Thermal history of the string universe

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The thermal history of the string universe based on the Brandenberger and Vafa scenario is examined. The analysis thereby provides a theoretical foundation of the string universe scenario. In particular, the picture of the initial oscillating phase is shown to be natural from the thermodynamical point of view. A precise description is also given of the transition process from the stringy phase to the radiation-dominated phase. Through the discussion it is shown that the well-known form of the string multistate density is incorrect in the interesting parameter region.

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I. INTRODUCTION

Since the Green and Schwarz anomaly cancellation [1] proved the importance of superstring theory (SST), detailed studies of it have been done. One unfortunate feature of SST is the fact that its typical energy scale is 10^{19} GeV which is far beyond our experimental access. However this does not necessarily imply that SST allows no experimental test. The most important feature of SST is that it unifies the theories of matter and gravity. This implies that SST in principle has an ability to determine the structure of space-time in which the strings themselves live. If SST is the true theory of the whole Universe, it is conceivable that SST has left some relics in our Universe observable even today. In fact the presently observed isotropic, uniform, and almost flat universe must have been determined by SST. From this point of view string cosmology has been studied by some authors [2–6].

Brandenberger and Vafa [4] proposed an interesting scenario of string cosmology. The starting point of their scenario is the heterotic string theory in the space of a nine-dimensional torus Universe of the Planckian size and a time dimension $T^9 \times R$. They argued that this small universe was oscillating in some period and eventually three dimensions out of nine began to expand resulting in the present large Universe.

In order to get a deeper theoretical understanding of their scenario, we perform a detailed thermodynamical analysis of it in this paper. Our strategy in this paper is to use the microcanonical formalism to follow an entire thermal history. So far several authors have employed the microcanonical formalism to examine the thermodynamical functions of the string gas. However the relation of their results to the thermal history of the string universe seems to be unclear to us.

In order to clear up these situations in this paper we give a concrete framework by which we can follow the thermal history of the string universe using the thermodynamical functions of the microcanonical formalism. According to this framework and based on some assumptions such as the local thermal equilibrium and others which we will state precisely later, we will determine the thermal history of the nine-dimensional torus universe of Brandenberger and Vafa as follows.

In the initial epoch during which the torus universe is oscillating, very high energy strings occasionally emit zero modes (massless point particles) due to cosmological expansion and sometimes absorb zero modes due to cosmological contraction. This process is shown to be adiabatic. This adiabaticity, or in other words reversibility, ensures that the oscillation is not damping. Thus our result is quite consistent with the picture of the initially oscillating universe which is followed by the three-dimensionally expanding epoch.

After some three-dimensional directions start to expand with the remaining six dimensions being kept Planckian, the energy of zero modes and strings having no winding along the three dimension (which we call nonwinding strings) grows roughly in proportion to the expanding volume. The temperature is shown to be fixed as the Hagedorn temperature in this period. This inflationlike energy growth is possible because the highest energy strings (which we call winding strings) continue to supply energy by their decay.

This epoch ends when the high energy strings decay away. At this stage what is left are the dominant zero modes along with a few nonwinding strings. From that time the redshift of the zero modes gets effective, resulting in a decrease in the temperature. The remaining nonwinding string modes are shown to quickly decay away because of their high specific heat. In this way the string universe is shown to transit to the conventional radiation-dominant universe. The exposition of this thermal history is the main result of this paper.

Our plan of the discussion is as follows. In the next section we explain on what setting we proceed with the discussion in this paper and summarize the approximations and assumptions used in this paper. In Sec. III we will give a brief review of the Brandenberger and Vafa scenario to fix the notation. In order to follow the thermal history of the string universe, we need to calculate the multistring state density. In the ideal string approximation the multistate density is evaluated by the single state density. In Sec. IV we give a detailed discussion on the single string state density. We especially explain the change occurring in the single state density induced by cosmological expansion, which is of importance in discussing thermal history. In Sec. V we give a remark on the value of the total energy of our microcanonical ensemble. This value turns out to be the key parameter which determines the thermal history. In Sec. VI we will provide a framework of the microcanonical formalism on which we can follow the thermal history. From Sec. VII to Sec. IX we will evaluate the multistate density from the single state densities. In these calculations a novel technique is introduced and used extensively. In Sec. X the thermal history of our string universe is deduced by gathering all the knowledge obtained in the preceding sections. The last section is devoted to some discussions. In the Appendix we will ascertain the validity of the Maxwell-Boltzmann approximation which will be used in this paper.

II. SETTING

In this section we are going to make sure of the tools and approximations we use in this paper. As is well known the thermodynamical treatment needs special care in string theory because of the exponentially growing state density [7, 1]. One method to treat such a system is to extend the temperature to a complex number [7–9] in the canonical formalism, and another is to quit using the canonical formalism and use the microcanonical formalism [4, 10–14].

Both methods are actually connected through the Laplace transformation. We will take the latter treatment in this paper. Our interest in this paper is fundamental string theory, not cosmic string theory. However so as to clarify our setting it is useful to review what is known in the studies of the cosmic string.

The ensemble of the cosmic or fundamental strings is in general subject to both statistical mechanics and the dynamics of the theory. In the case of cosmic string theory the dynamics is shown to prevail over the statistics [10, 11]. This is a consequence of the following settings. First of all for cosmic string theory one assumes Einstein gravity with a Robertson-Walker metric as a background since the relevant energy scale is not so close to the Planckian scale. One describes the string as a Nambu-Goto string in a radiation- or matter-dominated background. Then the description of the system simplifies well thanks to the one scale principle [10, 11]. This principle ensures that sooner or later the system will be attracted to a scaling solution irrespective of the initial configuration of the strings. It is shown that this behavior of the string ensemble is far from thermal equilibrium.

However in our case of fundamental string theory, it is no longer a natural assumption that simple Einstein gravity is applicable because the relevant energy scale is as high as the Planck mass. The dynamics of fundamental strings is poorly understood at present; thus, we cannot proceed further as in the case of cosmic strings. That is why we focus on the thermodynamical analysis and follow the thermal history using it in this paper. This strategy is essentially the one proposed by Brandenberger and Vafa [4].

Below we clarify our approximations used in this paper. Actually, we are going to investigate the properties of the string ideal gas in this paper. Thus we need some assumptions to identify our system with the real Universe. We summarize them here.

First we have to assume the validity of the local thermal equilibrium and that the ideal gas approximation is reasonably good for the string universe in the period of interest. If one of these is not a good one, our system of the string ideal gas is not guaranteed to be a good approximation of our Universe.

Next we have to assume that some special roles of (quantum) gravity, if any, are not important in considering the thermal history of our Universe. Of course string theory by birth is the quantum theory unifying gravity. Therefore, some special effects may well exist concerning gravity. But our present knowledge is so poor that we have to assume their unimportance.

Even under this difficult circumstance we do think that it is much more meaningful to do something rather than doing nothing. Some foothold may well be found by such trial. Our main purpose of this paper is to provide the zeroth approximation of the whole story of the string universe.

Within these approximations the thermodynamical functions have been calculated by some authors in several models of SST [7–10, 12, 13, 4]. In following the thermal history we actually need another assumption to determine the history uniquely. As a last assumption we require that the usual mechanism of the redshift works for massless particles even in the Planck time. We call this assumption a normal energy loss. In the initial epoch, this condition is shown to be equivalent to the equientropy condition which is also adopted by [4].

III. COSMOLOGICAL SCENARIO

In order to fix the notation we present a brief review of the Brandenberger and Vafa [4, 6] scenario in this section.

They started with the heterotic string theory [15] in the nine-dimensional torus $T^9 \times R$. For this model the single string spectrum reads [1]

$$\begin{aligned} \varepsilon^2 &= 2r^2 + 4(n_R + n_L)M_s^2, \\ r^2 &= \sum_{i=1}^{9} \left[\left(\frac{n_i}{a_i} \right)^2 + \left(m_i a_i M_s^2 \right)^2 \right], \end{aligned} \tag{1}$$

where

$$\begin{aligned} m_i = n_i &= 0, \pm 1, \pm 2, \dots, \\ n_R, n_L &= 0, 1, 2, \dots, \\ M_s &= 1/\sqrt{2\alpha'}. \end{aligned}$$

In these expressions $\sqrt{2}\pi a_i$ is a linear size of the torus. In string theory, M_s, which is of the order of the Planck mass, is the only dimensionful constant. We frequently set M_s to unity in the sequel.

The significant feature of this model is a duality $a \leftrightarrow 1/a$ which is manifest in the spectrum. This symmetry connects the large volume world with the small volume world. This is called the target space duality [16]. The self-dual point of this duality is $a = 1$ (in units of $1/M_s$). It is known that in the low energy limit of the closed string theory there emerges Einstein gravity [1]. However Einstein gravity does not respect this duality [4]. This means that the use of Einstein gravity is not legitimate in this realm.

They considered that Einstein gravity is modified in this realm so as to respect this duality. The winding mode is thought to play an essential role in this realm. Their approximate estimation showed that the winding mode works to slow down the expansion of the Universe while the momentum mode slows down the contraction. These effects make the Universe oscillate around the self dual point for awhile [6].

Their description of how a three-dimensional universe is born is as follows. Suppose that a d-dimensional space out of nine gets larger than average by accident. Then the winding modes along these d directions get more massive than the rest so that they tend to decay more than average. Consequently, the number of these modes would be reduced. Since the winding modes slow down the expansion, these d directions become easier to expand than the other directions. Thus the perturbation considered above has an unstable nature. They suggested that in this way d-dimensional space becomes much larger in size than the Planck length while the remaining $9 - d$ dimensions are kept at the Planckian scale. Since the Planckian scale is invisible at low energies, this mechanism effectively reduces the dimensionality of space-time; a($d + 1 < 10$)-dimensionally large universe arises out of a small $T^9 \times R$. They called this mechanism a decompactification.

