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# Close black-hole binary systems

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When the amount of gravitational radiation is small, a binary system of two orbiting black holes evolves in a quasistationary manner. This system can be approximated by a linear combination of geometries each of which has standing waves at spatial infinity; however, the combination has purely outgoing radiation. A variational principle, for the geometries with standing waves, provides information about the binding energy, the stability of orbits, and the amplitude and frequency of gravitational radiation. For holes of equal mass starting at a large separation, approximately 3% of the initial mass is emitted as gravitational radiation before the evolution becomes so rapid that the quasistationary approximation fails.

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## I. INTRODUCTION

Two black holes orbiting each other lose energy and angular momentum to gravitational radiation. The full set of Einstein equations for this system is currently amenable to neither analytical nor numerical solution. However, under rather specific and different circumstances, the post-Newtonian and test particle perturbation approximations are two effective approaches to studying this system. The post-Newtonian approximation allows comparable mass black holes as long as the speeds and gravitational fields are small; the test particle approximation allows for strong gravitational fields and fast speeds as long as one hole is much smaller than the other. An important condition which makes tractable each of these rather different approaches is that the effects of radiation reaction are small so that at the level of the approximation the dynamics conserve energy and angular momentum, and the holes move along geodesics. For both approaches the binding energy of the system as a function of separation yields important information about the cumulative effects of the emission of gravitational radiation.

In this paper we make the fundamental approximation that the amplitude of gravitational radiation is small enough that the time scale for the secular effect of radiation reaction is much longer than the dynamical time scale, $\omega_1 \ll \omega_R$ in the notation introduced below. This is true throughout most of the lifetime of a binary black-hole system. Furthermore, this approximation is less restrictive than either the post-Newtonian or test particle approximation and encompasses both of these as special cases.

The general solution to the Einstein equations for orbiting black holes contains a mix of incoming and outgoing gravitational radiation with each hole reacting to the radiation as well as the gravitational attraction of the companion. Those solutions which are physically reasonable, and most interesting, have no incoming radiation—the holes are in quasielliptical orbits which decay while the outgoing radiation carries away energy and angular momentum. But with the nonlinearity and complexity of the Einstein equations, these physically interesting solutions are currently impossible to examine.

However, when the evolution from radiation reaction is slow, there is a useful approach which does not involve the direct solution to the Einstein equations with outgoing wave boundary conditions. Instead, periodic solutions which have standing gravitational waves are combined to form approximate solutions with outgoing wave trains and all of the features of interest. These periodic solutions are much more easily analyzed.

In Sec. II we review the 3+1 view of general relativity and Thorne's [1] description of the local wave zone.

In Sec. III we categorize different types of binary black-hole geometries: (i) resonant geometries, (ii) those driven at frequencies slightly off resonance, and (iii) linear combinations of these which form approximate solutions of the Einstein equations with outgoing wave trains. Each resonant solution is seen to be the first term in a power-series expansion of an approximate solution with outgoing waves. Thus a slowly evolving geometry, in close to the black holes and at any given instant, is approximately one of the resonant solutions. And outgoing waves in the local wave zone have an amplitude dependent upon the geometry at the retarded time. Also, the error in the approximate solution is analyzed.

In Sec. IV the resonant and other periodic geometries are studied in more detail along with a summary of a method developed by Abbott and Deser [2], which gives meaning to the differences in mass and angular momentum of geometries which are asymptotically similar but not asymptotically flat. This allows us to define the concepts of "effective mass," the mass not counting the contribution from the radiation, and "effective angular momentum" for geometries which are periodic with standing waves in the local wave zone. Also, a variational principle for the effective mass, of a geometry which is time independent in a rotating frame of reference (a special case of periodic), is developed and shown to be able to provide accurate estimates of the relationships between the effective mass, effective angular momentum, angular velocity, and amplitude and phase of gravitational radiation.

In Sec. V a first application of the variational principle is described. A rough cut at a trial geometry yields estimates of the effective mass and angular momentum of black holes in binary system. The binding energy of these solutions gives an estimate of the total amount of energy lost by the system as it evolved to its current state, which is an upper limit to the energy emitted in gravitational waves.

In the Appendix we illustrate these same techniques with a straightforward example of a mass moving in one dimension under the influence of a nonlinear restoring force and connected to a string down which transverse waves are radiated. Much of our intuition about the black-hole problem was developed from this toy problem. And the organization and discussion of Secs. III and IV follow the Appendix very closely.

## II. BACKGROUND AND MOTIVATION

### A. Initial-value formalism and dynamics of general relativity

York [3] presents a careful, pedagogical treatment of the initial-value formalism and the dynamics of general relativity. We limit ourselves to the vacuum case and use the notation at Arnowitt, Deser, and Misner [4]. A four-dimensional spacetime with a metric $g_{ab}$ may be foliated into constant $t$, spacelike hypersurfaces, with a unit normal vector $n^a$ and a metric $\gamma_{ab}$:

$$ds^2 = g_{ab}dx^a dx^b = -N^2dt^2 + \gamma_{ab}(dx^a + N^a dt)(dx^b + N^b dt) . \quad (1)$$

The quantity $N$ is the lapse function, and $N^a$ is the shift vector. The three-dimensional metric $\gamma_{ab}$ has a derivative operator $D_a$ and Ricci tensor $R_{ab}$.

The constraint equations on a given hypersurface are restrictions from the Einstein equations upon $\gamma_{ab}$ and $\pi^{ab}$, which is the momentum conjugate to $\gamma_{ab}$. These are the Hamiltonian constraint

$$R + \gamma^{-1}(\tfrac{1}{2}\pi^2 - \pi_{ab}\pi^{ab}) = 0 , \quad (2)$$

and the momentum constraint

$$D_b(\gamma^{-1/2}\pi^{ab}) = 0 . \quad (3)$$

The time translation vector $t^a\partial/\partial x^a \equiv (Nn^a + N^a)\partial/\partial x^a$ points in the direction of increasing $t$ with all spatial coordinates held fixed. The extrinsic curvature $K^{ab}$ is related to $\pi^{ab}$ by

$$\pi^{ab} = -\gamma^{1/2}(K^{ab} - \gamma^{ab}K) , \quad (4)$$

so that

$$\mathcal{L}_t\gamma_{ab} = 2N\gamma^{-1/2}(\pi_{ab} - \tfrac{1}{2}\gamma_{ab}\pi) + \mathcal{L}_N\gamma_{ab} . \quad (5)$$

The dynamical part of the Einstein equations gives the derivative of $\pi^{ab}$:

$$\mathcal{L}_t\pi^{ab} = -N\gamma^{1/2}(R^{ab} - \tfrac{1}{2}\gamma^{ab}R) + \tfrac{1}{2}N\gamma^{-1/2}\gamma^{ab}(\pi^{cd}\pi_{cd} - \tfrac{1}{2}\pi^2) - 2N\gamma^{-1/2}(\pi^{ac}\pi_c^b - \tfrac{1}{2}\pi\pi^{ab}) + \gamma^{-1/2}(D^aD^bN - \gamma^{ab}D^cD_cN) + \gamma^{1/2}D_c(\gamma^{-1/2}\pi^{ab}N^c) - \pi^{ac}D_cN^b - \pi^{bc}D_cN^a . \quad (6)$$

### B. Local wave zone

Some of the spacetimes with which we are concerned contain gravitational standing waves far from the holes and are not asymptotically flat. This adds complications to our analysis, which are reduced by the use of the concept of the "local wave zone" introduced by Thorne [1]. The local wave zone is the region in which both waves are weak ripples on a background spacetime and also the effect of the background curvature on the propagation of the waves is negligible.

From Thorne's definition the inner edge of the local wave zone, $r_I$, is that location where $r$ is small enough that one of the following effects becomes important: (i) the waves become a near-zone field, $r \lesssim \lambda$ ($\lambda$ is the approximate wavelength of the radiation); (ii) the spacetime curvature from the source is significant, $r \approx M$ ($M$ is the Schwarzschild radius of the source); or (iii) the background curvature significantly distorts the wave fronts of the radiation and backscatters the waves, $(r^3/M)^{1/2} \lesssim \lambda$. Hence

$$r_I = \alpha \times \max\{\lambda, M, (M\lambda^2)^{1/3}\} , \quad (7)$$

where $\alpha$ is some suitably large number.

For our purposes Thorne's definition of the outer edge of the local wave zone, $r_O$, is the location where $r$ is big enough that a significant phase shift across the local wave zone has been produced either (i) by the $M/r$ gravitational field of the source, $(\pi M/\lambda)\ln(r/r_I)$ is no longer $\ll \pi$, or (ii) by the energy in the waves themselves, $(\pi Lr/\lambda)\ln(r/r_I)$ is no longer $\ll \pi$, where $L$ is the small dimensionless luminosity of the radiation averaged over the time it has taken for the radiation to reach the local wave zone. Hence

$$r_O = r_I \times \min\{\exp(\lambda/\beta M), \exp(\lambda/\gamma L r_O)\} , \quad (8)$$

where $\beta$ and $\gamma$ are suitably large numbers.

Finally, we require that the large numbers $\alpha$, $\beta$, and $\gamma$ be chosen so that the thickness of the local wave zone is many times the wavelength of the radiation:

$$r_O - r_I \gg \lambda . \quad (9)$$

There is a local wave zone only as long as $L$ is small enough. If $Lr_O > M$, then it follows from Eq. (8) that

$$\frac{1}{\gamma} = \frac{Lr_O}{\lambda}\ln(r_O/r_I) \gtrsim \frac{Lr_O(r_O - r_I)}{r_I\lambda} . \quad (10)$$

Hence a limit on $L$ is that

$$L \ll \frac{r_I \lambda}{r_O(r_O - r_I)} \ll \frac{r_I}{r_O} . \tag{11}$$

The great utility of the local wave zone is that the gravitational waves can be treated as perturbations of the metric which propagate on a flat background. The transverse, traceless part of the metric, $h_{ab}^{\rm TT}$, represents the gravitational radiation. When $h_{ab}^{\rm TT}$ is composed of a set of discrete Fourier components, it can be decomposed [1] as

$$\begin{aligned} h_{ab}^{\rm TT}(r,\theta,\phi,t) &= \sum_{n=-\infty}^{\infty} h_{nab}^{\rm TT}(r,\theta,\phi) e^{i\omega_n t} \\ &= r^{-1} \sum_{nJlm} (A_{nJlm}^{\rm in} e^{i\omega_n r} + A_{nJlm}^{\rm out} e^{-i\omega_n r}) \\ &\quad \times e^{i\omega_n t} T_{ab}^{J2lm} , \end{aligned} \tag{12}$$

where

$$\omega_n = -\omega_{-n} \quad \text{and} \quad T_{ab}^{TSlm*} = (-1)^m T_{ab}^{JSl-m} . \tag{13}$$

The coordinates $r$, $\theta$, and $\phi$ are spherical coordinates on the flat background, the $T_{ab}^{JSlm}$ are the pure spin $S$ tensor harmonics [1], and the index $J$ runs over electric- and magnetic-type parity, $l=2$ to $\infty$ and $m=-l$ to $l$. Henceforth we let $q$ represent a generic index for the set of $J$, $l$, and $m$ with spin $S=2$. For real $h_{ab}^{\rm TT}$ the gravitational wave amplitudes satisfy

$$A^{\rm in}_{-nJl-m} = (-1)^m A_{nJlm}^{\rm in*} , \tag{14}$$

and similarly for $A_{nq}^{\rm out}$; also, for standing waves,

$$A_{nq}^{\rm in} = A_{nq} e^{i\vartheta_{nq}} \quad \text{and} \quad A_{nq}^{\rm out} = A_{nq} e^{-i\vartheta_{nq}} , \tag{15}$$

for some complex amplitude $A_{nq}$ and real phase $\vartheta_{nq}$.

