Statistical mechanics of black holes

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We analyze the statistical mechanics of a gas of neutral and charged black holes. The microcanonical ensemble is the only possible approach to this system, and the equilibrium configuration is the one for which most of the energy is carried by a single black hole. Schwarzschild black holes are found to obey the statistical bootstrap condition. In all cases, the microcanonical temperature is identical to the Hawking temperature of the most massive black hole in the gas. U(1) charges in general break the bootstrap property. The problems of black-hole decay and of quantum coherence are also addressed.
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I. INTRODUCTION

It is by now well known that the area of a black-hole event horizon has received an interpretation in terms of thermodynamical entropy and that the black-hole mass has been related, in this thermodynamical picture, to a temperature called the Hawking temperature [1,2]. Black holes should then thermally evaporate, with the evaporation process ending when the black hole reaches its extreme limit.

This state of affairs has been viewed quite recently with a certain amount of criticism [3–5], particularly in view of the loss of quantum coherence in the black-hole decay process. The above interpretation of black-hole phenomena such as the decay (evaporation) process seems to imply that thermodynamics is more fundamental than quantum mechanics in this problem, and that pure states are converted into mixed states. This state of affairs has caused concern among researchers in this field.

Although such a problem originally arose in the mid-seventies, a recent surge of interest has occurred in connection with the discovery that black holes can carry quantum hair (for a nice discussion, see Ref. [4]). Quantum hair has the general effect of modifying the usual Hawking relation for temperature in terms of mass and affecting the expression for the entropy. It was hoped that black holes would carry quite a lot of quantum hair, enough to actually reduce the thermal attributes down to zero and thereby generate the recovery of quantum coherence.

Strings actually do carry quite a lot of quantum hair (recall the massive string excitations) and so attempts have been made to establish a connection between strings and black holes [6, 7].

The degeneracy of string states at mass level m is well known to be an exponentially growing function of mass. Typically, it is given as [8],

$$\rho_{\text{string}}(m) \sim c\, m^a e^{\beta_H m} , \tag{1.1}$$

where the constants c, a, and β_H (the inverse Hagedorn temperature) are model dependent. A microcanonical analysis reveals that strings obey the following bootstrap condition [9–11],

$$\frac{\Omega(E)}{\rho(E)} \to 1, \quad E \to \infty , \tag{1.2}$$

where $\Omega(E)$ is the microcanonical density of states.

Historically, the bootstrap condition was applied to the statistical model of hadrons in an effort to explain the ever-increasing number of nuclear resonances found at higher energies [9–11].

The degeneracy of states in mass space [Eq. (1.1)] was first arrived at as a solution of the self-consistent constraint [Eq. (1.2)], and only later was it recognized as a truly stringy attribute. In the context of the statistical model the bootstrap constraint implied that hadronic resonances could be viewed as being made of resonances, thereby replacing the elementary particle concept. Scattering theory was later developed and the property of duality was further demonstrated. A by-product of the duality symmetry in the scattering amplitudes is that the number of open channels in a scattering process rises in parallel with the degeneracy of states [Eq. (1.1)] as the energy is increased [10].

Like strings the degeneracy of states associated with a black hole increases exponentially, with the argument of the exponential now quadratic in mass, at least to leading order in a U(1) charge expansion.

As an example, in natural units ($\hbar = c = G = 1$), a Schwarzschild (neutral) black hole has the following density of states in mass space,

$$\rho_{\text{Schw}}(m) \sim c\, e^{4\pi m^2}, \tag{1.3}$$

where m is the mass of the black hole. This result is a nonperturbative quantum effect in the sense that it is obtained from the WKB method and that the argument of the exponential is of order $\hbar^{-1}$. (For a detailed derivation, see Ref. [4].) The constant c here represents the unknown effects from the purely perturbative quantum-field-theoretical sector of the theory.

