Bubble nucleation in first-order inflation and other cosmological phase transitions

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We address in some detail the kinematics of bubble nucleation and percolation in first-order cosmological phase transitions, with the primary focus on first-order inflation. We study how a first-order phase transition completes, describe measures of its progress, and compute the distribution of bubble sizes. For example, we find that the typical bubble size in a successful transition is of order 1% to 100% of the Hubble radius, and depends very weakly on the energy scale of the transition. We derive very general conditions that must be satisfied by Γ/H^4 to complete the phase transition (Γ=bubble nucleation rate per unit volume; H=expansion rate; physically, Γ/H^4 corresponds to the volume fraction of space occupied by bubbles nucleated over a Hubble time). In particular, Γ/H^4 must exceed $9/4\pi$ to successfully end inflation. To avoid the deleterious effects of bubbles nucleated early during inflation on primordial nucleosynthesis and on the isotropy and spectrum of the cosmic microwave background radiation, during most of inflation Γ/H^4 must be less than order 10^{-4}–10^{-3}. Our constraints imply that in a successful model of first-order inflation the phase transition must complete over a period of at most a few Hubble times and all but preclude individual bubbles from providing an interesting source of density perturbation. We note, though, that it is just possible for Poisson fluctuations in the number of moderately large-size bubbles to lead to interesting isocurvature perturbations, whose spectrum is not scale invariant. Finally, we analyze in detail several recently proposed models of first-order inflation.

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I. INTRODUCTION

The Universe may well have undergone several first-order phase transitions and important "fossils" of these phase transitions, including the isotropy and homogeneity of the Universe, gravitational waves, and possibly even the baryon asymmetry of the Universe, may exist today (see, e.g., Ref. [1]). Of current interest are first-order inflationary models [2] and first-order symmetry-breaking phase transitions (e.g., electroweak symmetry breaking [3]). During a first-order phase transition the Universe gets "hung up" in a metastable ("false-vacuum") state, and the transition to the new phase ("true vacuum") proceeds through the nucleation of bubbles of the new phase. Whether or not the Universe "recovers" from a first-order phase transition and any relics left behind depends upon the kinematics of bubble nucleation and how the eventual transition to the new phase is accomplished. We focus here primarily on inflationary phase transitions, in which the Universe gets hung up for at least 60 or so Hubble times (to solve the horizon and flatness problems). However, most of our results are also applicable to first-order cosmological phase transitions that are completed after only a moderate degree of supercooling below the critical temperature and proceed through the nucleation of thermal bubbles.

There are several requirements that must be satisfied in a successful inflationary transition: sufficient inflation, a "graceful exit," density perturbations of an appropriate size, the absence of unwanted relics such as magnetic monopoles, and so on [4]. Here we are concerned with the graceful exit from the false-vacuum phase, during which the expansion of the Universe is exponential (or at least accelerating), back to a very isotropic and homogeneous radiation-dominated phase. (Based upon primordial nucleosynthesis we are confident that the Universe was isotropic and homogeneous and radiation dominated by the time it was about 10^{-2} s old.) In first-order inflation the transition proceeds via the nucleation of true-vacuum bubbles, and whether or not the Universe successfully makes this transition depends crucially upon the relationship between the expansion rate H and the bubble-nucleation rate per unit volume Γ. Roughly speaking, when $\Gamma/H^4 \ll 1$ bubble nucleation is rare and the Universe is trapped in the false vacuum; when Γ/H^4 becomes of order unity bubble nucleation quickly completes the phase transition [5]. (The physical significance of this criterion will become clearer when we show that Γ/H^4 corresponds to the volume fraction of space occupied by bubbles nucleated over a Hubble time.) In "old inflation," the original example of first-order inflation, both Γ and H were constant, and thus it was impossible to both inflate (remain trapped in the false vacuum for at least 60 or so e-foldings of the scale factor) and gracefully exit [6].

In the more recent models of first-order inflation Γ and/or H can vary significantly during inflation, and thus it is possible for Γ/H^4 to be small at early times and large at late times. The variation of both Γ and H is frame dependent, as discussed in Ref. [7]; for our purposes this fact is not important. While the variation of Γ and/or H allows for both sufficient inflation and a graceful exit, a new worry arises: the production of bubbles relatively early during the inflationary phase which eventually reach astrophysically interesting sizes. These bubbles have the potential to leave unwanted "scars" (e.g., distortions of the microwave background or interference with primordial nucleosynthesis). We will find that the suppression of big bubbles requires that at early times Γ/H^4 must be less than about 10^{-4} or so; on the other hand, successful completion of the phase transition requires that Γ/H^4 must be greater than about $9/4\pi$ at late times. As we shall discuss, this poses a real challenge for those building models of first-order inflation.

The analysis of the most general model of first-order inflation, or of a first-order cosmological phase transition, is a formidable task. The approach of this paper is to begin with the most general considerations, and then to proceed to the specific. To wit, in the next section we discuss the history of a bubble of true vacuum from its nucleation to the homogenization of the energy it releases. In Sec. III we address the adverse effects of "big bubbles," and show that cosmological considerations based upon primordial nucleosynthesis and the cosmic microwave background radiation severely constrain their number and thereby the ratio Γ/H^4 during inflation. In Sec. IV we discuss bubble kinematics and measures of the progress, and ultimate completion, of a first-order phase transition, and in Sec. V we apply the results of Secs. III and IV to specific models of first-order inflation. We finish with a summary and concluding remarks in Sec. VI.

II. THE HISTORY OF A BUBBLE

In a strongly first-order cosmological phase transition the Universe gets hung up in a metastable, false-vacuum state, and its eventual evolution to the true vacuum proceeds through the quantum nucleation of true-vacuum bubbles. (By strongly first order, we mean a phase transition in which there is very significant supercooling; i.e., the temperature of the Universe when the transition completes is much less than the critical temperature.) Once nucleated, the evolution of true-vacuum bubbles is essentially classical; they expand and, in a successful first-order transition, eventually fill all of space, completing the transition from the old phase to the new phase. In the process entropy is produced; in first-order inflation this entropy production accounts for all of the thermal energy in the Universe today. Until we address the progress of a phase transition in detail in Sec. IV we will be somewhat vague about precisely when a phase transition is completed; as we shall see, in a successful phase transition the epoch can be fixed to within a few Hubble times. For the moment, we simply denote the end of the phase transition by a subscript asterisk.

[In many cases of cosmological interest the true vacuum and false vacuum can be identified with values of a scalar field ϕ, with the former corresponding to the global minimum of the scalar-field potential $V(\phi)$ and the latter to a higher-energy local minimum. Although we will occasionally find it convenient to refer to such a scalar field, it is irrelevant to our analysis whether or not the phase transition has anything to do with a fundamental scalar field.]

The decay of the false vacuum is a well-understood problem in quantum field theory [8], and the details of quantum bubble nucleation need not concern us here. Our interest is in bubble kinematics. To this end, it suffices to know that the quantum mechanical (i.e., nonthermal) bubble nucleation rate per unit volume is generally of the form

$$\Gamma = Ce^{-A} . \tag{2.1}$$

The prefactor has units of energy to the fourth power and is expected to be of the order of $\mathcal{M}^4$, where $\mathcal{M} \equiv \mathcal{M}_{14} 10^{14}$ GeV is the energy scale that characterizes the phase transition. For definiteness we define $\mathcal{M}$ to be the fourth root of the false-vacuum energy density (i.e., $\rho_{\rm vac} \equiv \mathcal{M}^4$). In first-order inflation the tunneling action A can be time dependent because of its dependence upon the evolution of other scalar fields. Although a vacuum bubble has a finite radius ($\equiv r_0$) at the time of its nucleation, its initial size can be neglected for our purposes. Likewise, we assume that the speed of expansion of a vacuum bubble is that of light, even though it only asymptotically approaches the speed of light, $v_{\rm bubble} = 1/\sqrt{1 + r_0^2/t^2}$. (If $\mathcal{M}$ is comparable to the Planck scale, $m_{\rm Pl} = 1.22 \times 10^{19}$ GeV, gravitational corrections to the bubble-nucleation rate and bubble-wall velocity can be significant. We shall assume throughout that $\mathcal{M} \ll m_{\rm Pl}$, so that gravitational effects are not important [9].)

At early times the interior of a bubble of true vacuum is empty; the false-vacuum energy ("latent heat") liberated by the bubble as it expands outward resides in the "kinetic energy" of the bubble wall (in both spatial and temporal gradients of the scalar field); the surface-energy density in the bubble wall $\sigma(t) \simeq \mathcal{M}^4 t/3$. When two bubbles collide the kinetic energy carried by their walls is converted into massive scalar particles which eventually decay into relativistic particles that then thermalize [10]. The time scale τ for both decay and thermalization is set by the mass of the scalar field and its couplings to other fields. Very roughly, we expect $\tau \sim m_\phi^{-1} \sim \mathcal{M}^{-1} \sim 10^{-38}$ s$/\mathcal{M}_{14}$. Schematically, the flow of the false-vacuum energy proceeds: false-vacuum energy$\rightarrow$bubble-wall energy$\rightarrow$scalar particles$\rightarrow$decay products$\rightarrow$thermal energy.

[Some aspects of this picture must be modified for the case of the thermal nucleation of bubbles. First, the action in Eq. (2.1) is replaced $E(T)/T$, where $E(T)$ is roughly the energy of a critical bubble, and the bubble-nucleation rate is thus explicitly temperature dependent. Second, thermal-bubble walls do not expand at the speed of light, but rather at some temperature-dependent velocity $v(T)<1$. Finally, not all of the latent heat is carried outward in the bubble walls; instead, some of it propagates into the bubble interior where it is thermalized. So far as the kinematics of bubble nucleation is concerned the only significant difference is the bubble-wall velocity.]

As we shall discuss, in a successful first-order phase transition most of the bubbles are nucleated and collide in a time interval comparable to or shorter than the Hubble time H^{-1}. After bubbles collide, their interiors, initially devoid of radiation, are rapidly filled as thermal radiation diffuses in, and the distribution of energy in the Universe becomes homogeneous. (Particle horizons play no role because the relevant bubbles are subhorizon size when they collide; moreover the Hubble radius is growing relative to them.) Any potential scars that could result from the nucleation and percolation of bubbles are on the scale of the Hubble radius at the end of inflation (or smaller), and this corresponds to a comoving scale of only about 10^{-21} Mpc/$\mathcal{M}_{14}$.

In this section and in the next section we discuss at length the rare, very large bubbles that were nucleated long before the end of inflation; these bubbles can grow to astrophysically interesting sizes. Their story line is very different and covers a much longer period of time; see Fig. 1. It is useful to characterize these bubbles by the epoch when they were nucleated. Specifically, for exponential inflation we label a bubble by the number of e-foldings N that the cosmic-scale factor $R(t)$ grows from the time it is nucleated until the end of inflation. The reason for this is simple. After a bubble is nucleated, it expands at the speed of light (and grows in comoving size); about a Hubble time after nucleation it crosses outside the "horizon" (more precisely, we mean that its physical size becomes larger than the Hubble radius H^{-1}). Thereafter it conformally stretches so that its physical size grows as the cosmic-scale factor R. At the end of inflation the radius of a "big bubble," up to factors of order unity, is e^N times the Hubble radius, which sets the scale for the size of typical bubbles being nucleated as the phase transition is being completed. (We will make these arguments more precise in Sec. IV.)

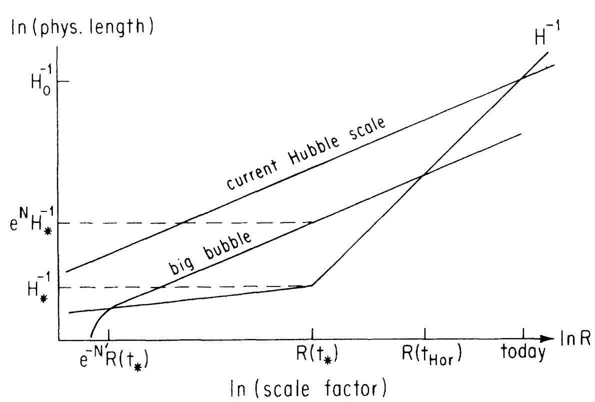

FIG. 1. The evolution of the physical size of a vacuum bubble from its nucleation to the time it reenters the horizon and for reference that of the comoving scale that corresponds to the current Hubble radius. For a time of order a Hubble time after its nucleation the coming size of a bubble grows; thereafter the size of a bubble simply stretches with the expansion.

If inflation is not exponential, the Hubble parameter varies with time, so that the ratio of the bubble size at the end of inflation to the Hubble radius at that time is not the same as the increase in the scale factor over the bubble's lifetime. For example, in power-law inflation, where $R(t)\propto t^p$ ($p>1$ is required for superluminal expansion), the Hubble parameter falls inversely with t. If we denote the value of the Hubble parameter at the end of inflation by H_*, then a bubble of size $e^N H_*^{-1}$ at the end of inflation would have had a radius equal to the Hubble radius at a time t when the scale factor was $N'\equiv Np/(p-1)$ e-foldings smaller than its value at the end of inflation (and would have been nucleated shortly before that). Note that as the growth of the scale factor becomes more rapid, i.e., $p\to\infty$, $N'\to N$.

Because a big bubble is superhorizon size at the end of inflation, it continues to be conformally stretched and to grow as $R(t)$; if it continued to grow conformally until today, its present size D would be

$$\frac{D}{1\text{ Mpc}}\simeq\left[\frac{\mathcal{M}}{3\text{ K}}\right]\frac{e^N H_*^{-1}}{1\text{ Mpc}}\simeq\frac{e^{N-48}}{\mathcal{M}_{14}}\equiv e^{N-n_{14}-48}\,,\qquad(2.2)$$

where $n_{14}=\ln\mathcal{M}_{14}$ and we have assumed that the Universe enters the radiation-dominated era with a temperature of order $\mathcal{M}$. If the cosmic-scale factor is normalized to be unity at the present, D is also the comoving size of the bubble.

