Method of effective charges and Brodsky-Lepage-Mackenzie criterion

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(Received 9 March 1992)

A renormalization-scheme-invariant method to improve the QCD perturbative series when the Brodsky-Lepage-Mackenzie (BLM) criterion indicates large perturbative corrections is presented. The method, which is a variant of the effective-charge scheme, is specific to QCD and relies on a new renormalization-group equation describing the response of a physical quantity to a change in the BLM next-to-leading-order coefficient r_1^*. Two alternative methods are also discussed: one of them consists in an improvement of the perturbative series for the effective charge β function, which also yields, as a by-product, an improvement of the series for the perturbative infrared fixed point of QCD at $f \approx 33/2$ flavors; the other represents an improvement of the perturbative series specific to the BLM choice of renormalization point.

PACS number(s): 12.38.Cy, 11.15.Bt, 12.38.Bx

I. INTRODUCTION

The ambiguities arising from the arbitrariness [1] in the choice of renormalization scheme (RS) are particularly annoying in perturbative QCD, where the coupling constant is usually not very small. They manifest themselves first at next-to-leading order as the "choice of renormalization point." Some time ago, Brodsky, Lepage, and Mackenzie (BLM) proposed [2] an interesting way to fix this scale ambiguity, by making use of the specific flavor dependence induced by fermion loops [3]. Their procedure can be summarized as follows. Consider the expansion of a physical quantity $R(Q)$ (depending upon an external scale Q) in powers of the coupling $a = a(\mu)$ of a given RS (μ is the renormalization point):

$$R = a(1 + r_1 a + r_2 a^2 + r_3 a^3 + r_4 a^4 + \cdots) . \tag{1.1}$$

Restricting the discussion to schemes where r_1 is linear in f (the number of fermion flavors), one can write

$$r_1 = r_1^* - \beta_0 \left[\ln \frac{Q^2}{\mu^2} - d_1^* \right] , \tag{1.2}$$

where β_0, the one-loop β-function coefficient, is linear in f: $\beta_0 = \beta_{00} + \beta_{01} f$, and r_1^* and d_1^* are f independent. BLM then propose to fix μ by the condition

$$L \equiv \ln \frac{Q^2}{\mu^2} - d_1^* = 0 \tag{1.3}$$

so that $r_1 = r_1^*$. They further suggest that the criterion for convergence of perturbation theory in the considered RS be that r_1^* is "small" [it has been pointed out [4] that r_1^* is RS dependent, so that a particular definition of the coupling must also be given; in practice, BLM suggest to use the modified minimal subtraction ($\overline{\text{MS}}$) scheme [5]]. I have already commented on this criterion in Ref. [6], and argued that large $|r_1^*|$ does not necessarily prevent the applicability of "renormalization-group- (RG-)improved" perturbation theory, as defined in Ref. [7], for the usually considered number of flavors ($f \leq 5$). The argument of Ref. [6] is based on the relation

$$r_1^* = -\frac{\bar{\beta}_2^* - \beta_2^*}{\beta_1^*} , \tag{1.4}$$

where the starred quantities refer to values of β_1, β_2, and $\bar{\beta}_2$ for the number of flavors $f = f^* = -\beta_{00}/\beta_{01}$ where β_0 vanishes ($f^* = 33/2$ in QCD); β_1 and β_2 are the two- and three-loop β-function coefficients,

$$\mu^2 \frac{da}{d\mu^2} = \beta(a) = -\beta_0 a^2 - \beta_1 a^3 - \beta_2 a^4 + \cdots , \tag{1.5}$$

and $\bar{\beta}_2$ is the three-loop effective charge [7] $\bar{\beta}$-function coefficient defined by

$$Q^2 \frac{dR}{dQ^2} = \bar{\beta}(R) = -\beta_0 R^2 - \beta_1 R^3 - \bar{\beta}_2 R^4 + \cdots . \tag{1.6}$$

Equation (1.6) follows from the RG invariance of R, and gives an RS-invariant differential equation which controls the Q^2 dependence of R. Consequently, $\bar{\beta}_2$ is an RS-invariant [8] (but process-dependent) quantity, which depends only upon the definition of R. I stressed in Ref. [7] that "RG-improved" perturbative QCD could be applied whenever the expansion in Eq. (1.6) is "well behaved" (in the usual sense of perturbation theory), in particular if the three-loop contribution $\bar{\beta}_2 R^4$ is small compared to the first two terms. Clearly, this condition depends on the magnitude of $\bar{\beta}_2$ only for the considered number of flavors f. There is no *a priori* reason that a large value of $|\bar{\beta}_2^*|$, following from Eq. (1.4) for large $|r_1^*|$, implies a correspondingly large value of $|\bar{\beta}_2|$ for $f \neq f^*$. This general argument however also indicates that some problem might arise for large $|r_1^*|$ with the perturbative expansion of $\bar{\beta}(R)$ at $f = f^*$, as well as away from $f = f^*$ if one assumes that $\bar{\beta}_2$ is weakly f dependent, i.e., $\bar{\beta}_2(f) \approx \bar{\beta}_2^*$. The aim of the present paper (which is a revised and expanded version of Ref. [9]) is to supply a method to deal precisely with these cases. Actually, I found three different ways to improve perturbation theory for large $|\bar{\beta}_2^*|$. The first method (Sec. VI A) represents an alternative implementation of the effective charge idea of Ref. [7]. It is based on a new RG equation (Secs. II, IV, and V), which makes some specific use of the general Stückelberg-Peterman RG (dealing with invariance under more general transformations than change of renormalization point). In the course of this investigation, some interesting properties related to the f dependence of the perturbative coefficients $\{r_i\}$ were discovered (Sec. III), including a new set of "universal" (i.e., process-independent) RS-invariant quantities [related to the critical exponent for the well-known infrared (IR) fixed point at $f \approx f^*$]. The second method (Sec. VI B) leads to an improvement of the perturbative series specific to the BLM choice of μ [Eq. (1.3)]. The third alternative yields an improved perturbation theory for the effective $\bar{\beta}$ function itself, as well as for the perturbative series of the IR fixed point, and is described in Sec. VII. This last method is in fact simpler, and to some extent more satisfactory, than the first two above, so the interested reader may eventually want to skip directly to this section. Concluding remarks are presented in Sec. VIII. Some results on the perturbative expansion of the IR fixed point at $f \to f^*$ are given in the Appendix.

II. A NEW RENORMALIZATION-GROUP EQUATION

The first improvement method is based on a new RG equation for R, motivated by the following observation [6]. Equation (1.4) shows explicitly that, although r_1^* is RS dependent [4] (through the β_2^*/β_1^* term), it does contain RS-invariant information (through the $\bar{\beta}_2^*/\beta_1^*$ contribution). In particular, the difference $r_1^* - \tilde{r}_1^*$ between the next-to-leading-order BLM coefficients associated with two physical quantities R and $\tilde{R}$ is RS invariant [2]. It is then natural to consider the change dR in R induced by a small change $dr_1^* = -d(\bar{\beta}_2^*/\beta_1^*)$ in r_1^* (with all RS-dependent parameters kept fixed); note that dr_1^* is RS invariant. To this change is associated a corresponding "effective" β function $\bar{B}(R)$:

$$\frac{dR}{d\bar{c}} = \bar{B}(R) = \bar{b}_0 R^2 + \bar{b}_1 R^3 + \bar{b}_2 R^4 + \bar{b}_3 R^5 + \bar{b}_4 R^6 + \cdots , \tag{2.1}$$

where I defined $\bar{c} \equiv -\bar{\beta}_2^*/\beta_1^*$. $\bar{B}(R)$ is expected to be an RS-invariant (but process-dependent) object. A similar quantity can be defined for the RS-dependent coupling a:

$$\frac{da}{dc} = B(a) = b_0 a^2 + b_1 a^3 + b_2 a^4 + b_3 a^5 + b_4 a^6 + \cdots , \tag{2.2}$$

where $c \equiv -\beta_2^*/\beta_1^*$, and describes the change in a induced by a change in the (RS-dependent) parameter c. The precise definition of the derivative in Eqs. (2.1) and (2.2) will be given in Sec. IV. I only note here that at next to leading order $dR/d\bar{c}$ is defined as a derivative at fixed L [Eq. (1.3)], and is thus *a priori* distinct from the $\bar{\beta}(R)$ function of Eq. (1.6). Similarly, da/dc does not correspond to a change of renormalization point, and is rather connected to the more general Stückelberg-Peterman RG transformations.