The above density of states has been given the interpretation [1, 2] of a thermodynamical system with entropy,

$$s(m) = 4\pi m^2 \tag{1.4}$$

and inverse temperature,

$$\beta_{\rm Hawking}(m) = \frac{ds}{dm} = 8\pi m \ . \tag{1.5}$$

Equation (1.5) however yields a negative specific heat,

$$(C_V)_{\rm Hawking} = -\beta^2 \frac{dm}{d\beta} = -\frac{\beta^2_{\rm Hawking}}{8\pi} \ . \tag{1.6}$$

It is important to recall that the results in Eqs. (1.4) – (1.6) can be derived (e.g., see Ref. [4]) by analytically continuing the expression for the black-hole metric to imaginary time and so to Euclidean space-time. The temperature emerges as the inverse period of the compact Euclidean time and its relation to the black-hole mass is determined from the requirement of the absence of conical singularities of the Euclidean space-time. This Matsubara-type method produces results which, in the thermodynamical interpretation, belong to the canonical ensemble.

Although usually interpreted as a sign of instability, the negative specific heat [Eq. (1.6)] represents nevertheless a flaw in the above thermal interpretation. The main point here is that, within the canonical (thermal) ensemble the specific heat is always a positive definite quantity [11, 12]. Gravitational systems (e.g., supernovae and galaxies) are known to exhibit a negative microcanonical specific heat [12]. However, in the canonical ensemble one sees only a phase transition, so that the canonical specific heat is always positive. The above results therefore point in our judgment to an inconsistency of the thermal interpretation, not just an instability.

The present work is based on the observation that two interpretations of the density of states [Eq. (1.3)] exist for the Schwarzschild black hole (or any black hole), one of which of course must be wrong. The above thermodynamical interpretation [1, 2] is one such interpretation. However, as was pointed out, it leads to undesirable features such as a negative canonical specific heat as well as a breakdown of the laws of quantum mechanics as pure states evolve into mixed states during the process of black-hole evaporation. This interpretation seems to run into problems with both thermodynamics and quantum mechanics.

In the other interpretation, we simply regard Eq. (1.3) as the degeneracy of states of a quantum Schwarzschild black hole at mass level m, in a way analogous to the degeneracy of states in string theory. In this way the laws of quantum mechanics remain untouched and the process of black-hole evaporation can be understood from an S-matrix–theory point of view.

One then naturally becomes interested in the problem of understanding the statistical mechanics of a gas of such objects.

In the following sections we shall find that the microcanonical ensemble is the unique sensible framework for analyzing this problem. At least for the Schwarzschild case it will be found that the bootstrap constraint [Eq. (1.2)] is met, hence providing us with the novel view of a black hole as an object itself made of other black holes, very much as in the old statistical model of hadrons.

The equilibrium state of such a system is not thermal. A quantum coherent view of black-hole decay (evaporation) can also be obtained in a way analogous to strings. Black-hole states decay into other black-hole states. Although the problem of constructing black-hole scattering amplitudes is not addressed in this paper, we shall nevertheless demonstrate that the number of open channels for n-body decay of a Schwarzschild black hole does indeed grow precisely in parallel with the density of states [Eq. (1.3)], thereby allowing duality symmetry for black-hole scattering amplitudes. Black holes may belong to a certain class of string theories, as conjectured previously [6].

Obvious applications of the considerations presented in this paper lie in cosmology and the very early Universe, as well as in galaxy formation and the general theory of gravitational collapse.

II. A SIMPLE CASE: THE SCHWARZSCHILD BLACK HOLE

In this section we analyze the statistical mechanics of a gas of Schwarzschild black holes with degeneracy given by Eq. (1.3). In this problem as a working hypothesis we shall assume that the equilibrium state (if any) can be achieved on time scales less than the individual life-times of the black holes in the gas. This is tantamount to neglecting the decay rates (hence collision processes) so that one remains conveniently in the ideal-gas approximation. There is a simple argument establishing the nonexistence of the canonical ensemble description of a gas of black holes with degeneracy given by Eq. (1.3). Recalling the form [Eq. (1.1)] for strings, it is well known that the canonical ensemble breaks down whenever the exponential factor in Eq. (1.1) wins over the Boltzmann factor $e^{-\beta m}$ in the statistical sum (integral over mass). For strings this occurs at temperatures above the Hagedorn temperature β_H^{-1}. For black holes however the exponential factor always dominates and so the canonical partition function diverges for all temperatures. One may then expect large disparities, which show up as unbounded fluctuations in the thermal ensemble, in the energy distribution among the components of the gas. This is in fact what happens.