Equally, if not more important, is the time when the bubble crosses back inside the horizon (again, to be precise, we mean inside the Hubble radius); it is at this epoch that it first becomes possible for light to cross the bubble, and thus for radiation to diffuse in, thereby allowing the bubble to become homogenized. The temperature of the Universe at horizon crossing is related to D by

$$T_{\text{hor}}\sim\frac{300\text{ eV}}{D/\text{Mpc}}\sim e^{48-N+n_{14}}(300\text{ eV})\,,$$
$$D\lesssim 13h^{-2}\text{ Mpc (or }N\lesssim 50+n_{14})\,,\qquad(2.3a)$$
$$T_{\text{hor}}\sim\frac{1000\text{ eV}}{(D/\text{Mpc})^2}\sim e^{96-2N+2n_{14}}(1000\text{ eV})$$
$$D\gtrsim 13h^{-2}\text{ Mpc (or }N\gtrsim 50+n_{14})\,,\qquad(2.3b)$$

where the present value of the Hubble parameter $H_0=100h$ km s^{-1} Mpc^{-1}, and the comoving scale $13h^{-2}$ Mpc separates bubbles that cross back inside the Hubble radius during the radiation-dominated epoch ($T_{\text{hor}}\gtrsim 10h^2$ eV) and those that do so during the matter-dominated epoch ($T_{\text{hor}}\lesssim 10h^2$ eV).

We are now ready to discuss the story line for a big bubble. When the phase transition ends via the rapid nucleation of Hubble-size (or smaller) bubbles, big bubbles are very much superhorizon size: size on the of order $e^N H_*^{-1}$. The energy carried by the expanding wall of a big bubble is thermalized through collisions with many Hubble-size bubbles. On the face of it a collision between a big bubble and a Hubble-size bubble is very different from that of two Hubble-size bubbles. However, when viewed in a frame that is Lorentz-boosted along the axis that connects their centers (with Lorentz factor $\gamma_{\rm big}=e^{N/2}$) the two colliding bubbles are of equal size. In the "equal-bubble" frame the conversion of bubble-wall energy into particles proceeds as before and takes a time of order τ; seen in the Friedmann-Robertson-Walker (FRW) frame (i.e., the rest frame of the Universe) it takes a factor of $\gamma_{\rm big}$ longer. Since τ is expected to be of order $\mathcal{M}^{-1}$ ($\sim 10^{-38}$ s/$\mathcal{M}_{14}$) and for the largest bubbles $\gamma_{\rm big}$ is at most 10^{11} or so, the time it takes to convert the kinetic energy of the expanding wall of even the largest bubble into particles is still very short ($\sim 10^{-27}$ s/$\mathcal{M}_{14}$).

If the particles produced in the equal-bubble frame are characterized by energy $\mathcal{M}$, in the FRW frame they have energy $\gamma_{\rm big}\mathcal{M}$ and move in the original direction of the motion of the big-bubble wall, thereby comprising a rapidly expanding, thin shell of very energetic particles. As these particles interact with the thermal bath of particles in the Universe their energies are quickly degraded. Even though the process of converting the false-vacuum energy to thermal radiation proceeds relatively quickly, the interior of a big bubble remains empty for a long time: On the basis of causality along the interior cannot be filled with thermal radiation until the bubble crosses back inside the horizon. Therein lies the fundamental danger of big bubbles: They represent highly inhomogeneous regions, a relatively empty region of space, surrounded by a region of higher energy density, that cannot be homogenized until a relatively late epoch, cf. Eq. (2.3) [11].

Before going on to discuss the perils associated with big bubbles and the constraints to Γ/H^4 that follow, let us relate the volume fraction occupied by big bubbles to the bubble nucleation rate Γ and expansion rate H. After a big bubble is nucleated and has grown to a size of about a Hubble radius, its size simply conformally stretches as the Universe expands. From that point on, the volume fraction of the Universe that it occupies remains constant. Thus the volume fraction occupied by the bubbles nucleated in the time interval t to $t+H^{-1}$ is roughly the volume of such a bubble when it begins conformally stretching, about $H^{-3}(t)$, times the number of such bubbles nucleated per unit volume, $\Gamma(t)H(t)^{-1}$. Noting that the bubbles nucleated over a Hubble time span roughly one e-folding in size, it follows that the volume fraction occupied per logarithmic interval in bubble size is

$$\frac{df_{\mathcal{V}}}{d\ln D}\approx\left[\frac{\Gamma}{H^4}\right]_N , \tag{2.4}$$

where the subscript N indicates that Γ/H^4 is to be evaluated at the time such bubbles were nucleated. Recall, e^N is the size of a big bubble at the end of inflation relative to the Hubble radius, while $e^{N'}$ is the increase in the scale factor from the time the bubble was nucleated until the end of inflation. If the scale factor increases exponentially during inflation, $N'=N$; if it increases as a power, $R\propto t^p$, then $N'=pN/(p-1)$. In any case, D and N are related by Eq. (2.2). Equation (2.4) makes the physical significance of $(\Gamma/H^4)_N$ manifest: It is the volume fraction of space occupied by bubbles nucleated over a Hubble time at a given epoch. We will derive a more precise version of this relationship in Sec. IV.

III. BIG BUBBLES ARE BAD

As a number of authors [12,13] have emphasized, big bubbles are a potentially dangerous relic of first-order inflation. The absence of scars in the postinflationary Universe associated with big bubbles can be used to sharply constrain the value of Γ/H^4 at various times during the inflationary epoch well before the end of inflation. The fact that Γ/H^4 must exceed $9/4\pi$ to successfully end inflation (derived in Sec. IV), together with our big-bubble constraints imply that Γ/H^4 must be very small for most of inflation, and then rapidly increase at the end of inflation. As we discuss in Secs. V and VI, this provides a considerable challenge for building models of first-order inflation.

Although some [14] have expressed the hope that the production of a few astrophysical-size bubbles might actually play a beneficial role, e.g., accounting for the voids seen in the distribution of galaxies, the big-bubble constraints derived here dash that hope. However, as we discuss at the very end of this section, it may still be possible that isocurvature density perturbations arising from Poisson fluctuations in the number of bubbles within a given volume could be of some relevance for the formation of structure in the Universe.

A. Primordial nucleosynthesis

The standard scenario of primordial nucleosynthesis is remarkably successful, accounting for the primeval abundances of D, ^{3}He, ^{4}He, and ^{7}Li, provided that the baryon-to-photon ratio η lies in the narrow interval 3×10^{-10} to 5×10^{-10} and the effective number of relativistic degrees of freedom at the time of nucleosynthesis corresponds to the equivalent of at most 3.4 light neutrino species [15]. Moreover, attempts to construct alternative scenarios for primordial nucleosynthesis have been remarkably unsuccessful. Consider, for example, nucleosynthesis in a model universe with large fluctuations in the local baryon-to-photon ratio that could have resulted, e.g., from a strongly first-order quark-hadron transition. When the assumed level of inhomogeneity in the baryon-to-photon ratio is raised to the point that it actually significantly affects the yields of nucleosynthesis, it is no longer possible to obtain agreement between the predicted and observed primordial abundances for any set of parameters [16]. With this fact in mind we use primordial nucleosynthesis to constrain deviations from homogeneity at this early epoch ($T\sim1$ MeV and $t\sim1$ s).

The basic idea underlying this constraint is that bubbles that are Hubble-size or larger at the epoch of nucleosynthesis are not yet homogenized and therefore have the potential to upset the successful predictions of nucleosynthesis. While we can be certain that nucleosynthesis in unhomogenized regions proceeds radically differently, precise statements are beyond the scope of the present work. The potential to overproduce D, ^{3}He, or ^{7}Li is great, since these light isotopes are produced in very small quantities in the standard scenario: a few parts in 10^5 for ^{3}He and D, and a few parts in 10^{10} for ^{7}Li. (In fact, in the aforementioned scenarios of inhomogeneous nucleosynthesis overproduction of all three of these light isotopes is a problem.) To be absolutely "^{7}Li safe" one would have to require that the fraction of space filled by superhorizon-size bubbles at the time of nucleosynthesis ($\equiv f_{\rm BBN}$) be less than about 10^{-10}. Then, even if in a big-bubble region the ^{7}Li production were 100%, after such bubbles homogenized and their nucleosynthesis yields were mixed with those of the rest of the Universe, ^{7}Li would necessarily be diluted to an acceptable level. For ^{3}He and D, the corresponding "absolutely safe limit" would be $f_{\rm BBN} \lesssim 10^{-5}$.

No doubt $f_{\rm BBN} \lesssim 10^{-10}$ or even $f_{\rm BBN} \lesssim 10^{-5}$ is unnecessarily stringent. In the absence of a more detailed analysis, one might prudently require $f_{\rm BBN} \lesssim 10^{-3}$ or so, which is the constraint that we adopt. This leads to the bound

$$f_{\rm BBN} \approx \int_{39+n_{14}}^{60+n_{14}} dN \left[\frac{\Gamma}{H^4}\right]_N \lesssim 10^{-3} \ . \tag{3.1a}$$

Here the lower limit corresponds to bubbles that are Hubble size when nucleosynthesis commences ($T_{\rm BBN} \sim 1$ MeV), while the upper limit corresponds to bubbles that are Hubble size today (since there are clearly no bubbles larger than this in our past light cone, they cannot be relevant for observational bounds). In models leading to successful transitions $(\Gamma/H^4)_N$ is generally a decreasing function of N, and the integral in Eq. (3.1) is dominated by the contribution from the lower end of the integration range, so that this bound is essentially a constraint to $(\Gamma/H^4)_{39+n_{14}}$:

$$\left[\frac{\Gamma}{H^4}\right]_{N \simeq 39+n_{14}} \lesssim c_a \times 10^{-3} \ , \tag{3.1b}$$

where c_a is of order unity.

B. CMBR distortions

The decoupling of (baryonic) matter and radiation occurs at a temperature of order 0.3 eV (provided that the Universe is not subsequently reionized, in which case final decoupling could occur as late as redshift $z \sim 60$). Needless to say the presence of a large, unhomogenized bubble near the last scattering surface would lead to a massive distortion of the temperature of the cosmic microwave background radiation (CMBR) [17]. The CMBR temperature is spatially uniform to almost a part in 10^5 on angular scales from less than 1° to 90° (aside from the dipole anisotropy, which is presumably due to our motion with respect to the cosmic rest frame) [18], and the CMBR spectrum is consistent with that of a blackbody at temperature 2.74 ± 0.01 K over almost three and one-half decades in wavelength ($\lambda \sim 100$ cm-0.03 cm) [19]. The horizon scale at decoupling corresponds to an angular scale of about 1° and a comoving length of about $100h^{-1}$ Mpc. (If decoupling occurred much later, say at redshift z_D, the decoupling horizon corresponds to $0.9° \sqrt{1000/z_D}$.) Thus bubbles of comoving size $D \gtrsim 100h^{-1}$ Mpc remain unhomogenized at decoupling; moreover, if such a bubble lay on the last scattering surface it would produce an order unity fluctuation in the CMBR temperature on an angular scale $(D/100h^{-1}$ Mpc$)°$, in clear conflict with current observations.

Based upon the absence of massive distortions in the CMBR temperature we can be certain that there could have been no bubble of size $100h^{-1}$ Mpc$-10^4\, h^{-1}$ Mpc (angular size 1° to 90°) on the last scattering surface [18,19]. Recall that the fraction of space occupied by bubbles in a unit logarithmic interval of size is about $(\Gamma/H^4)_N$ and that the size of a bubble on the microwave sky is $(D/100h^{-1}\ {\rm Mpc})^2$ square degrees, where D and N are related by Eq. (2.2). The entire sky contains of order 30 000 square degrees; thus the number of bubbles in a unit logarithmic interval around size D expected on the sky is 30 000 $(100h^{-1}\ {\rm Mpc}/D)^2(\Gamma/H^4)_N$. If we very conservatively assume that there could be no more than 10 or so large bubbles on the last scattering surface, we obtain the bound

$$\left[\frac{\Gamma}{H^4}\right]_N \lesssim 3\times 10^{-4}(D/100h^{-1}\ {\rm Mpc})^2 \ , \qquad 100h^{-1}\ {\rm Mpc} \lesssim D \lesssim 10^4\ h^{-1}\ {\rm Mpc} \ , \tag{3.2a}$$

which is most stringent on the scale $100h^{-1}$ Mpc:

$$\left[\frac{\Gamma}{H^4}\right]_{N \simeq 53+n_{14}} \lesssim 3\times 10^{-4} \ . \tag{3.2b}$$

This is in essence the original big-bubble constraint derived in Ref. [12].

Next, consider spectral distortions of the CMBR. These are usually referred to as μ distortions, because the CMBR is characterized by a Bose-Einstein distribution with nonzero chemical potential μT,

$$n(p) = \frac{1}{\exp(\mu + p/T) - 1} \ , \tag{3.3}$$

rather than a simple Planck distribution, $n(p) = 1/[\exp(p/T) - 1]$ [20]. The chemical potential μ can be either positive or negative, if it is negative, the form of $n(p)$ must be modified for $p \leq \mu T$.

Ordinarily one assumes that the chemical potential of the photon is zero. If there is nonthermal energy release in the Universe this will only be true provided photon-number-changing processes are occurring rapidly compared to the expansion rate of the Universe. When the temperature of the Universe drops below about 1 keV, photon-number changing processes, e.g., the double-Compton process $e^- + \gamma \leftrightarrow e^- + \gamma + \gamma$, become ineffective, and any energy injected after this epoch leads to a photon distribution with $\mu \neq 0$. Cosmic Background Explorer (COBE) and rocket-borne observations constrain $|\mu|$ to be less than about 10^{-3} [19].

Bubbles that are still unhomogenized at a temperature of about 1 keV will inevitably lead to a μ distortion as they homogenize. Again, a precise calculation of the size of such a distortion is beyond the scope of this paper. If we suppose that the μ distortion within a Hubble-size bubble at this epoch is order unity, a reasonable estimate for the μ distortion after such bubbles have been homogenized is the fraction of our Hubble volume occupied by such bubbles, or $(\Gamma/H^4)_N$. Requiring that $\mu \lesssim 10^{-3}$ results in the bound

$$\int_{47+n_{14}}^{60+n_{14}} dN \left[\frac{\Gamma}{H^4}\right]_N \lesssim 10^{-3} , \tag{3.4a}$$

where as before the lower limit corresponds to bubbles that were Hubble size when photon-number-changing processes became ineffective, while the upper limit corresponds to bubbles that are Hubble size today. As with constraint (3.1) we expect the integral to be dominated by the region near the lower limit, so that Eq. (3.4a) is essentially a constraint to $(\Gamma/H^4)_{N\simeq 47+n_{14}}$:

$$\left[\frac{\Gamma}{H^4}\right]_{N\simeq 47+n_{14}} \lesssim c_b \times 10^{-3} , \tag{3.4b}$$

with c_b of order unity.