## III. BINARY BLACK-HOLE GEOMETRIES

We are interested in solutions to the Einstein equations which represent two black holes whose orbits decay slowly under the influence of gravitational radiation reaction. Accordingly, we focus on spacetimes which may be foliated into a family of three-dimensional hypersurfaces which have the following properties. (i) A hypersurface $\mathcal{S}$ has three distinct spatial infinities; quantities associated with a region extending toward each spatial infinity are labeled with an index $i$, which is 0 for the region exterior to the two black holes and 1 and 2 for the regions inside each of the two holes. (ii) In each of the regions, there exists a local wave zone. (iii) The outgoing luminosity of the gravitational radiation is small in the local wave zone. (iv) Finally, the four-geometry is not necessarily asymptotically flat as it might contain standing gravitational waves at these spatial infinities.

### A. Periodic solutions

The physically reasonable geometries with outgoing radiation are described below in terms of the standing-wave geometries which are periodic in the sense that there exists a time coordinate $t$ with which the metric can be decomposed into a discrete set of Fourier components such as

$$g_{ab}(t) = \sum_{n=-\infty}^{\infty} g_{ab}^{n} e^{i\omega_n t} , \tag{16}$$

where

$$\omega_n = n\omega_1 . \tag{17}$$

These are the geometries which contain a steady stream of gravitational waves coming in from infinity which balances the waves going out from the source.

Given one periodic solution, there is a family of similar solutions with the same masses for the holes and the same spin and orbital angular momenta, but with the Fourier components being driven with radiation of different phases. Of all the periodic solutions, some are special and correspond to the resonances of the system; these are identified by the nature of the family of periodic solutions. We focus on the radiation in a particular component, i.e., a choice of $n$ and $q$. Let the frequency of a resonance be $\omega_{nqR}$ and the corresponding phase be $\vartheta_{nq}^{\rm res}$; also, let the frequency and phase of a similar solution be $\Omega_{nq}$ and $\vartheta_{nq}(\Omega_{nq})$. Then the signature of a resonance is a zero of $A_{nq}^{\rm in}$, considered an analytic function of $\Omega_{nq}$, at a complex frequency $\omega_{nqR} + i\omega_{nqI}$ close to the real axis. And the smaller the magnitude of $\omega_{nqI}/\omega_{nqR}$ is, then the sharper the resonance is. In other words,

$$\begin{aligned} A_{nq}^{\rm in}(\Omega_{nq}) &\equiv A_{nq}(\Omega_{nq}) e^{i\vartheta_{nq}} \\ &\approx \frac{i}{\omega_{nqI}} A_{nq}(\omega_{nqR}) e^{i\vartheta_{nq}^{\rm res}} (\Omega_{nq} - \omega_{nqR} - i\omega_{nqI}) , \end{aligned} \tag{18}$$

where $\vartheta_{nq}(\Omega_{nq})$ is given by

$$\vartheta_{nq}(\Omega_{nq}) - \vartheta_{nq}^{\rm res} = \tan^{-1}\left[\frac{\Omega_{nq} - \omega_{nqR}}{\omega_{nqI}}\right] . \tag{19}$$

It is useful to consider the independent variable to be the phase, $\vartheta_{n?}$, rather than the frequency; then,

$$A_{nq}^{\rm in}(\vartheta_{nq}) \approx i A_{nq}(\vartheta_{nq}^{\rm res}) e^{i\vartheta_{nq}^{\rm res}} [\tan(\vartheta_{nq} - \vartheta_{nq}^{\rm res}) - i] . \tag{20}$$

In the toy problem of the Appendix, we use perturbation analysis and find good approximations to this family of periodic solutions. But we have not been successful at applying similar methods to the binary black-hole problem, and we have no analysis that reveals the circumstances under which resonant solutions might exist. Nonetheless, on physical grounds we believe that for every state of a slowly evolving black-hole binary system, there is a nearby resonant solution and corresponding family of periodic solutions. And in Sec. IV B we discuss a variational technique for examining this family.

### B. Outgoing wave packets

Let $g_{ab}^{\rm res}$ be a resonant solution and $g_{ab}(\vartheta_{nq})$ be a particular member of its family with the further dependence upon spacetime coordinates understood. A linear combination of the $g_{ab}(\vartheta_{nq})$ creates an approximate solution to the Einstein equations and a generic wave packet:

$$g_{ab}^{W} \equiv \int_{-\pi/2}^{\pi/2} g_{ab}(\vartheta_{nq}) W(\vartheta_{nq}) d\vartheta_{nq} \ , \tag{21}$$

where $W(\vartheta_{nq})$ is a weighting function normalized so that

$$\int_{-\pi/2}^{\pi/2} W(\vartheta_{nq}) d\vartheta_{nq} = 1 \ . \tag{22}$$

To construct an outgoing wave packet, we choose

$$\theta \equiv \vartheta_{nq} - \vartheta_{nq}^{\mathrm{res}} = -\vartheta_{-nq} + \vartheta_{-nq}^{\mathrm{res}} \ ,$$

for $n > 0$, and a weighting function such that

$$g_{ab}^{W} \equiv \int_{-\infty}^{\infty} g_{ab}(\theta) \frac{d \tan\theta}{\pi(\tan^2\theta + 1)} \ . \tag{23}$$

In the local wave zone, the gravitational radiation in the $nq$ component is obtained via contour integration (just as in the Appendix) of the $nq$ component of the transverse, traceless part of Eq. (21); if $\omega_{nqI} > 0$, then, with the substitutions from Eqs. (12) and (20),

$$h_{nqab}^{\mathrm{TT}}(r,t) = \begin{cases} \frac{2}{r} A_{nq}(\omega_{nqR}) \exp[(i\omega_{nqR} + \omega_{nqI})(t+r) + i\vartheta_{nq}^{\mathrm{res}}] T_{ab}^{q} & \text{if } t < -r \ , \\ 0 \quad \text{if } -r < t < r \ , \\ \frac{2}{r} A_{nq}(\omega_{nqR}) \exp[(i\omega_{nqR} - \omega_{nqI})(t-r) + i\vartheta_{nq}^{\mathrm{res}}] T_{ab}^{q} & \text{if } t > r \ . \end{cases} \tag{24}$$

In other words, at early times a wave comes in from infinity; an incoming, trailing wave front is at $r = -t$; the wave front reflects off the black holes at $t = 0$ and thereafter is an outgoing, leading wave front at $r = t$, behind which is an outgoing wave. This wave packet is a physically reasonable, outgoing wave behind the wave front, $r < t$. If $\omega_{nqI} < 0$, then the evaluation and interpretation of the contour integral is slightly different; however, a region of spacetime which contains purely outgoing radiation is still obtained.

The weighting function of Eq. (23) may not fall off fast enough at either high or low frequencies to provide well-behaved solutions to the Einstein equations. For gravitational perturbations of a nonrotating neutron star in the Regge-Wheeler gauge, Price and Thorne [5] needed a general modification of $W(\theta)$ to obtain well-behaved solutions. They cut off the weighting function more rapidly at both ends and demonstrated that the effect on the superposition is only in the form of short-lived transients near the wave front. We are not interested in the details of the wave front and can use their technique to make $W(\theta)$ fall off away from the resonance as rapidly as needed to ensure well-behaved geometries.

To create a truly realistic approximation over a time sufficient to have substantial radiation-reaction effects, it is further necessary to consider a sequence of the wave-packet solutions, each of which evolves into the next. These solutions may be glued together smoothly so that the orbits of the holes are slowly but substantially modified and that at a given time $t$ the radiation in the wave zone at $r$ corresponds to that which was emitted at a retarded time $t - r$. However, we see below that the resonant solutions contain interesting information about frequencies and amplitude of radiation with errors proportional to $\omega_I/\omega_R$. Hence there seems little to be gained from the actual full construction of the dynamical solutions via wave packets.

### C. Analysis of error in the wave packet

Of course, the Einstein equations are not linear, hence, the linear combination of the metrics $g_{ab}(\theta)$, which creates $g_{ab}^{W}$, is not an exact solution, but rather a candidate for an approximate solution. When the effects of radiation reaction are small, the difference between two of these metrics is proportional to $\omega_I/\omega_R$ and satisfies the perturbed Einstein equations. This led us initially to believe that the linear combination in the wave packet would also satisfy the perturbed Einstein equations with a resultant error proportional to $(\omega_I/\omega_R)^2$. However, the error analysis in the Appendix shows that in general the contribution from the metrics far from resonance (large $\tan\theta$) makes the resultant error in the wave packet proportional to $\omega_I/\omega_R$. It follows that the damping of the radiation in the wave packet of Eq. (24) is of the same size as the expected error. Equation (24) does not, therefore, correctly show the effects of radiation reaction. We found this result so surprising that it substantially held up publication of this paper.

The resonant solution is easier to obtain than the other members of the family and alone provides a useful approximation to realistic, outgoing solutions. Reexpressing the wave packet as an integral over frequency rather than over phase, we obtain, from Eq. (23),

$$g_{ab}^{W} = \int_{-\infty}^{\infty} g_{ab}(\Omega_{nq}) \frac{|\omega_{nqI}| \pi^{-1} d\Omega_{nq}}{(\Omega_{nq} - \omega_{nqR})^2 + (\omega_{nqI})^2} \ . \tag{25}$$

And, except in the wave zone, a Taylor-series expansion of $g_{ab}(\Omega_{nq})$ yields

$$g_{ab}^{W} = \int_{-\infty}^{\infty} \left[ g_{ab}^{\mathrm{res}} + \sum_{n,q} (\Omega_{nq} - \omega_{nqR}) \frac{d g_{ab}(\Omega_{nq})}{d\Omega_{nq}} + \cdots \right] \times \frac{|\omega_{nqI}| \pi^{-1} d\Omega_{nq}}{(\Omega_{nq} - \omega_{nqR})^2 + (\omega_{nqI})^2} \ , \tag{26}$$

for small $(\Omega_{nq}-\omega_{nqR})$. Thus $g_{ab}^{\rm res}$ is the zeroth-order term in an expansion of $g_{ab}^{W}$, and the other terms in the Taylor series contribute higher powers of the small quantity $\omega_{nqI}$. And with the assumption of the linearization stability of the Einstein equations, $g_{ab}^{\rm res}$ is the zeroth-order term in an expansion of the physically realistic solution of the Einstein equations which has only outgoing radiation.