We now turn to the unique approach to this problem, namely, the microcanonical description. The microcanonical density of states is written as follows,

$$\Omega(E,V) = \sum_{n=1}^{\infty} \Omega_n(E,V) \ , \tag{2.1}$$

in which we have defined the density of states for the configuration with n black holes as,

$$\Omega_n(E,V) = \left[\frac{V}{(2\pi)^3}\right]^n \frac{1}{n!} \times \prod_{i=1}^{n} \left\{ \int_{m_0}^{\infty} dm_i \, \rho_{\rm BH}(m_i) \int_{-\infty}^{\infty} d^3p_i \right\} \times \delta\left(E - \sum_{i=1}^{n} E_i\right) \, \delta^3\left(\sum_{i=1}^{n} \mathbf{p}_i\right) \ , \tag{2.2}$$

where E is the total energy of the system and where V is the volume of the gas.

We can write the product of the degeneracy of states as,

$$\prod_{i=1}^{n} e^{a\, m_i^2} = e^{a \sum_{i=1}^{n} E_i^2} e^{-a \sum_{i=1}^{n} \mathbf{p}_i^2} , \quad (2.3)$$

in which we have made the working assumption that black holes obey the particlelike dispersion relation,

$$E^2 = \mathbf{p}^2 + m^2 . \quad (2.4)$$

As is clear from Eq. (2.3), the high-momentum states contribute negligibly (they are actually exponentially suppressed) to the microcanonical density of states. Hence at high energy the dominant contributions originate from mass. Following Frautschi and neglecting the momentum-conservation δ function in Eq. (2.2), the momentum integrations simply become n decoupled Gaussian integrals, the values of which can be absorbed into a redefinition of the volume factor. Equation (2.2) then reduces to,

$$\Omega_n(E,V) \simeq \left[\frac{V}{(2\pi)^3}\right]^n \frac{1}{n!} \prod_{i=1}^{n} \int_{m_0}^{\infty} dE_i\, \rho_{\rm BH}(E_i) \times \delta\left(\sum_{i=1}^{n} E_i - E\right) . \quad (2.5)$$

Again, Frautschi has made a general analysis of systems with degeneracy $\rho(m)$ of the following generic form,

$$\rho(m) = f(m)\, e^{b\, m^p} , \quad (2.6)$$

where $f(m)$ is a polynomial in m. Substituting this form into Eq. (2.5), we find the maximum value of the integrand to be at $E_i = E/n$ for any p (i.e., the total energy is evenly distributed among all parts). For $p < 1$ this is the dominant configuration. However for $p > 1$ contributions from the integration boundaries yield the dominant configuration in which most of the energy is carried by a single black hole. Such a configuration is the one for which, e.g., the nth black hole carries the energy $E_n = E - (n-1)m_0$ while the $n-1$ other black holes carry energies $E_i = m_0$ $(i = 1, ..., n-1)$. Since a Schwarzschild black hole corresponds to $p = 2$, the density of states [Eq. (2.5)] finally becomes,

$$\Omega_n(E,V) \simeq \left[\frac{cV}{(2\pi)^3}\right]^n \frac{1}{n!} e^{4\pi[E-(n-1)m_0]^2} e^{4\pi(n-1)m_0^2} , \quad (2.7)$$

an expression valid at high energy E.

The most probable equilibrium configuration is the one satisfying the condition,

$$\left.\frac{d\Omega_n(E,V)}{dn}\right|_{n=N(E,V)} = 0 . \quad (2.8)$$

We find

$$\exp[\Psi(N+1)] = \frac{cV}{(2\pi)^3} \exp[s(m_0) - m_0 \beta_{\rm Hawking}(E-(N-1)m_0)] , \quad (2.9)$$

where $s(x)$ is the Hawking entropy [Eq. (1.4)], $\beta_{\rm Hawking}(x) \equiv \frac{ds(x)}{dx}$, and $\Psi(x)$ is the psi function.