C. Isocurvature perturbations from finite bubble number

In slow-rollover inflation curvature ("true" density) perturbations arise from quantum fluctuations in the inflaton field. In first-order inflation the field that supplies the vacuum energy that drives inflation (the scalar field responsible for the phase transition) is anchored in the false vacuum and is not free to fluctuate; however, there is always another field that can fluctuate and produce curvature perturbations [21]. For example, in extended inflation it is the Brans-Dicke field [2]; in two-field inflation, it is the "trigger field" [22].

In addition to the perturbations associated with quantum fluctuations, one also expects perturbations to arise from the random distribution of bubbles. We have just considered the adverse consequences of individual large bubbles; we now consider the perturbations that arise due to Poisson fluctuations in the number of bubbles nucleated within a specified volume. To wit, we focus on a comoving scale λ: Different regions in the Universe of volume λ^3 once contained different numbers of bubbles (of various sizes), and one would expect Poisson fluctuations in the number of bubbles to give rise to density perturbations. As we shall see, bubble-number fluctuations lead to isocurvature perturbations, whose amplitudes we now estimate.

To illustrate why bubble-number fluctuations result in isocurvature, rather than curvature, perturbations we briefly review how curvature perturbations arise in inflation. As a given scale λ crosses outside the Hubble radius during inflation, fluctuations in the local energy density lead to fluctuations in the local curvature. These curvature fluctuations are characterized by a dimensionless, gauge-invariant quantity ζ whose amplitude remains constant while they are outside the horizon; when a given scale reenters the horizon during the postinflation epoch the amplitude of the resulting density perturbation is $(\delta\rho/\rho)_{\rm hor} \sim \zeta$. The gauge-invariant quantity ζ is given by

$$\zeta \simeq \frac{\delta\rho}{\rho+p} ; \tag{3.5}$$

here $\delta\rho$ is the magnitude of the density perturbation, ρ and p are the (mean) density and pressure, and ζ is to be evaluated as the scale crossed outside the Hubble radius [23]. Note that curvature perturbations only arise if there are density perturbations [24].

Now turn to the bubble-produced perturbations. As a point of reference we should keep in mind that in a successful model of first-order inflation the conversion of the bulk of the vacuum energy into radiation occurs through the nucleation of bubbles during the final few Hubble times of inflation and that these bubbles have comoving sizes that are very tiny by astrophysical standards, of the order of 10^{-21} Mpc/$\mathcal{M}_{14}$. The number of such bubbles within any interesting astrophysical volume is enormous, and so their Poisson fluctuations are completely negligible. Again, we are interested in the rarer bubbles that nucleated well before the end of inflation.

Consider the perturbations on the scale λ that arise from the Poisson fluctuations in the number of bubbles of comoving size D (necessarily smaller than λ) that are nucleated in a comoving volume λ^3. Noting that larger scales cross outside the horizon earlier (and reenter the horizon later) and recalling that bubbles simply conformally stretch from about a Hubble time after nucleation, we see that bubbles of size D must have been nucleated after the comoving scale λ has crossed outside the Hubble radius.

Since the quantity ζ that describes the evolution of a curvature perturbation remains constant after the scale λ crosses outside the Hubble radius, the nucleation of bubbles of size $D \lesssim \lambda$ cannot give rise to curvature perturbations on the scale λ. Instead, the fluctuation in the number of bubbles within a volume λ^3 gives rise to isocurvature or pressure perturbations; such perturbations have been discussed in some detail in Ref. [25]. Here we just state the key result: After the scale λ has reentered the horizon and the Universe has become matter dominated, the pressure perturbation will become a density perturbation of the same magnitude. (The underlying physics is straightforward to understand: The pressure fluctuation will move matter around resulting in a density perturbation.)

Within a single large bubble which went outside the horizon before its interior was homogenized the pressure contrast relative to the average is of order unity. The perturbation resulting from the Poisson fluctuations in the number of such bubbles nucleated in a volume λ^3 is then simple to estimate:

$$\frac{\delta p}{p} \sim \left[\frac{\delta N_D}{N_D}\right] f_D \sim \left[\frac{D}{\lambda}\right]^{3/2} \left[\frac{\Gamma}{H^4}\right]_D^{1/2} . \tag{3.6}$$

Here N_D is the mean number of bubbles of size D (per unit logarithmic interval) within a volume λ^3, $\delta N_D = \sqrt{N_D}$, and $f_D = \rho_D/\rho_{\rm tot}$ is the fraction of the energy density of the Universe that resulted from the nucleation of these bubbles (which is just the volume fraction of the Universe which they occupy). And of course, this estimate is only valid if $N_D \gtrsim 1$. The quantities f_D and N_D are related: $N_D \sim f_D(\lambda/D)^3$. From Eq. (2.4), $f_D = (\Gamma/H^4)_D d\ln D$, where as usual the subscript indicates that Γ/H^4 is to be evaluated at the time when bubbles of size D were nucleated. [From our previous arguments concerning the disastrous effects of big bubbles, we can be confident that big bubbles fill only a small fraction of space, and expect that $(\Gamma/H^4)_D$ is at most of order 10^{-3}.]

For bubble sizes D where the number of bubbles expected within a volume λ^3 is less than order unity, Poisson statistics do not apply. Bubbles of such size lead to rare perturbations (on the scale λ) of amplitude $(D/\lambda)^3$, which is just the relative contribution of a single bubble of size D to the energy in a region of size λ; the probability that a given λ^3 volume has such a perturbation is roughly N_D. In the previous subsection we discussed the potentially deleterious effects of such rare bubbles.

We now consider separately perturbations in the matter and in the radiation that result from isocurvature bubble fluctuations. Perturbations in the matter density on scales that reenter the horizon before matter-radiation equality, $\lambda \lesssim \lambda_{\rm eq} \simeq 13h^{-2}$ Mpc, reenter the horizon with amplitude $\delta\rho_m/\rho_m \sim \delta p/p$ [26]; however, these matter perturbations do not begin growing until the Universe becomes matter dominated, after which they grow as the scale factor $R(t)$. Perturbations in the matter density on scales that reenter the horizon after matter-radiation equality, $\lambda \gtrsim 13h^{-2}$ Mpc, also reenter the horizon with amplitude $\delta\rho_m/\rho_m \sim \delta p/p$, and immediately start growing, again as $R(t)$. Bringing this all together, the spectrum of matter perturbations at epoch of matter-radiation equality is given by

$$\left[\frac{\delta\rho_m}{\rho_m}\right]^2_{\rm eq} \sim \int \left[\frac{D}{\lambda}\right]^3 \left[\frac{\Gamma}{H^4}\right]_D d\ln D, \quad \lambda \lesssim 13h^{-2}\ \text{Mpc}\ , \tag{3.7a}$$

$$\left[\frac{\delta\rho_m}{\rho_m}\right]^2_{\rm eq} \sim \left[\frac{\lambda}{13h^{-2}\ \text{Mpc}}\right]^{-2} \int \left[\frac{D}{\lambda}\right]^3 \left[\frac{\Gamma}{H^4}\right]_D d\ln D\ , \quad \lambda \gtrsim 13h^{-2}\ \text{Mpc}\ ; \tag{3.7b}$$

where we have summed the contributions from bubbles of all sizes (in quadrature) and the integration only runs over bubble sizes D characterized by $N_D \gtrsim 1$, i.e., bubble sizes D such that $(\lambda/D)^3(\Gamma/H^4)_D \gtrsim 1$. [For scales larger than $\lambda_{\rm eq}$, the $(\lambda/\lambda_{\rm eq})^{-2}$ factor takes into account the fact that the amplitudes of these perturbations are being specified before they reenter the horizon: Because they grow from matter-radiation equality until they reenter the horizon by a factor of precisely $(\lambda/\lambda_{\rm eq})^2$, they reenter the horizon with amplitude $\delta p/p$, as noted above.]

For our present purpose the isocurvature matter perturbations are not of interest; they will be discussed elsewhere [27]. However, we mention some salient points: (i) Their spectrum is determined by the evolution of Γ/H^4 and thus in general is not scale invariant; (ii) if we suppose that Γ/H^4 only varies slowly with scale D, we see that the contribution to $\delta\rho_m/\rho_m$ on the scale λ increases with D, achieving its maximum value, roughly Γ/H^4, at the cutoff, i.e., where $N_D \sim 1$, which occurs for $D \sim (\Gamma/H^4)^{1/3}\lambda$; (iii) because the dominant contribution to $(\delta\rho_m/\rho_m)_\lambda$ comes from bubbles for which $N_D \sim 1$, the statistics of these isocurvature matter fluctuations should be non-Gaussian; and (iv) since our previous big-bubble bounds, as well as the new bounds we derive below, only constrain Γ/H^4 to be less than order 10^{-4}–10^{-3} or so and $(\delta\rho_m/\rho_m)_\lambda \sim (\Gamma/H^4)_{D\sim(\Gamma/H^4)^{1/3}\lambda}$, isocurvature bubble perturbations could possibly be a novel, if not important, mechanism for the origin of the fluctuations needed to initiate structure formation.

Now on to our main interest here, temperature fluctuations that arise from isocurvature perturbations. The horizon at decoupling corresponds to an angular scale of about 1°. Because patches on the sky of size greater than 1° were out of causal contact at the decoupling temperature fluctuations on these scales are simple to estimate. (This is not true for scales smaller than 1°, where the finite thickness of the last-scattering surface, photon diffusion, and the interaction of the radiation with baryons are all very important; see [28].) The temperature fluctuations on scales greater than about $100h^{-1}$ Mpc are up to factors of order unity equal to the initial pressure perturbation, cf. Eq. (3.6), which implies that

$$\left[\frac{\delta T}{T}\right]_{\theta\sim(\lambda/100h^{-1}\ \text{Mpc})^\circ} \sim \left[\int \left[\frac{D}{\lambda}\right]^3 \left[\frac{\Gamma}{H^4}\right]_D d\ln D\right]^{1/2}, \tag{3.8}$$

where again the integral over bubble sizes only extends over bubble sizes for which $N_D \gtrsim 1$, i.e., $D \lesssim (\Gamma/H^4)^{1/3}\lambda$.

These must be consistent with the observed isotropy of the CMBR. If for simplicity we summarize the current isotropy limits by the statement that $(\delta T/T)_\theta$ is a factor of a few smaller than 10^{-4} for $\theta \gtrsim 1°$, and insist that each logarithmic interval in bubble size satisfy this bound we obtain the constraint

$$\left[\frac{\Gamma}{H^4}\right]_D \lesssim 10^{-8}\left[\frac{\lambda}{D}\right]^3, \tag{3.9}$$

valid for $D/\lambda \lesssim (\Gamma/H^4)^{1/3}$ (bubble sizes for which $N_D \gtrsim 1$) and $\lambda \sim 100h^{-1}$ Mpc–10^4h^{-1} Mpc (corresponding to $\theta \sim 1°$–$90°$). Constraint (3.9) is illustrated in Fig. 2.

We see for a given angular scale θ that Eq. (3.9) provides the most stringent constraint on the largest bubbles, i.e., $D \sim (\Gamma/H^4)^{1/3}\lambda$:

$$\left[\frac{\Gamma}{H^4}\right]_{D\approx 0.05\lambda} \lesssim 10^{-4}\ , \tag{3.10}$$

valid for $\lambda \sim 100h^{-1}$ Mpc–$10^4\,h^{-1}$ Mpc. [This result can also be obtained by assuming that $(\Gamma/H^4)_D$ is slowly varying compared to D^2 and then approximately evaluating the integral, which is dominated by the upper end of the integration range. We will see that this assumption is valid in examples that just barely satisfy the big-bubble

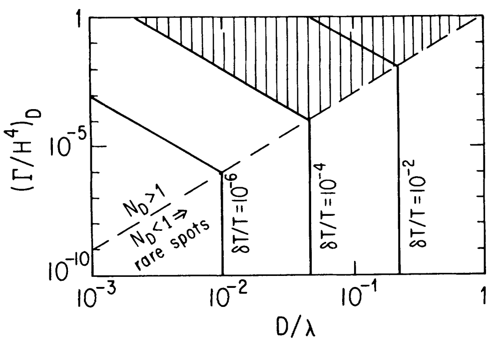

FIG. 2. Contours of the contribution to $(\delta T/T)_{(\lambda/100h^{-1}\,\mathrm{Mpc})}$, from Poisson fluctuations in the number of bubbles of size D per λ^3 volume as a function of $(\Gamma/H^4)_D$ and D/λ. The hatched region is forbidden because the contribution of a unit logarithmic interval of bubble sizes around D to $\delta T/T$ exceeds 10^{-4}, which is a generous upper limit to the measured isotropy on angular scales greater than 1°.