At any given time, the evolving black-hole system is well approximated by a particular resonant solution, as long as $\omega_{nqI} \ll \omega_{nqR}$. The system then progresses through a sequence of resonant solutions at a rate determined by the amplitude of the gravitational waves.

## IV. GEOMETRIES WITH STANDING WAVES IN THE LOCAL WAVE ZONE

Now that we recognize the utility of the standing-wave solutions, we focus on just those which are time independent as viewed from a rotating frame of reference (a special case of periodicity) so that $\partial/\partial t = Nn^a\partial/\partial x^a + N^a\partial/\partial x^a$ is a Killing vector. These approximate the slowly evolving black-hole systems when the orbits are slowly decaying circles.

### A. Conserved quantities in nonasymptotically flat geometries

The standing-wave geometries are not asymptotically flat, but they are asymptotically regular. In the local wave zones associated with each of the three spatial infinities, the gravitational waves can be treated in the short-wave approximation [6] with an effective stress tensor which mimics that of a null (traceless) fluid with equal amounts of flow in and out. The effective energy density $\rho_i$ falls off as $r^{-2}$. And our assumption that the luminosity is small is equivalent to $4\pi r^2\rho_i$ being small. But this density gives a divergent total mass and keeps the geometry from being asymptotically flat.

Such a geometry has no well-defined concept of mass or angular momentum. But Abbott and Deser [2] have a general method for comparing two geometries which are asymptotically similar but not necessarily asymptotically flat. To apply their methods to a standing-wave geometry, we introduce a new auxiliary manifold, for each local wave zone, which is similar asymptotically to the original and foliated into constant $t$ hypersurfaces with a mapping from $\mathcal{S}$ to $\mathcal{S}_i$. But the spatial hypersurface $\mathcal{S}_i$ is topologically $R^3$ and contains neither black holes nor stress energy. And, importantly, the differences between the geometries on $\mathcal{S}$ and $\mathcal{S}_i$,

$$M_{iab} \equiv \gamma_{ab} - \gamma_{iab}, \quad M_i \equiv \gamma^{ab} M_{iab} , \qquad P_i{}^{ab} \equiv \pi^{ab} - \pi_i{}^{ab} , \tag{27}$$

must satisfy the restrictions

$$M_{iab} = O(r^{-1}), \quad D_a M_{bc} = O(r^{-2}) , \qquad P_i{}^{ab} = O(r^{-1}) , \qquad N - N_i = O(r^{-1}), \quad N^a - N_i{}^a = O(r^{-1}) , \tag{28}$$

for large $r$ in the local wave zone, in a coordinate system which is close to Cartesian. Also, $N_i \approx n_i$, a constant, and $N_i{}^a \approx \Omega_i \partial/\partial\phi_i$ in the local wave zone, where $\Omega_i/n_i$ is the angular speed of the rotating frame of reference in which the geometries are time independent and $\phi_i$ is a local axial coordinate which runs from 0 to $2\pi$. These imply that the amplitude and phase of the gravitational radiation in the local wave zone of the auxiliary manifold match those of the original.

The Abbott-Deser method shows that a Killing vector common to both geometries has associated with it a quantity which depends upon the difference of the metrics and is conserved if the Einstein equations are satisfied and a particular flux integral vanishes sufficiently rapidly at infinity. For the special case that $Nn^a\partial/\partial x^a$ asymptotically approaches a Killing vector, the conserved quantity is the Killing energy:

$$16\pi n_i m_i \equiv \oint [ND_a(M_i{}^{ab} - \gamma^{ab}M_i) - (M_i{}^{ab} - \gamma^{ab}M_i)D_a N]\, d\Sigma_b , \tag{29}$$

with the evaluation done in the local wave zone. The restrictions of Eq. (28) are enough to guarantee that $m_i$ is invariant under an infinitesimal gauge transformation as can be shown with a process similar to that used by York [7] for asymptotically flat geometries. Again, $m_i$, the "effective" mass, should be thought of as the mass difference of the two geometries.

A similar technique, when $N^a\partial/\partial x^a \approx \Omega_i \partial/\partial\phi_i$ asymptotically approaches a Killing vector, leads to the angular momentum difference for the geometries:

$$16\pi\Omega_i J_i \equiv -\oint [2N_a P_i{}^{ab} + (2N^a\pi^{bc} - N^b\pi^{ac})M_{iac}]\gamma^{-1/2} d\Sigma_b . \tag{30}$$

That $J_i$ is also invariant under an infinitesimal coordinate transformation follows from the invariance of $m_i$ and the fact that, for the geometries which are invariant when viewed from a rotating frame of reference, $\partial/\partial t = (Nn^a + N^a)\partial/\partial x^a$ is an exact Killing vector with invariant, conserved quantity $16\pi(n_i m_i + \Omega_i J_i)$. In as formal a manner as possible, $m_i$ and $J_i$ are the mass and angular momentum of the radiating black-hole system with the divergent contribution from the standing waves subtracted off.

General questions of the existence and uniqueness of this auxiliary manifold with a metric are difficult to answer. However, at least in the limit of small amplitude, the standing gravitational waves satisfy the linearized Einstein equations throughout the auxiliary manifold. And the amplitude and phase in the local wave zone should, assuming the linearization stability of Einstein's equations, uniquely specify the auxiliary metric.

### B. Variational principle

The vacuum Einstein equations for an asymptotically flat geometry can be derived from Hamilton's principle. However, standing waves extending to spatial infinity not only complicate the analysis of the conserved quantities, requiring the method of Abbott and Deser, but also give a divergent contribution to the Hamiltonian and complicate the traditional method for dealing with boundary terms described by, say, Regge and Teitelboim [8]. A similar difficulty is encountered in the toy problem in the Appendix and in a variational principle for the phase shift in a one-dimensional potential scattering where the wave function is not quadratically square integrable [9]. In the toy problem and in wave mechanics, the difficulty is surmounted by the introduction of an auxiliary field which satisfies a simplified wave equation and by the use of a Hamiltonian for the combined system which is the difference of the Hamiltonia of the two fields. The variations of the original and auxiliary fields are independent except for a boundary condition which requires that the auxiliary field approaches the original field sufficiently rapidly at large $r$.

In general relativity for geometries with standing waves, the auxiliary geometry is just that which is introduced by the method of Abbott and Deser to define the effective mass and angular momentum. Thus we are led to define

$$\begin{aligned}H\equiv &-\int\left[N\left[R+\frac{1}{\gamma}\left(\tfrac{1}{2}\pi^2-\pi_{ab}\pi^{ab}\right)\right]+2N^aD_b(\gamma^{-1/2}\pi^b{}_a)\right]\gamma^{1/2}d^3x\\ &+\sum_i\int\left[N_i\left[R_i+\frac{1}{\gamma_i}\left(\tfrac{1}{2}\pi_i{}^2-\pi_{iab}\pi_i{}^{ab}\right)\right]+2N_i{}^aD_{ib}(\gamma_i{}^{-1/2}\pi_i{}^b{}_a)\right]\gamma_i{}^{1/2}d^3x_i\\ &+\oint_{R_0}[ND_a(M_0{}^{ab}-\gamma^{ab}M_0)-(M_0{}^{ab}-\gamma^{ab}M_0)D_aN]d\Sigma_b\ .\end{aligned}\tag{31}$$

The quantity $H$ is a functional of the lapse, shift, metric, and its canonical momentum on the hypersurface $\mathcal{S}$, as well as these same quantities on each auxiliary hypersurface $\mathcal{S}_i$ corresponding to each local wave zone of $\mathcal{S}$. These quantities and their variations are restricted only by the limiting conditions (28). In particular, $H$ is well defined whether or not the Einstein equations are satisfied. The volume integral over $\mathcal{S}$ extends out to $r=R_i$, which is in the local wave zone. The integrals over the $\mathcal{S}_i$ are bounded both at $R_i$ and at a sphere of small radius $r_i$ to avoid problems with regularity at the origin. The exact location of the inner boundary does not influence our analysis.

Note that when the constraint equations (2) and (3) and conditions (28) are satisfied exactly, then the volume integrals vanish and the value of $H$ is $16\pi n_0 m_0$.

The variation of $H$ under arbitrary independent variations of $N,N^a,\gamma_{ab},\pi^{ab}$ and $N_i{}^a,\gamma_{iab},\pi_i{}^{ab}$, restricted only by the limiting conditions (28) results in

$$\begin{aligned}\delta H=&-\int(\delta N\mathcal{N}\gamma^{1/2}+\delta N^a\mathcal{N}_a\gamma^{1/2}+\delta\gamma_{ab}\mathcal{P}^{ab}-\delta\pi^{ab}\mathcal{G}_{ab})d^3x\\ &+\sum_i\int(\delta N_i\mathcal{N}_i\gamma^{1/2}+\delta N_i{}^a\mathcal{N}_{ia}\gamma^{1/2}+\delta\gamma_{iab}\mathcal{P}_i{}^{ab}-\delta\pi_i{}^{ab}\mathcal{G}_{iab})d^3x_i\\ &-\sum_i\oint_{R_i}[ND_a(\delta\gamma^{ab}-\gamma^{ab}\delta\gamma)-(\delta\gamma^{ab}-\gamma^{ab}\delta\gamma)D_aN]d\Sigma_b\\ &+\sum_i\oint_{R_i}[N_iD_{ia}(\delta\gamma_i{}^{ab}-\gamma^{ab}\delta\gamma_i)-(\delta\gamma_i{}^{ab}-\gamma^{ab}\delta\gamma_i)D_{ia}N_i]d\Sigma_{ib}\\ &-\sum_i\oint_{R_i}[2N_a\delta\pi^{ab}+2N^a\pi^{bc}\delta\gamma_{ac}-N^b\pi^{ac}\delta\gamma_{ac}]\gamma^{-1/2}d\Sigma_b\\ &+\sum_i\oint_{R_i}[2N_{ia}\delta\pi_i{}^{ab}+2N_i{}^a\pi_i{}^{bc}\delta\gamma_{iac}-N_i{}^b\pi_i{}^{ac}\delta\gamma_{iac}]\gamma^{-1/2}d\Sigma_{ib}\\ &+\sum_i\oint_{r_i}[N_iD_{ia}(\delta\gamma_i{}^{ab}-\gamma^{ab}\delta\gamma_i)-(\delta\gamma_i{}^{ab}-\gamma^{ab}\delta\gamma_i)D_{ia}N_i]d\Sigma_{ib}\\ &+\sum_i\oint_{r_i}[2N_{ia}\delta\pi_i{}^{ab}+2N_i{}^a\pi_i{}^{bc}\delta\gamma_{iac}-N_i{}^b\pi_i{}^{ac}\delta\gamma_{iac}]\gamma^{-1/2}d\Sigma_{ib}\\ &+\delta\oint_{R_0}[ND_a(M_0{}^{ab}-\gamma^{ab}M_0)-(M_0{}^{ab}-\gamma^{ab}M_0)D_aN]d\Sigma_b\ .\end{aligned}\tag{32}$$

The capital script symbols appearing in the volume integrals in the first line are parts of the vacuum Einstein equations. In particular, $\mathcal{N}=0$ is the Hamiltonian constraint (2); $\mathcal{N}_a=0$ is the momentum constraint (3), and $\mathcal{L}_t\gamma_{ab}=\mathcal{G}_{ab}$ and $\mathcal{L}_t\pi^{ab}=\mathcal{P}^{ab}$ are the dynamical equations (5) and (6). The numerous surface integrals come from the integrations by parts and correspond directly to terms in the analysis of Regge and Teitelboim [8]. Note that the sum of the three surface terms at $R_i$ on $\mathcal{S}$ (such as the first, summed, surface integral) comes from its three spatial infinities, and the surface terms at $R_i$ on the three $\mathcal{S}_i$ (such as the second, summed, surface integral) come from the single spatial infinity of each $\mathcal{S}_i$.