Now since the lightest object in the gas is the extreme Schwarzschild black hole, we have $m_0 = 0$. Therefore,

$$\exp\Psi(N+1) = \frac{cV}{(2\pi)^3} . \quad (2.10)$$

The total entropy of the system is now given as,

$$S(E,V) \equiv \ln\Omega(E,V) \simeq \ln\Omega_N(E,V) \simeq N \ln\left[\frac{cV}{(2\pi)^3}\right] - \ln\Gamma(N+1) + s(E), \quad (2.11)$$

where $N(V)$ is given by Eq. (2.10). The inverse temperature β is obtained from the total entropy according to

$$\beta = \frac{dS}{dE} = \frac{\partial S}{\partial N}\frac{\partial N}{\partial E} + \frac{\partial S}{\partial E} . \quad (2.12)$$

The first term on the far right is zero at the maximum value of Ω [recall Eqs. (2.8) and (2.11)]. We find quite generally that

$$\beta = \frac{dS}{dE} = \frac{ds}{dE} = \beta_{\rm Hawking} = 8\pi E . \quad (2.13)$$

So the microcanonical temperature for a gas of Schwarzschild black holes is the same as the Hawking temperature of the black hole carrying the greatest amount of energy in the gas.

The microcanonical specific heat is likewise negative,

$$C_V = -\beta^2 \frac{dE}{d\beta} = \frac{-\beta^2}{8\pi} , \quad (2.14)$$

a result identical to Eq. (1.6). Although this result implies that instabilities will develop if the gas is brought in contact with a heat bath, it is not an inconsistent finding. The specific heat is allowed, in principle, to be negative in the microcanonical ensemble.

From the formula,

$$\beta P = \frac{\partial S}{\partial V} , \quad (2.15)$$

where P is the pressure, we obtain the following equation of state:

$$\beta P = \frac{N}{V} , \quad (2.16)$$

which is that of an ideal gas.

Finally, it is readily seen that the bootstrap condition is satisfied for a gas of Schwarzschild black holes, provided [cf. Eqs. (1.2), (1.3), and (2.11)],

$$\left[\frac{cV}{(2\pi)^3}\right]^N \frac{1}{\Gamma(N+1)} = c\,. \tag{2.17}$$

That this is so originates from the fact that an extreme Schwarzschild black hole is massless. Equation (2.17) together with Eq. (2.10) tells us that the size of the quantum corrections to the density of states Eq. (1.3) determines the volume of the gas in units of the Planck volume, as well as the size of the most probable configuration. Assuming $N \gg 1$ and c $\gg 1$, one gets,

$$V \sim \frac{(2\pi)^3}{c}\ln c\,,\ N \sim \ln c\,. \tag{2.18}$$

In the following sections we extend our analysis to gases of black holes carrying U(1) electric charge, namely, the Reissner-Nordström and dilaton black holes.

III. THE REISSNER-NORDSTRÖM BLACK HOLE

In this section we analyze the case of a gas of charged Reissner-Nordström black holes with identical individual charges Q. These individual charges are taken to be much smaller than the total energy E of the system. For such a case the density of states for the configuration with n black holes is given by the following generalization of Eq. (2.7),

$$\Omega_n(E,V,Q) \sim \left[\frac{cV}{(2\pi)^3}\right]^n \frac{1}{n!} e^{s(E-(n-1)m_0,Q)} e^{(n-1)s(m_0,Q)}, \tag{3.1}$$

which is valid at high energy E, and where $s(m,Q)$ is the Hawking entropy of a single Reissner-Nordström black hole of mass m,

$$s(m,Q) = \pi m^2 \left[1+\sqrt{1-\frac{Q^2}{m^2}}\right]^2 . \tag{3.2}$$

The corresponding Hawking (inverse) temperature is given by the relation

$$\beta_{\rm Hawking}(m,Q) = \frac{ds}{dm} = 2\pi m\left[1+\sqrt{1-\frac{Q^2}{m^2}}\right]^2 \times\left(1-\frac{Q^2}{m^2}\right)^{-1/2} . \tag{3.3}$$

The degeneracy of states for such a hole is given by

$$\rho_{\rm RN}(m,Q) = c\ \exp\ s(m,Q)\,. \tag{3.4}$$

Again for this case the dominant equilibrium configuration of the gas is not thermal, but given rather by the state with one very massive black hole and $(n-1)$ light ones with mass m_0.