To see why varying R at fixed L is a RS-invariant procedure, it is instructive to start from the solutions of the RG equations (1.5) and (1.6):

$$\beta_0 \ln\frac{Q^2}{\bar{\Lambda}^2} = \int^R dx \frac{\beta_0}{\bar{\beta}(x)} , \tag{2.3}$$

$$\beta_0 \ln\frac{\mu^2}{\Lambda^2} = \int^a dx \frac{\beta_0}{\beta(x)} , \tag{2.4}$$

where $\bar{\Lambda}$ (Λ) are scales, which play the role of boundary conditions, and depend only on the definition of R (a) [for instance, if $a = a_{\overline{\rm MS}}$, then $\Lambda = \Lambda_{\overline{\rm MS}}$ (where $\overline{\rm MS}$ denotes modified minimal subtraction)]. Taking the difference of Eqs. (2.3) and (2.4), one gets the RG-improved [7] relation for $R = R(Q/\mu, a)$:

$$-r_1 = \beta_0 \left[\ln\frac{Q^2}{\bar{\Lambda}^2} - \ln\frac{\mu^2}{\Lambda^2} \right] = \int^R dx \frac{\beta_0}{\bar{\beta}(x)} - \int^a dx \frac{\beta_0}{\beta(x)} , \tag{2.5}$$

where the first equality follows from the requirement that the power series Eq. (1.1) be a solution of Eq. (2.5). It is now useful to introduce for the RS-invariant quantity $\beta_0 \ln Q^2/\bar{\Lambda}^2$ (and its RS-dependent counterpart $\beta_0 \ln\mu^2/\Lambda^2$) the analogue of the BLM decomposition of Eq. (1.2), namely, one can write

$$\beta_0 \ln\frac{Q^2}{\bar{\Lambda}^2} = \beta_0 \ln\frac{Q^2}{\bar{\Lambda}^{*2}} - \bar{c} \tag{2.6}$$

and

$$\beta_0 \ln\frac{\mu^2}{\Lambda^2} = \beta_0 \ln\frac{\mu^2}{\Lambda^{*2}} - c \tag{2.7}$$

where $\bar{\Lambda}^*$ (Λ^*) are f-independent scales, which depend only on the definition R (a). One deduces

$$\begin{aligned} r_1 &= -\beta_0 \left[\ln\frac{Q^2}{\bar{\Lambda}^2} - \ln\frac{\mu^2}{\Lambda^2} \right] = -\beta_0 \left[\ln\frac{Q^2}{\bar{\Lambda}^{*2}} - \ln\frac{\mu^2}{\Lambda^{*2}} \right] + \bar{c} - c \\ &= -\beta_0 \left[\ln\frac{Q^2}{\mu^2} - d_1^* \right] + r_1^* \end{aligned} \tag{2.8}$$

so that

$$\ln\frac{\bar{\Lambda}^{*2}}{\Lambda^{*2}} = d_1^* \tag{2.9}$$

and

$$L = \ln\frac{Q^2}{\bar{\Lambda}^{*2}} - \ln\frac{\mu^2}{\Lambda^{*2}} . \tag{2.10}$$

Equation (2.10) shows that $\ln Q^2/\bar{\Lambda}^{*2}$ is the "RS-invariant invariant part" of L, just as $\bar{c}$ is the RS-invariant part of r_1^*. Therefore, a variation of R with $\bar{c}$ at fixed L (and fixed RS parameters) is simply a variation of R at fixed $\ln Q^2/\bar{\Lambda}^{*2}$, clearly a RS-invariant prescription. To complete the definition of $dR/d\bar{c}$ and compute the $\{\bar{b}_i\}$ in Eq. (2.1), it is necessary to exhibit the explicit $\bar{c}$ dependence of the $\{r_i\}$ which requires looking at their f dependence around $f=f^*$.

III. FLAVOR DEPENDENCE OF PERTURBATIVE COEFFICIENTS AND A SET OF RS INVARIANTS

To emphasize the structure near $f=f^*$, it is convenient to introduce the parameter $\alpha\equiv-\beta_0/\beta_1^*$ (which vanishes at $f=f^*$), and consider the r_i's as well as the β_i's and the $\bar{\beta}_i$'s as polynomials in α. Since $\alpha=-(\beta_{01}/\beta_1^*)(f-f^*)$ is linear in f, the α dependence just reflects the f dependence of the various coefficients. Equation (1.2) reads

$$r_1=r_1^*+(\beta_1^*L)\alpha\ . \tag{3.1}$$

I further write

$$\beta_1=\beta_1^*+\beta_{11}^*\alpha \tag{3.2}$$

and restrict the discussion to schemes where the r_i's are polynomials in f and, hence, in α. Then

$$\begin{aligned}\beta_2&=\beta_2^*+\beta_{21}^*\alpha+\beta_{22}^*\alpha^2+\beta_{23}^*\alpha^3\ ,\\ \bar{\beta}_2&=\bar{\beta}_2^*+\bar{\beta}_{21}^*\alpha+\bar{\beta}_{22}^*\alpha^2+\bar{\beta}_{23}^*\alpha^3\ .\end{aligned} \tag{3.3}$$

(In general, $O(f^3)$ and, hence, $O(\alpha^3)$ terms may be present [10,3] in β_2 and $\bar{\beta}_2$, although they are absent in $\beta_2^{\overline{\mathrm{MS}}}$.) Using the formula [7]

$$\frac{\partial R}{\partial a}=\frac{\bar{\beta}(R)}{\beta(a)} \tag{3.4}$$

one gets relations [7,11,12] between the r_i's, the β_i's, and the $\bar{\beta}_i$'s. The first of these is

$$r_2=r_1^2+\frac{\beta_1}{\beta_0}r_1+\frac{\bar{\beta}_2-\beta_2}{\beta_0} \tag{3.5}$$

from which I obtain

$$r_2=r_2^*+r_{21}^*\alpha+r_{22}^*\alpha^2 \tag{3.6}$$

with

$$\begin{aligned}r_2^*&=-(\beta_1^*L)+r_1^{*2}+\lim_{\alpha\to0}\frac{\beta_1r_1^*+\bar{\beta}_2-\beta_2}{\beta_0}\\ &=-(\beta_1^*L)+r_1^{*2}-\frac{\beta_{11}^*}{\beta_1^*}r_1^*-\frac{\bar{\beta}_{21}^*-\beta_{21}^*}{\beta_1^*}\ ,\end{aligned} \tag{3.7}$$

$$r_{21}^*=\left[2r_1^*-\frac{\beta_{11}^*}{\beta_1^*}\right](\beta_1^*L)-\frac{\bar{\beta}_{22}^*-\beta_{22}^*}{\beta_1^*}\ , \tag{3.8}$$

$$r_{22}^*=(\beta_1^*L)^2-\frac{\bar{\beta}_{23}^*-\beta_{23}^*}{\beta_1^*}\ . \tag{3.9}$$

In Eq. (3.7), I used the regularity of r_2 as $\alpha\to0$ (i.e., $f\to f^*$). The condition that the singular $O(1/\beta_0)$ terms on the right-hand side of Eq. (3.5) cancel at $f=f^*$ then gives Eq. (1.4). Similarly, putting

$$\begin{aligned}\beta_3&=\beta_3^*+\beta_{31}^*\alpha+\beta_{32}^*\alpha^2+\beta_{33}^*\alpha^3+\beta_{34}^*\alpha^4\ ,\\ \bar{\beta}_3&=\bar{\beta}_3^*+\bar{\beta}_{31}^*\alpha+\bar{\beta}_{32}^*\alpha^2+\bar{\beta}_{33}^*\alpha^3+\bar{\beta}_{34}^*\alpha^4\end{aligned} \tag{3.10}$$

and using the relation [7]

$$r_3=r_1^3+\frac{5}{2}\frac{\beta_1}{\beta_0}r_1^2+\frac{3\bar{\beta}_2-2\beta_2}{\beta_0}r_1+\frac{1}{2}\frac{\bar{\beta}_3-\beta_3}{\beta_0} \tag{3.11}$$

I get

$$r_3=r_3^*+r_{31}^*\alpha+r_{32}^*\alpha^2+r_{33}^*\alpha^3 \tag{3.12}$$

with

$$\begin{aligned}r_3^*=&\frac{2\bar{\beta}_2^*-3\beta_2^*}{\beta_1^*}(\beta_1^*L)+r_1^{*3}-\frac{5}{2}\frac{\beta_{11}^*}{\beta_1^*}r_1^{*2}-\frac{3\bar{\beta}_{21}^*-2\beta_{21}^*}{\beta_1^*}r_1^*\\ &-\frac{1}{2}\frac{\bar{\beta}_{31}^*-\beta_{31}^*}{\beta_1^*}\ ,\end{aligned} \tag{3.13}$$

$$\begin{aligned}r_{31}^*=&-\frac{5}{2}(\beta_1^*L)^2+\left[3r_1^{*2}-\frac{5\beta_{11}^*}{\beta_1^*}r_1^*-\frac{3\bar{\beta}_{21}^*-2\beta_{21}^*}{\beta_1^*}\right](\beta_1^*L)\\ &-\frac{3\bar{\beta}_{22}^*-2\beta_{22}^*}{\beta_1^*}r_1^*-\frac{1}{2}\frac{\bar{\beta}_{32}^*-\beta_{32}^*}{\beta_1^*}\ ,\end{aligned} \tag{3.14}$$

$$\begin{aligned}r_{32}^*=&\left[3r_1^*-\frac{5}{2}\frac{\beta_{11}^*}{\beta_1^*}\right](\beta_1^*L)^2-\frac{3\bar{\beta}_{22}^*-2\beta_{22}^*}{\beta_1^*}(\beta_1^*L)\\ &-\frac{3\bar{\beta}_{23}^*-2\beta_{23}^*}{\beta_1^*}r_1^*-\frac{1}{2}\frac{\bar{\beta}_{33}^*-\beta_{33}^*}{\beta_1^*}\ ,\end{aligned} \tag{3.15}$$

$$r_{33}^*=(\beta_1^*L)^3-\frac{3\bar{\beta}_{23}^*-2\beta_{23}^*}{\beta_1^*}(\beta_1^*L)-\frac{1}{2}\frac{\bar{\beta}_{34}^*-\beta_{34}^*}{\beta_1^*}\ , \tag{3.16}$$

where I used the condition that the singular $O(1/\beta_0)$ terms in Eq. (3.11) cancel at $f=f^*$, which gives [with the help of Eq. (1.4)] the relation

$$\beta_1^*\beta_3^*-\beta_2^{*2}=\beta_1^*\bar{\beta}_3^*-\bar{\beta}_2^{*2}\ . \tag{3.17}$$

Equation (3.17) shows that $\beta_1\beta_3-\beta_2^2$ is a "universal" RS invariant [13] at $f=f^*$.