We now describe the meaning of each surface integral in turn. The first two and last surface integrals together yield

$$-16\pi(n_1\delta m_1+n_2\delta m_2+n_0\delta m_0)+16\pi\delta(n_0 m_0) , \qquad (33)$$

from Eq. (29). Similarly, the third and fourth surface terms combine to yield $16\pi\sum_i \Omega_i \delta J_i$ from Eq. (30).

The surface terms at the inner boundary at $r_i$ on $\mathcal{S}_i$ are rather more difficult to handle. On $\mathcal{S}_i$ the geometry has no source of curvature but the gravitational waves. The dimensionless luminosity is small at the boundary at $R_i$ by assumption; hence, it will be small throughout the interior of $\mathcal{S}_i$ from the conservation of energy. It follows that the metric $\gamma_{iab}$ only deviates slightly from a flat metric $f_{ab}$ and that the deviation approximately satisfies the linearized Einstein equations out to the outer limit of the local wave zone—the general solutions to this are given in the Lorentz gauge by Thorne [1]. It is useful to equate the arbitrary variations in $\gamma_{iab}$ and $\pi_i{}^{ab}$ at $r_i$, which is in the near zone, to corresponding variations in the local wave zone which would be obtained if the linearized field equations for $\gamma_{iab}$ and $\pi_i{}^{ab}$ were solved exactly. Then the arbitrary variations are described in terms of wave amplitudes and phases and are easier to interpret. At $r=r_i$ on $\mathcal{S}_i$, the variation of the auxiliary metric is derived from

$$\begin{aligned} h_{iab} &\equiv \gamma_{iab}-f_{ab} \\ &= \sum_q \frac{2}{r} A_{iq}\cos[m\Omega_i r/n_i+\hat{\vartheta}_{iq}]T^q_{ab} , \end{aligned} \qquad (34)$$

with an arbitrary variation of $\gamma_{iab}$ and $\pi_i{}^{ab}$ being determined by the independent variations of $A_{iq}$ and $\hat{\vartheta}_{iq}$. The surface terms at $r_i$ can now be much simplified. In fact, the only part which contributes comes from $\Gamma^b{}_{ac}\delta\gamma_i{}^{ac}$ in the first term of the fifth surface integral. With the substitution from Eq. (34) we obtain, for the terms at $r_i$,

$$\begin{aligned} -\oint_{r_i}\tfrac{1}{2}n_i\delta h^{bc}\frac{\partial h_{bc}}{\partial x^a}d\Sigma^a &= -2\sum_q m\Omega_i A^*_{iq}\sin\vartheta_{iq}\delta(A_{iq}\cos\vartheta_{iq}) \\ &= -\sum_q[\Omega_i\delta(m|A_{iq}|^2\sin\vartheta_{iq}\cos\vartheta_{iq})-|A_{iq}|^2 m\Omega_i\delta\vartheta_{iq}] , \end{aligned} \qquad (35)$$

with $\vartheta_{iq}=m\Omega r_i/n_i+\hat{\vartheta}_q$.

With simplifications of the surface terms, Eq. (32) becomes

$$\begin{aligned} 16\pi n_0\delta m_0 = &-\int(\delta N\mathcal{N}\gamma^{1/2}+\delta N^a\mathcal{N}_a\gamma^{1/2}+\delta\gamma_{ab}\mathcal{P}^{ab}-\delta\pi^{ab}\mathcal{G}_{ab})d^3x \\ &+\sum_i\int(\delta N_i\mathcal{N}_i\gamma^{1/2}+\delta N_i{}^a\mathcal{N}_{ia}\gamma^{1/2}+\delta\gamma_{iab}\mathcal{P}_i{}^{ab}-\delta\pi_i{}^{ab}\mathcal{G}_{iab})d^3x_i \\ &-16\pi n_1\delta m_1-16\pi n_2\delta m_2+\sum_i 16\pi\Omega_i\delta j_i+\sum_{i,q}|A_{iq}|^2 m\Omega_i\delta\vartheta_{iq} , \end{aligned} \qquad (36)$$

where the "modified, effective" angular momentum is

$$j_i\equiv J_i-\frac{1}{16\pi}\sum_q m|A_{iq}|^2\sin\vartheta_{iq}\cos\vartheta_{iq} . \qquad (37)$$

Note that the difference between $J_i$ and $j_i$ is of the order of the angular momentum content in one wavelength of radiation, that the coefficient of $\delta\vartheta_{iq}/2\pi$, $2\pi|A_{iq}|^2 m\Omega_i$, is $16\pi n_i$ times the average energy in one wavelength of radiation, and that typically $n_1$ and $n_2$ are negative unless, perhaps, the holes are extremely close to each other.

From Eq. (36) we see that Eq. (31) yields a variational principle for $m_0$. Pick values of the parameters $m_1$, $m_2$, $j_0$, $j_1$, $j_2$, $\vartheta_{0q}$, $\vartheta_{1q}$, and $\vartheta_{2q}$ for all $q$. Consider $m_0$, as defined in Eq. (31), as a functional of $N, N^a, \gamma_{ab}, \pi^{ab}$ and $N_i, N_i{}^a, \gamma_{iab}, \pi_i{}^{ab}$. Now allow small, arbitrary, variations of these fields which leave the parameters unchanged and are restricted by Eq. (28). Then $m_0$ is an extremum if and only if both the constraint and dynamical Einstein equations are satisfied, with $t^a$ being a Killing vector, on $\mathcal{S}$ and $\mathcal{S}_i$ inside the region bounded at $R_i$.

An important use of this variational principle is to find an approximate value of $m_0$. Given a set of fields which are within $\delta$ of an exact solution to the Einstein equations with $t^a$ a Killing vector, Eq. (31) gives an approximation to $m_0$ which is accurate to order $\delta^2$. Also, by considering values of $m_0$, found from the variational principle, for slightly different values of $m_1$, $m_2$, $j_i$, and $\vartheta_{iq}$, we can even find the coefficients $n_i$, $\Omega_i$, and $|A_{iq}|^2 m\Omega_i$ to accuracy $O(\delta^2)$. For example,

$$n_1=-n_0\Delta m_0/\Delta m_1+O(\delta^2) , \qquad (38)$$

where $\Delta$ means let $m_1$ change and use the variational principle with all of the other parameters held constant.

Further consideration of the radiation term is necessary. A full solution to the Einstein equations can probably be found for each choice of the parameter $\vartheta_{iq}$ not too far from $\vartheta^{\rm res}_{iq}$ (cf. the analysis of the Appendix). With the other parameters held fixed, $\Omega_i$ is considered a function of the $\vartheta_{iq}$. Thus each choice of the $\vartheta_{iq}$ corresponds to a specific orbital frequency. Now, in general, the solution corresponds to the radiation driving the system at a frequency off resonance, $\Omega_i$ is not near the natural orbital frequency of the system, and the amplitude of the radiation is large. However, for some special choice of the $\vartheta_{iq}$, the frequency is the natural orbital frequency of the system, and the amplitude of the radiation is small. A minimum in $A_{iq}{}^2 m\Omega_i$, with the same functional dependence on $\vartheta_{iq}$ as seen in Eq. (20), identifies that solution which corresponds to the resonance of the slowly evolving system we wish to model.

## V. NUMERICAL RESULTS

The variational principle has now been successfully used to find estimates of the effective mass of a close binary system of black holes in this quasistationary approximation. For this first application, we consider only nonrotating black holes, so that $j_i=j_2=0$. In accordance with the variational principle, we fix the masses $m_1$ and $m_2$ and the modified angular momentum $j_0$, and then consider different trial geometries while looking for an extremum of $m_0$.

A choice of trial geometry is the specification of $N$, $N^a$, $\gamma_{ab}$, and $\pi^{ab}$ on $\mathcal{S}$ and $\mathcal{S}_i$. The trial geometry which we use is rather unsophisticated thus far, but it satisfies three important criteria. (i) In the limit that the holes are far apart, but move with any speed, the trial geometry reduces to two independent, boosted black holes. (ii) In the limit that the holes are slowly moving, but at any separation, the trial geometry reduces to the two-wormhole solution of the initial-value equations analyzed by Brill and Lindquist [10]. (iii) The auxiliary metrics are all flat, and at each infinity the trial geometry contains no gravitational waves and approaches as quickly as possible an asymptotically flat solution to the Einstein equations. Requirements (i) and (ii) are sufficient to force the results of the variational principle to match the Newtonian limit [11]. Requirement (iii) simplifies our numerical analysis at the expense of information about the gravitational-wave amplitude from the variational principle.

At a large distance from the system, the geometry has the quadrupole structure of two point masses $M_1$ and $M_2$ separated by a distance $S$. These masses are determined by the asymptotic behavior of the trial geometry outside the holes and differ from $m_1$ and $m_2$, which are determined in the local wave zones inside the holes. The quadrupole structure is used to define $S$, which is the only parameter varied, while we look for the extremum of $m_0$. This procedure is closely analogous to that of finding the circular orbits of Newtonian gravity or of a test particle orbiting a black hole. In these cases the Hamiltonian is considered a function of the angular momentum and the radius of the orbit and is called the effective potential. The circular orbit for a given angular momentum is located at an extremum of the Hamiltonian, and the extreme value itself is the energy of the orbit.

Figure 1 illustrates the application of the variational principle for $m_2=0.01m_1$. A comparison of these results with the known circular orbits of a test particle, $m_2 \ll m_1$, around a Schwarzschild black hole provides an important test of the trial geometry. In this test particle limit, $S$ is the radius of the orbit in isotropic coordinates. If our trial geometry exactly solved the constraint equations, then the ordinate would be the standard effective potential [12,13] for geodesics near a black hole.