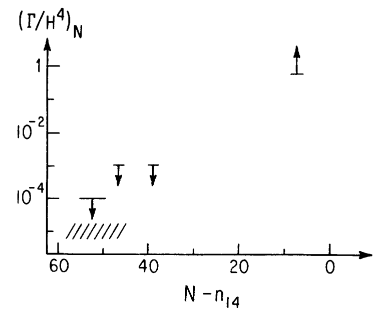

FIG. 3. Summary of the big-bubble constraints to $(\Gamma/H^4)_N$. Here $(\Gamma/H^4)_N$ refers to the value of Γ/H^4 when bubbles whose size at the end of inflation is e^N times the Hubble radius were nucleated and $n_{14}=\ln(\mathcal{M}/10^{14}\ \mathrm{GeV})$. The cross-hatched region indicates that such a value for (Γ/H^4) could lead to astrophysically interesting isocurvature fluctuations due to Poisson fluctuations in bubble number.

constraints.] Based upon the observed isotropy on angular scales from 1° to 90° we obtain the bound

$$\left[\frac{\Gamma}{H^4}\right]_{50+n_{14}\lesssim N\lesssim 54+n_{14}} \lesssim 10^{-4} . \qquad (3.11)$$

D. Epilogue

The quantity $(\Gamma/H^4)_N$ corresponds to the fraction of space filled by bubbles (in a unit logarithmic interval of bubble size) that were a factor e^N larger than the Hubble radius at the end of inflation. Our constraints apply to the value of Γ/H^4 when these bubbles were nucleated. In exponential inflation such bubbles were nucleated when the value of the cosmic-scale factor was e^N times smaller than its value at the end of inflation; for power-law inflation ($R \propto t^p$), such bubbles were nucleated when the value of the cosmic-scale factor was $e^{N'}$ was smaller than its value at the end of inflation, where $N'=pN/(p-1)$. Moreover, our constraints are actually integral constraints to $(\Gamma/H^4)_N$, though in most cases we expect the integral over $(\Gamma/H^4)_N$ to be dominated by the contribution near the lower end of the integration.

With these caveats stated let us summarize the prescribed behavior of Γ/H^4 in a successful model of first-order inflation: $(\Gamma/H^4)_N$ must be of order unity at the end of inflation (more precisely $\geq 9/4\pi$; see Sec. IV), less than about 10^{-3} for $N=39+n_{14}$ (primordial nucleosynthesis), less than about 10^{-3} for $N=47+n_{14}$ (μ distortion), less than a few times 10^{-4} for $N=53+n_{14}$ (CMBR temperature), and less than about 10^{-4} for $N=50+n_{14}$ to $54+n_{14}$ (CMBR isotropy). These constraints are illustrated in Fig. 3. As we shall see in the next two sections these conditions on Γ/H^4 are not easy to satisfy. Furthermore, if we understood the physics of the early Universe at temperatures well above that of primordial nucleosynthesis, e.g., the electroweak phase transition (which corresponds to $N \sim 27+n_{14}$), baryogenesis ($N \sim n_{14}$?), and so on, we could in principle place constraints on $(\Gamma/H^4)_N$ even closer to the end of inflation, which would prove even more challenging to model builders.

We have primarily focused on the adverse cosmological consequences of large bubbles; let us now discuss possible favorable consequences, e.g., accounting for large voids or seeding structure formation. In this regard it is an unfortunate fact that the constraints just derived imply that the fraction of space occupied by bubbles of astrophysically interesting size, say greater than of order 1 Mpc, is less than order 10^{-4}–10^{-3}. To be more specific, consider the number of big bubbles of comoving size of order $30h^{-1}$ Mpc expected within $100h^{-1}$ Mpc of the Milky Way (a volume about the size of the Harvard Smithsonian Center for Astrophysics (CfA_2) redshift survey [29]): The expected number is only about 3×10^{-2}. This it seems very unlikely that large, individual bubbles produced by first-order inflation have anything to do with the "bubbly" structure observed today on scales of order $30h^{-1}$ Mpc (large voids, walls, and so on), and even if they could be abundant enough, it is not clear that they could produce the desired structures [11]. On the other hand, it remains to be seen whether or not such structures need explanation beyond what inflation-produced curvature perturbations can account for. Finally, it is still possible that isocurvature fluctuations arising from Poisson fluctuations in the number of bubbles nucleated in different regions of space could be interesting; these fluctuations will be discussed further elsewhere [27].

IV. THE DEVELOPMENT OF A FIRST-ORDER PHASE TRANSITION: MEASURES OF PROGRESS

Several quantities provide useful measures of the progress of a first-order transition. The most fundamental of these is $p(t)$, the probability that a given point in space remains in the false vacuum at time t. While $p(t)$ measures the conversion of false-vacuum regions to true vacuum, it does not directly track the conversion of the latent heat of the transition into thermal energy. The crucial step here is the disappearance of the bubble walls, where the latent heat is initially stored, through collisions between bubbles. This leads us to examine a quantity $\mathcal{F}_E(t)$, which is the fraction of uncollided bubble wall, weighted by surface energy density. Although both of these quantities should approach zero as the transition is completed, we know from the example of old inflation that this is not a sufficient condition for successful transition [6]. We therefore also look at the physical volume in the false vacuum, $V_{\rm phys}(t) \propto R^3(t)p(t)$. For a successful transition this quantity must eventually approach zero, and we define a time t_e to be the time when $V_{\rm phys}$ begins to decrease.

It is straightforward to show that

$$p(t)=e^{-I(t)}\ , \tag{4.1}$$

where $I(t)$ is the expected volume of true-vacuum bubbles per unit volume of space at time t [30]. In calculating $I(t)$ the regions in which bubbles overlap are counted twice, and the "virtual bubbles" which would have nucleated had their point of nucleation not already been in a true-vacuum region are also included; the exponentiation of I corrects for these effects. If $r(t,t')$ is the coordinate radius at time t of a bubble that was nucleated at t', then

$$I(t)=\frac{4\pi}{3}\int_{t_0}^{t}dt'\Gamma(t')R^3(t')r^3(t,t')\ , \tag{4.2}$$

where t_0 is the time when the phase transition begins. If we make the approximation, valid in most cases of interest, of neglecting the bubble size at nucleation,

$$r(t,t')=\int_{t'}^{t}\frac{dt''v(t'')}{R(t'')}\ , \tag{4.3}$$

where $v(t)$ is the velocity at which the bubble walls are expanding. For bubbles nucleated at temperatures far below the critical temperature (i.e., vacuum-bubble transitions), as is the case in most scenarios for first-order inflation, v can generally be taken to be unity; as noted previously, this cannot be done for transitions which are completed at high temperature (thermal-bubble transitions).

It is instructive to view Γ and H as functions of the scale factor R rather than of t, and to rewrite Eq. (4.2) as

$$I(t)=\frac{4\pi}{3}\int_{R_0}^{R(t)}dy\frac{y^2\Gamma(y)}{H(y)}\left[\int_{y}^{R(t)}dz\frac{v(z)}{z^2H(z)}\right]^3\ . \tag{4.4}$$

In most cases of interest to us Γ and H^{-1} are increasing functions of time, so that the integrals in this expression are dominated by the upper end of the integration range. In such cases, it is clear that the integral over y is essentially an integral over Γ/H^4, and that $\Gamma(t)/H^4(t)$ sets the natural size for $I(t)$.

With this fact in mind, it is simple to see that, up to a numerical factor, $(\Gamma/H^4)_N$ measures the fraction of space occupied by bubbles larger than those of size e^N times the Hubble radius at the end of inflation (in accord with our assertion at the end of Sec. II). First note that $1-p(t_N)$ is precisely the fraction of space filled by these large bubbles (t_N is the time that a bubble whose size is e^N times that of the Hubble radius at the end of inflation was nucleated). Provided that this fraction is small, as it will be in all cases of interest, $1-p(t_N)\simeq I(t_N)$, which as discussed above should just be a numerical of order unity factor times $(\Gamma/H^4)_N$.

Note that we have used a Robertson-Walker scale factor $R(t)$ in these formulas even though the Universe ceases to be homogeneous and isotropic once bubbles begin to form. For low-temperature transitions where the bubbles expand at the speed of light, this is justified because the false-vacuum regions, not being in the light cone of any bubbles, have the same curvature as they would in the absence of bubble nucleation. Inside the bubbles the geometry is of course different from that outside (indeed, there does not even seem to be a naturally preferred way of continuing the hypersurfaces of constant t through the bubble walls), but we do not need the details of this geometry in order to follow the disappearance of the false-vacuum regions.

The situation is of course more complex for high-temperature transitions with $v<1$. However, we will see that for such transitions most of the bubbles are nucleated over a period much less than a Hubble time, so that the effects of the cosmic expansion are small.

The conversion of latent heat to thermal energy can be tracked by examining the fraction of bubble wall that remains uncollided at a given time. To that end, let us consider a reference bubble that was nucleated at time t_R and ask what fraction of its wall remains uncollided at some later time t. This is equivalent to asking what fraction of the points on the outer surface of the bubble remain in the false vacuum. At first thought, one might take this to be $p(t)=e^{-I(t)}$. However, this is not quite right, since a point on the bubble wall is not a random point. The existence of the reference bubble precludes the nucleation of "younger" bubbles (nucleated at times after t_R) in the interior of the reference bubble; however, these bubbles would never reach the surface of the reference bubble, and thus they have no effect on the calculation. Since the reference bubble must have nucleated in a false-vacuum region, its existence also precludes the existence of "older" bubbles in the past light cone of the point of nucleation. Because these excluded bubbles would have contained all of the points on the surface of the reference bubble, $I(t)$ must be corrected to reflect this absence. Of the bubbles nucleated at a time $t'<t_R$, those nucleated within a sphere of coordinate radius $r(t,t')$ would have reached a given point on the surface of the reference bubble by time t. Within this sphere, a region of coordinate radius $r(t_R,t')$ is excluded by the above considerations. Thus, the contribution of bubbles nucleated at time t' must be reduced by a factor of $[r(t_R,t')/r(t,t')]^3$ relative to their contribution to $I(t)$. It follows that the fraction of the reference bubble wall left uncovered is $e^{-f(t,t_R)}$, where

$$\begin{aligned}f(t,t_R)&=I(t)-\frac{4\pi}{3}\int_{t_0}^{t_R}dt'\Gamma(t')R^3(t')r^3(t_R,t')\\&=I(t)-I(t_R)\ .\end{aligned} \tag{4.5}$$

[At nucleation the fraction of the reference bubble's surface that is collided must of course vanish, which provides a heuristic argument that $f(t,t_R)=I(t)-I(t_R)$. One can also derive Eq. (4.5) by explicitly considering bubble nucleation in the vicinity of the reference bubble and computing the fraction of its surface that is covered by both younger and older bubbles.]

It is now a simple matter to compute the fraction $\mathcal{F}(t)$ of uncollided bubble wall at time t. The number of bubbles nucleated at time t_R is proportional to $\Gamma(t_R)R^3(t_R)p(t_R)$; the area of each of these is $4\pi r^2(t,t_R)$. Hence,

$$\mathcal{F}(t)=\frac{\int_{t_0}^{t}dt_R R^3(t_R)r^2(t,t_R)\Gamma(t_R)e^{-I(t_R)}e^{-f(t,t_R)}}{\int_{t_0}^{t}dt_R R^3(t_R)r^2(t,t_R)\Gamma(t_R)e^{-I(t_R)}} , \\ =\frac{e^{-I(t)}\int_{t_0}^{t}dt_R R^3(t_R)r^2(t,t_R)\Gamma(t_R)}{\int_{t_0}^{t}dt_R R^3(t_R)r^2(t,t_R)\Gamma(t_R)e^{-I(t_R)}} . \tag{4.6}$$

This calculation does not take into account the fact that the surface energy density of a bubble wall increases as the bubble expands. Because the false-vacuum energy liberated by a bubble is proportional to its volume and is, until collisions take place, stored entirely within the bubble walls, this surface energy density is proportional to the bubble radius [31]. By inserting factors of $r(t,t_R)/3$ into the integrals in both the numerator and denominator of Eq. (4.6), we obtain the energy-weighted fraction

$$\mathcal{F}_E(t)=\frac{e^{-I(t)}I(t)}{(4\pi/3)\int_{t_0}^{t}dt_R R^3(t_R)r^3(t,r_R)\Gamma(t_R)e^{-I(t_R)}} . \tag{4.7}$$

This expression suggests that the energy-weighted fraction of uncollided bubble wall decreases exponentially with $I(t)$. To obtain an explicit upper bound to $\mathcal{F}_E(t)$ we need a lower bound to the denominator. To this end we note that

$$I(t_R)\le\frac{4\pi}{3}\int_{t_0}^{t_R}dt' R^3(t')r^3(t,t')\Gamma(t') \\ \equiv J(t_R,t) . \tag{4.8}$$

Thus the integrand in the denominator must *be greater* than

$$R^3(t_R)r^3(t,t_R)\Gamma(t_R)e^{-J(t_R,t)} \\ =-\frac{3}{4\pi}\left[\frac{\partial}{\partial x}e^{-J(x,t)}\right]_{x=t_R} , \tag{4.9}$$

which is easily integrated to show that the denominator must be greater than $1-e^{-I(t)}$. This gives the interesting upper bound

$$\mathcal{F}_E(t)\le\frac{I(t)e^{-I(t)}}{1-p(t)} \\ \approx\begin{cases}1-I(t)/2, & I(t)\ll 1 ,\\ I(t)e^{-I(t)}, & I(t)\gg 1 .\end{cases} \tag{4.10}$$

To summarize, at time t a fraction $p(t)$ of the latent heat remains as vacuum energy in the false-vacuum regions, while a fraction $[1-p(t)]\mathcal{F}_E(t)$ is in uncollided bubble walls. Finally, a fraction

$$[1-p(t)][1-\mathcal{F}_E(t)]\ge 1-[1+I(t)]e^{-I(t)} \tag{4.11}$$

of the total energy has been released through the dissolution of bubble walls. We see that this last fraction approaches unity exponentially as $I(t)$ becomes large.

We recall that $p(t)$ is a misleading measure of progress in old inflation, since it tends to zero at large times even though the transition is never completed [6]. The bound of Eq. (4.10) shows that $\mathcal{F}_E$ is no better in this regard [32]. We therefore turn our attention to the actual physical volume in the false vacuum, $V_{\rm phys}(t)\propto R^3(t)p(t)$, and, more specifically, to the time t_e when

$$V_{\rm phys}^{-1}\frac{dV_{\rm phys}}{dt}=(3H-dI/dt) \tag{4.12}$$

becomes negative. At this epoch the fraction of physical space still remaining in the false vacuum begins to decrease.