IV. SINGLE STATE DENSITY

In this section we are going to investigate the fundamental properties of the single string state density $f(\varepsilon)$. This provides the theoretical foundation of our discussion of the thermal history of the string universe. In fact, as we see in the later sections, the thermal history is deduced from the functional form of the multistate density and the multistate density is calculated from the single state density. In the high energy range the functional form of $f(\varepsilon)$ has already been estimated analytically. We first review this result and explain how its volume dependence comes out. Later this volume dependence will prove to have key importance in observing the thermal history of the string universe. Next we will give our numerical estimation of $f(\varepsilon)$ by the direct counting of the single string states in the low energy range. This clarifies the explicit number distribution of the strings. Lastly we remark that an interesting effect in the single state density is induced as the cosmological expansion proceeds.

A. High energy behavior

For the general closed superstring theory in a compact space which is multiply connected, the single state density is written as [7–13, 4]

$$f(\varepsilon) = \frac{CV}{\varepsilon^{\eta+1}} e^{\beta_H \varepsilon} \tag{2}$$

for large enough energy ε. In this expression $\eta = D/2$ with D being the number of noncompact dimensions, V denotes a D-dimensional volume and $1/\beta_H$ is a constant called the Hagedorn temperature. The constants C and β_H depend on the string model. The above form of the single state density is uniquely implied by the single string spectrum (1). From (1) we learn that the energy of the string consists of a kinetic part r and an oscillation part $n_R + n_L$. For the kinetic part r^2 we have winding modes $a_i m_i$ in addition to the usual momentum modes because it is the closed string theory in the nonsimply connected manifold.

Before considering the volume dependence of (2), let us consider what form of the single state density is implied in the usual relativistic particle. Such a system does not have an oscillation mode and a winding mode; the spectrum (1) reduces to a simpler form $\varepsilon^2 = \sum_{i=1}^{d} (n_i/a_i)^2$ on the d-dimensional torus. The single state density in this case is proportional to the surface area of the elliptic sphere in d dimensions having axes $\varepsilon a_1, \ldots, \varepsilon a_d$. Namely, we get

$$f(\varepsilon) \propto \frac{d}{d\varepsilon} \left(\prod a_i \right) \varepsilon^d = V \varepsilon^{d-1}, \tag{3}$$

where V is a d-dimensional volume.

This represents a simple fact: the single state density is an extensive quantity. One of the peculiar phenomena in string thermodynamics is that f is no longer an extensive quantity. In fact the number D in (2) is not the total space dimension but the noncompact dimension. This means that f is not extensive. In the extreme case of a totally compact space, f is volume independent.

Let us see how this peculiar behavior comes out. Only the kinetic part reflects the structure of the space, so that we concentrate ourselves on the degeneracy of the kinetic part. Just like the case of the usual point particles, the degeneracy is obtained as the surface area of the elliptic sphere having the axes $a_1 r, \ldots, a_9 r, r/a_1, \ldots, r/a_9$, see (1). The state density is therefore proportional to

$$\frac{d}{d\varepsilon} \left(a_1 r \times a_2 r \times \cdots \times a_9 r \times \frac{r}{a_1} \times \frac{r}{a_2} \times \cdots \times \frac{r}{a_9} \right). \tag{4}$$

This shows that $f(\varepsilon)$ is certainly a_i independent. This property is essentially a consequence of the cancellation between the momentum mode and the corresponding winding mode. As the volume expands the phase space of the former increases while that of the the latter decreases. Now let us see what happens if D dimensions are open. This time D momentum modes miss their partner to cancel, so that a_i dependence remains. Accordingly we get $f(\varepsilon) \propto a_1 \cdots a_D$. This surely explains the peculiar volume dependence shown in (2).

B. Numerical analysis in low energy range

We present here the result of our numerical analysis. In Fig. 1 is presented the plot of $f(\varepsilon)e^{-\beta_H \varepsilon}$ in the totally compact case $D = 0$. In the current situation the kinetic

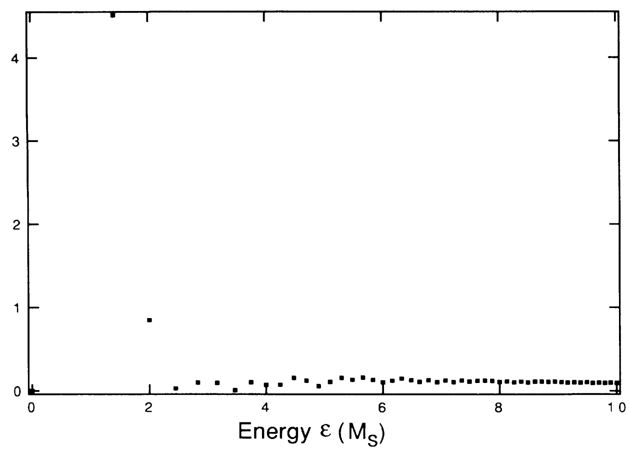

FIG. 1. The plot of the single string state density of the totally compact model normalized as $f(\varepsilon)e^{-\beta_H \varepsilon}$. This is predicted to tend to $1/\varepsilon$.

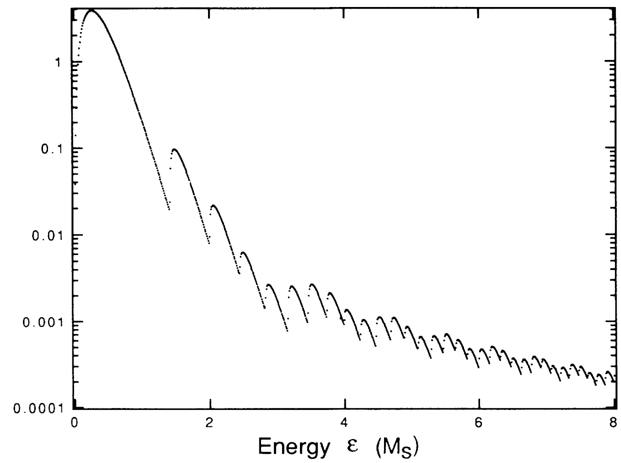

FIG. 3. The semilog plot of $f(\varepsilon)e^{-\beta_H \varepsilon}/V$ in the case of the space having $D = 3$ open dimensions.

energy is discrete by the finiteness of the space, which makes the energy spectrum discrete as seen in Fig. 1. We also plot $\varepsilon f(\varepsilon)e^{-\beta_H \varepsilon}$ in Fig. 2. The asymptotic behavior $f(\varepsilon)e^{-\beta_H \varepsilon} \to 1/\varepsilon$ is clearly seen in Fig. 2. This is the first time that this aysmptotic behavior is shown to set in already at $\sim 10M_s$.

These two figures in fact have a clear physical meaning. It is shown [12] in the microcanonical formalism that $f(\varepsilon)e^{-\beta_H \varepsilon}$ and $\varepsilon f(\varepsilon)e^{-\beta_H \varepsilon}$ represent the number distribution and the energy distribution of the strings respectively.

As we will recognize later, the $D = 3$ case is relevant to our discussion. The value of $f(\varepsilon)e^{-\beta_H \varepsilon}/V$ versus ε is presented in Fig. 3. The spiky behavior therein represents the opening of various modes. The analytic estimation indicates that the quantity tends to behave like $C/\varepsilon^{5/2}$ for large ε [see (2)]. The plot $\varepsilon^{5/2} f(\varepsilon)e^{-\beta_H \varepsilon}/V$ is shown in Fig. 4 which justifies this asymptotic behavior and tells us where this behavior sets in.

C. Change in the single state density

In this subsection we examine what occurs to $f(\varepsilon)$ when three accidentally chosen directions expand while the remaining dimensions are kept Planckian. Because we will restrict ourselves to the case in which three directions are expanding at an equal rate, we set $a_1 = a_2 = a_3 = a$ and $a_4 = \cdots = a_9 = b \sim 1$ [see (1)] from now on.

In the preceding section we saw that the high energy behavior of the single state density $f(\varepsilon)$ is independent of a since $D = 0$. This is a consequence of the cancellation between the momentum mode and the winding mode. However this cancellation becomes incomplete at low energies for the following reason.

As a gets larger the winding mode along the a direction is getting heavier. Eventually the winding mode in that direction becomes too heavy to be excited especially in the low energy range. This means the winding modes along the expanding three directions are effectively frozen. Then the cancellation between the momentum mode and the winding mode breaks down in the

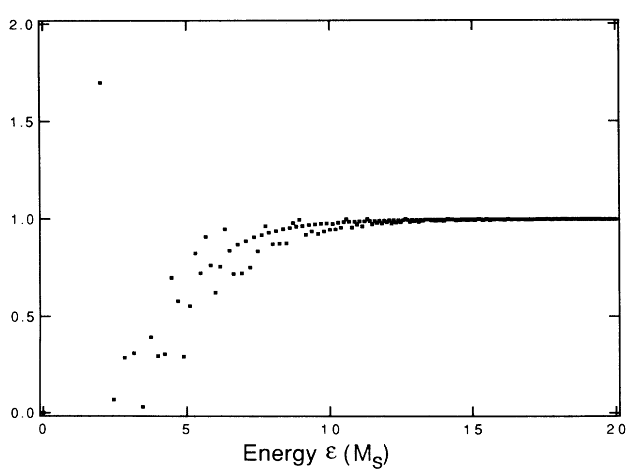

FIG. 2. The plot of $\varepsilon f(\varepsilon)e^{-\beta_H \varepsilon}$. This shows the predicted asymptotic behavior.