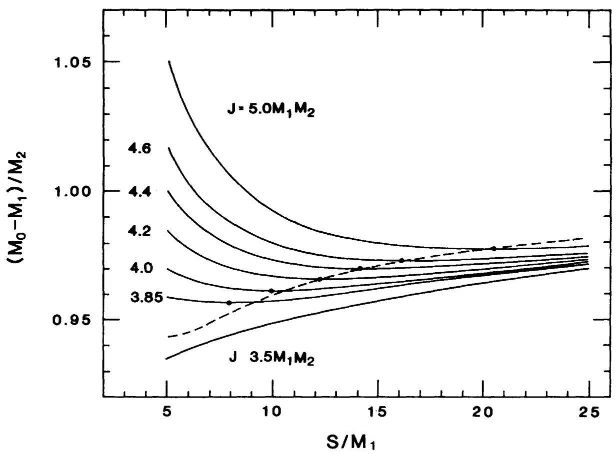


FIG. 1. Effective potential $(m_0-m_1)/m_2$ vs $S$ for fixed angular momentum and with $m_2=0.01m_1$. The minimum in each curve is marked by a dot and is at the radius of the circular orbit for the appropriate value of $j_0$. The known analytic position of the orbit of a test particle moving on a circular geodesic is represented by the dashed line. Curves are given, from bottom to top, for $j_0/m_1m_2=3.5$, 3.85, 4.0, 4.2, 4.4, 4.6, and 5.0.

For the innermost stable orbit which we find, the angular momentum is $j_0=3.85m_1m_2$, the binding energy is $0.0435m_2$, and $S=8.0m_1$. For this amount of $j_0$, our numbers should be compared with the analytic values of $0.0417m_2$ and $S=9.6m_1$. Our estimate of the mass-angular momentum relationship is quite good (5%), but the trial geometry, and in particular $S$, is not well known (20%), as we would expect for a variational calculation. The analytic values for the innermost stable orbit are $j_0=3.46m_1m_2$, $S=4.97m_1$, and the binding energy is $0.0572m_2$; hence, we miss the innermost stable orbit by 10% in $j_0$.

Another test of our analysis is the post-Newtonian limit of two equal-mass black holes. For $j_0=7m_1^2$, the Newtonian value of the binding energy of a circular orbit is $0.00496m_1$, the post-Newtonian [14] value is $0.005\,32m_1$, and we find $0.005\,36m_1$. We show 1% agreement with the post-Newtonian value when the Newtonian approximation is good to only 7%. Figure 2 illustrates $m_0(j)$ for equal-mass black holes. When the holes are very far apart, the total mass is $2m_1$. In the absence of other dissipative effects, gravitational radiation must carry away mass energy $2m_1-m_0$ for the system to evolve to a given separation. This amount can be read off Fig. 2.

From the trial geometry we can calculate the quadrupole moment and its time derivatives. The resultant dimensionless luminosity of gravitational radiation,

$$L=\frac{32}{5}\Omega^6 S^4\left[\frac{M_1M_2}{M_1+M_2}\right]^2, \tag{39}$$

is plotted in Fig. 3. This expression for the luminosity is formally similar to that for a Newtonian system [14]; however, we obtain $\Omega$ from the trial geometry rather than from Kepler's laws and the $M_i$ differ from the $m_i$ as

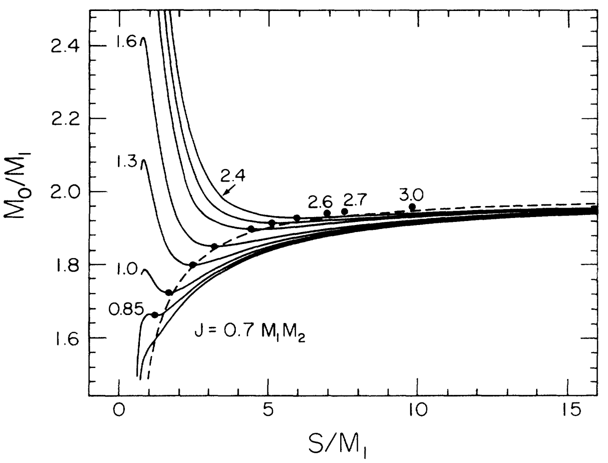


FIG. 2. Total mass as a function of separation for equal-mass black holes and fixed angular momentum. The dots are at the minimum for $j_0/m_1m_2=0.85$, 1.0, 1.3, 1.6, 2.0, 2.2, 2.4, 2.6, 2.7 and 3.0. Also included is a curve for 0.70, which has no minimum and no circular orbit. The dashed line represents a Newtonian analysis.

mentioned above. This is the true quadrupole luminosity of our trial geometry and is not found from the variational principle. We use it as an indicator of the validity of our assumption of the small effects of gravitational radiation over dynamical time scales. In particular, from Sec. III C, we expect the error from the creation of the outgoing wave packet to be proportional to $\omega_I/\omega_R$ for the quadrupole radiation or $L/(2\Omega\times E_B)$ where $E_B$ is the binding energy of the system. Equation (39) and the Newtonian values for the dynamics gives an approximate error of $\frac{8}{5}(2M/S)^{5/2}$.

An approximate picture of the stages of evolution of the equal-mass black-hole binary system emerges from an analysis of Fig. 2. The minimum of each curve pinpoints $m_0(j_0)$ and reveals $S$, with less accuracy, for each stable orbital configuration. Radiation reaction drives the binary along the sequence of minima at a rate which can be estimated from a comparison of Fig. 3 with the binding energy derived from Fig. 2. When the holes are far apart, Newtonian physics determines the dynamics, and the dissipative effects of gravitational radiation are well approximated by the quadrupole moment formalism. When $S\approx 23m_1$, $j_0\leq m_0^2$. The angular momentum is small enough that a rotating black hole could form with these amounts of mass and angular momentum and be a member of the Kerr sequence. The quadrupole normal-mode frequency for a Schwarzschild black hole [12,15] is $0.38m_0^{-1}$. The frequency of the radiation matches this when $S\approx 7.5m_1$ and $j_0\approx 2.7m_1^2$, $L=5\times 10^{-3}$, and $m_0\approx 1.94m_1$, and at this separation $\omega_I/\omega_R\approx 0.06$ from the previous paragraph. Our analysis appears firm up until this separation. We conclude, using the conservation of energy, that 3% of the initial mass of the holes is emitted in the form of gravitational radiation before this orbit is reached and with a frequency less than the quadrupole normal-mode frequency of the final black hole.

At $S\approx 3.5m_1$, $m_0\approx 1.86m_1$, and so the binding energy is 7% of the energy at infinite separation, and the dimensionless luminosity is about 0.1, at which point the approximation of small luminosity clearly fails.

The innermost stable orbit which we find has $S=1.2m_1$, $j_0=0.85m_1^2$, and the binding energy equals $0.35m_1$. However, for such a close orbit, the approximate luminosity is greater than unity, and our analysis is without foundation. For comparison, two holes at a moment of time symmetry [10] are encompassed by a single apparent horizon when $S\leq 0.79m_0\approx 1.5m_1$. For our trial geometry, the actual location of the apparent horizon has not been found.

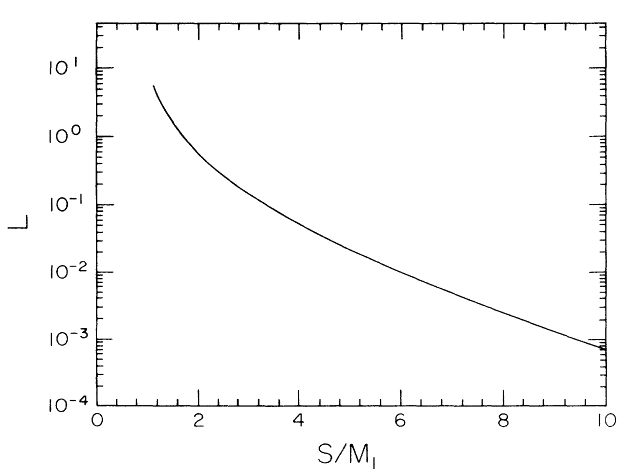


FIG. 3. Approximate luminosity $L$ of gravitational radiation as a function of the separation of equal-mass black holes as calculated in Eq. (39).

## VI. CONCLUSIONS

Almost none of this approach to the binary black-hole problem is original. However, the juxtaposition of the many diverse ideas is. There remain weak points in our analysis. These include the existence of the family of standing-wave solutions which corresponds to the system being driven slightly off resonance, the uniqueness of the geometry on the auxiliary manifold, the necessity of Eq. (28) for the invariance of the effective mass and angular momentum, and an incomplete understanding of the role that the Abbott-Deser conserved quantities play in the Hamiltonian treatment of the standing-wave geometries. The foremost difficulty is, perhaps, the dependence of our approach upon the existence of the local wave zone, which ultimately restricts the gravitational-wave luminosity as in Eq. (11)—we are currently working on this problem and expect to be able to remove this restriction.

In a paper in preparation, we will describe a similar approach for nonvacuum geometries and a variational principle for all geometries which are periodic, including quasielliptic orbits. In the future we plan to apply more sophisticated trial geometries to the variational principle, which will lead to firm predictions of radiation amplitudes.

## ACKNOWLEDGMENTS

This work has been in progress for some time. Over the past few years, we have benefited greatly from conversations, comments, and encouragement from many colleagues, but most importantly from Doug Eardley, David Garfinkle, Jim Ipser, Bernard Whiting, and Jim York. Part of this research has been supported by the National Science Foundation, Grants Nos. PHY-8906915 and PHY-9107007.

## APPENDIX

Much of the analysis of the main body of this paper is derived from techniques developed in nuclear scattering theory. For pedagogical reasons it is useful to illustrate all of these techniques with an example involving just classical physics. We focus on a single mass $M$, which moves in one dimension, but is attached to a light string which stretches off to infinity in a direction perpendicular to the motion of $M$. For simplicity, let the string have a small constant density $\rho$ and a tension $T$, with $\rho/T=1$ so that the motion of the mass is weakly damped by transverse waves traveling down the string at the speed of light.

The mass $M$ is constrained to move along the $x$ axis with a displacement $X(t)$ from the origin and under the influence of a restoring force $F(X)$ in the $x$ direction. The displacement of the string from the $z$ axis is a distance $\xi(z,t)$. The momenta conjugate to $X$ and $\xi$ are $P$ and $p$, respectively. The dynamical equations for this system are

$$\dot{X}=P/M, \quad \dot{P}=F(X)+T\frac{\partial\xi}{\partial z}\bigg|_{z=0} \tag{A1}$$

and

$$\dot{\xi}=p/\rho, \quad \dot{p}=T\frac{\partial^2\xi}{\partial z^2}\ , \tag{A2}$$

with the boundary condition that

$$\xi|_{z=0}=X\ . \tag{A3}$$

The general solution to these equations represents a mix of incoming and outgoing radiation on the string with the motion of $M$ responding to the reaction forces of the radiation as well as the force $F(X)$. Those solutions which are physically reasonable, and most interesting, have no incoming radiation—the mass undergoes damped oscillations about the origin, while the outgoing radiation carries away energy.