In an inflationary transition a bubble "outruns" the general cosmic expansion only during the first Hubble time after nucleation. Our expectation therefore is that $V_{\rm phys}$ can decrease in such a transition only if the nucleation rate is greater than one bubble per Hubble volume per Hubble time. We can make this more precise by examining the right-hand side of Eq. (4.12). For $v=1$ the quantity dI/dt is given by

$$\frac{dI}{dt}=4\pi\int_{t_0}^{t}dt'\Gamma(t')R^3(t')r^2(t,t')\frac{\partial}{\partial t}r(t,t') \\ =4\pi\int_{t_0}^{t}dt'\Gamma(t')R^3(t')R^{-1}(t)r^2(t,t') . \tag{4.13}$$

By changing the integration variable from t to R, we can rewrite this expression as

$$\frac{dI}{dt}=4\pi\int_{R(t_0)}^{R}dy\frac{\Gamma(y)y^2}{RH(y)}\left[\int_y^R\frac{dz}{z^2H(z)}\right]^2 . \tag{4.14}$$

An upper bound to dI/dt can be obtained by replacing $\Gamma(y)$ by the maximum value that the nucleation rate achieves during the interval t_0 to t and by replacing $H(y)$ and $H(z)$ by the minimum value of H during this interval. The integrals are then easily evaluated. In almost any scenario for first-order inflation the nucleation rate will be an increasing function of time. We also expect H to be either constant or a decreasing function of time. If we therefore assume that the maximum of Γ and the minimum of H are both reached at the upper end of the range of integration, we obtain the bound

$$\frac{dI}{dt}\le\frac{4\pi}{3}\left[\frac{\Gamma}{H^3}\right]\left[1-\frac{R_0}{R}\right]^3<\frac{4\pi}{3}\left[\frac{\Gamma}{H^3}\right]_t , \tag{4.15}$$

where $R_0=R(t_0)$ and all other quantities are to be evaluated at time t. Combining this with the condition $I'(t_e)=3H$, we obtain the lower bound

$$\left[\frac{\Gamma}{H^4}\right]_{t_e}>\frac{9}{4\pi} . \tag{4.16}$$

[Following a similar line of reasoning, but with the less stringent requirement that $\Gamma(t)/H^3$ is monotonic one can obtain the slightly weaker bound $(\Gamma/H^4)_{t_e} \geq 3/8\pi$.]

While t_e defines one "milestone" which must be reached in the course of the transition, it is not the only one. One cannot reasonably say that a transition has completed until $p(t)$, the fraction of comoving volume remaining in the false vacuum, has become small. This suggests that we define another milestone t_*, by the condition that

$$p(t_*)=e^{-M}, \tag{4.17}$$

where M is some suitably large number. Defined in this manner, this would be a somewhat misleading criteria in old inflation, where $p(t)$ becomes arbitrarily small without the transition ever being completed [6]. However, for transitions that do in fact complete $p(t)$ will generally become small near the time of completion. We can deal with the exceptional cases where this is not so by noting that it is only the bubbles nucleated during the last 60 or so e-foldings of inflation which are of relevance for us. Any bubbles produced earlier would correspond to regions larger than the currently observed Universe. Such bubbles were evidently not nucleated in our past light cone, and so could not have contributed to the completion of the transition in this portion of the Universe. We therefore append to the definition of t_* the proviso that only bubbles nucleated during the last 60 e-foldings are to be included in the calculation of $p(t)$. Similarly, we define a quantity δt, which measures the duration of the transition, as the time that it takes this restricted $p(t)$ to go from being nearly unity to being nearly zero. To be specific, we take $\delta t \equiv t_* - t_m$, where $p(t_m)=e^{-m}$ with m some appropriately small number. If, for example, we choose $m=0.01$ and $M=5$, then δt will be the time over which the Universe goes from 1% true vacuum to 99% true vacuum.

Depending on the speed of the transition, either t_e or t_* can come first. For a rapid transition $t_e < t_*$, as can be seen by noting that during the course of a sufficiently rapid transition the expansion of the Universe can be neglected, in which case $V_{\rm phys}$ declines from the very onset of the transition. On the other hand, if the transition is slow enough $t_* < t_e$. Again, this can be seen by referring to the limiting case, old inflation, where t_e is never reached. By considering these limiting cases, we see that neither of these times gives a definition of the "end" of the transition that is adequate for all situations; since the conditions associated with both are necessary for completion, we will always choose the latter of the two. We will find that for transitions that meet the big-bubble constraints of the previous section $|t_* - t_e|$ is never much more than a few Hubble times. As a result, the eventual constraints that big-bubble considerations place on the parameters of the theory are rather insensitive to which time we use as the end of inflation.

Another milestone that is often discussed is the percolation of the low-temperature phase. This is clearly necessary for the homogeneous completion of the transition. Indeed, proving the absence of percolation was the crucial step in demonstrating the failure of old inflation. However, since percolation by itself is not sufficient for a successful transition, the onset of percolation is of less interest for us here [33].

Also of interest is the distribution of bubble sizes. Bubbles that have a physical radius ρ at time t were nucleated at a time t_ρ determined by the relation

$$\rho=R(t)r(t,t_\rho)=R(t)\int_{t_\rho}^{t}\frac{dt''v(t'')}{R(t'')} . \tag{4.18}$$

The number density of such bubbles is determined by the nucleation rate $\Gamma(t_\rho)$ at that time, and by the fraction $p(t_\rho)$ of space still in the false vacuum at that time. Taking into account the expansion of the Universe between t_ρ and t, the number of bubbles per unit physical volume with physical radius ρ is given by

$$\begin{aligned}\frac{dN}{d\rho}&=\Gamma(t_\rho)\left[\frac{R(t_\rho)}{R(t)}\right]^3 p(t_\rho)\frac{dt_\rho}{d\rho}\\ &=\Gamma(t_\rho)\left[\frac{R(t_\rho)}{R(t)}\right]^4\frac{p(t_\rho)}{v(t_\rho)} .\end{aligned} \tag{4.19}$$

A related quantity of interest is the size distribution of the bubbles that have collided with a given "reference bubble" by a time t. Let us assume that the reference bubble was nucleated at time t_R at position $r=0$, so that its physical size at time t is $\rho_R=R(t)r(t,t_R)$. In order that a second bubble of size ρ (and hence nucleated at time t_ρ) could have collided with the reference bubble, two conditions must hold. First, it must have been nucleated close enough for it and the reference bubble to have reached each other. In other words, its nucleation must have occurred at a point whose coordinate distance r_C from the origin obeys

$$R(t)r_C<\rho_R+\rho . \tag{4.20a}$$

Second, since the younger of the two bubbles cannot have nucleated within the older bubble,

$$R(t)r_C>|\rho_R-\rho| . \tag{4.20b}$$

Thus, the center of the colliding bubble must lie in a spherical shell of physical volume

$$\mathcal{V}(\rho_R,\rho)=4\pi(\rho_>^2\rho_<+\rho_<^3) , \tag{4.21}$$

where we have defined $\rho_>$ and $\rho_<$ to be the greater and lesser of ρ_R and ρ.

Naively, we would just multiply this volume by $dN/d\rho$ to obtain the expected number of colliding bubbles of a given size. This is not quite right. The difficulty lies in the factor of $p(t_\rho)$ in Eq. (4.19). This is the probability that a random point is still in the false vacuum at time t_ρ. However, the points in the region we are considering are not random, since the existence of the reference bubble precludes the nucleation of any bubbles in the past light cone of its point of nucleation. This is essentially the same problem as we encountered in the calculation of $\mathcal{I}$ and $\mathcal{I}_E$, but the correction needed here, while going in the same direction [i.e., increasing the effective $p(t_\rho)$] is somewhat more complicated to calculate. In any case, the correction only becomes important for bubbles nucleating late in the transition, after $p(t)$ has begun to deviate significantly from unity. In fact, since the correction arises from the absence of potential bubbles which would have nucleated before *both* the reference bubble and the colliding bubble, we may write the distribution of colliding bubble sizes as

$$\frac{dN_C}{d\rho} = \mathcal{V}(\rho_R,\rho)\frac{dN}{d\rho}[1+O(1-p(t_{\rho>}))] \ . \tag{4.22}$$

We will go on shortly to evaluate the various quantities defined above for several specific models. Before doing so, we point out a useful approximation that can be applied in any inflationary transition. In such a transition, the integral in Eq. (4.3) for $r(t,t')$ remains finite as $t \to \infty$, and is dominated by a time interval of order H^{-1} at the lower end of the integration range. As we have discussed previously, this corresponds to the fact that a bubble expands freely at the speed of light for a time of order H^{-1} after which it simply conformally stretches as the Universe expands. This suggests that for $(t-t') \gg H^{-1}$ the coordinate size of the bubble is simply

$$r(t,t') = \frac{kH^{-1}(t')}{R(t')} \quad \text{(for inflationary transitions)} , \tag{4.23}$$

with k of order unity. Explicit evaluation of the integral gives $k=1$ for exponential expansion and $k=p/(p-1)$ for $R(t) \propto t^p$ with $p>1$. Using this approximation in Eq. (4.2) for $I(t)$ and then differentiating gives

$$\left[\frac{\Gamma}{H^4}\right]_{t_e} \approx \frac{9}{4\pi} \quad \text{(for inflationary transitions)} , \tag{4.24}$$

suggesting that the lower bound of Eq. (4.16) is actually reached in inflationary transitions.

Furthermore, using this approximation and setting $v=1$ in Eq. (4.19) leads to

$$\rho^3 \frac{dN}{d\ln\rho} \approx \left[\frac{\Gamma}{H^4}\right]_{t_\rho} \quad (\text{valid for } \rho \gg H^{-1}) , \tag{4.25}$$

where t_ρ is the time at which bubbles that have physical size ρ at time t were nucleated. Thus, as was asserted at the end of Sec. II, the volume fraction occupied by large bubbles in a unit logarithmic interval of bubble size is essentially equal to the value of Γ/H^4 at the time of their nucleation.

Comparing Eqs. (4.24) and (4.25) we see that within this approximation the big-bubble constraints of Sec. III are simply lower bounds to the net increase in Γ/H^4 from t_N to t_e and are thus insensitive to the behavior of Γ and H at intermediate times.

V. SPECIFIC EXAMPLES

A. Model 1: Exponentially growing nucleation rate in a static universe [34]

In this section we apply the general formulas of Sec. IV to some specific examples. As a first example, we consider a transition where the bubble-nucleation rate Γ increases very rapidly near the time of completion, with essentially all of the bubbles being nucleated within much less than a Hubble time, so that the expansion of the Universe can be neglected. Such a transition could be an inflationary transition completing at essentially zero temperature, or it could be a noninflationary first-order transition taking place at some finite temperature [35]. While the bubble-wall velocity v can be set equal to unity in the former case, this will not in general be true for the latter. However, since we expect v to be determined by the temperature, which by our assumptions changes little over the time that the bubbles are nucleated, we will take v to be a constant.

Because the nucleation rate is typically proportional to the exponential of some action, we again write it as

$$\Gamma = Ce^{-A(t)} , \tag{5.1}$$

with C assumed to be of the order of $\mathcal{M}^4$, where $\mathcal{M}$ is the mass scale characterizing the transition. In order that the usual treatment of bubble nucleation be valid, the tunneling action $A(t)$ must be greater than order unity. (If the tunneling action is of order unity or smaller, the phase transition proceeds via spinodal decomposition, formation of very irregularly shaped fluctuation regions, rather than via the nucleation of true-vacuum bubbles.) We expand $A(t)$ about time t_*, which we take here to be the time that the phase transition completes,

$$A(t) = A_* - \beta(t-t_*) + \cdots , \tag{5.2}$$

and keep only the first two terms in the expansion. Here $A_* \equiv A(t_*)$ and $\beta \equiv -(dA/dt)_{t_*} > 0$. We will find that the functional form of our results is independent of both β and v, which only serve to set the scales of time and distance. For our assumption of a "fast" transition to be valid, β must be large compared to H; we will discuss the magnitude of β further at the end of our calculation.

With the cosmic expansion neglected, the factors of R cancel in the evaluation of Eq. (4.2) for $I(t)$. We can with little error set $t_0 = -\infty$ to obtain

$$I(t) = \frac{8\pi v^3}{\beta^4}\Gamma(t_*)e^{\beta(t-t_*)} = I(t_*)e^{\beta(t-t_*)} . \tag{5.3}$$

Exponentiating this gives $p(t)$, which is shown in Fig. 4. In the same figure we have also plotted $\mathcal{F}_E(t)$; note that it tracks $p(t)$ rather closely, but with a time lag of about $0.06\beta^{-1}$. The time scale for the completion of the transition is of order β^{-1}; specifically, the time for $p(t)$ to decrease from e^{-m} to e^{-M} is

$$\delta t = t_* - t_m = \ln\left[\frac{M}{m}\right]\beta^{-1} . \tag{5.4}$$

To obtain the distribution of bubble sizes we note that a bubble which has a radius ρ at time t must have been nucleated at a time $t_\rho = t - \rho/v$. From Eqs. (4.1) and (4.19), the bubble distribution at time t is then

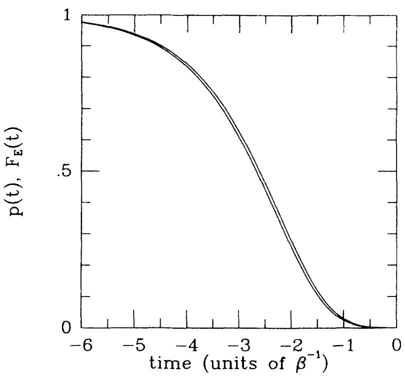

FIG. 4. The evolution of $p(t)$ (lower curve) and $\mathcal{F}_E(t)$ (upper curve) for model 1. Time is measured in units of β^{-1}, which is the inverse of the rate of change of the tunneling action at the completion of the phase transition. Time zero corresponds to the time that $p(t)=e^{-10}$.

$$\left[\frac{dN}{d\rho}\right]_t = \Gamma(t-\rho/v)e^{-I(t-\rho/v)}/v \; ;$$
$$= \frac{\beta^4}{8\pi v^4}\exp\left[-\left[I(t)e^{-\beta\rho/v}+\frac{\beta\rho}{v}-\ln I(t)\right]\right] . \quad (5.5)$$

This distribution of bubble sizes, shown in Fig. 5, achieves its maximum at

$$\bar{\rho}(t)=\frac{v}{\beta}\ln I(t) \; , \quad (5.6)$$

and has a width of order v/β. Note that as t increases the shape of this distribution is unchanged, aside from an overall shift to larger ρ (simply due to the expansion of each bubble). The total number of bubbles (per unit volume) at time t is $N_{\rm tot}=\beta^3[1-p(t)]/8\pi v^3$, and moments of the distribution can be expressed in terms of the incomplete gamma function and its derivatives.