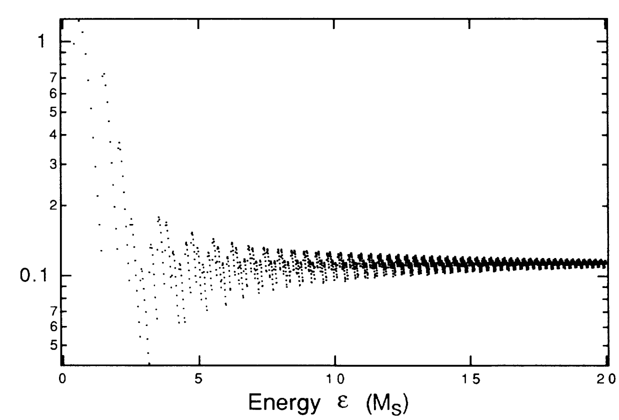

FIG. 4. The semilog plot of $\varepsilon^{5/2} f(\varepsilon)e^{-\beta_H \varepsilon}/V$. This shows the expected asymptotic behavior.

low energy region. As a result the single state density behaves as if $D = 3$ instead of $D = 0$ at low energies. Namely $f(\varepsilon)e^{-\beta_H \varepsilon}$ behaves as $CV/\varepsilon^{5/2}$ at low energies and as $1/\varepsilon$ in the high energy range, respectively. We denote this energy m_0 which separates the low and high energy ranges. As a becomes large, the strings with higher energies behave as if $D = 3$. Namely the effective $D = 3$ range extends as the Universe expands.

In fact it can be shown by examining the functional form of the state density that as a grows m_0 grows at the rate $m_0(a) \propto a^2$. This is justified by the numerical analysis of Allega *et al.* and ours. This phenomena has been discussed also by other authors, in different ways by Slomonson *et al.* in [9] and by the authors of [17, 14].

Before closing this section we summarize the behavior of the single state density f in an expanding epoch. Below the first excitation energy $\varepsilon < m_1$, f describes the zero modes which are regarded as the usual point particles. In the range between m_1 and m_0 the state density behaves as a string gas in open three space dimensions. We call the string in this range a nonwinding string because such a string does not wind along the expanding directions. Note that the nonwinding string in general winds along the remaining six directions.

Lastly, for an energy greater than $m_0(a)$, f behaves as a string gas in the totally compact space. We call such a string a winding string. We can express the single state density using the step function θ as:

$$f(\varepsilon) = f_z(\varepsilon) + \frac{CV}{\varepsilon^{\eta+1}} e^{\beta_H \varepsilon} \theta(\varepsilon - m_1)\theta(m_0(a) - \varepsilon) + \frac{1}{\varepsilon} e^{\beta_H \varepsilon} \theta(\varepsilon - m_0(a)) \quad (5)$$

with $\eta = 3/2$, where f_z denotes the state density of the zero modes. Although our interest in this paper is in the $\eta = 3/2$ case only, we keep η arbitrary in the subsequent expressions for better understanding of the structure of our treatment. We again stress here that f is proportional to V below $m_0(a)$.

V. HOW BIG SHOULD THE TOTAL ENERGY BE?

In this section we will give an important remark on the question of how big the total energy E of our microcanonical ensemble should be.

Let us consider the string distribution in the initial epoch. The number distribution of strings are displayed in Fig. 1; see the second paragraph of Sec. IV B. The first excited state opens at $m_1 = \sqrt{8}(\times M_s)$ corresponding to $N = n_R + n_L = 2$ instead of $N = 1$ since the latter is inconsistent with the level matching condition of heterotic string theory [1].

The quantity E is the total energy of the microcanonical ensemble which we have to introduce at the beginning of the discussion. What we can find from Fig. 1 is that if we take E of the order of M_s as in the usual dimensional analysis, we have no string modes from the beginning since the distribution terminates at $\varepsilon = E$. In such a situation our Universe is no longer a model of string cosmology.

You may say that we only have to take E as large as we like. However in cosmology with causality we can only have a finite region in thermal equilibrium because the speed of light is finite. We are dealing with equilibrium thermodynamics in this paper. Therefore it is implicitly assumed that the spatial region having energy E must be in thermal equlibrium. Consequently we cannot take E as large as we like.

Now we define E to be the maximally allowed energy in thermal equilibrium and discuss how big E can be.

Since our present knowledge of string theory does not allow us to determine the value of E, we are left with two possibilities. The first one is that the region having E (defined as above) is smaller than the whole Universe; the second one is that the whole torus universe is in thermal equilibrium.

First we consider the former case. Because the value of E is considered to be the maximal energy which a single string can occupy, it appears that E must be large enough in order for our Universe to be regarded as a model of string cosmology. But it is not necessarily the case. In fact there is a loophole in this argument. A string extending over beyond the causal region can have an energy much greater than E. Such a noncausal fundamental string may well be produced if the Universe itself is born through quantum tunneling or something like that. However such a situation is beyond the control of our present technology. We do not and cannnot go further in such a case in this paper.

Next we consider the second case when the whole Universe is in thermal equilibrium. Now we simply conclude that the value of E must be large for the Universe to be full of strings. There is no loophole this time.

Because the loophole in the former possibility is out of our control we assume that E is large enough in this paper. Of course the alternative case that our Universe has no strings even in the initial epoch is another possibility. However we will not treat this case since the purpose of this paper is to explore the possibility of a universe full of strings.

As we stated before our scenario is based on that of Brandenberger and Vafa in which the string universe is supposed to oscillate around the self dual point for an initial period. This picture nicely fits the second case mentioned above, since the oscillation over many periods tends to thermalize the whole Universe. Even if we had started in the loophole case, this oscillation makes the noncausal strings causal. Moreover, we point out that this picture has another advantage from the cosmological viewpoint. This picture is plausible from the viewpoint of the horizon problem. If the whole Universe would be thermalized during the initial period of oscillation, we would go through history with the background radiation of the same temperature at any part of the Universe.

VI. FRAMEWORK TO FOLLOW THE THERMAL HISTORY

In this section we intend to give a concrete framework to follow the thermal history of the string universe. Many authors have discussed the behavior of a string gas in the microcanonical formalism so far [7–13, 4]. The stringy phase has been examined in various ways and the differences lying between the stringy phase (which is frequently referred to as a high-density phase) and the low-temperature phase (which is referred to as a low-density phase) have been exposed.

However very little attention has been paid to the transition of these two phases. The problem how a stringy universe evolves into a radiation-dominated universe is still an open problem. In order to treat the transient period we present a concrete framework based on the microcanonical formalism.

The multistate density is written by the single state density under the Maxwell-Boltzmann (MB) approximation as [12]

$$\Omega(E) = \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^{\infty} \prod_{j=1}^{n} d\varepsilon_j f(\varepsilon_j) \delta \left(E - \sum_{j=1}^{n} \varepsilon_j \right). \tag{6}$$

We discuss the validity of the MB approximation in the Appendix.

We recall that the single state density f is expressed as a sum of the state densities of zero modes, the nonwinding strings and the winding strings. We define the multistate densities associated with these regions by imitating (6) as

$$\begin{aligned}
\omega_z(\varepsilon_z) &= \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^{\infty} \prod_{j=1}^{n} d\varepsilon_j f_z(\varepsilon_j) \delta \left(\varepsilon_z - \sum_{j=1}^{n} \varepsilon_j \right), \\
\omega_N(\varepsilon_N) &= \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^{\infty} \prod_{j=1}^{n} \frac{CV d\varepsilon_j}{\varepsilon_j^{\eta+1}} e^{\beta_H \varepsilon_j} \theta(\varepsilon_j - m_1) \theta(m_0(a) - \varepsilon_j) \delta \left(\varepsilon_N - \sum_{j=1}^{n} \varepsilon_j \right), \\
\omega_W(\varepsilon_W) &= \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^{\infty} \prod_{j=1}^{n} \frac{d\varepsilon_j}{\varepsilon_j} e^{\beta_H \varepsilon_j} \theta(\varepsilon_j - m_0(a)) \delta \left(\varepsilon_W - \sum_{j=1}^{n} \varepsilon_j \right).
\end{aligned} \tag{7}$$

The similarity between (6) and the Taylor expansion of the exponential function enables us to anticipate that Ω is expressed as a product form of ω's. Actually, we can prove by a straightforward calculation that

$$\begin{aligned}
\hat{\Omega}(E) = \int_0^{\infty} & d\varepsilon_z d\varepsilon_N d\varepsilon_W \; \hat{\omega}_z(\varepsilon_z) \hat{\omega}_N(\varepsilon_N) \hat{\omega}_W(\varepsilon_W) \\
& \times \delta(E - \varepsilon_z - \varepsilon_N - \varepsilon_W).
\end{aligned} \tag{8}$$

The carets on the top of ω's mean the addition of the delta function: $\hat{\omega}(\varepsilon) = \omega(\varepsilon) + \delta(\varepsilon)$. These carets enable us to express the equation in a simple form as above. The necessity of the delta functions is readily understood if we recall that the right-hand side of (8) counts the number of all the composite states of three kinds of substances: the zero modes, the nonwinding strings, and the winding strings. For example there are also the states having no zero modes, which must be counted. The term $\delta(\varepsilon_z)\omega_N(\varepsilon_N)\omega_W(\varepsilon_W)$ is responsible for these states to be taken into account in the integration.