In this appendix we examine the details of how the radiation reaction slowly drives the evolution in a quasistationary fashion from one nearly equilibrium configuration to another. In Sec. 1 of this appendix, we consider a formal perturbation expansion of $X(t)$ in powers of the small parameter $\epsilon\equiv T/M$; this system is simple enough that we explicitly see how outgoing radiation damps the motion of the mass. In Sec. 2 we examine the periodic solutions to the dynamical equations which contain standing waves in the radiation. In Sec. 3 a specific linear combination of these periodic solutions is constructed and shown to approximate a solution with an outgoing wave front. And in Sec. 4 a variational principle for a generic periodic solution is derived. Thus, in the appendix, we reveal how a much more complicated nonlinear system coupled to weak radiation might be studied—the variational principle provides an analysis of the generic periodic solutions, and an appropriate linear combination of these approximates the physically interesting situation with an outgoing wave packet.

### 1. Damped oscillations

This dynamical system is simple enough that a rather straightforward analysis is successful. The dynamical equations (A1) and (A2) reduce to

$$\frac{d^2X}{dt^2}-f(X)=\epsilon\frac{\partial\xi}{\partial z}\bigg|_{z=0}\ , \tag{A4}$$

where

$$f(X)\equiv F(X)/M\ . \tag{A5}$$

The general outgoing wave on the string is

$$\xi=\xi(t-z)\ , \tag{A6}$$

so that

$$\frac{\partial\xi}{\partial z}\bigg|_{z=0}=-\frac{\partial\xi}{\partial t}\bigg|_{z=0}=-\frac{dX}{dt}\ , \tag{A7}$$

from the boundary condition (A3). Hence the dynamical equation is just

$$\frac{d^2X}{dt^2}-f(X)=-\epsilon\frac{dX}{dt}\ . \tag{A8}$$

Now we substitute a formal perturbation expansion for $X(t)$,

$$X(t)=X_0(t)+\epsilon X_1(t)+\epsilon^2X_2(t)+\cdots\ , \tag{A9}$$

into Eq. (A8) and separate off each power of $\epsilon$,

$$\epsilon^0:\quad \frac{d^2X_0}{dt^2}-f(X_0)=0\ ; \tag{A10}$$

$$\epsilon^1:\quad \frac{d^2X_1}{dt^2}-\frac{df}{dX}\bigg|_{X_0(t)}X_1=-\frac{dX_0}{dt}\ ; \tag{A11}$$

$$\epsilon^2:\quad \frac{d^2X_2}{dt^2}-\frac{df}{dX}\bigg|_{X_0(t)}X_2=-\frac{1}{2}\frac{d^2f}{dt^2}\bigg|_{X_0(t)}X_1-\frac{dX_1}{dt}\ . \tag{A12}$$

The solutions of Eq. (A10) are periodic, and we assume that these can be found (perhaps through numerical means) for the restoring force $f(X)$ at hand. Let a specific periodic solution of this equation be

$$X_0(t)=\sum_n x_n e^{i\omega_n t}\ , \tag{A13}$$

where $n$ runs from $-\infty$ to $+\infty$ and $\omega_n=n\omega_1$. For $X_0$ to be real, it is necessary that $x_{-n}=x_n^*$.

Equation (A11) for $X_1$ is a linear, inhomogeneous differential equation. Its general solution requires two solutions of its homogeneous form—the equation for a perturbation of the oscillator with no string attached. And two independent solutions are $\chi_1(t)$ and $\chi_2(t)$, where $\chi_1(t)$ is the difference between $X_0(t)$ and a similar solution to Eq. (A10) shifted slightly in phase but with the same energy:

$$\chi_1(t) \equiv \sum_n i\omega_n x_n e^{i\omega_n t} , \tag{A14}$$

and the difference between $X_0(t)$ and a solution with slightly different energy, but the same initial phase, gives

$$\chi_2(t) \equiv \frac{t}{\omega_1}\frac{d\omega_1}{dE}\sum_n i\omega_n x_n e^{i\omega_n t} + \sum_n \frac{dx_n}{dE} e^{i\omega_n t} , \tag{A15}$$

where $d\omega_1/dE$ and $dx_n/dE$ are assumed to be known from an analysis of the undamped problem.

The Wronskian of these two solutions is constant, $\chi_1\dot{\chi}_2 - \dot{\chi}_1\chi_2 - 1$. And the general solution to the inhomogeneous equation (A11) is

$$X_1(t) = -A(t)\chi_1 + B(t)\chi_2 + C\chi_1 + D\chi_2 , \tag{A16}$$

where

$$A(t) = \int_0^t \chi_2 \frac{dX_0}{dt} dt = \int_0^t \chi_2 \left[ \sum_m i\omega_m x_m e^{i\omega_m t} \right] dt , \tag{A17}$$

$$B(t) = \int_0^t \chi_1 \frac{dX_0}{dt} dt = \int_0^t \chi_1 \left[ \sum_m i\omega_m x_m e^{i\omega_m t} \right] dt , \tag{A18}$$

and where $C$ and $D$ are arbitrary constants.

These integrals are elementary, and one solution is

$$\begin{aligned} X_1(t) = {} & \sum_{n,m\neq n} \left[ \frac{1}{\omega_1}\frac{d\omega_1}{dE}\frac{\omega_m\omega_n x_m^* x_n}{(\omega_n-\omega_m)^2} - \frac{\omega_m x_m^*\, dx_n/dE}{\omega_n-\omega_m} \right] e^{i(\omega_n-\omega_m)t} \sum_j i\omega_j x_j e^{i\omega_j t} \\ & + \sum_{n,m\neq n} \frac{i\omega_m\omega_n x_m^* x_n}{\omega_n-\omega_m} e^{i(\omega_n-\omega_m)t} \sum_j \frac{dx_j}{dE} e^{i\omega_j t} \\ & + \sum_m \left[ -\frac{t^2}{2\omega_1}\frac{d\omega_1}{dE}\omega_m^2 x_m^* x_m - it\omega_m x_m^* \frac{dx_m}{dE} \right] \sum_j i\omega_j x_j e^{i\omega_j t} - t\sum_m \omega_m^2 x_m^* x_m \sum_j \frac{dx_j}{dE} e^{i\omega_j t} . \end{aligned} \tag{A19}$$

This expression can be recombined with $X_0(t)$ to form an approximation to $X(t)$ which is accurate through order $\epsilon$:

$$X(t) = \sum_n \left[ x_n + \epsilon a_n + t\frac{dx_n}{dE}\frac{dE}{dt} \right] \exp\left[ i(\omega_n + \Delta\omega_n)t + \frac{i}{2}t^2 \frac{d\omega_n}{dE}\frac{dE}{dt} \right] + O(\epsilon^2) . \tag{A20}$$

The constants $a_n$ are the constant coefficients of $e^{i\omega_n t}$ in the first two terms of Eq. (A19) and cause a conservative change in the dynamics which does not grow in time; also,

$$\frac{dE}{dt} = -\epsilon \sum_m \omega_m^2 x_m^* x_m \tag{A21}$$

is the rate at which energy is lost from the oscillator through the radiative damping and cause the secular effect of slowly evolving the system from one quasiequilibrium configuration to the next. A constant-frequency shift caused by the damping is $\Delta\omega_n$, which is $O(\epsilon)$ and found from the part of the third sum over $m$ in Eq. (A19), which is linear in $t$.

Expression (A20) nicely captures all of the features of the radiation reaction with a remainder of $O(\epsilon^2)$. However, this approximation is not uniformly valid and the remainder grows with $t$. Consideration of Eq. (A12) reveals the size of the error involved. The homogeneous form of Eq. (A12) is the same as that of (A11), and the source on the right-hand side is of the generic form

$$\sum_n (\alpha_n + t\beta_n + t^2\gamma_n) e^{i\omega_n t} , \tag{A22}$$

which yields

$$X_2(t) = \sum_n (\mu_n + t\nu_n + t^2\lambda_n + t^3\sigma_n) e^{i\omega_n t} . \tag{A23}$$

Thus, by stopping the original expansion at $\epsilon^1$, we ignore terms which grow as $\epsilon^2 t^3$. And it is easy to see that if the approximation is stopped before $\epsilon^p$, then we ignore terms which grow as $\epsilon^p t^{(p+1)}$. Thus the approximation of Eq. (A20) is accurate only as long as $t \ll \epsilon^{-2/3}$.

### 2. Standing-wave solutions

The simple identification of the outgoing waves leading to Eq. (A8) is not possible for a system slightly more complicated, say, when $T$ is a function of $z$. Then it is interesting to look at those periodic solutions which have standing waves in the radiation.

The formal perturbation expansion differs only slightly from that of Sec. 1. The displacement of the string is expanded as

$$\xi(t) = \xi_0(t) + \epsilon\xi_1(t) + \cdots , \tag{A24}$$

so that the coefficients of $\epsilon^j$ in the boundary condition (A3) give

$$\xi_j(t)=X_j(t)\ . \tag{A25}$$

Now $\xi(z,t)$ is generally a mix of incoming and outgoing radiation, and the phase of the radiation, $\vartheta_n$, can be specified arbitrarily at the zeroth order in $\epsilon$. Thus, for a periodic solution, $\xi$ is a function of $\vartheta_n$, and we consider the decomposition

$$\xi_0(\vartheta_n,z,t)=\sum_n A_n(e^{i\omega_n z+i\vartheta_n}+e^{-i\omega_n z-i\vartheta_n})e^{i\omega_n t}\ , \tag{A26}$$

where the boundary condition (A25) implies that

$$x_n=A_n(e^{i\vartheta_n}+e^{-i\vartheta_n})=2A_n\cos\vartheta_n\ . \tag{A27}$$

Also, $\xi_0(\vartheta_n,z,t)$ is a real function, and so

$$A_{-n}=A_n^*\ \text{ and }\ \vartheta_{-n}=-\vartheta_n^*\ . \tag{A28}$$

Furthermore, if the amplitudes of the incoming and outgoing radiation are the same, then $\vartheta_n$ is a real number.