As the last example, I consider r_4, which presents some new features. One has [7,14]

$$\begin{aligned}r_4=&r_1^4+\frac{13}{3}\frac{\beta_1}{\beta_0}r_1^3+\left[\frac{6\bar{\beta}_2-3\beta_2}{\beta_0}+\frac{3}{2}\frac{\beta_1^2}{\beta_0^2}\right]r_1^2\\ &+\left[\frac{2\bar{\beta}_3-\beta_3}{\beta_0}+\frac{3}{2}\frac{\beta_1}{\beta_0}\frac{\bar{\beta}_2-\beta_2}{\beta_0}\right]r_1\\ &+\frac{1}{3}\frac{\bar{\beta}_2-\beta_2}{\beta_0}\frac{5\bar{\beta}_2-4\beta_2}{\beta_0}-\frac{1}{6}\frac{\beta_1}{\beta_0}\frac{\bar{\beta}_3-\beta_3}{\beta_0}+\frac{1}{3}\frac{\bar{\beta}_4-\beta_4}{\beta_0}\ .\end{aligned} \tag{3.18}$$

The new feature here is the presence of the $O(1/\beta_0^2)$ double poles. For simplicity, I only give the expression for $r_4^*=r_4(\alpha=0)$. Some algebra yields

$$r_4^* = \frac{3}{2}(\beta_1^* L)^2 + \left[-13 r_1^{*2} - \frac{12\bar\beta_2^* - 6\beta_2^* - 6\beta_{11}^*}{\beta_1^*} r_1^* - \frac{2\bar\beta_3^* - \beta_3^* - 3(\bar\beta_{21}^* - \beta_{21}^*)}{\beta_1^*} + 3\frac{\beta_{11}^*}{\beta_1^*}\frac{\bar\beta_2^* - \beta_2^*}{\beta_1^*} \right](\beta_1^* L) + r_1^{*4} - \frac{13}{3}\frac{\beta_{11}^*}{\beta_1^*} r_1^{*3}$$
$$+ \left[\frac{-6\bar\beta_{21}^* + 3\beta_{21}^*}{\beta_1^*} + \frac{3}{2}\frac{\beta_{11}^{*2}}{\beta_1^{*2}} \right] r_1^{*2} + \left[-\frac{2\bar\beta_{31}^* + \beta_{31}^*}{\beta_1^*} + 3\frac{\beta_{11}^*}{\beta_1^*}\frac{\bar\beta_{21}^* - \beta_{21}^*}{\beta_1^*} \right] r_1^* + \frac{1}{3}\frac{\bar\beta_{21}^* - \beta_{21}^*}{\beta_1^*}\frac{5\bar\beta_{21}^* - 4\beta_{21}^*}{\beta_1^*}$$
$$- \frac{1}{6}\frac{\beta_{11}^*}{\beta_1^*}\frac{\bar\beta_{31}^* - \beta_{31}^*}{\beta_1^*} + \frac{1}{3}\left[\frac{\bar\beta_{22}^*}{\beta_1^*}\frac{\bar\beta_2^*}{\beta_1^*} - \frac{\bar\beta_{41}^* + \frac{1}{2}\bar\beta_{32}^*}{\beta_1^*} \right] - \frac{1}{3}\left[\frac{\beta_{22}^*}{\beta_1^*}\frac{\beta_2^*}{\beta_1^*} - \frac{\beta_{41}^* + \frac{1}{2}\beta_{32}^*}{\beta_1^*} \right], \tag{3.19}$$

where I used the conditions that the $O(1/\beta_0^2)$ double poles and the $O(1/\beta_0)$ single poles in Eq. (3.18) cancel at $f = f^*$. The cancellation of the double poles follows automatically from Eqs. (1.4) and (3.17). That of the single poles gives a new constraint, namely, the quantity

$$I_3 = -\frac{1}{3}\left[\frac{\beta_2^*}{\beta_1^*}\right]^3 + \frac{1}{6}\frac{\beta_{11}^*}{\beta_1^*}\left[\frac{\beta_2^*}{\beta_1^*}\right]^2 - \left[I_2 + \frac{1}{3}\frac{\beta_{21}^*}{\beta_1^*}\right]\left[\frac{\beta_2^*}{\beta_1^*}\right]$$
$$+ \frac{1}{6}\frac{\beta_{31}^*}{\beta_1^*} + \frac{1}{6}\frac{\beta_{11}^*}{\beta_1^*} I_2 + \frac{1}{3}\frac{\beta_4^*}{\beta_1^*} \tag{3.20}$$

should be a "universal" RS invariant, such that $I_3(\bar\beta_i^*, \bar\beta_{ik}^*) = I_3(\beta_i^*, \beta_{ik}^*)$ where

$$I_2 \equiv -\left[\frac{\beta_2^*}{\beta_1^*}\right]^2 + \frac{\beta_3^*}{\beta_1^*} \tag{3.21}$$

is also a universal RS invariant [see Eq. (3.17)]. The requirement that the r_i's be regular at $f = f^*$ thus generates a set of invariants $\{I_n\}$ $(n \geq 2)$, where I_n is a polynomical of degree n in β_2^*/β_1^* (whose coefficients depend on the $\alpha = 0$ derivatives β_{ik}).

The existence of these invariants may be understood from another viewpoint: one can check (see the Appendix) that they are related to the expansion coefficients $\{\omega_i\}$ in powers of α of the critical exponent

$$\omega = \left.\frac{\partial \beta}{\partial a}\right|_{a = a^*} = +\omega_0 \alpha^2 + \omega_1 \alpha^3 + \omega_2 \alpha^4 + \omega_3 \alpha^5 + \cdots \tag{3.22}$$

(well known to be a universal RS-invariant quantity), where a^* is the nontrivial IR fixed point [15] of $\beta(a)$,

$$\beta(a^*) = 0, \tag{3.23}$$

and is also perturbatively calculable as a power series in α as $\alpha \to 0$ (i.e., $f \to f^*$), thanks to the vanishing of β_0 at $f = f^*$: $a^* = \alpha + O(\alpha^2)$. One finds

$$\omega_0 = -\beta_1^*, \quad \frac{\omega_1}{\omega_0} = -\frac{\beta_{11}^*}{\beta_1^*},$$
$$\frac{\omega_2}{\omega_0} = I_2 + \left[\frac{\beta_{11}^*}{\beta_1^*}\right]^2, \tag{3.24}$$
$$\frac{\omega_3}{\omega_0} = 6 I_3 - \left[\frac{\beta_{11}^*}{\beta_1^*}\right]^3 - 5\frac{\beta_{11}^*}{\beta_1^*} I_2 .$$

I further note an interesting property of the r_i's at $L = 0$: they are polynomial in r_1^* with a similar structure as the one following from Eq. (3.4) [see Eqs. (3.5), (3.11), and (3.18)], but with the β-function coefficients β_1/β_0, β_2/β_0, β_3/β_0, and β_4/β_0 replaced, respectively, by

$$\tilde b_1 = -\frac{\beta_{11}^*}{\beta_1^*}, \quad \tilde b_2 = -\frac{\beta_{21}^* + \beta_{22}^* \alpha + \beta_{23}^* \alpha^2}{\beta_1^*},$$
$$\tilde b_3 = -\frac{\beta_{31}^* + \beta_{32}^* \alpha + \beta_{33}^* \alpha^2 + \beta_{34}^* \alpha^3}{\beta_1^*} \tag{3.25}$$

and (for $\alpha = 0$)

$$\tilde b_4|_{\alpha=0} = -\frac{\beta_{41}^*}{\beta_1^*} - \frac{1}{2}\frac{\beta_{32}^*}{\beta_1^*} + \frac{\beta_{22}^*}{\beta_1^*}\frac{\beta_2^*}{\beta_1^*}$$

and with the similar replacements for the $\bar\beta$ function coefficients. This fact suggests a possible RG improvement of the series for R $(L = 0,\ f = f^*)$, useful for large $|r_1^*|$, based upon these alternative β functions. These coefficients turn out to be those of a function $\tilde B(a)$ [different from $B(a)$] whose definition will be clarified at the end of Sec. VI. Starting at 4 loops, the new coefficients appear to depend on β_2^*/β_1^* $(\bar\beta_2^*/\beta_1^*)$, which may cause a problem for large β_2^*/β_1^* $(\bar\beta_2^*/\beta_1^*)$ (this may not be a severe one, however, due to the high order in the perturbation expansion where this feature appears). In any case, these relations do suggest that the one-loop BLM coefficient r_1^* (as well as $\bar\beta_2^*/\beta_1^*$ and β_2^*/β_1^*) be put on a different footing from the α derivatives β_{ik}^* and $\bar\beta_{ik}^*$. In particular, they show the genuine RS-invariant information contained in r_2 consists in the value of the derivative $\bar\beta_{2i}/\beta_1^*$, whereas $\bar\beta_2^*/\beta_1^*$ should be looked upon as an essentially one-loop quantity (related to r_1^*); furthermore, they suggest a large value of r_1^* $(\bar\beta_2^*/\beta_1^*)$ does not necessarily imply correspondingly large values of the $\bar\beta_{ik}^*/\beta_1^*$ coefficients.