Based on the equation we argue as follows. Because the delta function ensuring energy conservation is included in the right-hand side of (8), we can perform an integration over ε_W to obtain a two-dimensional integration over $(\varepsilon_z, \varepsilon_N)$. In many cases of interest the integrand has a sharp peak at some single point on the $(\varepsilon_z, \varepsilon_N)$ plane, and the contribution from this point dominates the integral. The position of the peak is dependent on a since the ω's have an implicit dependence on a. We denote the position of the peak as $(e_z(a), e_N(a))$.

If we recall the fundamental principle of the equal *a priori* probability, we conclude that we find the subsystem in the energies $(e_z(a), e_N(a))$ when the size of the Universe is a. This is because this state has an overwhelming probability. This is nothing but the essence of the microcanonical formalism.

Therefore, once we find the functional form of $e_z(a)$ and $e_N(a)$, we can follow the thermal history of the Universe. This is our strategy — to determine the thermal history in the microcanonical formalism. To carry out this program we need to calculate the multistring state densities. In the next three sections we will evaluate the multistate densities associated with nonwinding strings, winding strings and the zero modes successively.

VII. MULTISTATE DENSITY OF THE NONWINDING STRINGS

The evaluation of the multistate densities $\omega_N(\varepsilon)$ and $\omega_W(\varepsilon)$ have been given by several authors [10, 12, 13, 4]. The method used there is the Laplace transformation and the saddle-point approximation. We note that the latter is reliable only for large ε. However, as we will see later, the small ε behavior of $\omega_N(\varepsilon)$ is necessary in the analysis of thermal history.

In the following we will employ a completely new method to examine the form of ω_N and ω_W. That is the characterization of ω's by a differential-difference equation. We will show that ω's are solutions to some linear differential-difference equations, and solve them to find the functional form of ω's.

A. Evaluation

First we remark that we can factor out the exponential part of ω_N as $\omega_N(\varepsilon) = A(\varepsilon, v)e^{\beta_H \varepsilon}$ with

$$A(\varepsilon, v) = \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^{\infty} \prod_{j=1}^{n} \frac{v d\varepsilon_j}{\varepsilon_j^{\eta+1}} \theta(\varepsilon_j - m_1)\theta(m_0(a) - \varepsilon_j)\delta\left(\varepsilon - \sum_{j=1}^{n} \varepsilon_j\right), \tag{9}$$

where $v = CV$. This is possible since the integration is constrained by the delta function. If we denote

$$A_n(\varepsilon, v) = \int_0^{\infty} \prod_{j=1}^{n} \frac{d\varepsilon_j}{\varepsilon_j^{\eta+1}} \theta(\varepsilon_j - m_1)\theta(m_0(a) - \varepsilon_j)\delta\left(\varepsilon - \sum_{j=1}^{n} \varepsilon_j\right), \tag{10}$$

(9) is rewritten as

$$A(\varepsilon, v) = \sum_{n=1}^{\infty} \frac{v^n}{n!} A_n(\varepsilon, v). \tag{11}$$

We change the variables as $\varepsilon_j \to \varepsilon x_j$ and obtain

$$A_n(\varepsilon, v) = \frac{1}{\varepsilon^{\eta n+1}} \int_0^{\infty} \prod_{j=1}^{n} \frac{dx_j}{x_j^{\eta+1}} \theta(x_j - m_1/\varepsilon)\delta(m_0(a)/\varepsilon - x_j)\theta\left(1 - \sum_{j=1}^{n} x_j\right). \tag{12}$$

By operating $\varepsilon\partial_\varepsilon$ and $\eta v \partial_v$ on it we have

$$\varepsilon\partial_\varepsilon A_n(\varepsilon, v) = -(\eta n + 1)A_n(\varepsilon, v) + \frac{n}{m_1^\eta} A_{n-1}(\varepsilon - m_1, v) - \frac{n}{m_0^\eta} A_{n-1}(\varepsilon - m_0, v)$$

and

$$\eta v \partial_v A_n(\varepsilon, v) = \frac{n}{m_0^\eta} A_{n-1}(\varepsilon - m_0, v), \tag{13}$$

respectively. Here we made use of the fact that $m_0(a) \propto a^2 \propto v^{1/\eta}$ (see Sec. IV C).

From these we obtain

$$(1 + \varepsilon\,\partial_\varepsilon + \eta v \partial_v)A_n(\varepsilon, v) = -\eta n A_n(\varepsilon, v) + \frac{n}{m_1^\eta} A_{n-1}(\varepsilon - m_1, v). \tag{14}$$

Summing up this equation over all n we get the equation

$$(1 + \varepsilon\partial_\varepsilon + \eta v \partial_v)A(\varepsilon, v) = \frac{v}{m_1^\eta} A(\varepsilon - m_1, v). \tag{15}$$

This is the equation exactly satisfied by A.

When ε is large compared with m_1, we can use an approximation $A(\varepsilon - m_1, v) = A(\varepsilon, v) - m_1\partial_\varepsilon A(\varepsilon, v)$ to rewrite the equation as

$$\left(1 + \varepsilon\partial_\varepsilon + \frac{v}{m_1^\eta} m_1 \partial_\varepsilon + \eta v \partial_v\right) A(\varepsilon, v) = \frac{v}{m_1^\eta} A(\varepsilon, v). \tag{16}$$

In the case that a string has a high energy density $\varepsilon/m_1 >> v/\eta m_1^\eta$ (which is the case studied in Refs. [10, 12, 13]), we can neglect the third term of the left-hand side of (16) in comparison with the second term. The equation reduces to

$$(1 + \varepsilon\partial_\varepsilon + \eta v \partial_v)A(\varepsilon, v) = \frac{v}{m_1^\eta} A(\varepsilon, v). \tag{17}$$

This can be solved with ease to obtain

$$A(\varepsilon, v) = \frac{v}{\varepsilon^{\eta+1}} g'\left(-\frac{v}{\eta\varepsilon^\eta}\right) \times \exp\left[g\left(-\frac{v}{\eta\varepsilon^\eta}\right) + \frac{v}{\eta m_1^\eta}\right]\theta(\varepsilon - m_1), \tag{18}$$

with some analytic function $g(x)$. The last step function means that $A(\varepsilon, v) = 0$ for $\varepsilon \le m_1$ by definition. Comparing this with the $\eta = 0$ solution which will be obtained in the next section, we can determine g up to x as

$$g(x) = 0 + [\eta O(\eta) + 1]x + \cdots, \tag{19}$$

where $O(\eta)$ represents the function of η vanishing at $\eta = 0$.

Consequently for the string gas which has an energy ε_N such that $v/\eta\varepsilon_N^\eta \sim 0$, the multistate density is expressed as

$$\omega_N(\varepsilon, v) = \text{const} \times \frac{CV}{\varepsilon^{\eta+1}} \exp\left(\frac{CV}{\eta m_1^\eta} + \beta_H \varepsilon\right). \tag{20}$$

This reproduces the known result [10, 12, 13].

We have examined the high density region above. However, we will realize that for our string universe the region with low energy density $\varepsilon/m_1 << v/\eta m_1^\eta$ is relevant. Thus we have to evaluate $\omega_N(\varepsilon, v)$ in this low energy density. Before carrying it out we estimate the position of the peak of A and the height of it which are of importance in thermal history.

Setting $\varepsilon = e_z(a)$ in (16) this equation is reduced to an ordinary differential equation

$$(1 + \eta v \partial_v) A(e_N(a), v) = \frac{v}{\eta m_1^\eta} A(e_N(a), v), \tag{21}$$

since $\partial_\varepsilon A$ vanishes on that point. This ordinary differential equation is readily solved and we get the height of the peak:

$$A(e_N(a), v) = \frac{\text{const}}{v^{1/\eta}} \exp\left[\frac{v}{\eta {m_1}^\eta}\right]. \tag{22}$$

The logarithm of this is nothing but the entropy of the nonwinding strings. Therefore, the expression simply tells us that the entropy produced when D-dimensional volumes out of nine expands is proportional to the expanding volume (note that $v = CV$).