Since extreme Reissner-Nordström black holes have mass $m = Q$, and since these are the lightest elements of the gas, we have the identification

$$m_0 = Q. \tag{3.5}$$

The most probable configuration $N(E,V,Q)$ is the one which maximizes the density of states [Eq. (3.1)]. We find

$$e^{\Psi(N+1)} = \frac{cV}{(2\pi)^3} e^{s(Q,Q)-Q\beta_{\rm Hawking}(E-(N-1)Q,Q)}\,. \tag{3.6}$$

Now since $E \gg (N-1)Q$, we get the following approximate relation:

$$\Psi(N+1) \sim \ln\left[\frac{cV}{(2\pi)^3}\right] - 8\pi QE + O(Q^2)\,. \tag{3.7}$$

For large N the condition

$$\frac{cV}{(2\pi)^3} \gg e^{8\pi QE} \tag{3.8}$$

must be satisfied. Clearly at high enough energy the above condition cannot be met. The most probable configuration at high energy is the one for which N is as small as possible, i.e., $N = 1$. In the statistical bootstrap model of hadrons the configuration $N = 1$ corresponds to an elementary particle and is usually ruled out. In the present microcanonical formulation there is no logical argument to exclude such a case.

The most probable equilibrium configuration of a gas of Reissner-Nordström black holes is then described by the conditions,

$$\begin{aligned}(N-1)Q \ll E \ll E_c\,,\ N \gg 1,\\ E = E_c\,,\ N = 1\,,\end{aligned} \tag{3.9}$$

where E_c is an "ionization point" determined by the following formula:

$$Q\beta_{\rm Hawking}(E_c,Q) = s(Q,Q) + \ln\left[\frac{cV}{(2\pi)^3}\right] - \Psi(2)\,. \tag{3.10}$$

For small charge Q we get

$$E_c \sim \frac{1}{8\pi Q}\left\{\ln\left[\frac{cV}{(2\pi)^3}\right] - \Psi(2)\right\}\,. \tag{3.11}$$

The total entropy of the gas is given as follows,

$$\begin{aligned}S(E,V,Q) &\simeq \ln\Omega_N(E,V,Q)\\ &= N\ln\left[\frac{cV}{(2\pi)^3}\right] - \ln\Gamma(N+1)\\ &\quad + s(E-(N-1)Q,Q) + (N-1)\ s(Q,Q)\,.\end{aligned} \tag{3.12}$$

As in the Schwarzschild case the microcanonical temperature is the same as the Hawking temperature of the most massive black hole in the gas,

$$\beta(E,V,Q) = \beta_{\rm Hawking}(E-(N-1)Q,Q)\,, \tag{3.13}$$

with $N(E,V,Q)$ given by Eq. (3.6). The equation of state is that of an ideal gas,

$$\beta P = \frac{N}{V}. \tag{3.14}$$

Clearly the bootstrap constraint cannot be met for $Q \neq 0$ and/or $N \neq 1$. At the "ionization point", however, the bootstrap constraint is trivially met (there is a single black hole). Inserting $N = 1$ and $E = E_c$ into Eq. (3.12) leads to the volume constraint,

$$V = V_c \equiv (2\pi)^3. \tag{3.15}$$

The lone black hole occupies a region of space with a size of the order of the Planck volume.

Inserting Eq. (3.15) into expression (3.11) for E_c we find

$$E_c \sim \frac{1}{8\pi Q}[\ln c - \Psi(2)] \, . \tag{3.16}$$

Of course, consistency requires that $\ln c > \Psi(2)$.

For $E > E_c$ there is no equilibrium configuration. It seems plausible to speculate that some kind of phase transition may occur at the "ionization point."

We close this section by providing approximate formulas in the large-N ($E \ll E_c$) region. For large N, Eq. (3.7) becomes

$$N \simeq \frac{cV}{(2\pi)^3} e^{-8\pi QE}. \tag{3.17}$$

In this approximation the total entropy [Eq. (3.12)] becomes

$$S \simeq 4\pi E^2 + 8\pi QE + \frac{cV}{(2\pi)^3} e^{-8\pi QE} + O(Q^2). \tag{3.18}$$

This expression gives for β,

$$\beta \simeq 8\pi E + 8\pi Q \left[1 - \frac{cV}{(2\pi)^3} e^{-8\pi QE}\right] + O(Q^2) \, , \tag{3.19}$$

which is, to the same approximation, the inverse Hawking temperature. In this case again the specific heat is negative,

$$C_V = -\beta^2 \left(\frac{d\beta}{dE}\right)^{-1} \simeq -8\pi \left[E^2 + 2QE \left(1 - \frac{cV}{(2\pi)^3} e^{-8\pi QE}\right)\right] + O(Q^2) \, . \tag{3.20}$$

Thus the charged black hole, like the Schwarzschild black hole, cannot reach thermal equilibrium with its surroundings.