Since β plays a crucial role in determining the typical size of bubbles at the time the transition is completed, let us examine it a bit more closely. First, consider the case

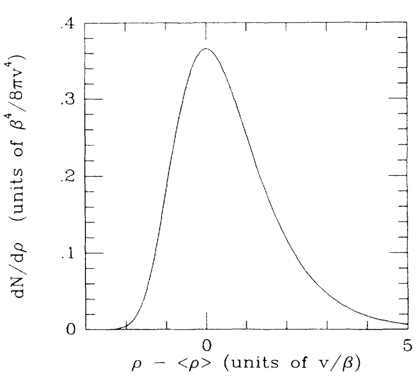

FIG. 5. Distribution of bubble sizes in model 1 as a function of bubble radius ρ at time t minus the mean bubble radius $\langle\rho\rangle = v\ln I(t)/\beta$ (v is the speed at which the bubble wall expands).

where the time dependence of Γ is due solely to the variation of temperature with time, so that

$$\beta = -\left[\frac{dA}{dt}\right]_{t_*} = \left[\frac{T}{A}\frac{dA}{dT}\right]_{t_*} A_* H_* \equiv \gamma(T_*)A_*H_* \; , \quad (5.7)$$

where we have defined $T_*=T(t_*)$, $H_*=H(T_*)$, and used the fact that $\dot{T}/T=-H$.

The first factor on the right-hand side is a dimensionless quantity which is defined in such a way as to be at most weakly dependent on the mass scale of the transition. Except for a possible divergence near the critical temperature [in a region where $A(T)$ would be too large for significant nucleation in any case] $\gamma(T)$ is of order unity for most plausible estimates of the nucleation rate. (For a further discussion of this point see Ref. [34].) To estimate the second factor we note that Eq. (5.3) implies that

$$\Gamma(t_*) = \frac{\beta^4}{8\pi v^3} I(t_*) = \frac{\beta^4 M}{8\pi v^3} \; . \quad (5.8)$$

Inserting Eqs. (5.1) and (5.7) then gives

$$A_* + 4\ln A_* = \ln\left[\frac{8\pi v^3}{\gamma^4(T_*)M}\frac{C}{H_*^4}\right] . \quad (5.9)$$

The logarithm on the right-hand side is controlled by the only factors that are not of order unity, namely, $C\sim\mathcal{M}^4$ and $H_*\sim\mathcal{M}^2/m_{\rm Pl}$. Hence

$$A_* \approx 4\ln\left[\frac{m_{\rm Pl}}{\mathcal{M}}\right] . \quad (5.10)$$

As long as $\mathcal{M}$ is at least 1 or 2 orders of magnitude below the Planck scale, this is large enough for the standard nucleation picture to be valid and implies that βH^{-1} is large enough to justify our neglect of the cosmic expansion. On the other hand, A_* cannot be too large. Even for $\mathcal{M}$ as low as 100 MeV, A_* is only of the order of 200. Hence, the characteristic bubble size at the end of the transition lies roughly between 1% and 100% of the Hubble radius, and we see that the bubble size is not set by $\mathcal{M}^{-1}$, as one might naively have guessed, but rather by H^{-1} which, by determining $\dot{T}$, fixes the rate of change of the nucleation rate. Similarly, the value of Γ/H^4 at t_* is not arbitrary, but rather must be of the order of $A_*^4\sim[\ln(m_{\rm Pl}/\mathcal{M})]^4$.

We will also be interested in cases where the time variation of the nucleation rate arises from the effects of some slowly varying scalar field ψ (other than the scalar field that is involved in the phase transition). Equation (5.7) is then replaced by

$$\beta = \left[\frac{\psi}{A}\frac{dA}{d\psi}\right]_{t_*}\left[\frac{\dot{\psi}}{\psi}\right]_{t_*} A_* \; . \quad (5.11)$$

As before, the first factor is in general of order unity, while A_* can be determined with the aid of Eq. (5.3). The second factor depends upon the time evolution of the field ψ which, through its equation of motion, depends upon the expansion rate H. As before, we can anticipate that the value of β will be determined by H. We will discuss this case further when we consider two-field inflation.

B. Model 2: Exponentially growing nucleation rate with exponential cosmic expansion

We now extend the previous model to include the cosmic expansion. For simplicity we take the expansion to be exponential, $R(t)=R_0 e^{Ht}$, with constant H. Also, since we expect the effects of the expansion to be significant only for the low-temperature bubbles produced in an inflationary transition, we set $v=1$ throughout. As before, we approximate $A(t)$ by a linear function

$$A(t)=A_* - \frac{H}{\alpha}(t-t_>) \ , \tag{5.12}$$

where $t_>$ is the greater of t_* and t_e, and $A_* \equiv A(t_>)$. For large α the nucleation rate grows slowly, and the completion of the transition takes many Hubble times; we will use the big-bubble constraints of Sec. III to place an upper limit to α (of order 6). On the other hand, for small α the transition completes in much less than a Hubble time, and this model reduces to the previous one, with $\beta=H/\alpha$.

It is now straightforward to evaluate $I(t)$ for times not very different from $t_>$. As in the previous example, we set the lower limit of the integration to $-\infty$. (In doing so, we need not worry about excluding bubbles produced before the last 60 e-foldings of inflation since the contribution from these is negligible for all values of α consistent with the big-bubble constraints.) Evaluation of the integral in Eq. (4.2) for $I(t)$ then yields

$$I(t)=\frac{8\pi\alpha^4}{(1+\alpha)(1+2\alpha)(1+3\alpha)}\frac{\Gamma(t)}{H^4} \ . \tag{5.13}$$

Differentiating this equation reveals that $dI/dt=HI/\alpha$. From the definition of t_e, it follows that

$$I(t_e)=3\alpha \ . \tag{5.14}$$

Since $I(t)$ increases monotonically, we see that t_e is later than t_* if Γ increases sufficiently slowly, $\alpha > M/3 \sim 1$, in accordance with our previous remarks. More specifically, Eq. (5.13) gives the time needed for $I(t)$ to vary between the two given values. Thus,

$$t_* - t_e = \alpha H^{-1} \ln\left[\frac{M}{3\alpha}\right] ; \tag{5.15}$$

while δt, the time interval over which I increases from m to M, is

$$\delta t = \alpha H^{-1} \ln\left[\frac{M}{m}\right] . \tag{5.16}$$

To obtain the distribution of bubble sizes, we note that with constant Hubble parameter Eq. (4.18) takes the form

$$\rho = H^{-1}\{\exp[H(t-t_\rho)]-1\} \ . \tag{5.17}$$

Using this relation, as well as Eqs. (4.19), (5.12), and (5.13), the distribution of bubble sizes at time t is found to be

$$\left[\frac{dN}{d\rho}\right]_t = \frac{\Gamma(t)}{(1+H\rho)^{4+1/\alpha}}\exp\left[-\frac{I(t)}{(1+H\rho)^{1/\alpha}}\right] . \tag{5.18}$$

It achieves its maximum at

$$\bar{\rho}(t) = H^{-1}\left[\left[\frac{I(t)}{4\alpha+1}\right]^\alpha - 1\right] . \tag{5.19}$$

For $I(t) < 4\alpha+1$, $\bar{\rho}$ is negative, corresponding to a distribution of bubble sizes that is monotonically decreasing for all $\rho > 0$.

For large α (i.e., a relatively slow transition) Eqs. (5.13) and (5.14) lead to

$$\left[\frac{\Gamma}{H^4}\right]_{t_e} = \frac{9}{4\pi}\left[1+\frac{11}{6\alpha}+\cdots\right] , \tag{5.20}$$

which should be compared with the approximation (4.24), $(\Gamma/H^4)_{t_e} \approx 9/4\pi$. The big-bubble constraints are bounds to

$$\left[\frac{\Gamma}{H^4}\right]_N = e^{-N/\alpha}\left[\frac{\Gamma}{H^4}\right]_{t_e} . \tag{5.21}$$

Using Eq. (5.20) to evaluate the right-hand side, and taking $\mathcal{M}=10^{14}$ GeV, we find that the bounds of Sec. III give an upper limit $\alpha \le \alpha_{\max} \simeq 5-6$.

From Eq. (5.14) it follows that $p(t_e)=e^{-3\alpha}$, indicating that for $\alpha \gtrsim 1$ all but a small fraction of space is converted to true vacuum before V_{phys} begins to decrease. This is reminiscent of old inflation, and indeed the bubble distribution bears some resemblance to that of old inflation. Small bubbles dominate numerically: $dN/d\rho$ decreases monotonically from $\rho=0$ and falls to one-half of its peak value by $\rho=H^{-1}$. On the other hand, large bubbles are responsible for the bulk of the volume. The volume-weighted bubble distribution at t_e,

$$\left[\rho^3\frac{dN}{d\ln\rho}\right]_{t_e} = \frac{\Gamma(t_e)}{H^4}\frac{(H\rho)^4}{(1+H\rho)^{4+1/\alpha}} \times \exp\left[-\frac{3\alpha}{(1+H\rho)^{1/\alpha}}\right] , \tag{5.22}$$

achieves its maximum at

$$\rho_{\text{peak}} \approx (3\alpha)^\alpha H^{-1} \ , \tag{5.23}$$

and as a function of $\ln\rho$ has a width of order $\alpha\sqrt{e}$. Because the interior of a large bubble is not homogenized until it comes back within the horizon, a relatively homogeneous radiation-dominated universe cannot be established until the temperature has fallen to about $\mathcal{M}/(H\rho_{\text{peak}})$; for $\alpha=6$ this is about 8 orders of magnitude below the critical temperature.

To illustrate the development of the transition at the ragged edge of the big-bubble constraint, we show $p(t)$, $\mathcal{F}_E(t)$, and the distribution of bubble sizes for $\alpha=6$ in Figs. 6 and 7.

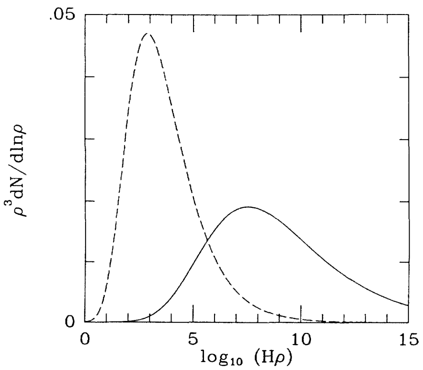

FIG. 6. The volume-weighted distribution of bubble sizes, $\rho^3 dN/d\ln\rho$, for model 2 with $\alpha=6\simeq\alpha_{\max}$ (solid curve) and $\alpha=3$ (broken curve). Note that the distribution peaks at a bubble radius of order $\rho_{\rm peak}\sim(3\alpha)^\alpha$ and that for $\rho\gg\rho_{\rm peak}$, $\rho^3 dN/d\ln\rho$ only decreases as $\rho^{-1/\alpha}$.

We can also recover the case of a rapid transition by taking the limit $\alpha\ll1$. The end of the transition is now signaled by t_*, rather than t_e, with the duration δt of the transition becoming much less than a Hubble time. As expected, in this limit the various formulas go over into the corresponding results for model 1, with $\beta=H/\alpha$ and $v=1$.

C. Model 3: Power-law nucleation rate with power-law expansion

For our third model we take

$$R(t)=R_0 t^p,\quad \frac{\Gamma}{H^4}=\gamma t^q\ . \tag{5.24}$$

Note that this implies $\Gamma=\gamma p^4 t^{q-4}$. This form is motivated in particular by Brans-Dicke extended inflation [2] and generalizations [36] of this model involving dilaton fields. We require $p>1$, so that the expansion be inflationary, and $q>0$, so that the transition eventually complete; moreover, we only require that these functional forms be good approximations near the end of inflation. Again, we set $v=1$ because we expect this model to apply primarily to inflationary transitions.

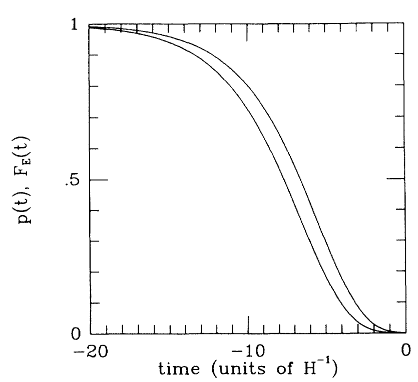

FIG. 7. For model 2 and $\alpha=6\simeq\alpha_{\max}$, $p(t)$ (lower curve) and $\mathcal{F}_E(t)$ (upper curve). The zero of time corresponds to t_e; i.e., the epoch when the volume of physical space in the false vacuum begins to decrease.

Evaluation of Eq. (4.2), with the lower limit t_0 set equal to zero, gives

$$I(t)=\frac{8\pi p^4}{q(q+p-1)(q+2p-2)(q+3p-3)}\left[\frac{\Gamma}{H^4}\right]_t\ . \tag{5.25}$$

Differentiating this reveals that

$$I(t_e)=\frac{3p}{q}\ . \tag{5.26}$$

Equation (5.25) shows that times of interest during the course of the transition are related by

$$\frac{t_1}{t_2}=\left[\frac{I(t_1)}{I(t_2)}\right]^{1/q}\ . \tag{5.27}$$

In particular, for a fast transition, where $q\gg p>1$,

$$\delta t=t_*-t_m=t_*\left[1-\left[\frac{M}{m}\right]^{1/q}\right]\approx\tau\ln\left[\frac{M}{m}\right]\ , \tag{5.28}$$

where $\tau=q/t_*$ is the characteristic time scale for the variation of Γ/H^4 at the end of the transition. Since $\tau=(d\ln\Gamma/dt)_{t*}$, this reproduces the corresponding result for model 1, cf. Eq. (5.4), as should be expected.