Next we determine the functional form of $e_N(a)$. From a numerical determination of $A(\varepsilon, v)$ which we will give later, we see that $v/\eta {m_1}^\eta$ is much greater than $e_z(a)/m_1$. In this case we can neglect the second term instead of the third one in (16). This reduces the equation to

$$\left(1 + \frac{v}{{m_1}^\eta} m_1 \partial_\varepsilon + \eta v \partial_v\right) A(\varepsilon, v) = \frac{v}{{m_1}^\eta} A(\varepsilon, v). \tag{23}$$

By solving it we can determine the functional form of $A(\varepsilon, v)$ in this region as

$$A(\varepsilon, v) = \frac{1}{v^{1/\eta}} \exp\left[h\left(\frac{\varepsilon}{m_1} - \frac{v}{\eta m_1^\eta}\right) + \frac{v}{\eta m_1^\eta}\right], \tag{24}$$

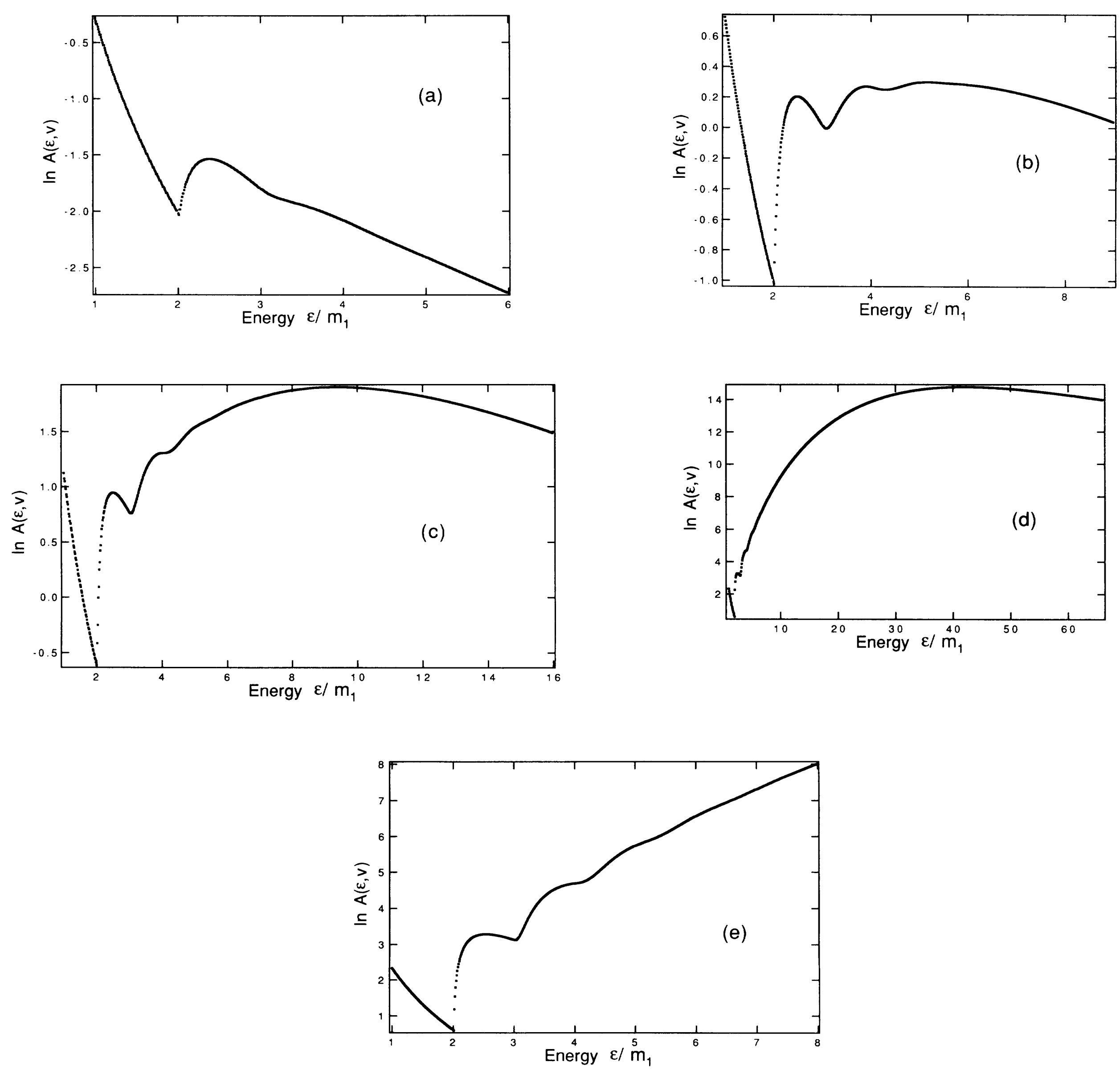

FIG. 5. The plots of $\ln A(\varepsilon, v)$ versus ε at growing values of a's. (a) is for $a = 1$, (b) is for $a = 1.4$, (c) is for $a = 1.6$, (d) is for $a = 2.4$, and (e) is the close view around $\varepsilon \sim m_1$ of (d).

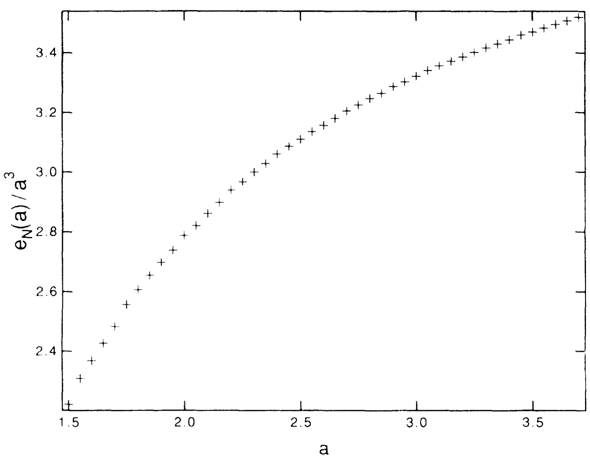

FIG. 6. The plot of the favored energy of the nonwinding strings $e_N(a)$ normalized by a^3 versus a. This is consistent with the analytic estimate.

where h is some function to be determined by a boundary condition. But the explicit form of h is unnecessary for our present purpose. The function $A(\varepsilon, v)$ is maximized when the function $h(x)$ is maximal. Denoting the position of the peak of $h(x)$ as $x = c$, we can express the position of the peak of A, $e_N(a)$ as

$$e_N(a)/m_1 = \frac{v}{\eta m_1^{\eta}} + c. \tag{25}$$

Numerically it can be expressed as $e_N(a) = 4.00a^3$+const. Namely $e_N(a)$ increases in proportion to the volume for large a.

We solved (15) for small ε, at growing values of a. Figure 5 shows the plot of $\ln A(\varepsilon, v)$ versus ε. As the energy increases, new modes start to open. This fact is exhibited in the low energy behavior of $A(\varepsilon, v)$ as the emergence of several peaks. We see that the position of the peak moves to higher energy as a grows. In the high energy region, the spiky behavior seen at low energies is smeared, resulting in a smooth curve. Figures 6 and 7 show the

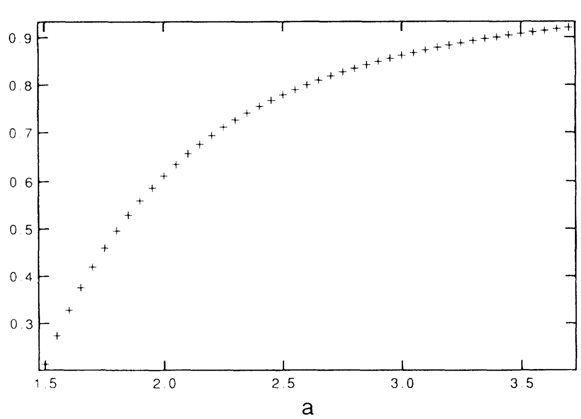

FIG. 7. The plot of $\ln A(e_N(a), v)/v$ versus a. This is consistent with the analytic estimate.

plots of $[e_N(a)/m_1]/a^3$ and $\ln(A(\varepsilon_N, v))/(v/\eta m_1^{\eta})$ versus a, respectively. The behavior in both figures is consistent with the above analytic estimate.

To prepare for later usage we present here the plots of a microcanonical temperature of the nonwinding strings. This quantity, defined as

$$\beta_N(\varepsilon, v) = \partial_\varepsilon \ln \omega_N(\varepsilon, v) = \partial_\varepsilon \ln A(\varepsilon, v) + \beta_H, \tag{26}$$

measures the rate of entropy increase due to energy increase. We show the plots of β_N/β_H versus ε in Fig. 8.

The global decreasing behavior means that the specific heat is globally positive. The point of ε on which $\beta_N = \beta_H$ is where $\ln A(\varepsilon, v)$ peaks. The local spiky behavior around $\varepsilon \sim m_1$ means there is thermodynamical instability in this region.

VIII. MULTISTATE DENSITIES OF THE WINDING STRINGS

Next we calculate the multistate density of the winding strings. We only have to make an analogue of the previous discussion. Since $\eta = 0$ in the present case we get

$$(1 + \varepsilon\partial_\varepsilon)A(\varepsilon) = A(\varepsilon - m_0) \tag{27}$$

instead of (15). Using a similar approximation we have $\partial_\varepsilon A(\varepsilon) = 0$. Determining the normalization by

$$A(m_0) = \int_{m_0}^{\varepsilon} \frac{d\varepsilon_1}{\varepsilon_1}\theta(m_0 - \varepsilon_1)\delta(\varepsilon - \varepsilon_1)\big|_{\varepsilon=m_0} = 1/m_0, \tag{28}$$

we finally obtain

$$\omega_W(\varepsilon) = \frac{1}{m_0} e^{\beta_H \varepsilon}\theta(\varepsilon - m_0). \tag{29}$$

This reproduces the known result [4]. One particular feature of this functional form is that $\omega_W(\varepsilon)e^{-\beta_H\varepsilon}$ has no peak. This is not the case for the nonwinding strings and the zero modes.

IX. MULTISTATE DENSITY OF THE ZERO MODES

In this section we estimate the multistate density of the zero modes. When we enter into the thermal history of the string universe, the knowledge of the position of the peak of the function $\omega_z(\varepsilon, v)e^{-\beta_H\varepsilon}$ will be necessary. We will determine this in the last part of this section.

The multistate density $\omega_z(\varepsilon, v)$ in (8) can be rewritten as

$$\omega_z(\varepsilon, v) = e^{\beta\varepsilon} \times \sum_1^{\infty} \frac{1}{n!} \int_0^{\infty} \prod_1^{n} d\varepsilon_j f_z(\varepsilon_j) e^{-\beta\varepsilon_j} \delta\left(\varepsilon - \sum_{j=1}^{k} \varepsilon_j\right), \tag{30}$$

with an arbitrary positive β parameter noting the delta function constraint.