IV. HIGHER-ORDER COEFFICIENTS OF THE $\bar B(R)$ FUNCTION

To compute the $\bar b_i$'s in Eq. (2.1) I make the definition of $dR/d\bar c$ further precise by making the natural assumption that the derivatives $dr_i/d\bar c$ are to be taken with the α derivatives $\bar\beta_{ik}^*$ kept fixed; i.e., one performs an f-independent variation of the $\bar\beta_i$'s. I show below that this prescription ensures interesting properties of $\bar B(R)$; in particular, it guarantees a smooth $\alpha \to 0$ limit of the $\bar b_i$'s, the $d/d\bar{c}$ variation being "neutral" with respect to α dependence. Since $r_1^*=\bar{c}-c$, I thus get, from Eqs. (3.1) and (3.7)–(3.9),

$$\frac{dr_1}{d\bar{c}}=1\ ,$$

$$\frac{dr_2}{d\bar{c}}=2r_1^*-\frac{\beta_{11}^*}{\beta_1^*}+2(\beta_1^*L)\alpha \tag{4.1}$$

and, from Eqs. (3.13)–(3.16),

$$\begin{aligned}\frac{dr_3}{d\bar{c}}=&-2(\beta_1^*L)+3r_1^{*2}-5\frac{\beta_{11}^*}{\beta_1^*}r_1^*-\frac{3\bar{\beta}_{21}^*-2\beta_{21}^*}{\beta_1^*}\\&+\left[\left(6r_1^*-5\frac{\beta_{11}^*}{\beta_1^*}\right)(\beta_1^*L)-\frac{3\bar{\beta}_{22}^*-2\beta_{22}^*}{\beta_1^*}\right]\alpha\\&+\left[3(\beta_1^*L)^2-\frac{3\bar{\beta}_{23}^*-2\beta_{23}^*}{\beta_1^*}\right]\alpha^2\ .\end{aligned} \tag{4.2}$$

Then Eq. (1.1) yields

$$\bar{B}(R)=\frac{dr_1}{d\bar{c}}a^2+\frac{dr_2}{d\bar{c}}a^3+\frac{dr_3}{d\bar{c}}a^4+\cdots\ . \tag{4.3}$$

On the other hand, reexpanding Eq. (2.1) in powers of a using Eq. (1.1), and comparing with Eq. (4.3), I obtain

$$\bar{b}_0=1,\quad \bar{b}_1=-\frac{\beta_{11}^*}{\beta_1^*} \tag{4.4}$$

and

$$\bar{b}_2=-\frac{\bar{\beta}_{21}^*+\bar{\beta}_{22}^*\alpha+\bar{\beta}_{23}^*\alpha^2}{\beta_1^*}\ . \tag{4.5}$$

I note the following.

(i) $\bar{b}_0$ and $\bar{b}_1$ are universal RS invariants, independent of the definition of R (I shall therefore suppress the overbar).

(ii) The $\bar{b}_i$'s are independent of L, as well as of the β_i^* and β_{ik}^* RS-dependent parameters, as expected from the RS invariance of $\bar{B}(R)$.

(iii) More remarkably, the $\bar{b}_i$'s ($i\le 2$) are also independent of $\bar{c}=-\beta_2^*/\beta_1^*$. This feature (which persists for $i\le 4$), crucial for the RG improvement for large $\bar{c}$ (see Sec. VI), will be explained in Sec. V, and related to a fourth observation.

(iv) The $\bar{b}_i$'s are indeed regular as $\alpha\to 0$. Furthermore, b_0 and b_1 are independent of α, whereas $\bar{b}_2$ is a quadratic polynomical in α (hence does not contain f^3 terms, contrary to $\bar{\beta}_2$). Finally, let us derive $\bar{b}_3$ ($\alpha=0$). From Eq. (3.19) I get

$$\begin{aligned}\frac{dr_4^*}{d\bar{c}}=&\left[-2r_1^*+3\frac{\beta_{11}^*}{\beta_1^*}-6\frac{\beta_2^*}{\beta_1^*}+4\frac{\bar{\beta}_2^*}{\beta_1^*}\right](\beta_1^*L)+4r_1^{*3}\\&-13\frac{\beta_{11}^*}{\beta_1^*}r_1^{*2}+\left[\frac{-12\bar{\beta}_{21}^*+6\beta_{21}^*}{\beta_1^*}+3\frac{\beta_{11}^{*2}}{\beta_1^{*2}}\right]r_1^*\\&+\frac{-2\bar{\beta}_{31}^*+\beta_{31}^*}{\beta_1^*}+3\frac{\beta_{11}^*}{\beta_1^*}\frac{\bar{\beta}_{21}^*-\beta_{21}^*}{\beta_1^*}-\frac{1}{3}\frac{\bar{\beta}_{22}^*}{\beta_1^*}\ ,\end{aligned} \tag{4.6}$$

where I considered $\bar{\beta}_3^*/\beta_1^*$ to be a function of $\bar{\beta}_2^*/\beta_1^*$ [see Eq. (3.21)],

$$\frac{\bar{\beta}_3^*}{\beta_1^*}=\left[\frac{\bar{\beta}_2^*}{\beta_1^*}\right]^2+I_2\ , \tag{4.7}$$

since I_2 is a universal RS invariant, which should be kept fixed as $\bar{c}$ is varied, so that I used

$$\frac{d(\bar{\beta}_3^*/\beta_1^*)}{d\bar{c}}=-2\frac{\bar{\beta}_2^*}{\beta_1^*}\ . \tag{4.8}$$

Following the same steps as above, one gets

$$\bar{b}_3(\alpha=0)=-\frac{\bar{\beta}_{31}^*}{\beta_1^*}-\frac{1}{3}\frac{\bar{\beta}_{22}^*}{\beta_1^*} \tag{4.9}$$

which is again independent of $\bar{c}$. Note that, at the difference of $\bar{b}_i$ ($\alpha=0$) for $i\le 2$, $\bar{b}_3$ ($\alpha=0$) is not equal to the value $-\bar{\beta}_{31}^*/\beta_1^*$ naively expected from the structure of r_3^* ($L=0$) [see Eq. (3.25) and the remarks at the end of Sec. III]. In Sec. V, I shall derive the full $\bar{b}_3$ for $\alpha\neq 0$, as well as $\bar{b}_4$, by a different, easier method.

It is also straightforward to express the $\bar{b}_i$'s in terms of the more familiar derivatives of $\bar{\beta}_i$ around $f=0$. For instance, putting

$$\beta_1=\beta_{10}+\beta_{11}f,\quad \bar{\beta}_2=\bar{\beta}_{20}+\bar{\beta}_{21}f+\bar{\beta}_{22}f^2+\bar{\beta}_{23}f^3\ ,$$

and

$$\bar{b}_2=\bar{b}_{20}+\bar{b}_{21}f+\bar{b}_{22}f^2$$

one has

$$b_1=\frac{\beta_{11}}{\beta_{01}} \tag{4.10}$$

and

$$\begin{aligned}\bar{b}_{20}&=\frac{\bar{\beta}_{21}+\bar{\beta}_{22}f^*+\bar{\beta}_{23}f^{*2}}{\beta_{01}}\ ,\\ \bar{b}_{21}&=\frac{\bar{\beta}_{22}+\bar{\beta}_{23}f^*}{\beta_{01}}\ ,\quad \bar{b}_{22}=\frac{\bar{\beta}_{23}}{\beta_{01}}\ .\end{aligned} \tag{4.11}$$

For completeness, I also note the relations

$$r_2=r_{20}+r_{21}f+r_{22}f^2$$

with

$$r_{20|L=0}=r_1^{*2}+b_1r_1^*+\bar{b}_{20}-b_{20}\ , \tag{4.12}$$

$$r_{21|L=0}=\bar{b}_{21}-b_{21}\ , \tag{4.13}$$

$$r_{22|L=0}=\bar{b}_{22}-b_{22} \tag{4.14}$$

which show in particular that $\bar{b}_{21}$ and $\bar{b}_{22}$ and, hence, $\bar{\beta}_{22}$ and $\bar{\beta}_{23}$, follow from the (easier to compute) fermionic part of r_2, whereas $\bar{b}_{20}$ and, *hence*, $\bar{\beta}_{21}$, depend essentially on the *gluonic* part of r_2.

V. INTEGRAL REPRESENTATION OF $\bar{B}(R)$

Some interesting properties of $\bar{B}(R)$ can be understood from an integral representation I now derive. I shall simplify the notation in the following and drop all overbars, so that $\bar{\beta}(R)\to\beta(R)$, $\bar{B}(R)\to B(R)$, $\bar{c}\to c$, etc., since no reference will be made in this section to another effective coupling than R itself. I start with a differential equation for $B(R,c)$. Differentiating both sides of Eq. (1.6) with respect to c, I get, using Eq. (2.1),

$$\frac{d}{dc}\left[\frac{dR}{d\ln Q^2}\right]=\frac{d}{dc}\beta(R,c)=\frac{\partial\beta}{\partial R}B(R,c)+\frac{\partial\beta}{\partial c}\ , \qquad (5.1)$$

whereas differentiating Eq. (2.1) with respect to $\ln Q^2$ gives

$$\frac{d}{d\ln Q^2}\left[\frac{dR}{dc}\right]=\frac{dB}{dR}\beta(R,c)\ . \qquad (5.2)$$

Assuming one can interchange the double derivatives, I deduce the equation

$$\frac{\partial\beta}{\partial R}B(R,c)+\frac{\partial\beta}{\partial c}=\frac{dB}{dR}\beta(R,c)\ , \qquad (5.3)$$

whose solution, when $\beta_0\neq 0$, is

$$B(R,c)=\beta(R,c)\left[\int_0^R dx\frac{(\partial\beta/\partial c)(x,c)}{[\beta(x,c)]^2}+C\right],$$

where C is an integration constant, determined below to be equal to $-1/\beta_0$ from the condition that $B(R,c)\approx R^2$ for $R\to 0$. I thus get

$$B(R,c)=\beta(R,c)\left[\int_0^R dx\frac{(\partial\beta/\partial c)(x,c)}{[\beta(x,c)]^2}-\frac{1}{\beta_0}\right]. \qquad (5.4)$$

