Phi\Phi\rangle/2$. The usual procedure, beginning from (A6), yields

$$\frac{1}{2}\langle\Phi\dot\Phi+\dot\Phi\Phi\rangle\simeq\frac{2^{2\nu-4}}{\pi^3}\Gamma(\nu)\Gamma(\nu-1)\eta_0^{1-2\nu}\eta\times\int_{\eta_0^{-1}}^{\epsilon\eta^{-1}}dk\,k^{4-2\nu}. \tag{A32}$$

The integral in (A32) is infrared divergent for $\nu\geq\frac{5}{2}$ (i.e., $1<p\leq 2$). For $\nu=\frac{5}{2}$,

$$\frac{1}{2}\langle\Phi\dot\Phi+\dot\Phi\Phi\rangle\simeq\frac{3}{4\pi^2}\eta_0^{-4}\eta\ln\left[\frac{\epsilon\eta_0}{\eta}\right] \tag{A33}$$

which vanishes at late times. In terms of the cosmic time,

$$\frac{1}{2}\langle\Phi\Phi'+\Phi'\Phi\rangle\simeq\frac{3}{4\pi^2}\eta_0^{-3}\left[\frac{t_0}{t}\right]^3\ln\left[\frac{\epsilon t}{t_0}\right]. \tag{A34}$$

After an initial period of growth, at late times this expectation value vanishes.

When $p\neq 2$ we obtain, from (A32),

$$\frac{1}{2}\langle\Phi\Phi'+\Phi'\Phi\rangle\simeq\frac{2^{2\nu-1}}{8\pi^3(5-2\nu)}\Gamma(\nu)\Gamma(\nu-1)\eta_0^{-3}\left[\frac{t_0}{t}\right]^3\times\left[\epsilon^{5-2\nu}-\left[\frac{t_0}{t}\right]^{2p-4}\right]. \tag{A35}$$

It is trivial to check that this quantity is always positive. At late times it vanishes but the dependence on ϵ is a function of the power law. For large p, (A35) becomes

$$\frac{1}{2}\langle\Phi\Phi'+\Phi'\Phi\rangle\simeq\frac{\eta_0^{-3}}{8\pi^2}\left[\frac{\eta}{\eta_0}\right]^{3/p}\left[\epsilon^2-\left[\frac{\eta}{\eta_0}\right]^{2-2/p}\right] \tag{A36}$$
$$\simeq\frac{\epsilon^2}{8\pi^2}\eta_0^{-3}\left[\frac{t_0}{t}\right]^3. \tag{A37}$$

If $p\to\infty$, (A36) reproduces the de Sitter result (A9) for a massless field (with $H_0=\eta_0^{-1}$).

Finally we compute $\langle\dot\Phi^2\rangle$. The standard calculation yields

$$\langle\dot\Phi^2\rangle\simeq\frac{\eta_0^{1-2\nu}}{8\pi^3}2^{2(\nu-1)}\Gamma(\nu-1)^2\eta^2\int_{\eta_0^{-1}}^{\epsilon\eta^{-1}}dk\,k^{6-2\nu}. \tag{A38}$$

The k integral is infrared divergent provided $\nu\geq\frac{7}{2}$ (i.e., $1<p\leq\frac{3}{2}$). For the special case $\nu=\frac{7}{2}$ corresponding to $p=\frac{3}{2}$, we have

$$\langle\dot\Phi^2\rangle\simeq\frac{9}{4\pi^2}\eta_0^{-6}\eta^2\ln\left[\frac{\epsilon\eta_0}{\eta}\right] \tag{A39}$$

or

$$\langle\Phi'^2\rangle\simeq\frac{9}{8\pi^2}\eta_0^{-4}\left[\frac{t_0}{t}\right]^4\ln\left[\frac{\epsilon^2 t}{t_0}\right]. \tag{A40}$$

At late times this expectation value vanishes.

In the general case ($\nu\neq\frac{7}{2}$) we find

$$\langle\Phi'^2\rangle\simeq\frac{2^{2(\nu-1)}\eta_0^{-4}}{8\pi^3(7-2\nu)}\Gamma(\nu-1)^2\left[\frac{t_0}{t}\right]^4\times\left[\epsilon^{7-2\nu}-\left[\frac{t_0}{t}\right]^{2(2p-3)}\right] \tag{A41}$$

which also vanishes at late times. When $p\gg 1$, (A41) gives, to leading order,

$$\langle\Phi'^2\rangle\simeq\frac{\eta_0^{-4}}{16\pi^2}\left[\frac{\eta}{\eta_0}\right]^{4/p}\left[\epsilon^4-\left[\frac{\eta}{\eta_0}\right]^{4-2/p}\right] \tag{A42}$$
$$\simeq\frac{\epsilon^4}{16\pi^2}\eta_0^{-4}\left[\frac{t_0}{t}\right]^4. \tag{A43}$$

The limit $p\to\infty$ taken in (A42) gives back the de Sitter result (A12) for a massless field.