In order to estimate the power series above, we note two facts here. First the sum of the exponential function $\sum x^k/k!$ receives its dominant contribution from the $k = x$ term. In fact it can be verified that the expression $x^k/k!$ if seen as a function of k has a sharp peak at $k = x$.

Next we can ascertain that the power series of $\omega_z(\varepsilon)$ can be regarded as $\sum x^k/k!$. In order for it to be valid it is sufficient that the important contribution to the integral of ε's comes from the diagonal region $\varepsilon_1 \sim \varepsilon_2 \sim \cdots \sim \varepsilon_k$. As mentioned before f_z behaves as a positive power of ε; see Sec. IV A. This enables us to apply the well-known inequality $\sqrt[k]{\varepsilon_1\varepsilon_2\cdots\varepsilon_k} \leq \frac{1}{k}\sum \varepsilon_j$ to conclude that $\prod f_z(\varepsilon)$ maximizes on the diagonal region justifying our claim.

Because x of $\sum x^k/k!$ corresponds to $x_0 = \int_0^\infty f_z(\varepsilon)e^{-\beta\varepsilon}d\varepsilon$ of (30), we only have to focus on the

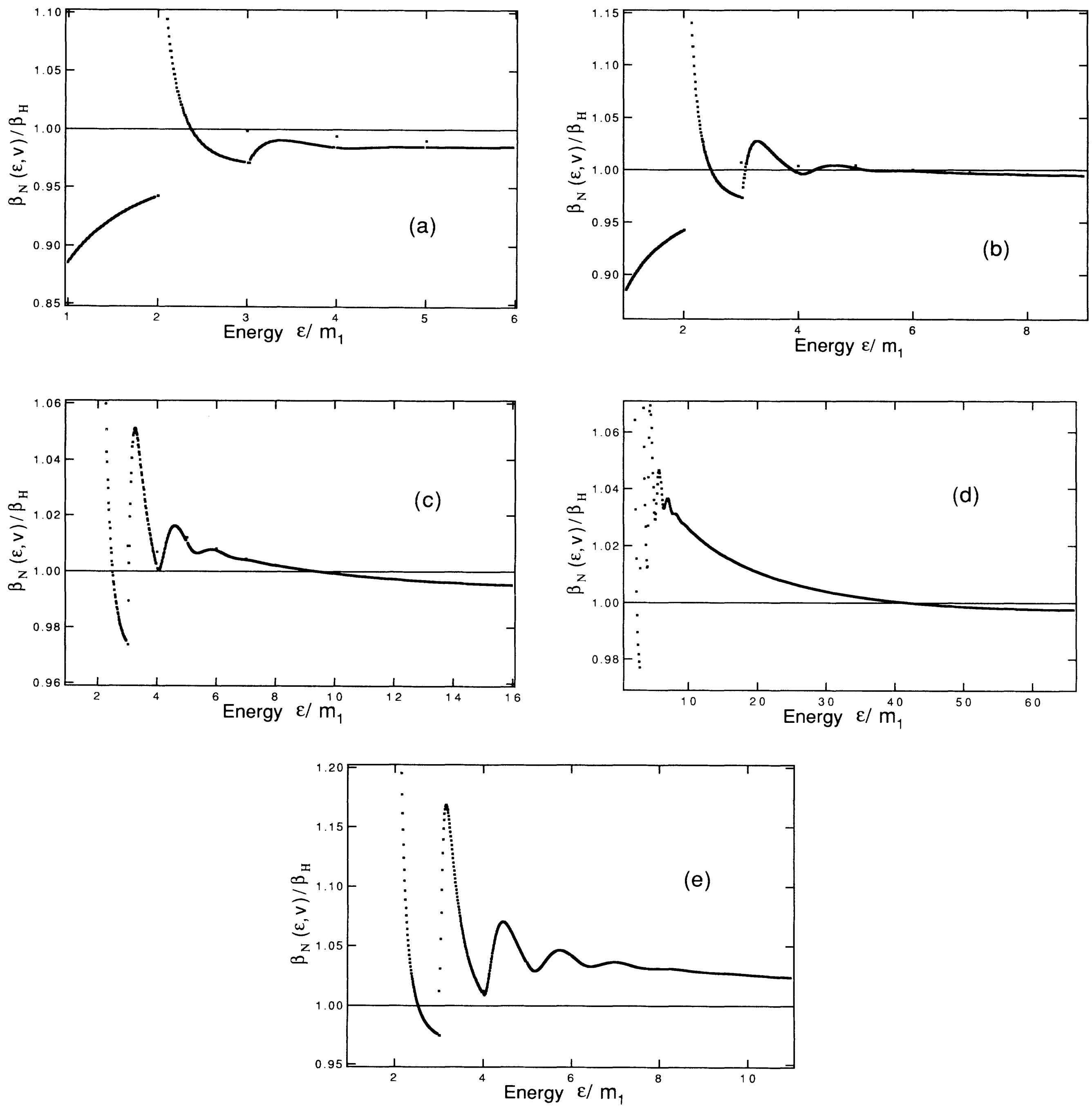

FIG. 8. The plots of the microcanonical temperature of the nonwinding strings $\beta_N(\varepsilon, v)$ versus ε at growing values of a's. The figure (a) is for $a = 1$, (b) is for $a = 1.4$, (c) is for $a = 1.6$, (d) is for $a = 2.4$, and (e) is the close view around $\varepsilon \sim m_1$ of (d).

$k = x_0$ term. Let us examine for which energy ε this term does not vanish. For the delta function not to vanish, the argument of it must vanish. So let us see what is the typical value of $\sum_{j=1}^{k} \varepsilon_j$ in the delta function of (30). This is estimated as

$$\left\langle \sum_{j=1}^{k} \varepsilon_j \right\rangle = k\langle \varepsilon_j \rangle = k\frac{\int_0^\infty d\varepsilon\, \varepsilon f_z(\varepsilon)e^{-\beta\varepsilon}}{\int_0^\infty d\varepsilon\, f_z(\varepsilon)e^{-\beta\varepsilon}} = \int_0^\infty d\varepsilon\, \varepsilon f_z(\varepsilon)e^{-\beta\varepsilon}. \tag{31}$$

We define here h and n as

$$h(\beta, v) = \int_0^\infty \varepsilon f_z(\varepsilon)e^{-\beta\varepsilon}d\varepsilon, \qquad n(\beta, v) = \int_0^\infty f_z(\varepsilon)e^{-\beta\varepsilon}d\varepsilon \tag{32}$$

for later convenience. As a result we realize that $\omega_z(\varepsilon, v)e^{-\beta\varepsilon}$ has a sharp peak at $\varepsilon = h(\beta, v)$ with the height $\exp[n(\beta, v)]$ with some width $q(\beta, v)$.

Making use of a function $G(x)$ having a peak at $x = 0$ with a unit width and a unit height we can express $\omega_z(\varepsilon, v)e^{-\beta\varepsilon}$ as

$$\omega_z(\varepsilon, v)e^{-\beta\varepsilon} = G\left(\frac{\varepsilon - h(\beta, v)}{q(\beta, v)}\right)\exp[n(\beta, v)]. \tag{33}$$

Especially the case $\beta = \beta_H$ will be relevant in later applications. We write it here with an explicit numerical coefficient:

$$h(\beta_H, v) = 65.0a^3. \tag{34}$$

It is worth noting that this argument does not apply to the evaluation of $\omega_N(\varepsilon, v)$. As we saw in (2) the single state density was a product of the exponential part and the negative power of ε. The exponential part is irrelevant since it can be factored out as usual. While the fact that the remaining part is a negative power of ε implies the situation opposite to the above case since $1/(\varepsilon_1\varepsilon_2\cdots\varepsilon_k)$ maximizes in the boundary region such as $\varepsilon_1 \sim \varepsilon$, $\varepsilon_1 \sim \varepsilon_2 \sim \cdots \sim \varepsilon_1 \sim m_1$. The is the essence of what is called the Frautshi-Carlitz picture.

X. THERMAL HISTORY

In this section we explicitly follow the thermal history of our system using the multistate densities calculated in the previous sections. The history that we are going to describe below consists of two distinct epochs which we refer to as epoch I and epoch II, respectively.

We insert ω's obtained above into (8). Among the constituents of Ω, ω_N and ω_W have the same exponential dependence $e^{\beta_H\varepsilon}$. As for the zero modes, we can formally factor out the same exponential using the previous formula (33) with $\beta = \beta_H$.

If we insert these ω_z, ω_N, and ω_W into (8), the exponential factors can be combined to form $\exp(\beta_H E)$ thanks to the delta function $\delta(\varepsilon - \varepsilon_W - \varepsilon_z - \varepsilon_N)$ and then be picked out from the integral as

$$\hat{\Omega}(E, v) = \frac{1}{m_0}\exp\left[\beta_H E + n(\beta_H, v)\right]\int_0^\infty d\varepsilon_z d\varepsilon_N d\varepsilon_W G\left(\frac{\varepsilon_z - h(\beta_H, v)}{q(\beta_H, v)}\right)\hat{A}(\varepsilon_N, v) \times[\theta(\varepsilon_W - m_0) + \delta(\varepsilon_W)]\delta(E - \varepsilon_z - \varepsilon_N - \varepsilon_W). \tag{35}$$

A. Oscillating epoch and epoch I

As we mentioned before we find the system with energies ε_z, ε_N, and ε_W are determined as the position of the peak of the integrand. The functions G and $\hat{A}$ prefer that ε_z and ε_N take their most probable values, respectively. The winding string energy ε_W is adjusted to meet the requirement by the delta function since the integrand of (35) does not have other dependence on ε_W.