Equation (5.4) may also be obtained from the solution Eq. (2.3) of the standard RG equation for R, by differentiating both sides with respect to $\bar{c}$, taking into account Eq. (2.6). This representation for $B(R,c)$ is analogous to the ones derived in Ref. [8] for alternative RG functions corresponding to the "full" Stückelberg-Peterman RG, obtained by varying one by one the higher-order coefficients of the β function. In the present case, however, the derivative $(\partial\beta/\partial c)(x,c)$ involves changing simultaneously *all* higher-order coefficients of the β function: we already know [cf. Eqs. (4.7) and (3.20)] that β_3^*/β_1^* and β_4^*/β_1^* must be considered to be functions of $\beta_2^*/\beta_1^*\equiv -c$; the same statement holds for all higher order β_i^*'s, and expresses the condition that the critical exponent $\omega(\alpha)$ be the same for any value of c. To determine $B(R,c)$ from Eq. (5.4), it remains to compute $\partial\beta/\partial c$. Since c is varied at fixed α derivatives, $\partial\beta/\partial c$ is independent of α (i.e., as mentioned earlier, one performs an f-independent variation of β), and we have

$$\frac{\partial\beta}{\partial c}=-\frac{\partial\beta_2^*}{\partial c}x^4-\frac{\partial\beta_3^*}{\partial c}x^5-\frac{\partial\beta_4^*}{\partial c}x^6-\frac{\partial\beta_5^*}{\partial c}x^7+\cdots\ .$$

Clearly, $-\partial\beta_2^*/\partial c=\beta_1^*$; hence,

$$\frac{\partial\beta}{\partial c}=\beta_1^*x^4\left[1-\frac{\partial(\beta_3^*/\beta_1^*)}{\partial c}x-\frac{\partial(\beta_4^*/\beta_1^*)}{\partial c}x^2-\frac{\partial(\beta_5^*/\beta_1^*)}{\partial c}x^3+\cdots\right]. \qquad (5.5)$$

Expanding Eq. (5.4) in powers of R, one finds the first two terms depend only upon the leading $O(x^4)$ contribution to $\partial\beta/\partial c$ in Eq. (5.5). I get

$$B(R,c)=R^2+\frac{\beta_1-\beta_1^*}{\beta_0}R^3+O(R^4)\ ;$$

hence, $b_0=1$ as required, and also

$$b_1=\frac{\beta_1-\beta_1^*}{\beta_0} \qquad (5.6)$$

which agrees with Eq. (4.4). The higher order b_i's ($i\geq 2$) depend upon the derivatives $\partial(\beta_i^*/\beta_1^*)/\partial c$ ($i\geq 3$). One gets, from Eqs. (5.4) and (5.5)

$$\begin{aligned}B(R,c)=&R^2+\frac{\beta_1-\beta_1^*}{\beta_0}R^3+\left[\frac{\beta_2}{\beta_0}+\frac{\beta_1^*}{\beta_0}\frac{1}{2}\frac{\partial(\beta_3^*/\beta_1^*)}{\partial c}\right]R^4+\left[\frac{\beta_3}{\beta_0}-\frac{\beta_1^*}{\beta_0}\frac{1}{3}\left[\frac{\beta_2}{\beta_0}+\frac{1}{2}\frac{\beta_1}{\beta_0}\frac{\partial(\beta_3^*/\beta_1^*)}{\partial c}-\frac{\partial(\beta_4^*/\beta_1^*)}{\partial c}\right]\right]R^5\\&+\left\{\frac{\beta_4}{\beta_0}-\frac{\beta_1^*}{\beta_0}\left[\frac{1}{6}\frac{\beta_1^2}{\beta_0^2}\left[-\frac{1}{2}\frac{\partial(\beta_3^*/\beta_1^*)}{\partial c}-\frac{\beta_2}{\beta_1}\right]+\frac{1}{2}\left[\frac{1}{3}\frac{\partial(\beta_4^*/\beta_1^*)}{\partial c}+\frac{\beta_3}{\beta_1}\right]-\frac{1}{4}\frac{\partial(\beta_5^*/\beta_1^*)}{\partial c}\right]\right\}R^6+\cdots\\&\equiv R^2+b_1R^3+b_2R^4+b_3R^5+b_4R^6+\cdots\ . \end{aligned} \qquad (5.7)$$

It appears from Eq. (5.7) that the b_i's may contain $1/\beta_0$ singularities in the limit $\beta_0\to 0$ (i.e., $\alpha\to 0$). Remarkably, they turn out to cancel if the correct definition of $\partial(\beta_i^*/\beta_1^*)/\partial c$ is used. Indeed, using Eq. (4.8), Eq. (5.7) gives

$$b_2=\frac{\beta_2-\beta_2^*}{\beta_0} \qquad (5.8)$$

which agrees with Eq. (4.5). The coefficient b_3 depends upon $\partial(\beta_4^*/\beta_1^*)/\partial c$, which can be obtained from Eq. (3.20):

$$\frac{\beta_4^*}{\beta_1^*}=\left[\frac{\beta_2^*}{\beta_1^*}\right]^3-\frac{1}{2}\frac{\beta_{11}^*}{\beta_1^*}\left[\frac{\beta_2^*}{\beta_1^*}\right]^2+\left[\frac{\beta_{21}^*}{\beta_1^*}+3I_2\right]\left[\frac{\beta_2^*}{\beta_1^*}\right]-\frac{1}{2}\frac{\beta_{31}^*}{\beta_1^*}-\frac{1}{2}\frac{\beta_{11}^*}{\beta_1^*}I_2+3I_3\ ; \qquad (5.9)$$

hence, since I_2 and I_3 are universal RS invariants,

$$\frac{\partial(\beta_4^*/\beta_1^*)}{\partial c}=-3\left[\frac{\beta_2^*}{\beta_1^*}\right]^2+\frac{\beta_{11}^*}{\beta_1^*}\frac{\beta_2^*}{\beta_1^*}-\frac{\beta_{21}^*}{\beta_1^*}-3I_2=-3\frac{\beta_3^*}{\beta_1^*}+\frac{\beta_{11}^*}{\beta_1^*}\frac{\beta_2^*}{\beta_1^*}-\frac{\beta_{21}^*}{\beta_1^*}\ , \qquad (5.10)$$

where in the last step I used Eq. (4.7). Equation (5.7) then yields

$$b_3=-\frac{\beta_{31}^*+\frac{1}{3}\beta_{22}^*}{\beta_1^*}-\frac{\beta_{32}^*+\frac{1}{3}\beta_{23}^*}{\beta_1^*}\alpha-\frac{\beta_{33}^*}{\beta_1^*}\alpha^2-\frac{\beta_{34}^*}{\beta_1^*}\alpha^3 \tag{5.11}$$

[b_3 ($\alpha=0$) agrees with Eq. (4.9)].

The calculation of b_4 requires knowledge of $\partial(\beta_5^*/\beta_1^*)/\partial c$. This is very cumbersome to obtain directly (one needs to compute r_5 and the invariant I_4). However, as argued in Sec. IV, I expect the b_i's to be regular in the $\alpha\to 0$ limit, and therefore I conjecture the correct value of $\partial(\beta_i^*/\beta_1^*)/\partial c$ is always such that the $1/\beta_0$ singularities in the b_i's cancel. I have checked that this requirement correctly reproduces both $\partial(\beta_i^*/\beta_1^*)/\partial c$ and b_i for $i\le 3$, starting from Eq. (5.7). For $i=4$, it determines

$$\frac{\partial(\beta_5^*/\beta_1^*)}{\partial c}=-4\frac{\beta_4^*}{\beta_1^*}+2\frac{\beta_{11}^*}{\beta_1^*}\frac{\beta_3^*}{\beta_1^*}-\frac{2}{3}\frac{\beta_{22}^*}{\beta_1^*}-2\frac{\beta_{31}^*}{\beta_1^*} \tag{5.12}$$

and fixes b_4:

$$b_4=-\left[\frac{\beta_{41}^*}{\beta_1^*}+\frac{1}{2}\frac{\beta_{32}^*}{\beta_1^*}+\frac{1}{6}\frac{\beta_{22}^*}{\beta_1^*}\frac{\beta_{11}^*}{\beta_1^*}+\frac{1}{6}\frac{\beta_{23}^*}{\beta_1^*}\right]-\left[\frac{\beta_{42}^*}{\beta_1^*}+\frac{1}{2}\frac{\beta_{33}^*}{\beta_1^*}+\frac{1}{6}\frac{\beta_{23}^*}{\beta_1^*}\frac{\beta_{11}^*}{\beta_1^*}\right]\alpha-\left[\frac{\beta_{43}^*}{\beta_1^*}+\frac{1}{2}\frac{\beta_{34}^*}{\beta_1^*}\right]\alpha^2-\frac{\beta_{44}^*}{\beta_1^*}\alpha^3-\frac{\beta_{45}^*}{\beta_1^*}\alpha^4\,. \tag{5.13}$$

I note that b_3 and b_4 are, respectively, at most cubic and quartic polynomials in α (hence in f), even if β_3 and β_4 contain, respectively, f^4 and f^5 terms. Furthermore, one finds that b_3 and b_4 are independent of c, as the lower orders b_i's. *The b_i's start however to depend upon c for $i\ge 5$* (contrary to the statement in Ref. [9]), unless the β_i's are linear in f. This feature appears to be a consequence of the regular behavior of the b_i's in the $\beta_0\to 0$ limit. Indeed Eqs. (5.4) and (5.5) give

$$B(R,c)=-R^2\left[1+\frac{\beta_1}{\beta_0}R+\frac{\beta_2}{\beta_0}R^2+\cdots\right]\left[\int_0^R dx\frac{\beta_1^*}{\beta_0}\frac{1+\sum_{n\ge 1}\frac{\partial(\beta_{n+2}^*/\beta_1^*)}{\partial c}x^n}{\left[1+\frac{\beta_1}{\beta_0}x+\frac{\beta_2}{\beta_0}x^2+\cdots\right]^2}-1\right]. \tag{5.14}$$

Inspection of Eq. (5.14) [see also Eq. (5.7)] shows a necessary condition to have cancellation of the $1/\beta_0$ singularities in the b_i's is to assume the quantities $(\beta_1^*/\beta_0)[\partial(\beta_n^*/\beta_1^*)/\partial c]$ depend linearly upon the ratios β_i^*/β_0, since among the potentially singular terms in Eq. (5.14), those which involve the β_i^*'s appear only through these ratios. Such terms should cancel against each other in the b_i's, which will therefore be independent of the β_i^*'s (and in particular of c), unless they are multiplied by some sufficiently large power of α which makes them finite as $\alpha\to 0$. In the latter case, the β_i^*'s will appear in the b_i's multiplied by some β_{ik}^* derivative ($k\ge 2$). Explicit calculation, as well as a general argument (omitted here for brevity), indicate such c-dependent terms do occur in the b_i's for $i\ge 5$ (unless the β function is linear in f).