In the next section we analyze a somewhat more general case, the so-called dilaton black hole.

IV. THE DILATON BLACK HOLE

The dilaton black hole [13, 14] is somewhat similar to the Reissner-Nordström black hole with the added complexity of the effects of an additional dilaton field coupling.

The microcanonical analysis of a gas of dilaton black holes is identical to the case of the Reissner-Nordström black-hole gas. The Hawking entropy and temperature however are now given by

$$s(m,Q) = \pi m^2 \left[1 + \sqrt{1 - \frac{(1-a^2)Q^2}{m^2}}\right]^2 \left(1 - \frac{(1+a^2)Q^2}{m^2\left(1+\sqrt{1-\frac{(1-a^2)Q^2}{m^2}}\right)^2}\right)^{\frac{2a^2}{1+a^2}} \tag{4.1}$$

and

$$\beta_{\text{Hawking}}(m,Q) = 4\pi m \left[1 + \sqrt{1 - \frac{(1-a^2)Q^2}{m^2}}\right] \left(1 - \frac{(1+a^2)Q^2}{m^2\left(1+\sqrt{1-\frac{(1-a^2)Q^2}{m^2}}\right)^2}\right)^{\frac{a^2-1}{a^2+1}} \, . \tag{4.2}$$

Again the degeneracy of states is given as

$$\rho(m,Q) = c \, \exp[s(m,Q)] \, . \tag{4.3}$$

Notice that the case a=0 reduces to the Reissner-Nordström black hole.

The extreme dilaton black hole has a mass

$$m_0^2 = \frac{Q^2}{1+a^2} \, . \tag{4.4}$$

Repeating the analysis of the previous section, the "ionization point" ($N = 1$) at high energy E_c is found to be given by the formula

$$\frac{Q}{\sqrt{1+a^2}} \beta_{\text{Hawking}}(E_c, Q) = s\left(\frac{Q}{\sqrt{1+a^2}}, Q\right) + \ln\left[\frac{cV}{(2\pi)^3}\right] - \Psi(2) \, . \tag{4.5}$$

Again, for small Q, the result is found to be similar to the Reissner-Nordström case,

$$E_c \sim \frac{\sqrt{1+a^2}}{8\pi Q}\left\{\ln\left[\frac{cV}{(2\pi)^3}\right] - \Psi(2)\right\} \, . \tag{4.6}$$

For large N the density of states Ω is a maximum at

$$N \simeq \frac{cV}{(2\pi)^3} e^{-8\pi QE/(1+a^2)^{1/2}} \left(\frac{(N-1)Q}{\sqrt{1+a^2}} \ll E \ll E_c \right) . \quad (4.7)$$

The total entropy of the gas is

$$\begin{aligned} S(E,V,Q) &\simeq \ln \Omega_N \\ &\simeq N \ln \left[\frac{cV}{(2\pi)^3} \right] - \ln \Gamma(N+1) \\ &\quad + s\left(E - \frac{(N-1)Q}{\sqrt{1+a^2}}, Q \right) \\ &\quad + (N-1)\, s\left(\frac{Q}{\sqrt{1+a^2}}, Q \right) . \end{aligned} \quad (4.8)$$

The microcanonical temperature is again easily derived. We find

$$\begin{aligned} \beta(E,V,Q) &= \frac{dS}{dE} = \frac{\partial S}{\partial N}\frac{\partial N}{\partial E} + \frac{\partial S}{\partial E} \\ &= \frac{\partial s(E - \frac{(N-1)Q}{\sqrt{1+a^2}}, Q)}{\partial E} \\ &= \beta_{\text{Hawking}} \left(E - \frac{(N-1)Q}{\sqrt{1+a^2}}, Q \right) , \end{aligned} \quad (4.9)$$

in which we once again used the fact that $N(E,V,Q)$ is the most probable configuration, i.e., $\frac{\partial S}{\partial N} = 0$.