The function for the zero modes strongly favors

$$\varepsilon_z = e_z(a) = h(\beta_H, v) = \int_0^\infty d\varepsilon\, \varepsilon f_z(\varepsilon)e^{-\beta_H\varepsilon}, \tag{36}$$

[see (32)], while the favorable value for the nonwinding strings is similarly determined as $\varepsilon_N = e_N(a) = m_1 v/(\eta m_1^\eta)$+const, which is derived in (25). Accordingly the most probable value of ε_W is determined as $\varepsilon_W = e_W(a) = E - e_z(a) - e_N(a)$.

Because both $e_z(a)$ and $e_N(a)$ grows as a^3 for large a, the ratio of $e_N(a)$ to $e_z(a)$ approaches a constant. Numerically, however, we see that $e_N(a) << e_z(a)$ from (25) and (34). Namely the zero modes are always dominant over the nonwinding strings. In view of $e_z(a)$ in (36), we can find that the zero modes are distributed in the canonical distribution with the Hagedorn temperature $1/\beta_H$. Namely the temperature in this period is fixed to be the Hagedorn temperature. Therefore, the total energy of the zero modes grows in proportion to the volume. The reason why it is possible is that the winding strings continuously supply the energy by decay. The supplied energy is also given to the nonwinding strings, resulting in the growth of $e_N(a)$ found at (25). The conversion of the energy into the nonwinding strings and the zero modes continues until the winding strings disappear [i.e., $e_W(a) = 0$]. We call the period epoch I before their disappearence.

Here we give an important comment on the change of the total energy E. Generally in cosmology the total energy is not a constant quantity [18]. For example the energy density of the radiation-dominated universe scales as

$$e_z(a)/a^3 \propto 1/a^4 \tag{37}$$

implying $e_z(a) \propto 1/a$. This energy loss is attributed to the redshift of radiation due to the expansion of the Universe. If the size of the Universe is multiplied by a factor a, the wavelength is multiplied by the same factor. The radiation loses its energy by this effect. The energy lost is given to the gravitational field. We call it a normal energy loss.

This energy loss is deduced from Einstein gravity. It is true that in the period in question Einstein gravity is not a reliable approximation because of the possible string corrections. However, even in this period, we consider that this energy loss works for the zero-mode sector. Then we decide to take the normal energy loss as an assumption based on which we follow the thermal history of the Universe. It is amazing to observe that this normal energy loss nicely fits to our scenario. We will show it below.

In the pure radiation case the normal energy loss is described in the differential equation as $de_z(a)/d\varepsilon = -e_z(a)/a$. In order to apply this to our case we have to take the existence of the other modes into account. The energy exchange between the other modes and the zero modes is allowed while the energy loss is only through the redshift of the zero modes $-e_z(a)/a$. Therefore the normal energy loss now means

$$\frac{d}{da}[e_z(a) + e_N(a) + e_W(a)] = \frac{d}{da}E(a) = -\frac{1}{a}e_z(a). \tag{38}$$

Adding this equation to the previously given conditions we can uniquely determine the functional form of $e_z(a)$, $e_N(a)$, $e_W(a)$, and $E(a)$ throughout thermal history.

In order to reveal what (38) means we examine the entropy change of the system under this assumption; the entropy of this system is estimated from (35) as

$$S = \beta_H E(a) + n(\beta_H, v) + \ln A(e_N(a), v). \tag{39}$$

The last term comes from the nonwinding strings. Let us first examine the case without it. Then the change of S due to the growth of a reads

$$\frac{d}{da}S = -\beta_H \frac{e_z(a)}{a} + \frac{3n(\beta_H, v)}{a}. \tag{40}$$

Surprisingly we can show that it vanishes. This is verified by combining the fact that $-\partial_{\beta_H} n(\beta_H, v) = h(\beta_H, v) = e_z(a)$ and $n(\beta_H, v) \propto 1/{\beta_H}^3$ [see (32)]. These two imply $e_z(a) = \frac{3}{\beta_H} n(\beta_H, v)$ which implies $dS/da = 0$.

Consequently, we have proven that as far as we neglect the entropy of the nonwinding strings, the assumption of the normal energy loss is equivalent to the equientropy. Let us consider below what this fact means.

The generation of nonwinding strings is rephrased as the unknotting of the winding strings along the expanding direction. It was the necessary condition for our Universe to exit out of the oscillation epoch and enter into the three-dimensionally expanding universe due to the Brandenberger and Vafa instability. It is natural to suppose the Universe is oscillating around the self-dual point, until the condition for the unknotting along some three directions is met.

On the other hand what we proved above is that the process in such an epoch is adiabatic. In other words the process is reversible. Sometimes winding strings decay by generating the zero modes as the Universe expands and sometimes the winding strings absorb the energy from the zero modes as the contraction proceeds. These processes can repeat themselves because they are reversible processes. This situation is naturally identified with the oscillating epoch of the Brandenberger and Vafa scenario.

Once the unknotting in some three directions proceeds enough, the entropy generation occurs as we have shown above. This time we cannot go back because the entropy is generated. Namely the Universe is destined to be a three-dimensionally expanding universe. This is natural as we naively expect that the birth of the three-dimensional universe is an irreversible process. Consequently, we have recognized that the result of our thermal analysis is perfectly consistent with our cosmological scenario thus far.

The epoch I ends when all the winding strings decay away, in other words when all the winding along the three directions unknot. The point $a = a_0$ when epoch I ends is determined by solving the equation $E(a) = e_z(a)+e_N(a)$. We can easily determine the functional form of $E(a)$ from (38) as

$$E(a) = E_0 + \frac{1}{3}[e_z(1) - e_z(a)], \tag{41}$$

where $E_0 = E(1)$ is the initially given total energy. Using this, (25), and (34) we obtain

$$a_0 = \left(\frac{E_0 + \frac{1}{3}e_z(1)}{\frac{4}{3}e_z(1) + e_N(1)}\right)^{1/3} \cong \left(\frac{3E_0}{4e_z(1)}\right)^{1/3}. \tag{42}$$

B. Epoch II

We call the period $a \geq a_0$ epoch II. In epoch II there is no energy supply from the winding strings. The nonwinding strings and the zero modes compete for the limited amount of total energy $E(a)$ in this time. The three functions $E(a)$, $e_z(a)$, and $e_N(a)$ in this epoch are uniquely determined by the three conditions

$$E(a) = e_z(a) + e_N(a), \tag{43}$$

$$\beta_z(e_z(a), v) = \beta_N(e_N(a), v), \tag{44}$$

$$\frac{d}{da}E(a) = -\frac{1}{a}e_z(a). \tag{45}$$

The functions in the second lines are the microcanonical temperature defined as $\beta_z(\varepsilon, v) = \partial_\varepsilon \ln \omega_z(\varepsilon, v)$ and (26). The second equation is an equitemperature condition. As we stated before the most probable value of the energy is determined as a meeting point of the competition between the zero modes and the nonwinding strings. The functions β_z and β_N measure how strongly the respective modes compete for the limited total energy.

These equations provide the rule for the competition between the winding strings and the zero modes. Now the thermal history in epoch II can be understood qualitatively with the knowledge of β_N, which is given in Fig. 8.

As a grows, the temperature of the zero modes decreases due to the redshift. For the equitemperature condition to be met, the nonwinding strings must cool by the same amount. From Fig. 8 we can readily find that the nonwinding strings have a very large specific heat. A very slight change of the temperature corresponds to a large energy change. Hence a decrease of the temperature signifies a violent loss of its energy. For this reason the string modes surviving in epoch II quickly decay away as the expansion proceeds. After the strings die away the Universe comes into the usual radiation-dominated universe with the usual redshift $E(a) = e_z(a) = 1/a$.

XI. DISCUSSION

In this paper we have performed the thermodynamical analysis of string cosmology focusing on the Brandenberger and Vafa scenario. Our analysis was based on the following assumptions: local thermal equilibrium; ideal gas approximation; normal energy loss; no unexpected effect due to the non-Einstein correction.

As a result our analysis has presented the following thermal history of the string universe. In the very initial epoch the Brandenberger and Vafa scenario suggested that the Universe oscillates around the self-dual point of the target space duality. Our analysis has shown that the emission of the zero modes from the strings due to cosmological expansion and the absorption of it due to contraction in the initial Universe are adiabatic processes. Then these could be repeated, resulting in the oscillatory behavior. These observations clarify the nature of the oscillating period from the thermodynamical point of view.

Once the accidental three-dimensional expansion is triggered, nonwinding strings are produced with the entropy production. In this process the winding strings decay producing nonwinding strings and zero modes. During this period the temperature is fixed to be the Hagedorn temperature, realizing the inflationlike situation.

This period ends when the winding modes are exhausted. At that time the temperature begins to fall due to the assumed redshift of the zero modes. The surviving string modes quickly die away by this cooling process and there emerges the usual radiation-dominant universe of the standard big bang cosmology.

Brandenberger and Vafa also addressed the problem of how our space-time dimensionality is determined to be four within their scenario. In the rest of this paper we consider this interesting problem and we propose an alternative idea to determine the dimensionality of space time. We first review the Brandenberger and Vafa discussion shortly and give our reconsideration next.