For completeness, I mention that a representation similar to Eq. (5.14) also holds at $\beta_0=0$ (i.e., $\alpha=0$), but with a different integration constant:

$$B^*(R,c)=\beta^*(R,c)\int^R dx\frac{(\partial\beta^*/\partial c)(x,c)}{[\beta^*(x,)]^2}\,, \tag{5.15}$$

where the asterisk refers as usual to $\alpha=0$ quantities, and I left the lower limit of integration and, hence, the integration constant, undetermined since the integral is now singular as $R\to 0$. Noting that $\partial\beta^*/\partial c=\partial\beta/\partial c$, and is given in Eq. (5.5) and using $\beta^*(x,c)=-\beta_1^*x^3-\beta_2^*x^4-\beta_3^*x^5+\cdots$, one can expand Eq. (5.15) in powers of R. One gets

$$B^*(R,c)=R^2\left[1+\left[\frac{\beta_2^*}{\beta_1^*}+D\right]R+\left[\frac{\beta_2^{*2}}{\beta_1^{*2}}+\frac{3\beta_3^*}{\beta_1^*}+\frac{\beta_2^*}{\beta_1^*}D+\frac{\partial(\beta_4^*/\beta_1^*)}{\partial c}\right]R^2+\cdots\right], \tag{5.16}$$

where D is the integration constant. Comparing with Eq. (4.4) yields $D=-(\beta_2^*+\beta_{11}^*)/\beta_1^*$. The expression Eq. (4.8) for $\partial(\beta_3^*/\beta_1^*)/\partial c$ was used implicitly in Eq. (5.16) to prevent the appearance of nonanalytic $\ln R$ terms which would otherwise be expected from Eq. (5.15) and the general expansion Eq. (5.5). Note also comparison of $O(R^4)$ terms in Eqs. (5.7) and (5.16) allows one to determine both $\partial(\beta_3^*/\beta_1^*)/\partial c$ [by the condition the $O(1/\beta_0)$ singularities cancel] and $\partial(\beta_4^*/\beta_1^*)/\partial c$ (from the remaining finite contribution), without having to compute the $O(R^5)$ term.

VI. RENORMALIZATION-GROUP-IMPROVED PERTURBATION THEORY FOR LARGE $|r_1^*|$

In this section, two different methods for improving the QCD perturbative series for large $|r_1^*|$ are presented. The first one, which makes use of the $\bar{B}(R)$ function, is a completely RS-invariant alternative to the standard method of effective charges, and is not tied to any peculiar choice of μ. The second method is simpler, but is tied to the BLM choice of μ Eq. (1.3), i.e., is designed to improve specifically the perturbative series at $L=0$: it can be seen as a RG improvement of the BLM scheme.

A. RS-invariant method

The RG equation (2.1) yields a new method to improve QCD perturbation theory, especially suitable to deal with those cases where the standard method of effective charges [7] might fail because of a large value of $|r_1^*|$ or $|\bar{\beta}_2^*|$. Since the $\bar{b}_i$'s are independent of $\bar{\beta}_2^*$ for $i\leq 4$, perturbation theory could still be useful for the $\bar{B}(R,\bar{c})$ function even then. In particular, Eq. (5.8) shows that the magnitude of $\bar{b}_2$ is a measure of the deviation of $\bar{\beta}_2$ from its $f=f^*$ value, which could remain small even for large $|\bar{\beta}_2^*|$ and $|\bar{\beta}_2|$.

It is instructive to consider first the special case where the B and $\bar{B}$ functions do not depend explicitly on c and $\bar{c}$ to all orders. Then the solution of Eqs. (2.1) and (2.2) is straightforward:

$$\bar{c}+\bar{\delta}=\int^R \frac{dx}{\bar{B}(x)}\ , \tag{6.1}$$

$$c+\delta=\int^a \frac{dx}{B(x)}\ , \tag{6.2}$$

where δ and $\bar{\delta}$ are integration "constants" (which however depend, respectively, on μ and Q). An improved perturbation theory may be based, similarly to the standard case (which makes use of the β and $\bar{\beta}$ functions [cf. Eqs. (2.3)–(2.5)]), on the relation obtained by subtracting Eq. (6.2) from Eq. (6.1):

$$r_1^*+\Delta=\int^R \frac{dx}{\bar{B}(x)}-\int^a \frac{dx}{B(x)}\ , \tag{6.3}$$

where $\Delta=\bar{\delta}-\delta$. Note that for $R\to 0$, Eq. (6.1) gives

$$\bar{c}+\bar{\delta}=-\frac{1}{R}+\frac{\beta_{11}^*}{\beta_1^*}\ln R+\cdots \tag{6.4}$$

so that $R\to 0^+$ for $\bar{c}\to-\infty$, i.e., for $\bar{\beta}_2^*/\beta_1^*\to+\infty$, which indicates the domain of validity of this improved perturbation theory includes the case of *large and negative* $\bar{c}$, i.e., of *large and negative* r_1^* (which typically occurs for the process $\Upsilon\to$hadrons, see Ref. [2]).

However, a complication arises here with respect to the standard case; namely the "constant" Δ, although explicitly independent of $\bar{c}$, is a function $\Delta(a,c,L)$ (computable as a power series in a: $\Delta=\Delta_0+\Delta_1 a+\cdots$) such that $d\Delta/dc=\partial\Delta/\partial c+B(a)\partial\Delta/\partial a=0$. This feature reflects the fact that $R=R(a,r_1^*,c)$ does not depend on c and $\bar{c}$ solely through their difference r_1^*. Consequently, an improved perturbation theory based on Eq. (6.3) will still depend on the choice of the RS coupling a, through the expansion of the Δ term. A different, completely RS-invariant approach is preferable, which consists in focusing on the Q dependence of $\bar{\delta}$. Furthermore, this approach also applies to the general case where $\bar{B}(R,c)$ does depend explicitly on $\bar{c}$ in high orders. For this purpose, it is useful to keep defining a function $\bar{\delta}(Q,\bar{c})$ through Eq. (6.1), but with $\bar{B}(x)$ replaced by $\bar{B}(x,\bar{c})$: i.e. (I drop all overbars since all quantities refer exclusively to R),

$$\delta(Q,c)\equiv -c+\int^R \frac{dx}{B(x,c)}\ . \tag{6.5}$$

Note that $\delta(Q,c)$, being no more an integration constant, does depend on c now. Differentiating both sides of Eq. (6.5) with respect to Q gives

$$\frac{d\delta}{d\ln Q^2}=\frac{\beta(R,c)}{B(R,c)}\equiv-\beta_1^*(\rho-\alpha)\ , \tag{6.6}$$

where I defined an auxiliary effective charge $\rho(R,c)$ through the relation

$$\frac{dR}{d\ln Q^2}=\beta(R,c)\equiv-\beta_1^*(\rho-\alpha)B(R,c)\ . \tag{6.7}$$

ρ can be computed as a power series in R:

$$\rho=R(1+\rho_1 R+\rho_2 R^2+\cdots)\ . \tag{6.8}$$

One finds

$$\rho_1=\frac{\beta_2^*}{\beta_1^*}+\frac{\beta_{11}^*}{\beta_1^*}=-c+\frac{\beta_{11}^*}{\beta_1^*}\ , \tag{6.9}$$

$$\begin{aligned}\rho_2&=\frac{\beta_3^*}{\beta_1^*}+\frac{\beta_{21}^*}{\beta_1^*}+\frac{\beta_{11}^*}{\beta_1^*}\frac{\beta_2^*+\beta_{11}^*}{\beta_1^*}+\frac{2}{3}\frac{\beta_{22}^*\alpha+\beta_{23}^*\alpha^2}{\beta_1^*}\\&=c^2+\frac{\beta_{11}^*}{\beta_1^*}c+I_2+\left[\frac{\beta_{11}^*}{\beta_1^*}\right]^2+\frac{\beta_{21}^*}{\beta_1^*}+\frac{2}{3}\frac{\beta_{22}^*\alpha+\beta_{23}^*\alpha^2}{\beta_1^*}\ ,\end{aligned} \tag{6.10}$$

where Eq. (4.7) has been used in the last step. It is convenient to determine δ as a function of ρ, the Q dependence of ρ itself following from its associated RG equation:

$$\frac{d\rho}{d\ln Q^2}\equiv\tilde{\beta}(\rho,c)\equiv-\beta_1^*(\rho-\alpha)\psi(\rho,c)\ . \tag{6.11}$$

Equation (6.11) introduces the auxiliary function $\psi(\rho,c)$ which can be computed as a power series in ρ (see below):

$$\psi(\rho,c)=\psi_0\rho^2+\psi_1\rho^3+\psi_2\rho^4+\cdots\ . \tag{6.12}$$

Then Eq. (6.6) gives

$$\tilde{\beta}\frac{d\delta}{d\rho}=-\beta_1^*(\rho-\alpha)\ ;$$

hence,

$$\frac{d\delta}{d\rho}=\frac{1}{\psi(\rho,c)}\ . \tag{6.13}$$

The motivation for introducing the ψ function is that it does not depend on c if the B function does not. Indeed, in this case $\delta(Q,c)$ in Eq. (6.5) does not depend on c either (it is a genuine integration constant), and Eq. (6.6) implies the *total* derivative $d\rho/dc=\partial\rho/\partial c+B(R)\partial\rho/\partial R=0$. It then follows from Eq. (6.11) or (6.13) that $\psi(\rho,c)$ also does not depend on c.