Now the $\epsilon^1$ part of Eq. (A4) is

$$\begin{aligned}\epsilon^1:\quad \frac{d^2X_1}{dt^2}-\left.\frac{df}{dX}\right|_{X_0(t)}X_1&=\left.\frac{\partial\xi_0(z,t)}{\partial z}\right|_{z=0}\\ &=\sum_n -2\omega_n A_n\sin\vartheta_n e^{i\omega_n t}\\ &=\sum_n -\omega_n x_n\tan\vartheta_n e^{i\omega_n t}\ .\end{aligned} \tag{A29}$$

This is a linear, inhomogeneous differential equation for $X_1$ similar to that studied above—in fact, it becomes the one studied above if $\tan\vartheta_n=i$. One solution is

$$\begin{aligned}X_1(\vartheta_n,t)=&\sum_{n,m\neq n}\left[\frac{1}{\omega_1}\frac{d\omega_1}{dE}\frac{i\tan\vartheta_m^*\omega_m\omega_n x_m^* x_n}{(\omega_n-\omega_m)^2}-\frac{i\tan\vartheta_m^*\omega_m x_m^*\,dx_n/dE}{\omega_n-\omega_m}\right]e^{i(\omega_n-\omega_m)t}\sum_j i\omega_j x_j e^{i\omega_j t}\\ &-\sum_{n,m\neq n}\frac{\tan\vartheta_m^*\omega_m\omega_n x_m^* x_n}{\omega_n-\omega_m}e^{i(\omega_n-\omega_m)t}\sum_j\frac{dx_j}{dE}e^{i\omega_j t}\\ &+\sum_m\left[-\frac{i\tan\vartheta_m^* t^2}{2\omega_1}\frac{d\omega_1}{dE}\omega_m^2 x_m^* x_m+\tan\vartheta_m^* t\omega_m x_m^*\frac{dx_m}{dE}\right]\sum_j i\omega_j x_j e^{i\omega_j t}\\ &-t\sum_m i\tan\vartheta_m^*\omega_m^2 x_m^* x_m\sum_j\frac{dx_j}{dE}e^{i\omega_j t}\ .\end{aligned} \tag{A30}$$

The boundary condition (A25) gives

$$\begin{aligned}\xi_1(\vartheta_n,z,t)=&\sum_n\frac{\epsilon b_n}{2\cos\vartheta_n}(e^{i\omega_n z+i\vartheta_n}+e^{-i\omega_n z-i\vartheta_n})e^{i\omega_n t}\\ &+\frac{\epsilon}{2\omega_1}\frac{d\omega_1}{dE}\sum_m[\tfrac{1}{2}i\omega_m^2(\tan\vartheta_m-\tan\vartheta_m^*)x_m x_m^*]\\ &\qquad\times\sum_n\frac{i\omega_n x_n}{2\cos\vartheta_n}[(t+z)^2e^{i\omega_n z+i\vartheta_n}+(t-z)^2e^{-i\omega_n z-i\vartheta_n}]e^{i\omega_n t}\\ &+\epsilon\sum_m\left[\tfrac{1}{2}\omega_m\left[\tan\vartheta_m x_m\frac{dx_m^*}{dE}-\tan\vartheta_m^* x_m^*\frac{dx_m}{dE}\right]\right]\\ &\quad\times\sum_n\frac{i\omega_n x_n}{2\cos\vartheta_n}[(t+z)e^{i\omega_n z+i\vartheta_n}+(t-z)e^{-i\omega_n z-i\vartheta_n}]e^{i\omega_n t}\\ &+\epsilon\sum_m\left[\frac{i}{2}\omega_m^2(\tan\vartheta_m-\tan\vartheta_m^*)x_m x_m^*\right]\\ &\quad\times\sum_n\frac{1}{2\cos\vartheta_n}\frac{dx_n}{dE}[(t+z)e^{i\omega_n z+i\vartheta_n}+(t-z)e^{-i\omega_n z-i\vartheta_n}]e^{i\omega_n t}\ .\end{aligned} \tag{A31}$$

The constants $b_n$ come from the constant coefficients of $e^{i\omega_n t}$ in the first two terms on the right-hand side of Eq. (A30). The combination of $\xi_0(z,t)$ and $\xi_1(z,t)$ is split into ingoing and outgoing parts as

$$\xi(\vartheta_n,z,t)=\sum_n \tfrac{1}{2}e^{i\omega_n(t+z)}i\,(\tan\vartheta_n-i)(x_n+\epsilon b_n+\cdots)$$
$$\times\exp\left\{-\frac{\epsilon(t+z)^2\omega_n}{4\omega_1}\frac{d\omega_1}{dE}\sum_m\omega_m^2(\tan\vartheta_m-\tan\vartheta_m^*)x_m x_m^*\right.$$
$$+\tfrac{1}{2}i\epsilon(t+z)\omega_n\sum_m\omega_m\left[\tan\vartheta_m x_m\frac{dx_m^*}{dE}-\tan\vartheta_m^* x_m^*\frac{dx_m}{dE}\right]$$
$$\left.+\tfrac{1}{2}\epsilon(t+z)\frac{1}{x_n}\frac{dx_n}{dE}\sum_m i\omega_m^2(\tan\vartheta_m-\tan\vartheta_m^*)x_m x_m^*\right\}$$
$$-\sum_n\tfrac{1}{2}e^{i\omega_n(t+z)}i(\tan\vartheta_n+i)(x_n+\epsilon b_n+\cdots)\exp[(t+z)\to(t-z)]\ . \tag{A32}$$

This procedure could be continued at higher orders of $\epsilon$. At order $p$, $X_p$ is found from an inhomogeneous equation, which is the $\epsilon^p$ part of Eq. (A4), with a source involving $X_{p-1}$, $\xi_{p-1}$, and the phase $\vartheta_n$, where the boundary condition at $z=0$ determines $\xi_{p-1}$ in terms of $X_{p-1}$ and the phases. It is clear that, for any choice of the $\vartheta_n$, there is a solution to this system of equations which is similar to the resonant solution ($\vartheta_n=0$) initially ($t=0$). For small $\epsilon$ and $t$, the above equations show how the exact, periodic, nonresonant solution slowly evolves away from the resonant solution. Thus the phase $\vartheta_n$ is the natural quantity which distinguishes a resonant solution to the dynamical equations $\vartheta_n=0$ from a nonresonant one.

For the special case that $\vartheta_n=-\vartheta_{-n}$, with $\vartheta_n$ being real, the first and third sums in the exponent of solution (A32) drop out. What remains is

$$\xi(\vartheta_n,z,t)=\sum_n\tfrac{1}{2}i(\tan\vartheta_n-i)(x_n+\epsilon b_n+\cdots)\exp\left\{i\omega_n(t+z)\left[1+\tfrac{1}{2}\epsilon\sum_m\omega_m\tan\vartheta_m\frac{dx_m x_m^*}{dE}\right]\right\}$$
$$-\sum_n\tfrac{1}{2}i(\tan\vartheta_n+i)(x_n+\epsilon b_n+\cdots)\exp\left\{i\omega_n(t-z)\left[1+\tfrac{1}{2}\epsilon\sum_m\omega_m\tan\vartheta_m\frac{dx_m x_m^*}{dE}\right]\right\}, \tag{A33}$$

which is strictly periodic and represents the standing-wave solution, which is accurate through first order in $\epsilon$.

For a system more complicated than this one, we might not be able to find the approximation for $\xi(\vartheta_n,z,t)$ similar to Eq. (A32) and the resonances might occur at $\vartheta_n\neq 0$. In that case the variational techniques discussed below, or perhaps a direct numerical approach, might still provide $\xi(\vartheta_n,z,t)$ and allow for the identification of a resonance. We would expect that the amplitude of the incoming and outgoing parts of $\xi(\vartheta_n,z,t)$ would still be proportional to $\tan(\vartheta_n-\vartheta_{nR})-i$ and $\tan(\vartheta_n-\vartheta_{nR})+i$, respectively, where $\vartheta_{nR}$ is the phase at resonance.

The important role of the phases is discussed again below from the point of view of the variational principle.

### 3. Outgoing wave packets

For a linear problem, such as the Schrödinger equation, it is common to construct an outgoing wave packet by a linear superposition of the standing-wave solutions. The system we consider here is nonlinear. However, the wave equation and the boundary condition (A3) are linear and the perturbation analysis in powers of $\epsilon$ reduces our consideration of the motion of the mass to a linear analysis. Hence we can take linear combinations of the periodic solutions as long as we include a careful analysis of the error.

We construct a generic wave packet for $\xi$ as

$$\xi_W(z,t)=\int_{-\pi/2}^{\pi/2}\xi(\vartheta_n,z,t)W(\vartheta_n)d\vartheta_n\ , \tag{A34}$$

where $W(\vartheta_n)$ is a weighting function normalized so that

$$\int_{-\pi/2}^{\pi/2}W(\vartheta_n)d\vartheta_n=1\ . \tag{A35}$$

To construct an outgoing wave packet, we let $\theta=\vartheta_n=-\vartheta_{-n}$, for $n>0$, change the variable of integration to $\tan\theta$ and choose $W(\theta)$ so that

$$\xi_W(z,t)=\int_{-\infty}^{\infty}\xi(\theta,z,t)\frac{d\tan\theta}{\pi(\tan^2\theta+1)}\ . \tag{A36}$$

This integration can be done as a contour integral, and each term in the sum over $n$ in Eq. (A33) should be considered separately. We assume that the system is stable in the sense that

$$\sum_m|\omega_m|\frac{dx_m x_m^*}{dE}<0\ ,$$

so that the oscillations get smaller when the system loses energy. Then, for the region of spacetime behind the outgoing wave front, where $t-z>0$, the contour is closed in the upper half of the complex $\tan\theta$ plane for $n>0$ and in the lower half plane for $n<0$. For the outgoing terms in the sum over $n$, the contour integral picks up a regular singularity at $\tan\vartheta_n=i$, and the wave packet takes the form

$$\xi_W(z,t)=\sum_n (x_n+\epsilon b_n+\cdots)\times\exp\left\{i\omega_n(t-z)\times\left[1+\frac{i}{2}\epsilon\sum_m \omega_m\frac{dx_m x_m^*}{dE}\right]\right\}. \tag{A37}$$

Note that this differs at order $\epsilon$ from the damped oscillation (A20) found by direct methods.

It is important to analyze carefully the error in this approximate solution to the dynamical equations. The wave packet is created by taking a linear combination of approximations to the dynamical equations, each of which has an error of order $\epsilon^2$. The only nonlinearity in this system is in the restoring force $F(X)$. In fact, the straight substitution of Eq. (A37) for $\xi_W(z,t)$ into Eq. (A4), with use of the boundary condition (A3) at $z=0$, shows that the amount by which $\xi_W(z,t)$ fails to satisfy the dynamical equation is

$$f\left[\int_{-\infty}^{\infty}X(\theta,t)\frac{d\tan\theta}{\pi(\tan^2\theta+1)}\right]-\int_{-\infty}^{\infty}f(X(\theta,t))\frac{d\tan\theta}{\pi(\tan^2\theta+1)}. \tag{A38}$$

For nonlinear $F(X)$, this error is of order $\epsilon$. The quantities $X_W(t)$ and $X(\theta,t)$ differ only by order $\epsilon$, and it might seem that a Taylor-series expansion of $f(X)$ leads to an error of order $\epsilon^2$; but the approximation is not uniformly valid in $\tan\theta$ and the integral samples $X(\theta,t)$ for large $\tan\theta$, which causes the error to be of order $\epsilon$. This explains why the damping of the oscillation as implied by Eq. (A37) is in error and differs from that of Eq. (A20). Thus the wave packet of Eq. (A36) correctly approximates the oscillations with outgoing radiation, but with the error of order $\epsilon$, the effects of radiation reaction are not seen.