For large N we have the following approximate expression

$$\begin{aligned} \beta &\simeq 8\pi E + \frac{8\pi Q}{\sqrt{1+a^2}} \left[1 - \frac{cV}{(2\pi)^3} e^{-\frac{8\pi QE}{\sqrt{1+a^2}}} \right] \\ &\quad + O\left(\frac{Q^2}{\sqrt{1+a^2}} \right) \quad \left(\frac{(N-1)Q}{\sqrt{1+a^2}} \ll E \ll E_c \right) . \end{aligned} \quad (4.10)$$

Finally, the microcanonical specific heat is given by

$$C_V = -\beta^2 \left(\frac{d\beta}{dE} \right)^{-1} = -\beta^2 \left[\frac{\partial \beta}{\partial E} + \frac{\partial \beta}{\partial N}\frac{\partial N}{\partial E} \right]^{-1} . \quad (4.11)$$

Recalling Eq. (4.9), we find,

$$\begin{aligned} C_V &= -\beta^2 \left[\frac{\partial \beta}{\partial E} + \frac{\partial}{\partial E}\left(\frac{\partial s}{\partial N} \right) \frac{\partial N}{\partial E} \right]^{-1} \\ &= -\beta^2 \left(\frac{\partial \beta}{\partial E} \right)^{-1} \left[1 - \frac{Q}{\sqrt{1+a^2}} \frac{\partial N}{\partial E} \right]^{-1} . \end{aligned} \quad (4.12)$$

Therefore,

$$\begin{aligned} C_V(E,V,Q) &= (C_V)_{\text{Hawking}} \left(E - \frac{(N-1)Q}{\sqrt{1+a^2}}, Q \right) \\ &\quad \times \left[1 - \frac{Q}{\sqrt{1+a^2}} \frac{\partial N}{\partial E} \right]^{-1} . \end{aligned} \quad (4.13)$$

Recalling Eq. (4.7) one finds that $\frac{\partial N}{\partial E} < 0$ so that the sign of the microcanonical specific heat is determined by that of the Hawking specific heat.

Although $\frac{\partial N}{\partial E} < 0$ is formally valid at high energy, it is tempting and probably valid to extrapolate to the low-energy domain where the total energy approaches the extreme limit $E \to \frac{NQ}{\sqrt{1+a^2}}$ of the gas with total charge NQ. It is known that, for $a < 1$, the dilaton black hole has a positive Hawking specific heat as it approaches its extreme limit, whereas for $a > 1$ the specific heat is negative [5]. It may be an interesting problem to analyze the implications of such properties from the viewpoint of physical processes in the very early Universe.

V. QUANTUM COHERENT BLACK-HOLE DECAY

The previous sections were devoted to the statistical mechanics of black holes. In this section we attempt to uncover a few facts about (quantum coherent) black-hole decay and scattering.

A property particular to strings and dual models is the duality of the scattering amplitudes. The four-point amplitude, for instance, can be expressed as a sum over resonances either in the s- or t-channel, even at very high energy. Frautschi [10] has pointed out that in order for duality to hold, the number of n-body channels open in the statistical model of hadrons, and so the total number of open channels, must rise precisely in parallel with the number of resonances as the center-of-mass energy is increased,

$$N_n(m) \sim \rho_{\text{string}}(m), \quad m \to \infty , \quad (5.1)$$

where $N_n(m)$ is the number of open n-body channels at center-of-mass energy m. An explicit expression for the two-body case is given by

$$N_2(m) = \frac{1}{2!} \int_{m_0}^{m-m_0} dm_2\, \rho(m_2) \int_{m_0}^{m-m_2} dm_1\, \rho(m_1) . \quad (5.2)$$

Consequently, if duality can be argued to be a symmetry of, say, Schwarzschild black-hole scattering amplitudes, one should be able to support it by a direct calculation of Eq. (5.2), making use of the degeneracy [Eq. (1.3)]. Inserting Eq. (1.3) into Eq. (5.2), we obtain ($m_0 = 0$)

$$N_2(m) = \frac{c^2}{2!} \int_0^m dm_2 \int_0^{m-m_2} dm_1\, e^{4\pi(m_1^2+m_2^2)} . \quad (5.3)$$

Clearly, contributions from the region $m_2 \simeq m$ give negligible results. The dominant contribution is obtained when $m \gg m_2$ and $m_1 \simeq m$. This is the same situation which occurred in the evaluation of the density of states with n black holes, in which most of the energy was carried by a single black hole and the $n-1$ others shared the tiny remnants. In this approximation we obtain

$$\begin{aligned} N_2(m) &\simeq \frac{c^2}{2} e^{4\pi m^2} \\ &= \frac{c}{2}\, \rho_{\text{Schw}}(m) , \quad m \to \infty . \end{aligned} \quad (5.4)$$

This argument can be extended to any n, and one finds the same result, namely, that as in the string-theory case, the number of open channels does grow in parallel with the degeneracy of states as energy is increased.