We now explain the origin of the cutoffs in the momentum integrals. To prevent infrared divergences we follow the strategy of Ford and Parker [17] by assuming that for $\eta_0 > \eta$ the Universe is radiation dominated and that the quantum state is the conformal vacuum. Matching the field modes and their time derivatives at $\eta = \eta_0$, one finds

$$|C_1(k) - C_2(k)|^2 = \frac{1}{1 + (2^{2\nu}/2\pi)(k\eta_0)^{1-2\nu}\Gamma(\nu)^2} . \quad \text{(A44)}$$

Since the upper cutoff forces $k\eta < 1$, we can use the small argument form (A15) of the Hankel function and compute

$$|\phi_k|^2 \simeq \frac{2^{2\nu}}{4\pi}\Gamma(\nu)^2 \eta_0^{1-2\nu} k^{-2\nu} |C_1(k) - C_2(k)|^2 . \quad \text{(A45)}$$

We are now in a position to compute the preinflationary contributions to the various expectation values of interest. First, consider

$$\begin{aligned}\langle \Phi^2 \rangle_{\text{pi}} &= \frac{1}{2\pi^2} \int_0^{\eta_0^{-1}} dk\, k^2 |\phi_k|^2 \\ &\simeq \frac{1}{4\pi^2} \int_0^{\eta_0^{-1}} dk\, k \\ &= \frac{\eta_0^{-2}}{8\pi^2} ,\end{aligned} \quad \text{(A46)}$$

which is a constant *independent* of ν. In a similar fashion we can evaluate

$$\frac{1}{2}\langle \Phi\Phi' + \Phi'\Phi \rangle_{\text{pi}} \simeq \frac{\eta_0^{-3}}{32\pi^2(\nu - 1)} \left[\frac{\eta}{\eta_0} \right]^{\nu + 1/2} \quad \text{(A47)}$$

and

$$\langle \Phi'^2 \rangle_{\text{pi}} \simeq \frac{\eta_0^{-4}}{96\pi^2(\nu - 1)^2} \left[\frac{\eta}{\eta_0} \right]^{2\nu + 1} . \quad \text{(A48)}$$

If we restrict attention to de Sitter space, then (A46)–(A48) specialize to

$$\langle \Phi^2 \rangle_{\text{pi}} \simeq \frac{H_0^2}{8\pi^2} , \quad \text{(A49)}$$

$$\frac{1}{2}\langle \Phi\Phi' + \Phi'\Phi \rangle_{\text{pi}} \simeq \frac{H_0^3}{16\pi^2} e^{-2H_0(t - t_0)} , \quad \text{(A50)}$$

$$\langle \Phi'^2 \rangle_{\text{pi}} \simeq \frac{H_0^4}{24\pi^2} e^{-4H_0(t - t_0)} . \quad \text{(A51)}$$

Comparison with (A5), (A9), and (A12) shows that the late time results are unaffected: the dominant contribution to these expectation values comes from the inflationary sector.

In the case of a power-law inflation, we find

$$\langle \Phi^2 \rangle_{\text{pi}} \simeq \frac{\eta_0^{-2}}{8\pi^2} , \quad \text{(A52)}$$

$$\frac{1}{2}\langle \Phi\Phi' + \Phi'\Phi \rangle_{\text{pi}} \simeq \frac{\eta_0^{-3}}{16\pi^2} \left[\frac{p+1}{p-1} \right] \left[\frac{t_0}{t} \right]^{2p-1} , \quad \text{(A53)}$$

$$\langle \Phi'^2 \rangle_{\text{pi}} \simeq \frac{\eta_0^{-4}}{24\pi^2} \left[\frac{p-1}{p+1} \right]^2 \left[\frac{t_0}{t} \right]^{2(2p-1)} . \quad \text{(A54)}$$

The late time constant value of $\langle \Phi^2 \rangle$ does get shifted due to (A52) but for large p this shift is negligible as comparison with (A29) shows. Even for relatively small values of p, this term is relatively unimportant [compare with (A31) for $p = 2$]. From (A37) and (A43) we know that for $p \gg 1$, the contributions from the inflationary sector to $\langle \Phi\Phi' + \Phi'\Phi \rangle/2$ and $\langle \Phi'^2 \rangle$ fall off as $(t_0/t)^3$ and $(t_0/t)^4$ respectively. These falloffs are much slower than those given by (A53) and (A54): again the preinflationary contributions are insignificant. It is only for weak power-law expansions ($p \sim 1$) that this sector is of any significance.