The physical radius of a bubble that was nucleated at time t_ρ at time t is

$$\rho=\rho_0\left[\left[\frac{t}{t_\rho}\right]^{p-1}-1\right]\ , \tag{5.29}$$

where

$$\rho_0=\frac{1}{H(t)}\left[\frac{p}{p-1}\right]\ . \tag{5.30}$$

Using this one finds that

$$\left[\frac{dN}{d\rho}\right]_t=\Gamma(t)\left[\frac{\rho_0}{\rho_0+\rho}\right]^{4+q/(p-1)}\times\exp\left[-I(t)\left[\frac{\rho_0}{\rho_0+\rho}\right]^{q/(p-1)}\right]\ . \tag{5.31}$$

This formula is the same as Eq. (5.18) if we make the replacements

$$(p-1)/q\to\alpha\ ,\quad \rho_0^{-1}\to H\ . \tag{5.32}$$

Thus, the qualitative features of the bubble distribution at the end of the transition are the same as in model 2 (see Fig. 6). In a slow transition small bubbles predominate numerically, but most of the volume is in large bubbles, with $\rho^3 dN/d\ln\rho$ peaking at $\rho\approx(3p/q)^{p/q}\rho_0$. In a fast transition the bubble distribution is sharply peaked about $\rho\approx\tau\ln M$.

To apply the big-bubble constraints we note that for a slow transition, where $p\gg q\geq0$, Eqs. (5.25) and (5.26) imply that

$$\left[\frac{\Gamma}{H^4}\right]_{t_e}=\frac{9}{4\pi}\left[1+\frac{11q-18}{6p}+\cdots\right] . \tag{5.33}$$

The bounds are to be applied to bubbles with radius $e^N(1-1/p)\rho_0$ at the end of inflation. From Eq. (5.29) we see that these were nucleated at a time $t_p=t_e[p/(p-1)]^{1/(p-1)}e^{-N/(p-1)}$, so that

$$\left[\frac{\Gamma}{H^4}\right]_N=e^{-qN/(p-1)}\left[\frac{p}{p-1}\right]^{q/(p-1)}\left[\frac{\Gamma}{H^4}\right]_{t_e} . \tag{5.34}$$

Comparing this with Eq. (5.21) for model 2, we see that the bounds for the two models are related by

$$\left[\frac{p-1}{q}\right]_{\max}\approx\alpha_{\max}\simeq 5-6 . \tag{5.35}$$

In Brans-Dicke extended inflation $q=4$ and $p=\omega+\frac{1}{2}$. This upper bound then translates into $\omega\lesssim 20-25$, which is essentially the bound of Ref. [12]. There is also a lower bound to ω from the requirement of sufficient inflation: The effective Planck mass in this theory is given by the square root of the Brans-Dicke field Φ, and grows linearly with time during inflation, reaching essentially its present value of 10^{19} GeV at the end of inflation. In order that quantum gravity effects be negligible, $\sqrt{\Phi}$ must be larger than the mass scale $\mathcal{M}$ of the transition throughout the inflationary period. For any given degree of inflation, this upper limit to the growth of Φ places a lower limit to ω. To obtain 60 e-foldings of inflation, $\omega\gtrsim 5$ if $\mathcal{M}=10^{14}$ GeV; as the amount of inflation desired rises, so does the lower limit to ω.

Related models involving dilaton fields have also been proposed [36]. For these, our bounds agree with those obtained by Wang [37]. In these models, also, the big-bubble constraints and the requirement of sufficient inflation work in opposite directions.

D. Two-field inflation and related models

Two-field models of first-order inflation contain an inflaton field σ, which is trapped in a false-vacuum state with $\sigma=\sigma_f$ (and is the field responsible for the phase transition), and a trigger field ψ, which slowly evolves toward the minimum of its potential. The coupling between these fields is arranged to make the tunneling action A for the inflaton field decrease as ψ evolves. When ψ reaches a critical value A is sufficiently small that bubbles are rapidly nucleated and the transition completes; for our purposes we define this critical value to be $\psi_e\equiv\psi(t_e)$.

In order that this be first-order inflation, rather than slow-rollover inflation, the evolution of ψ must have little effect on the vacuum energy, which can therefore be viewed as arising from the σ field alone. (A consequence of this is that H may be treated as constant during inflation.) Hence, $[\partial V(\sigma,\psi)/\partial\psi]|_{\sigma_f}$ must be small. This is also needed in order that the trigger field be a "slow roller," with equation of motion

$$\dot{\psi}\simeq\frac{-V'}{3H} , \tag{5.36}$$

where $V'=\partial V/\partial\psi$. (Here, and henceforth, V and its derivatives are assumed to be evaluated with $\sigma=\sigma_f$.)

The amount of inflation is determined by the time it takes ψ to evolve from its initial value ψ_i to ψ_e. In a successful model this must be long enough to ensure at least 60 or so e-foldings of the scale factor:

$$\mathcal{N}(\psi_i,\psi_e)\equiv\int H\,dt=-3H^2\int_{\psi_i}^{\psi_e}\frac{d\psi}{V'}\gtrsim 60 . \tag{5.37}$$

With the aid of Eq. (5.36), this may be rewritten as

$$\mathcal{N}(\psi_i,\psi_e)=\frac{3H^2\psi_e}{|V'(\psi_e)|}f(\psi_i,\psi_e)\gtrsim 60 , \tag{5.38}$$

where we have defined the dimensionless quantity

$$f(\psi_i,\psi_e)=\int_{\psi_e}^{\psi_i}\frac{d\psi}{\psi_e}\frac{|V'(\psi_e)|}{V'(\psi)} . \tag{5.39}$$

Typically f is expected to be of order unity. For example, if $V(\psi)$ can be approximated by the form $a\pm b^2\psi^p$:

$$f(\psi_i,\psi_e)=\begin{cases}\pm\ln\left[\dfrac{\psi_i}{\psi_e}\right] , & p=2\\[2ex] \pm\dfrac{1}{p-2}\left[1-\left[\dfrac{\psi_e}{\psi_i}\right]^{p-2}\right] , & p\neq 2 .\end{cases} \tag{5.40}$$

[Note that for the upper (lower) choice of sign ψ_i is greater (less) than ψ_e.]

We must also require that the big-bubble constraints be satisfied. If the tunneling action is approximated by a linear function of time, as in Eq. (5.12), these constraints place an upper bound to $\alpha=H/\dot{A}(t_e)$. Defining

$$g=\left[\frac{\psi}{A}\left|\frac{dA}{d\psi}\right|\right]_{t_e} , \tag{5.41}$$

we may write

$$\alpha^{-1}=gA(t_e)H^{-1}\left[\frac{|\dot{\psi}|}{\psi}\right]_{t_e}=\frac{gA(t_e)|V'(\psi_e)|}{3H^2\psi_e} , \tag{5.42}$$

where the second equality is obtained with the aid of Eq. (5.36). Comparing this with Eq. (5.38) we obtain

$$\alpha=\frac{\mathcal{N}(\psi_i,\psi_e)}{A(t_e)}f(\psi_i,\psi_e)g^{-1} . \tag{5.43}$$

Arguments similar to those leading to Eq. (5.9) show that $A(t_e)\approx 4\ln(m_{\rm Pl}/\mathcal{M})\approx 46+4n_{14}$. Using the bound (5.38) on $\mathcal{N}$ we may write the constraint $\alpha\le\alpha_{\max}\simeq 5-6$ as

$$\frac{\mathcal{N}(\psi_i,\psi_e)}{46+4n_{14}}f(\psi_i,\psi_e)g^{-1}\lesssim 5-6 . \tag{5.44}$$

Since g, like f, will typically be of order unity, while $\mathcal{N}$ is bounded from below by 60 or so, we see that the generic two-field model will always be close to the edge when it comes to satisfying the big-bubble constraints. Furthermore, since α increases with $\mathcal{N}$ and, in the spirit of inflation, there is no reason for $\mathcal{N}$ to be close to 60 or so, satisfying this constraint is likely to be very difficult indeed.

A successful model must also be such that the density perturbations arising from quantum fluctuations in the ψ field during its slow roll phase are not too great [38]. The amplitude of these density perturbations at the post inflationary horizon crossing is easy to estimate using the fact that while a given mode is outside the horizon the gauge-invariant quantity $\zeta \equiv \delta\rho/(\rho+p)$ remains constant. The inflaton field makes no contribution to $\rho+p$ (since $\rho_\sigma + p_\sigma = 0$) or to $\delta\rho$ (σ-field fluctuations are suppressed because in the false-vacuum state $m_\sigma^2 \gg H^2$). For the ψ field, $\rho_\psi + p_\psi = \dot{\psi}^2$ and $\delta\rho = \Delta\psi |V'| \simeq H|V'|/2\pi$. From this it follows that

$$\left(\frac{\delta\rho}{\rho}\right)_{\rm hor} \simeq \frac{3H^3}{|V'|} , \tag{5.45}$$

where as usual the right-hand side of the equation is to be evaluated when the scale of interest crossed outside the horizon during inflation. For astrophysically interesting scales, which went outside the horizon 40–60 e-foldings before the end of inflation, this quantity can be at most $\sim 3\times 10^{-5}$.

This is the familiar formula for the amplitude of inflation-produced density perturbations, though there are some important differences. First, we should remember that the vacuum energy that fixes the value of H is determined by the σ field rather than by the ψ field. This increases the value of $(\delta\rho/\rho)_{\rm hor}$ relative to that expected for the case of slow-rollover inflation driven by the ψ field. Second, inflation ends when the σ field makes the transition to the true vacuum, and not when the ψ field leaves the slow-rolling regime. Hence, there is no need for the denominator on the right-hand side of Eq. (5.45) to be increasing near the end of inflation. This too works in the direction of increasing the density fluctuations. In pure slow-rollover inflation the issue of density perturbations is a difficult one; in two-field models it is even more so.

Another interesting relation follows from the consideration of density perturbations. Equation (5.45) relates the density perturbations to the value of $V'(\psi)$ at times 40–60 e-foldings before the end of inflation, while Eq. (5.42) involves the same quantity at the time the transition is completed. As remarked above, in ordinary slow-rollover inflation V' is significantly larger at the later time, since the end of inflation occurs as the inflaton field leaves the flat part of the potential. There is no reason to expect this to be true in two-field first-order inflation, since the value of ψ needed to trigger the transition is not related to any special property of V or its derivatives. Instead, one might plausibly guess that the values of V' entering Eqs. (5.42) and (5.45) are roughly the same. Assuming this to be the case we obtain the relation

$$\alpha = \frac{1}{g}\frac{1}{A(t_e)}\left(\frac{\delta\rho}{\rho}\right)_{\rm hor}\frac{\psi_e}{H} . \tag{5.46}$$

Given the magnitude of the density fluctuations, this translates the upper bound on α to an upper bound on ψ_e. For example, if we take $(\delta\rho/\rho)_{\rm hor} \simeq 3\times 10^{-5}$ and choose $\mathcal{M} = 10^{14}$ GeV, so that $A(t_e) \simeq 46$, we find that $\psi_e \lesssim 3\times 10^{16}$ GeV; it cannot be much less than this value, or else $\mathcal{N}$ will be too small.

A specific realization of two-field inflation is provided by the "ramp-potential" model of Ref. [22]. In this model the coupling of the inflaton and trigger fields is such that the energy difference between the true and false vacua is of the form

$$\varepsilon = \varepsilon_0(1+\eta\psi^3/\psi_e^3) . \tag{5.47}$$

In the thin-wall approximation the tunneling action varies as ε^{-3}. Using this approximation, the authors of Ref. [22] write

$$A = \frac{A_0}{(1+\eta\psi^3/\psi_e^3)^3} . \tag{5.48}$$

Substitution of this form into Eq. (5.41) yields

$$g = \frac{9\eta}{1+\eta} . \tag{5.49}$$

Unless the value of A_0 is just such that the Universe is on the verge of nucleating bubbles long before ψ reaches ψ_e, η cannot be too small nor can the initial value of ψ be close to ψ_e (i.e., $\psi_i \ll \psi_e$). Furthermore, since we did not want the variation of the trigger field to affect the energy density significantly, η should be at most of order unity. Together, these two considerations suggest that the value of η should be 0.1–1 (which is what the authors of Ref. [22] also conclude), and hence that g is in the range 0.8–5. [Actually, when the thin-wall approximation is valid the tunneling action tends to be much larger than the value we have deduced for $A(t_e)$. We therefore expect the transition to complete after ε has become too big to justify the thin-wall approximation. The dependence of A upon ε is then expected to be milder, which will tend to exacerbate the problem of satisfying the big-bubble constraint.]

In this model, when the σ field is at its false-vacuum value the trigger field "feels" a linear potential

$$V(\psi) = V(\psi_0) - \mathcal{F}(\psi - \psi_0) . \tag{5.50}$$

This gives

$$f(\psi_i, \psi_e) = \frac{\psi_e - \psi_i}{\psi_e} , \tag{5.51}$$

so that the amount of inflation is given by

$$\mathcal{N}(\psi_i, \psi_e) = \frac{3H^2}{\mathcal{F}}(\psi_e - \psi_i) \approx \frac{3H^2\psi_e}{\mathcal{F}} . \tag{5.52}$$

The constraint (5.44) can be rewritten as

$$\frac{\mathcal{N}(\psi_i,\psi_e)}{46+4n_{14}}\left(\frac{\psi_e-\psi_i}{\psi_i}\right)\left(\frac{1+\eta}{9\eta}\right) \lesssim 5\text{–}6 . \tag{5.53}$$

Since we expect the initial value of the trigger field to be much smaller than its final value ψ_e, we conclude that unless the value of $\mathcal{N}$ is close to the minimum required to solve the horizon and flatness problems, the big-bubble constraints cannot be satisfied. Put another way, the ramp model can, at best, marginally satisfy the big-bubble constraints.

VI. CONCLUDING REMARKS

We have studied the kinematics of bubble nucleation in first-order inflation and other cosmological phase transitions. In particular we have identified several measures that track the progress of a phase transition, the probability that a point in space remains in the false vacuum, the volume of physical space in the false vacuum, and the fraction of uncollided bubble wall, and we have discussed the distribution of bubble sizes and the duration of the transition. In a successful phase transition the end of the transition as determined by our three different measures of progress agree to within a few Hubble times. In general, the characteristic bubble size and duration of the phase transition are determined by a characteristic time $\tau=(d\ln\Gamma/dt)^{-1}$, which for a very wide class of models is of the order of 1% to 100% of the Hubble time, depending only logarithmically on the energy scale of the phase transition.