It is now very plausible to argue that the black-hole (at least the Schwarzschild black hole) scattering amplitudes do share the property of duality with the string scattering amplitudes.

The above results seem to support earlier conjectures [6] that black holes do belong to a certain class of string theories.

VI. DISCUSSION

In this work we analyzed the statistical mechanics of a gas of black holes from the standpoint of the microcanonical ensemble. In fact, as we showed in Sec. I, this is the only approach to the problem, because the energy is not thermally distributed among the elements of the gas. The lack of a thermal distribution of the energy is reflected by the nonexistence of the canonical partition function.

A gas of Schwarzschild (neutral) black holes naturally obeys the bootstrap condition, a property related to the fact that extreme Schwarzschild black holes are massless. We found that quantum corrections to the degeneracy of states play an important role in selecting the most probable number of black holes in the gas. The equilibrium configuration is the one for which most of the energy is carried by a single black hole, a situation somewhat analogous to strings.

For charged black holes such as the Reissner-Nordström or its dilaton generalization, the bootstrap condition is in general not realized, mainly because the extreme case is not massless. However, for such models there is a high-energy "ionization point" at which the gas does obey trivially the bootstrap condition. At such a "critical" point the gas is composed of a single supermassive black hole whose size is of the order of the Planck volume.

For all models analyzed here we find that the microcanonical temperature is identical to the Hawking temperature of the most massive black hole in the gas.

One motivation for the present work was the realization that the thermodynamical interpretation of the single black-hole event horizon leads to, in our judgment, the inconsistent result of negative canonical specific heat. We contend that the views presented here represent an improvement in the understanding of black-hole phenomena. In particular, at least viewed from the present statistical model, black-hole states can decay into or scatter with other black-hole states. This lends support to a completely quantum-coherent view of black-hole "evaporation." Furthermore, arguments presented in Sec. V. seem to indicate that Schwarzschild black-hole scattering amplitudes may obey a duality symmetry very similar to that of strings. One could then argue that black holes belong to a special class of string theories, as conjectured by 't Hooft [6].

As an example of a physical application, the very early Universe can be regarded as a black hole consisting of one very massive black hole surrounded by countless massless others; a typical equilibrium configuration, enclosed in a small volume of the order of the Planck volume. It is possible that such an inhomogeneous initial equilibrium energy distribution gave rise to structures and thus to galaxy formation. Such a suggestion was put forward by Carlitz [11] long ago in the context of hadronic matter where an inhomogeneous equilibrium distribution also occurs.

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[1] J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973).
[2] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975); Phys. Rev. D **13**, 191 (1976), and references therein.
[3] J. Preskill, P. Schwarz, A. Shapere, S. Trivedi, and F. Wilczek, Mod. Phys. Lett. A **6**, 2353 (1991).
[4] S. Coleman, J. Preskill, and F. Wilczek, Nucl. Phys. **B378**, 175 (1992), and references therein.
[5] C. F. E. Holzhey and F. Wilczek, Report No. IASSNS-HEP-91/71, 1991 (unpublished).
[6] G. 't Hooft, Nucl. Phys. **B355**, 138 (1990), and references therein.
[7] H. Verlinde and E. Verlinde, Report No. IASSNS-HEP-91/60, 1991 (unpublished).
[8] M. Bowick and L. Wijewardhana, Phys. Rev. Lett. **54**, 2485 (1985).
[9] R. Hagedorn, Nuovo Cimento Suppl. **3**, 147 (1970).
[10] Steven Frautschi, Phys. Rev. D **3**, 2821 (1971).
[11] Robert D. Carlitz, Phys. Rev. D **5**, 3231 (1972).
[12] W. Thirring, Z. Phys. **235**, 339 (1970).
[13] G. W. Gibbons and K. Maeda, Nucl. Phys. **B298**, 741 (1988).
[14] D. Garfinkle, G. T. Horowitz, and A. Strominger, Phys. Rev. D **43**, 3140 (1991).