The rationale for the upper cutoff is simple. In quantum field theory in curved spacetime nontrivial ultraviolet divergences can arise because of the spacetime curvature. In principle one has to apply an appropriate regularization scheme (point splitting, for example) followed by an ultraviolet subtraction. Following this more sophisticated procedure one obtains terms proportional to the curvature in $\langle \Phi^2 \rangle$. However, these terms either vanish at late times (power-law inflation) or are constants which are small compared to the contribution from the infrared sector (exponential inflation). In this sense our procedure is justified. (The fact that $k\eta \sim 1$ separates the low- and high-frequency sectors is due to the following behavior of the Hankel functions: oscillatory for $k\eta \gg 1$, and power law for $k\eta \ll 1$.)

[1] A. A. Starobinsky, in *Fundamental Interactions* (MGPI Press, Moscow, 1983); *Current Topics in Field Theory, Quantum Gravity, and Strings,* Proceedings of the Seminar, Meudon and Paris, France, 1984–1985, edited by H. J. de Vega and N. Sanchez, Lecture Notes in Physics Vol. 226 (Springer, New York, 1986). See also A. Vilenkin, Nucl. Phys. **B226**, 527 (1983).

[2] D. S. Salopek and J. R. Bond, Phys. Rev. D **43**, 1005 (1991); G. V. Chibisov and Yu. V. Shtanov, Int. J. Mod. Phys. A **13**, 2625 (1990).

[3] A. S. Goncharov, A. D. Linde, and V. F. Mukhanov, Int. J. Mod. Phys. A **2**, 561 (1987).

[4] D. Coule and M. Mijić, Int. J. Mod. Phys. A **3**, 617 (1988).

[5] A. Ortolan, F. Lucchin, and S. Matarrese, Phys. Rev. D **38**, 465 (1988).

[6] H. E. Kandrup, Phys. Rev. D **39**, 2245 (1989).

[7] M. Mijić, Phys. Rev. D **42**, 2469 (1990).

[8] S. Habib and M. Mijić, UBC report, 1991 (unpublished).

[9] B.-L. Hu, J. P. Paz, and Y. Zhang, Maryland report, 1991 (unpublished).

[10] For a similar comment, but for somewhat different reasons, see E. A. Calzetta, Ann. Phys. (N.Y.) **190**, 32 (1989).

[11] S. A. Fulling, Gen. Relativ. Gravit. **10**, 807 (1979); *Aspects of Quantum Field Theory in Curved Spacetime* (Cambridge University Press, Cambridge, England, 1989).

[12] N. Weiss, Phys. Rev. D **34**, 1768 (1986).

[13] This is discussed by J. R. Ray, Am. J. Phys. **47**, 626 (1979).

[14] See, e.g., Fulling, *Aspects of Quantum Field Theory in Curved Spacetime* [11]; N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).
[15] A. Vilenkin and L. H. Ford, Phys. Rev. D **26**, 1231 (1982); A. D. Linde, Phys. Lett. **116B**, 335 (1982); A. A. Starobinsky, *ibid.* **117B**, 175 (1982).
[16] V. Sahni, Class. Quantum Grav. **5**, L113 (1988); C. Pathinayake and L. H. Ford, Phys. Rev. D **37**, 2099 (1988).
[17] L. H. Ford and L. Parker, Phys. Rev. D **16**, 245 (1977).
[18] T. S. Bunch and P. C. W. Davies, Proc. R. Soc. London **A360**, 117 (1978).
[19] See, e.g., M. Sasaki, Y. Nambu, and K. Nakao, Nucl. Phys. **B308**, 868 (1988).
[20] E. Merzbacher, Physica **96A**, 263 (1979).
[21] S. Habib (in preparation).
[22] E. P. Wigner, Phys. Rev. **40**, 749 (1932).
[23] See, e.g., L. Cohen, J. Math. Phys. **7**, 781 (1966).
[24] M. Hillery, R. F. O'Connell, M. O. Scully, and E. P. Wigner, Phys. Rep. **106**, 121 (1984).
[25] F. J. Narcowich, in *Seminars in Mathematical Physics* (Texas A&M University, College Station, Texas, 1986).
[26] R. Kubo, J. Math. Phys. **4**, 174 (1963).
[27] R. P. Feynman and F. L. Vernon, Ann. Phys. (N.Y.) **24**, 118 (1963).
[28] F. R. Graziani, Phys. Rev. D **38**, 1122 (1988); **38**, 1131 (1988); **38**, 1802 (1988).
[29] A discussion of entropies based on the Wigner function is given by S. Habib and H. E. Kandrup, Ann. Phys. (N.Y.) **191**, 335 (1989).
[30] See, e.g., S. Habib and R. Laflamme, Phys. Rev. D **42**, 4056 (1990), J. P. Paz and S. Sinha, *ibid.* **44**, 1038 (1991), and references therein.
[31] See, e.g., H. Risken, *The Fokker-Planck Equation* (Springer, New York, 1984); N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981).
[32] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2738 (1977).
[33] M. Morikawa, Phys. Rev. D **42**, 1027 (1990).
[34] Y. Nambu, Hiroshima report, 1991 (unpublished).