In any first-order phase transition the crucial quantity that determines essentially all aspects of the transition is Γ/H^4, the number of bubbles nucleated per Hubble time per Hubble volume. The value of Γ/H^4 at time t determines the fraction of space occupied by bubbles nucleated between t and $t+H^{-1}$. In a successful model of first-order inflation Γ/H^4 must ultimately exceed $9/4\pi$ in order that the volume of physical space remaining in the false vacuum decreases. It must also be quite small only a short time earlier: In first-order inflation bubbles nucleated well before the end of inflation grow to astrophysical size and remain as unhomogenized regions until late in the history of the Universe. Such big bubbles can leave scars on the Universe by interferring with primordial nucleosynthesis, by distorting the spectrum of the CMBR, or by leading to large anisotropies in the temperature of the CMBR. The empirical requirement that big bubbles be suppressed constrains the value of Γ/H^4 to be less than $10^{-4}-10^{-3}$ at the time that such bubbles were nucleated, which for exponential inflation is between about 39 and 54 e-foldings in the scale factor before the end of inflation. That dashes the hope [14] that individual bubbles have anything to do with the apparent bubbly nature of the distribution of galaxies [29], although the possibility remains that Poisson fluctuations in the number of moderate-size bubbles nucleated within a given volume could give rise to isocurvature perturbations that are of interest for structure formation and whose spectrum is not scale invariant.

In all the models of first-order inflation that we have considered we have seen that there is a conflict between having sufficient inflation and adequately suppressing big bubbles. To achieve the former one wants Γ/H^4 to increase slowly, while to satisfy the latter one wants Γ/H^4 to increase rapidly (see Fig. 3). In models where Γ/H^4 varies smoothly the "window" defined by these constraints is narrow; e.g., in Brans-Dicke extended inflation ω must exceed 5 to achieve sufficient inflation, while suppression of big bubbles requires that ω be less than about 20. There is, of course, nothing that forbids models in which Γ/H^4 changes from varying slowly to varying rapidly. However, such models lose one of the attractive features of the first successful model of first-order inflation (extended inflation), in that the critical value of the "spectator field" (Brans-Dicke field, dilaton, or trigger field) which brings Γ/H^4 above unity must be correlated with a mass scale set by the spectator dynamics.

The big-bubble constraints can be even more difficult to satisfy in more exotic models. For example, some authors have proposed a model in which a period of first-order inflation is followed by a second period of slow-rollover inflation [39]. The big-bubble constraints that we have discussed place limits on the value of Γ/H^4 when the value of the cosmic scale was a specified number of e-foldings smaller than at the end of inflation (about 39 to 54); if we assume that the second period of inflation lasts P e-foldings, then the big-bubble constraints place limits to Γ/H^4 of $10^{-4}-10^{-3}$ at $39-P$ to $54-P$ e-foldings before the end of the first epoch of inflation, and of course to successfully terminate the first period of inflation Γ/H^4 must exceed $9/4\pi$. (For simplicity, we have set $n_{14}=0$ and assumed both epochs of inflation were exponential.) This means that Γ/H^4 must grow from order $10^{-4}-10^{-3}$ to order unity in a shorter time than in the absence of the second inflationary epoch, and tightens the constraints to the rate of change of Γ/H^4; e.g., to α in model 2, or to $(p-1)/q$ in model 3.

Finally, in our discussions of the deleterious effects of big bubbles, we have focused on large bubbles that are homogenized at the time of nucleosynthesis or later. We have seen that the suppression of such bubbles to an acceptably low level is a rather nontrivial constraint on model building. It is therefore interesting to consider the possibility that the density of these bubbles is in fact close to the bounds we have derived. If this is the case, one might expect to find significant effects from regions that homogenize at times just before nucleosynthesis. One place to look for such effects might be the electroweak phase transition. The presence of large, unhomogenized regions would imply a departure from thermal equilibrium, one of the necessary conditions for baryogenesis to proceed, and a condition that seems to be at best marginally satisfied in the standard scenario of the electroweak phase transition [3]. Perhaps big bubbles could have good effects too.

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[1] See, e.g., E. W. Kolb and M. S. Turner, *The Early Universe* (Addison-Wesley, Redwood City, CA, 1990), Chap. 7. Gravitational radiation from vacuum-bubble collisions is discussed by M. S. Turner and F. Wilczek, Phys. Rev. Lett. **66**, 5 (1991) and A. Kosowsky, M. S. Turner, and R. Watkins, Phys. Rev. D **45**, 4514 (1992).

[2] By first-order inflation we mean an inflationary model wherein the scalar field that supplies the false-vacuum energy which powers inflation is trapped in a metastable, local minimum of its potential and must tunnel to the true vacuum. In the simplest models of first-order inflation this scalar field is responsible for a symmetry-breaking phase transition. By contrast, in slow-rollover inflation the scalar field that supplies the vacuum energy that powers inflation evolves to the minimum of its potential classically (by slowly rolling down its potential). The first model of first-order inflation was Guth's original model of inflation, A. H. Guth, Phys. Rev. D **23**, 347 (1981). This model was a cosmological disaster; starting with extended inflation, D. La and P. J. Steinhardt, Phys. Rev. Lett. **62**, 376 (1989), there has been a revival of interest in first-order inflation; see, e.g., the review, E. W. Kolb, Phys. Scr. **T36**, 199 (1991).

[3] There has been a great deal of interest in the electroweak phase transition recently, particularly the possibility that the baryon asymmetry of the Universe could have risen during this epoch; see, e.g., A. D. Dolgov, Phys. Rep. (to be published); M. E. Shaposhnikov, Phys. Scr. **T36**, 183 (1991); G. W. Anderson and L. J. Hall, Phys. Rev. D **45**, 2685 (1992); M. Gleiser and E. W. Kolb, Fermilab Report No. FERMILAB-Pub-91-305-A, 1991 (unpublished); D. E. Brahm and S. D. H. Hsu, Caltech Reports No. CALT-68-1705 and No. CALT-68-1762, 1991 (unpublished); M. Dine, R. Leigh, P. Huet, A. Linde, and D. Linde, Phys. Rev. **46**, 550 (1992); N. Turok, Phys. Rev. Lett. **68**, 1803 (1992).

[4] See, e.g., E. W. Kolb and M. S. Turner, *The Early Universe* [1], Chap. 8.

[5] The reason that $\Gamma/H^4 \approx 1$ is the dividing point between sufficient and insufficient bubble nucleation is simple to see: In de Sitter space a bubble grows in comoving size for only about a Hubble time; thereafter, its size simply stretches as the Universe expands. Thus, in order to transform all of comoving space to the true vacuum, of the order of one bubble must be nucleated per Hubble volume in less than a Hubble time.

[6] A. H. Guth and E. J. Weinberg, Nucl. Phys. **B212**, 321 (1983).

[7] See, e.g., R. Holman *et al.*, Phys. Lett. B **237**, 37 (1990).

[8] S. Coleman, Phys. Rev. D **15**, 2929 (1977); C. Callan and S. Coleman, *ibid.* **16**, 1762 (1977); S. Coleman and F. De Luccia, *ibid.* **21**, 3305 (1980).

[9] It has been claimed that effects such as these can be made large in Brans-Dicke extended inflation by the appropriate choice of coupling constants, see D. S. Goldwirth and H. W. Zaglauer, Phys. Rev. Lett. **67**, 3639 (1991). However, this choice gives a value for Γ/H^4 at the end of inflation that is far too small for the transition to have completed. Also see A. R. Liddle and D. Wands, Phys. Rev. D (to be published).

[10] For a discussion of how the false-vacuum energy is thermalized by bubble collisions, see, e.g., S. W. Hawking, J. Stewart, and I. Moss, Phys. Rev. D **26**, 2681 (1982); or R. Watkins and L. M. Widrow, Nucl. Phys. **B374**, 446 (1992).

[11] The detailed cosmological evolution of a very large bubble will be discussed by S. Vadas (in preparation).

[12] E. J. Weinberg, Phys. Rev. D **40**, 3950 (1989); D. La, P. J. Steinhardt, and E. Bertschinger, Phys. Lett. B **231**, 231 (1989).

[13] Various aspects of the effect of big bubbles on the microwave background in extended inflation have been discussed; see, e.g., A. R. Liddle and D. Wands, Mon. Not. R. Astron. Soc. **253**, 637 (1991); A. Wu, Phys. Rev. D **45**, 2653 (1992); A. R. Liddle and D. Wands, *ibid.* **45**, 2665 (1992); Phys. Lett. B **276**, 18 (1992).

[14] See, e.g., D. La, Phys. Lett. B **265**, 232 (1991).

[15] See, e.g., the recent review by T. P. Walker *et al.*, Astrophys. J. **376**, 51 (1991).

[16] See, e.g., the recent review by R. A. Malaney and G. J. Matthews, Phys. Rep. (to be published).

[17] One might also worry about the effect of a large bubble between us and the last scattering surface; so long as the perturbation associated with the bubble is in the linear regime there is no effect because the redshift of a CMBR photon only depends upon the difference in the Newtonian potential between the last-scattering surface and the observer; see R. K. Sachs and A. Wolfe, Astrophys. J. **147**, 73 (1967). A distortion can arise if a CMBR photon traverses a region where the gravitational potential is varying rapidly and the perturbation is in the nonlinear regime; see E. Martinez-Gonzalez, J. L. Sanz, and J. Silk, *ibid.* **355**, L5 (1990).

[18] See, e.g., J. Silk, Phys. Scr. **T36**, 16 (1991); or Kolb and Turner, *The Early Universe* [1], Chaps. 1 and 9. and 9.

[19] H. Gush *et al.*, Phys. Rev. Lett. **65**, 537 (1990); J. Mather *et al.*, Astrophys. J. **354**, L37 (1990).

[20] For a discussion of the μ distortion, see, e.g., L. Danese and G. De Zotti, Nuovo Cimento **7**, 277 (1977).

[21] E. W. Kolb, D. Salopek, and M. S. Turner, Phys. Rev. D **42**, 3925 (1990); A. H. Guth and B. Jain, *ibid.* **45**, 426 (1992).

[22] See, e.g., F. Adams and K. Freese, Phys. Rev. D **43**, 353 (1991).

[23] J. M. Bardeen, P. J. Steinhardt, and M. S. Turner, Phys. Rev. D **28**, 679 (1983).

[24] In the Bardeen formalism spatial variations in the equation of state can affect the amplitude of ζ when the perturbation scale is still outside the horizon; however, the approach we take here will lead to the same final result; see J. M. Bardeen, Phys. Rev. D **22**, 1882 (1980).

[25] W. H. Press and E. T. Vishniac, Astrophys. J. **239**, 1 (1980); G. Efstathiou and J. R. Bond, Mon. Not. R. Astron. Soc. **218**, 103 (1986); Kolb and Turner, *The Early Universe* [1], Chap. 9.

[26] One could imagine a scenario where $\delta\rho_m/\rho_m$ is much greater than $\delta p/p$, e.g., if dark matter particles were preferentially produced within large bubbles, or where $\delta\rho_m/\rho_m$ is much smaller. It is a simple matter to modify the results we discuss here.

[27] M. S. Turner *et al.* (in preparation).

[28] See, e.g., G. Efstathiou, in *Physics of the Early Universe*, Proceedings of the Scottish Universities Summer School, 1989, edited by J. A. Peacock, A. F. Heavens, and A. T. Davies, Scottish Universities Summer School in Physics No. 36 (Hilger, Bristol, 1989); or J. Silk, Phys. Scr. **T36**, 15 (1991).

[29] V. de Lapparent, M. J. Geller, and J. Huchra, Astrophys. J. **302**, L1 (1986); M. J. Geller and J. Huchra, Science **246**, 897 (1989).

[30] A. H. Guth and H. Tye, Phys. Rev. Lett. **44**, 631 (1980); **44**, 963(E) (1980).

[31] Because of the differences in the expansion of the bubble interior, the bubble walls, and the outside false vacuum, there may be corrections to this statement in an expanding universe. We expect these corrections to be relatively small for the transitions we consider, for which significant bubble nucleation lasts for at most a few Hubble times.

[32] On the other hand, $\mathcal{F}$ does give an indication that old inflation is unsatisfactory. For exponential expansion with constant $\Gamma \ll H^4$, $\mathcal{F}$ always remains close to unity, with $1-\mathcal{F}$ approaching $22\pi\Gamma/9H^4$ at large times.

[33] After the onset of percolation there will be regions of false vacuum surrounded completely by true vacuum. It has been suggested that the existence of such regions could lead to the formation of black holes and baby universes during the course of extended inflation; see, S. D. H. Hsu, Phys. Lett. B **251**, 343 (1990). Because these false-vacuum regions are rapidly being converted to true vacuum (we expect Γ/H^4 to be of order unity at the time of percolation) we do not believe this to be the case. In this regard, we note that the singularity theorems cannot be applied, since the false vacuum does not obey the required energy conditions.

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[35] This model would not, however, describe a finite-temperature transition with slowly expanding bubbles in which the release of the latent heat was efficient enough to maintain a constant temperature during the course of the transition.

[36] T. Damour, G. W. Gibbons, and C. Gundlach, Phys. Rev. Lett. **64**, 123 (1990); R. Holman, E. W. Kolb, and Y. Wang, *ibid.* **65**, 17 (1990); R. Holman, F. W. Kolb, S. L. Vadas, and Y. Wang, Phys. Rev. D **43**, 3833 (1991).

[37] Y. Wang, Phys. Rev. D **44**, 991 (1992).

[38] Because the ψ field is very weakly coupled, one also has to worry about the energy density left in the coherent oscillations of the ψ field, which behave as nonrelativistic matter: the "Polonyi problem." See, e.g., G. D. Coughlan *et al.*, Phys. Lett. **131B**, 59 (1983). Unless the ψ field is unstable and has a relatively short lifetime, this can be a serious problem.

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