In their scenario, an expanding universe begins as a result of the accidental growth of the torus universe along some d directions in nine-dimensional space. In order for the accidental perturbations to grow, the strings should collide with each other frequently so as to diminish the number of their winding modes along d directions, because the winding modes slow down the expansion. If they intersect they would probably unwind.

On the other hand the string is a two-dimensional entity if seen in space-time. They argued that in order for two strings to collide with finite probability, $d+1$ must be smaller than or equal to $2+2=4$; otherwise space-time is too broad for two world sheets to have an intersection. This is their derivation of the relation $d+1 \leq 4$. They said that the Universe continues to make trials and errors until they learn that less than or equal to three spatial directions only become large. When the Universe finished the course, a $(d \leq 3)$-dimensionally large universe would have been born. Thus, while they presented the intriguing idea for understanding why $d \leq 3$, they did not succeed in giving compelling reasons why d should be 3.

Now we reconsider their argument. We cannot fully agree with their discussion to determine the dimensionality because of the following reasons. First, it is true that, in the point particle case, especially in ϕ^4 theory in R^d, the correlation functions are known [19] to be represented in terms of random walks in R^d. From this, it is rigorously proven that the theory is free if $d+1 \geq 5$. However this is not the case for string theory. We have repeated the same analysis as ϕ^4 theory in the light cone string field theory [20]. We have found that a Φ^3 interaction term prevents us from constructing an analogous representation as the ϕ^4 theory.

Second, it is true that the low-energy point particles cannot travel to the compact direction since the unit momenta are too heavy in such directions. This enforces the point particles effectively confined in the d-dimensional space. In this case it is only d directions that the point particles can utilize to collide. However, we are now concerned with the collision of the strings long enough to wind the torus universe many times. One can imagine with ease that these strings need not be confined in d-dimensional space and can move in any direction. Then what aspects are relevant to the string case?

Let us recall what occurs to the point particles in the expanding universe. The expansion makes the mean separation of particles larger. If the time scale of expansion time gets shorter than that of interaction time, the interaction is effectively frozen.

This consideration gives rise to the following interesting possibility. Let us assume that the accidental expansion in the $d(\leq 9)$ directions is always too fast to keep their mutual intersections. If it is true, the expansion in the d directions plays a negative role with respect to the unwinding of the winding strings.

To put it another way, the winding strings only can use the rest of the $9-d$ dimensions to unwind. Hence the greater value of $9-d$ is favored from the viewpoint of killing the winding modes along d directions. This implies the inequality opposite in direction from that of Brandenberger and Vafa. One may say that this implies that $d=1$ is preferred. However the story is not so simple.

As we have observed in expression (22), the entropy produced in the process of the unwinding is proportional to the expanding volume. This means that the Universe prefers to expand as many dimensions as possible. The larger value of d is preferred from the second law of thermodynamics, the entropy increase. This effect will compete with the preceding one. Our idea is that the number of the expanding space dimensions is determined to be three as a result of this competetion. This idea is not yet formulated on a rigorous ground at present. However we think it is one of the plausible candidates of a mechanism to fix the space-time dimension. We are planning to make a numerical simulation of the strings to obtain some indications to this idea.

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APPENDIX

In this appendix we examine what kind of correction is added when we include the full quantum statistics. A rough estimation of it has been done in [13]. We will consider this in our framework. As a result we can show that the correction in our case is very small.

It is known [12] that the thermodynamical functions in the canonical formalism and the microcanonical formalism are connected through the Laplace transformation. If we denote the single canonical partition function for the nonwinding string as $\tilde{f}_N(\beta)$, this can be explicitly written as

$$\int_{-i\infty}^{i\infty} \bar{d}\beta \exp\left(\tilde{f}_N(\beta)\right) e^{\beta\varepsilon} = \hat{A}(\varepsilon, v) e^{\beta_H \varepsilon} \tag{A1}$$

within the MB approximation, where $\bar{d}\beta = d\beta/2\pi i$.

The multistring state density with all the statistical effects $\omega_N(\varepsilon)$ is given by a similar integral, but the integrand should be changed to

$$Z(\beta) = \prod_{r=1, r:\text{odd}}^{\infty} \exp\left[\frac{1}{r}\tilde{f}(r\beta)\right]. \tag{A2}$$

Namely the integrand is a product of the functions $\frac{1}{r}\tilde{f}(r\beta)$. Equation (A1) readily implies

$$\int_{-i\infty}^{i\infty} \bar{d}\beta \exp\left(\frac{1}{r}\tilde{f}_N(r\beta)\right) e^{\beta\varepsilon} = \frac{1}{r}\hat{A}(\varepsilon/r, v/r) e^{\beta_H \varepsilon/r}. \tag{A3}$$

Here we remind the readers of a well-known formula that the inverse Laplace transform of the product of two functions is equal to the convolution of the inverse Laplace transforms of the two functions. Namely we have

$$\int_{-i\infty}^{i\infty} \bar{d}\beta\, \tilde{\phi}_1(\beta)\tilde{\phi}_2(\beta) e^{\beta\varepsilon} = \int_0^{\infty} d\varepsilon_1 d\varepsilon_2 \phi_1(\varepsilon_1)\phi_2(\varepsilon_2)\delta(\varepsilon - \varepsilon_1 - \varepsilon_2). \tag{A4}$$

Now the repeated use of this formula leads us to the multistate density with full quantum statistical corrections:

$$\hat{\omega}_N(\varepsilon, v) = \lim_{k\to\infty} \int_0^{\infty} d\varepsilon_1 d\varepsilon_3 \cdots d\varepsilon_k \times \hat{\alpha}(\varepsilon_1, v)\hat{\alpha}(\varepsilon_3, v/3)\cdots\hat{\alpha}(\varepsilon_k, v/k) \times \delta\left(\varepsilon - \sum_{r=1}^{k} r\varepsilon_r\right), \tag{A5}$$

where we have set $\alpha(\varepsilon, v) = A(\varepsilon, v)e^{\beta_H \varepsilon}$. All the summation and product indices in this appendix mean to run only odd integers unless otherwise stated.

Now we are going to examine the size of the corrections to the MB approximation using (A5). Because $\alpha(\varepsilon, v)$ does not vanish only for $\varepsilon \geq m_1$, the product of (A5) is actually a finite product.

For the range $km_1 \leq \varepsilon < (k+1)m_1$ with k being an integer we consider the sequence of odd numbers such that $1 \leq r_0 < r_1 < \cdots < r_l$ and $r_0 + r_1 + \cdots + r_l = k$. Only r_0 has the possibility to become unity. All the corrections acquired by $\omega_N(\varepsilon, v)$ in this range are written as the form

$$L = \int d\varepsilon_{r_0} \cdots d\varepsilon_{r_l}\, \alpha(\varepsilon_{r_0}, V/r_0) \cdots \alpha(\varepsilon_{r_l}, V/r_l) \times \delta(\varepsilon - (r_0\varepsilon_{r_0} + \cdots + r_l\varepsilon_{r_l})). \tag{A6}$$

If we set $v/\eta m_1^{\eta} = w$ and $\varepsilon/m_1 = x$, α is rewritten as

$$\alpha = \frac{1}{m_1}\exp[h(x - w) + w + \beta_H m_1 x]. \tag{A7}$$

Using it and making a variable change $\varepsilon_{r_j} = m_1 x_j$ (here j runs even and odd integer) enables us to rewrite $Le^{-\beta_H \varepsilon}$ as

$$Le^{-\beta_H \varepsilon} = {m_1}^l \int dx_0 \cdots dx_l \exp\left[\sum_{j=0}^{l}(h(x_j - w/r_j) + w/r_j - \beta_H m_1(r_j - 1)x_j)\right] \delta\left(x - \sum_{j=0}^{l} r_j x_j\right). \tag{A8}$$

This is a typical form of the corrections added to $A(\varepsilon, v) = \alpha(\varepsilon, v)e^{-\beta_H \varepsilon}$.

This function is expressed as an integration of a product of functions of the form

$$\exp\left[h(x_j - w/r_j) + w/r_j - \beta_H m_1(r_j - 1)x_j\right]. \qquad \text{(A9)}$$

First we consider the case $r_0 = 1$. We already know that this peaks at $x_0^{\max} = w + c$ with the height $\frac{1}{w^{1/\eta}}\exp(w)$, where c is the point such that $h'(c) = 0$.

Next we consider the general case $r_j \neq 1$. In this case the function is strongly damped by the exponential suppression $\exp[-\beta_H m_1(r_j - 1)x_j]$. Actually the position of the peak is now located around $x_j^{\max} \sim 1$. It can be verified by examining where the derivative of the exponent of (A9) changes its sign from positive to negative. The derivative in question is written as $h'(x_j - w/r_j) - \beta_H m_1(r_j - 1)$. We recall here that we had the information on the derivative $h'(x)$ because it is essentially the microcanonical temperature examined before, see (26). We know that

$$\begin{aligned} h'(x - w) = \partial_x \ln A &= m_1 \beta_N(\varepsilon, v) \\ &= \beta_H m_1(\beta_N(\varepsilon, v)/\beta_H). \qquad \text{(A10)} \end{aligned}$$

From our previous numerical analysis (see Fig. 7) we know that β_N/β_H is very close to unity except when $x \sim 1$; we observe that the above derivative is negative at the entire range except for the very edge $x_j \sim 1$. This means that the position of the peak is $x_j^{\max} \sim 1$, and accordingly its height is negligibly small. It is no longer exponentially large like $\sim \exp(v/\eta m_1^\eta)$. This is always true if (A6) contains the factor with $r_j \neq 1$. Consequently we conclude that any corrections to MB approximation are very small.

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