Furthermore, in the general case where the b_i's do depend on c for $i\geq 5$, it can be shown that the ψ_i's also depend on c only for $i \geq 5$, which makes the use of perturbation theory for the ψ function very reasonable, even at large $|c|$. Integration of Eq. (6.13) yields δ as a function of ρ:

$$\delta = K + \int^{\rho} \frac{dx}{\psi(x,c)} . \tag{6.14}$$

The integration constant K is determined by eliminating δ between Eqs. (6.5) and (6.14). One gets

$$-c - K = \int^{\rho} \frac{dx}{\psi(x,c)} - \int^{R} \frac{dx}{B(x,c)} . \tag{6.15}$$

Requiring the series Eq. (6.8) to be a solution of Eq. (6.15), one finds $-c-K=\rho_1$ and, hence, $K=-\beta_{11}^*/\beta_1^*$, which leads to an "improved" relation between ρ and R:

$$-c + \frac{\beta_{11}^*}{\beta_1^*} = \int^{\rho} \frac{dx}{\psi(x,c)} - \int^{R} \frac{dx}{B(x,c)} . \tag{6.16}$$

The ψ_i's in Eq. (6.12) can be computed from the relation

$$\frac{\partial \rho}{\partial R} = \frac{\psi(\rho,c)}{B(R,c)} \tag{6.17}$$

which follows either from the ratio of Eqs. (6.7) and (6.11), or by varying ρ with R in Eq. (6.16) at fixed c. Expanding both sides of Eq. (6.17) in powers of R, one finds $\psi_0=b_0=1$, $\psi_1=b_1=-\beta_{11}^*/\beta_1^*$, and the analogues of Eqs. (3.5) and (3.11), namely,

$$\rho_2 = \rho_1^2 + \frac{b_1}{b_0}\rho_1 + \frac{\psi_2 - b_2}{b_0} , \tag{6.18}$$

$$\rho_3 = \rho_1^3 + \frac{5}{2}\frac{b_1}{b_0}\rho_1^2 + \frac{3\psi_2 - 2b_2}{b_0}\rho_1 + \frac{1}{2}\frac{\psi_3 - b_3}{b_0} \tag{6.19}$$

from which one determines

$$\psi_2 = \left[I_2 + \left[\frac{\beta_{11}^*}{\beta_1^*} \right]^2 \right] - \frac{1}{3}\frac{\beta_{22}^*\alpha + \beta_{23}^*\alpha^2}{\beta_1^*} \tag{6.20}$$

and (for simplicity, I quote only the $\alpha=0$ value)

$$\psi_3|_{\alpha=0} = \left[6I_3 - \left[\frac{\beta_{11}^*}{\beta_1^*} \right]^3 - 5\frac{\beta_{11}^*}{\beta_1^*} I_2 \right] + \frac{1}{3}\frac{\beta_{22}^*}{\beta_1^*} . \tag{6.21}$$

Note that ψ_2 and ψ_3 are indeed independent of c. It is also interesting that $\psi_2|_{\alpha=0}$ coincides with ω_2/ω_0, the $O(\alpha^4)$ critical exponent coefficient [Eq. (3.24)]. Furthermore, although $\psi_3|_{\alpha=0}$ is not a universal RS invariant (because of the $\frac{1}{3}\beta_{22}^*/\beta_1^*$ term), its RS-invariant part coincides with ω_3/ω_0. These features follow from Eq. (6.11), which shows that $\tilde{\beta}$ vanishes for $\rho=\alpha$, so that the IR fixed point is $\rho^* \equiv \alpha$ (without higher-order corrections) and gives $\omega(\alpha) = \partial\tilde{\beta}/\partial\rho|_{\rho=\alpha} = -\beta_1^*\psi(\rho=\alpha)$. In the peculiar case where the β_i's are linear in f, one finds that $\psi(\rho)$ does not depend explicitly on α, and coincides with the critical exponent function $-\beta_1^*(\rho) \equiv \omega(\rho)$. The corresponding coupling ρ is then identical to the *universal* coupling $\tilde{\alpha}$, defined by [see Eq. (6.11)]

$$\frac{d\tilde{\alpha}}{d \ln Q^2} = (\tilde{\alpha} - \alpha)\omega(\tilde{\alpha}) \equiv \tilde{\beta}(\tilde{\alpha}) . \tag{6.22}$$

In general, however, $\rho \neq \tilde{\alpha}$ and does depend upon the definition of R (an alternative improvement procedure making use of the coupling $\tilde{\alpha}$ is described in Sec. VI B below).

An improved perturbation theory follows from Eq. (6.16). The resulting $\rho(R,c)$ function may be inserted into Eq. (6.7) [this yields an improvement of the $\beta(R,c)$ series], or, equivalently (but more conveniently), one can use Eq. (6.16) together with Eq. (6.11) to get the Q dependence of R. As an example, consider the two-loop approximation to Eq. (6.16), where $\psi(x)=B(x)=x^2-\beta_{11}^*/\beta_1^* x^3$. One gets

$$\frac{\beta_2^* + \beta_{11}^*}{\beta_1^*} = \frac{1}{R} - \frac{1}{\rho} + \frac{\beta_{11}^*}{\beta_1^*} \ln \left[\frac{1/R - \beta_{11}^*/\beta_1^*}{1/\rho - \beta_{11}^*/\beta_1^*} \right] , \tag{6.23}$$

whereas Eq. (6.11) gives

$$\frac{d\rho}{d \ln Q^2} = -\beta_0\rho^2 - \beta_1\rho^3 + \beta_{11}^*\rho^4 . \tag{6.24}$$

Integration of Eq. (6.24) yields, neglecting the $O(\rho^4)$ term,

$$\beta_0 \ln \frac{Q^2}{\Lambda_{\overline{\rm MS}}^2} = \rho_{1,\overline{\rm MS}} + \frac{1}{\rho} - \frac{\beta_1}{\beta_0} \ln \left[\frac{1}{\beta_0} \left[\frac{1}{\rho} + \frac{\beta_1}{\beta_0} \right] \right] , \tag{6.25}$$

where I used for definiteness $\Lambda_{\overline{\rm MS}}$ as the reference QCD scale parameter, and $\rho_{1,\overline{\rm MS}}$ is the coefficient in the series: $\rho = a_{\overline{\rm MS}}(1 + \rho_{1,\overline{\rm MS}} a_{\overline{\rm MS}} + \cdots)$, with $a_{\overline{\rm MS}} = a_{\overline{\rm MS}}(\mu = Q)$. It is given by

$$\rho_{1,\overline{\rm MS}} = \rho_1 + r_{1,\overline{\rm MS}}(\mu = Q)$$
$$= \frac{\beta_{2,\overline{\rm MS}}^*}{\beta_1^*} + \frac{\beta_{11}^*}{\beta_1^*} + \beta_0 d_{1,\overline{\rm MS}}^* \tag{6.26}$$

where $r_{1,\overline{\rm MS}}$ is the coefficient r_1 of Eq. (1.1) for $a = a_{\overline{\rm MS}}$, and $d_{1,\overline{\rm MS}}^*$ is the corresponding parameter of Eq. (1.2). Equations (6.23) and (6.25) relate R to $\Lambda_{\overline{\rm MS}}$. Note the present method is a completely RS-invariant one, and is not tied to any peculiar choice of the RS coupling a. It is also worth pointing out that, in the 2-loop approximation, the coupling ρ is universal (and coincides with $\tilde{\alpha}$). One can then eliminate ρ between Eq. (6.23) and its counterpart where $R \to a_{\overline{\rm MS}}$ and $\beta_2^* \to \beta_{2,\overline{\rm MS}}^*$; i.e., one gets an improved relation between R and $a_{\overline{\rm MS}}$ (at $L=0$):

$$r_{1,\overline{\rm MS}}^* = \frac{1}{a} - \frac{1}{R} + \frac{\beta_{11}^*}{\beta_1^*} \ln \left[\frac{1/a - \beta_{11}^*/\beta_1^*}{1/R - \beta_{11}^*/\beta_1^*} \right] , \tag{6.27}$$

where $a = a_{\overline{\rm MS}}$ ($L=0$). One can then relate $a_{\overline{\rm MS}}$ to $\Lambda_{\overline{\rm MS}}$ in the standard way [Eq. (2.4)], with $\beta_{\overline{\rm MS}}(x)$ truncated at 2-loop [the resulting value of $\Lambda_{\overline{\rm MS}}$ will slightly differ from the one obtained from the first procedure, if one does not use the "improved" form of $\beta_{\overline{\rm MS}}(x)$].