Under some circumstances the weighting function of Eq. (A36) may not fall off fast enough at large phase shifts to provide well-behaved solutions to the dynamical equations. Price and Thorne [5] describe a general modification of $W(\theta)$, cutting off the weighting function more rapidly, and demonstrate that the effect on the superposition is only in the form of short-lived transients near the wave front. We are not interested in the details of the wave front and can use their technique to make $W(\theta)$ fall off away from the resonance as rapidly as needed.

#### 4. Variational principle

So far, in this appendix, we have revealed the utility of the periodic solutions for an understanding of the outgoing wave solutions to the dynamical equations. Now, for these periodic solutions, we demonstrate a variational principle for the most interesting of the physical quantities which describe the dynamics of the system: the amplitude, phase, and frequency of the radiation and the total energy of the system. With a good approximation to a periodic solution of the dynamical equations, the variational principle yields a better approximation to these interesting physical quantities.

Morse and Feshback [9] give a variational principle for the phase shift of the scattering amplitude for the Schrödinger equation with a one-dimensional potential. When the application of such a variational principle was in vogue, the phase-shift information was used to determine the energy and width of resonances in a manner similar to ours. We use a modest variation of their variational principle.

We are interested in solutions which have standing waves at infinity and assume that all of the dynamical variables are periodic in time with a fundamental frequency $\Omega$.

The Hamiltonian of the system plays a fundamental role in the variational principle. But the integral of the Hamiltonian density of a string diverges at infinity. To avoid this divergence we must consider the difference between the Hamiltonian and another integral which diverges in the same way. Accordingly, we introduce a new dynamical variable $\xi_1(z,t)$ and its conjugate momentum $p_1(z,t)$, whose dynamics are governed by

$$\dot{\xi}_1=p_1/\rho \quad\text{and}\quad \dot{p}_1=T\frac{\partial^2\xi_1}{\partial z^2} \tag{A39}$$

and satisfy the boundary conditions

$$\xi_1(z,t)\to\xi(z,t) \quad\text{and}\quad p_1(z,t)\to p(z,t) \quad \text{as } z\to\infty \tag{A40}$$

sufficiently rapidly that

$$H\equiv\frac{P^2}{2M}+U(X)+\int_0^{\infty}\left[\frac{p^2}{2\rho}+\tfrac{1}{2}T\left[\frac{\partial\xi}{\partial z}\right]^2\right]dz-\int_0^{\infty}\left[\frac{p_1^2}{2\rho}+\tfrac{1}{2}T\left[\frac{\partial\xi_1}{\partial z}\right]^2\right]dz \tag{A41}$$

is finite. And $H$ is a functional of $X$, $P$, $\xi$, $p$, $\xi_1$, and $p_1$, where these dynamical variables are periodic functions of the dimensionless phase $\Omega t$. Then the average of $H$ over a full period is

$$\langle H\rangle=\frac{\Omega}{2\pi}\int_0^{2\pi/\Omega}H(t)dt\ . \tag{A42}$$

Now the variation of $\langle H\rangle$ under arbitrary, independent variations of the dynamical variables, which respect the periodicity and boundary conditions, is

$$\delta\langle H\rangle = \frac{\Omega}{2\pi}\int_0^{2\pi/\Omega}\delta H(t)dt$$
$$= \frac{\Omega}{2\pi}\int_0^{2\pi/\Omega}\Bigg\{\delta P\left[\frac{P}{M}-\dot X\right]+\delta P\dot X-\delta X\dot P+\delta X\left[\frac{\partial U}{\partial X}+\dot P-T\frac{\partial\xi}{\partial z}\bigg|_{z=0}\right]$$
$$+\int_0^\infty\left[\delta p(p/\rho-\dot\xi)+\delta p\dot\xi-\delta\xi\dot p-\delta\xi\left[T\frac{\partial^2\xi}{\partial z^2}-\dot p\right]\right]dz$$
$$+\int_0^\infty\left[-\delta p_1(p_1/\rho-\dot\xi_1)-\delta p_1\dot\xi_1+\delta\xi_1\dot p_1+\delta\xi_1\left[T\frac{\partial^2\xi_1}{\partial z^2}-\dot p_1\right]\right]dz+T\delta\xi_1\frac{\partial\xi_1}{\partial z}\bigg|_{z=0}\Bigg\}dt\ , \quad \text{(A43)}$$

where we have used the boundary conditions (A3) and (A40).

A number of simplifications of this expression can be made. Note that

$$\int_0^{2\pi/\Omega}(\delta P\dot X-\delta X\dot P)dt=\int_0^{2\pi/\Omega}\left[\delta P\dot X-\frac{d}{dt}(P\delta X)+\delta\dot X P\right]dt$$
$$=\int_0^{2\pi/\Omega}(\delta P\dot X+\delta\dot X P)dt$$
$$=\delta\int_0^{2\pi/\Omega}P\dot X dt\ . \quad \text{(A44)}$$

The terms involving the pairs $p,\xi$ and $p_1,\xi_1$ can be treated similarly. The boundary term at $z=0$ which involves $\xi_1$ simplifies if we represent $\xi_1$ at the boundary by

$$\xi_1(z,t)=\sum_n(A_n^{\text{in}}e^{in\Omega z}+A_n^{\text{out}}e^{-in\Omega z})e^{in\Omega t}\ , \quad \text{(A45)}$$

with

$$A_n^{\text{in}}=A_n e^{i\vartheta_n}\ \text{and}\ A_n^{\text{out}}=A_n e^{-i\vartheta_n}\ . \quad \text{(A46)}$$

Then the quantities $A_n$ and $\vartheta_n$ describe $\xi_1$ at $z=0$ and are to be varied independently. The boundary term becomes

$$\frac{\Omega}{2\pi}\int_0^{2\pi/\Omega}T\,\delta\xi_1\frac{\partial\xi_1}{\partial z}\bigg|_{z=0}dt$$
$$=T\sum_n 2n\Omega[\delta(A_n^2\sin\vartheta_n\cos\vartheta_n)-A_n^2\delta\vartheta_n]\ . \quad \text{(A47)}$$

Finally, we define $S$ by

$$S\equiv\int_0^{2\pi/\Omega}\left[P\dot X+\int_0^\infty(p\dot\xi-p_1\dot\xi_1)dz\right]dt$$
$$+T\sum_n 4\pi n A_n^2\sin\vartheta_n\cos\vartheta_n\ . \quad \text{(A48)}$$

With all these simplifications, Eq. (A43) becomes

$$\delta\langle H\rangle=\frac{\Omega}{2\pi}\int_0^{2\pi/\Omega}[\![\text{dyn. eqn.}]\!]dt$$
$$+\frac{\Omega}{2\pi}\delta S-\frac{\Omega}{2\pi}T\sum_n 4\pi n A_n^2\delta\vartheta_n\ , \quad \text{(A49)}$$

where "[[dyn. eqn.]]" represents terms which vanish if and only if the dynamical equations are satisfied.

Equation (A49) justifies the following variational principle: *Choose values of $S$ and each of the $\vartheta_n$ and consider the class of functions $X(\Omega t)$, $P(\Omega t)$, $\xi(z,\Omega t)$, $p(z,\Omega t)$, $\xi_1(z,\Omega t)$, and $p_1(z,\Omega T)$ which satisfy the boundary conditions (A3) and (A40), which are periodic in $\Omega t$ with period $2\pi$ and which have the chosen values of $\vartheta_n$ and $S$ from Eqs. (A46) and (A48). The quantity $\langle H\rangle$ in Eq. (A42) is stationary with respect to small variations of the functions if and only if the functions satisfy the dynamical equations (A1), (A2), and (A39).* Thus an approximate solution to the dynamical equations which is accurate to $O(\delta)$ will provide an estimate of $\langle H\rangle$ which is accurate to $O(\delta^2)$ for a true solution to the dynamical equations.

When the dynamical variables satisfy the dynamical equations, the value of $\langle H\rangle$ is what we call the "effective" energy of the system in the main body of this paper, which is the total energy without the contribution from the radiation. It can be shown easily, from the time derivative of Eq. (A41), averaged over a few cycles, and a process similar to that leading to Eq. (A49), that $\langle H\rangle$ decreases appropriately when outgoing radiation carries energy away from the rest of the system. In fact, the energy in one wavelength of the radiation is

$$E_\lambda\equiv\int_z^{z+\lambda}\left[\frac{p^2}{2\rho}+\tfrac{1}{2}T\left[\frac{\partial\xi}{\partial z}\right]^2\right]dz\ , \quad \text{(A50)}$$

so that, from

$$\xi_n(z,t)=A_n[e^{i(n\Omega z+\vartheta_n)}+e^{-i(n\Omega z-\vartheta_n)}]e^{in\Omega t}\ , \quad \text{(A15)}$$

it follows that

$$E_\lambda=T4\pi n\Omega A_n^2\ , \quad \text{(A52)}$$

which is just the coefficient of $\delta\vartheta_n/2\pi$ in Eq. (A49).

Although $\Omega$ and $t$ appear separately above, in fact the previous dozen equations could have been written entirely in terms of the dimensionless phase $\Omega t$ of the oscillation. Thus it is not necessary to know $\Omega$ before applying the variational principle. In fact, all of the variables in the variational principle should be thought of as functions of the phase, and then $\Omega$ can be found by application of the variational principle.

This variational principle can be used to find not only the effective energy of the system, as described above, but also the frequency, amplitude, and phase of radiation. Given trial functions accurate to $O(\delta)$, the variational principle already yields the relationship between $\langle H\rangle$, $S$, and the $\vartheta_n$ accurate to $O(\delta^2)$ because the latter quantities had to be specified before $\langle H\rangle$ could be found. To find $\Omega$, then, repeat the process with a slightly different value of $S$, $S+\Delta S$, to find $\langle H \rangle + \Delta \langle H \rangle$. From Eq. (A49) it follows that

$$\Omega = 2\pi \Delta \langle H \rangle / \Delta S \tag{A53}$$

is an accurate estimate of $\Omega$ with an error of $O(\delta^2)$. In a similar manner, a small change in one of the $\vartheta_n$ yields an estimate of $T 4\pi n \Omega A_n^2$ and, therefore, of the amplitude of radiation with an error of $O(\delta^2)$. Thus the variational principle can be used to map out the amplitude and phase of the radiation as functions of frequency for the periodic solutions. And this information can be used to construct accurate approximations to the outgoing wave packets as described above.

In this example the two integrals in $H$ in Eq. (A41) cancel each other exactly when the dynamical equations and boundary conditions are satisfied. For a more complicated problem, $\rho$ might be a function of $z$, but with a nice limit for large $z$. Then we might choose to let the auxiliary string have just a constant $\rho_1$, which would be the limiting value of $\rho$. The formal use of the variational principle would be unchanged, but the integrals would not cancel exactly, and the interpretation of the value of $\langle H \rangle$ as the effective energy of the system would be clouded.

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