As an application, consider the process $\Upsilon \to$ hadrons. The quantity of interest is the ratio

$$R \equiv \left[\frac{81 e_b^2}{10(\pi^2 - 9)} \alpha^2 \frac{\Gamma_g(\Upsilon \to \text{hadrons})}{\Gamma(\Upsilon \to \mu^+\mu^-)} \right]^{1/3}$$

where Γ_g is the gluonic width of the Υ. One finds [2]

$$R=\alpha_{\overline{\rm MS}}(M_\Upsilon)\left[1+\frac{1}{3}\frac{\alpha_{\overline{\rm MS}}}{\pi}(2.77\beta_0-14.0)+\cdots\right].$$

One deduces [2] $\mu_{\rm BLM}=0.157\,M_\Upsilon$ and also [6] (I use a normalization where $a_{\overline{\rm MS}}\equiv\alpha_{\overline{\rm MS}}/4\pi$) $r^*_{1,\overline{\rm MS}}=-18.67$. Hence, knowing $\beta^*_{2,\overline{\rm MS}}/\beta^*_1=14.45$, one finds [6] $\overline{\beta}^*_2/\beta^*_1=33.12$. Using the experimental value $R=0.163$, one deduces from Eq. (6.27) (with $\beta^*_{11}/\beta^*_1=-19$) $a_{\overline{\rm MS}}\approx0.019$; hence, for $f=4$, $\Lambda_{\overline{\rm MS}}=0.13$ GeV [from Eqs. (6.23) and (6.25), one would find instead $\rho=0.017$ and $\Lambda_{\overline{\rm MS}}=0.12$ GeV]. Note also the standard method of effective charges, based on Eq. (2.3) (with $Q=M_\Upsilon$), gives instead $\Lambda_{\overline{\rm MS}}=0.106$ GeV, a smaller value.

B. Renormalization-group improvement of the BLM scheme

I next turn to an alternative approach based on the RG improvement of the $L=0$ series mentioned at the end of Sec. III. This alternative is made possible owing to the existence of the universal coupling $\tilde{\alpha}$ introduced in Eq. (6.22), which suggests one to define an alternative $\tilde{B}(R,c)$ function through the analogue of Eq. (6.7), with ρ replaced by $\tilde{\alpha}$: namely (a factor of β^*_1 is now included in $\tilde{B}$),

$$\frac{dR}{d\ln Q^2}=\beta(R,c)\equiv(\tilde{\alpha}_R-\alpha)\tilde{B}(R,c)\ , \qquad (6.28)$$

where $\tilde{\alpha}_R\equiv\tilde{\alpha}(R)|_{L=0}$ [$\tilde{\alpha}$ and R must be related by the BLM choice of renormalization point $L=0$, in order that $\tilde{B}(R,c)$ has no explicit L dependence]. Taking the ratio of Eqs. (6.22) and (6.28), one gets

$$\left.\frac{dR}{d\tilde{\alpha}}\right|_{L=0}=\frac{\tilde{B}(R,c)}{\omega(\tilde{\alpha})}\ . \qquad (6.29)$$

Equation (6.29) allows one to compute $\tilde{B}$, and check that its coefficients coincide with those given in Eq. (3.25). This result can be derived without further calculation by specializing Eq. (3.25) to the case $a\equiv\tilde{\alpha}$. I note that the results of Sec. III apply to this case, since Eq. (6.22) shows that the coefficients of $\tilde{\beta}$ are (linear) polynomial in f, and furthermore $\tilde{\beta}$ yields the correct critical exponent $\partial\tilde{\beta}/\partial\tilde{\alpha}|_{\tilde{\alpha}=\alpha}=\omega(\alpha)$ (where I used the fact that the fixed point $\tilde{\alpha}^*\equiv\alpha$). It is clear that in the special case where the β_i's are linear in f (such as the $\tilde{\beta}_i$'s), the coefficients in Eq. (3.25) are just $-\beta^*_{i1}/\beta^*_1$. But for the $\tilde{\beta}$ of Eq. (6.22), we have

$$-\frac{\beta^*_{i1}}{\beta^*_1}=\frac{\omega_i}{\omega_0}\ ; \qquad (6.30)$$

i.e., in this case the coefficients in Eq. (3.25) are those of the critical exponent function, which completes the proof. The $\tilde{B}(R,c)$ function may be used to improve the $L=0$ series. Taking the ratio of Eq. (6.29) and its analogue for the RS coupling a, one gets

$$\left.\frac{dR}{da}\right|_{L=0}=\frac{\tilde{\overline{B}}(R,c)}{\tilde{B}(a,c)}\ , \qquad (6.31)$$

where I have reinstalled the overbar. The $\omega(\tilde{\alpha})$ function has canceled out in the ratio, since it is universal, and taken for the same value of its argument $\tilde{\alpha}$ (corresponding to the $L=0$ prescription). Integration of Eq. (6.31) gives an improved relation between R and a at $L=0$ [compare with Eq. (6.3)]:

$$r^*_1=\int^R\frac{dx}{\tilde{\overline{B}}(x,\overline{c})}-\int^a\frac{dx}{\tilde{B}(x,c)}\ , \qquad (6.32)$$

where the $\tilde{b}_i$ ($\tilde{\overline{b}}_i$) coefficients exhibit c ($\overline{c}$) dependence for $i\geq4$ (i.e., one order earlier than in the case of the B and $\overline{B}$ functions). Note that of the two-loop level, Eq. (6.32) coincides with Eq. (6.27).

VII. IMPROVED PERTURBATION THEORY FOR THE β FUNCTION

The improvement methods of Sec. VI may be criticized on the grounds they make use of RG functions (B and $\tilde{B}$) which depend explicitly on the parameter c at sufficiently high order. In this section, I describe a different (and in fact simpler) approach, which is free of this problem. The idea is the following: the relations (4.7) and (5.9) show that β^*_3/β^*_1 and β^*_4/β^*_1 behave, respectively, as c^2 and $-c^3$ for large c, and suggest that an improvement of the $\beta(R,c)$ series themselves should be possible, which would resum the leading c contributions to the β^*_i/β^*_1 coefficients. In fact, these relations suggest that it should be possible to determine the entire $\beta(R)$ function at $f=f^*$ in terms of the f derivatives β^*_{ik}, given the invariants $\{I_i\}$, i.e., given the critical exponent function $\omega(\alpha)$. A simple argument shows that this is indeed the case and yields as a by-product an improved perturbation expansion of the IR fixed point at $f\to f^*$ (Sec. III). One starts from the definition of the fixed point:

$$\beta(R^*,\alpha)=0\ . \qquad (7.1)$$

Equation (7.1) is an implicit equation for the function $R^*(\alpha)$ (it has been determined as a power series in α in the Appendix). Taking the total derivative of Eq. (7.1) with respect to α gives

$$\frac{\partial\beta}{\partial R^*}\frac{dR^*}{d\alpha}+\frac{\partial\beta}{\partial\alpha}=0\ ;$$

hence, since $\partial\beta/\partial R^*\equiv\omega(\alpha)$,

$$\omega(\alpha)\frac{dR^*}{d\alpha}=-\frac{\partial\beta}{\partial\alpha}(R^*,\alpha)\ . \qquad (7.2)$$

Equation (7.2) is a differential equation for $R^*(\alpha)$. Let us now split the $\alpha=0$ part of $\beta(R,\alpha)$, i.e., $\beta^*(R)$, from the part which contains the f derivatives,

$$\beta(R,\alpha)=\beta^*(R)-\alpha\gamma(R,\alpha)\ , \qquad (7.3)$$

which defines the $\gamma(R,\alpha)$ function:

$$\gamma(R,\alpha)=-\frac{\beta(R,\alpha)-\beta^*(R)}{\alpha}=-\beta_1^*R^2\left[1-\frac{\beta_{11}^*}{\beta_1^*}R-\frac{\beta_{21}^*+\beta_{22}^*\alpha+\beta_{23}^*\alpha^2}{\beta_1^*}R^2+\cdots\right]. \quad (7.4)$$

Equation (7.2) becomes

$$\frac{dR^*}{d\alpha}=\frac{\partial/\partial\alpha[\alpha\gamma(R^*,\alpha)]}{\omega(\alpha)}. \quad (7.5)$$

Next, I note that the knowledge of $R^*(\alpha)$ and of $\gamma(R,\alpha)$ determines $\beta^*(R)$. Indeed, let $\alpha^*(R)$ be the inverse function of $R^*(\alpha)$: for a given R, α^* is the value of α where $\beta(R,\alpha)=0$. From Eq. (7.3) one deduces

$$\beta^*(R)=\alpha^*(R)\gamma[R,\alpha^*(R)] \quad (7.6)$$

and

$$\beta(R,\alpha)=\alpha^*(R)\gamma[R,\alpha^*(R)]-\alpha\gamma(R,\alpha). \quad (7.7)$$

Equations (7.5) and (7.6) show that $\beta^*(R)$ is determined [given $\omega(\alpha)$] by the function $\gamma(R,\alpha)$. Furthermore, $\gamma(R,\alpha)$ does not depend on c, so that these equations can be used to determine $R^*(\alpha)$ [or $\alpha^*(R)$] and $\beta^*(R)$ even for large c, using ordinary perturbation theory for γ; i.e., one obtains an *improved perturbation theory for the fixed point $R^*(\alpha)$, as well as for $\beta^*(R)$*. It is more convenient to consider the equation for $\alpha^*(R)$, obtained by taking the inverse of both sides of Eq. (7.5):

$$\frac{d\alpha^*}{dR}=\frac{\omega(\alpha^*)}{\partial/\partial\alpha^*[\alpha^*\gamma(R,\alpha^*)]}. \quad (7.8)$$

The solution $\alpha^*(R)$ of Eq. (7.8) needs a boundary condition to be uniquely determined. This is provided by perturbation theory, according to which (see the Appendix)

$$\alpha^*(R)=R(1+\delta_1R+\cdots)$$

with

$$\delta_1=\frac{\beta_2^*+\beta_{11}^*}{\beta_1^*}. \quad (7.9)$$

Indeed, it is easy to check that Eqs. (7.8) and (7.9) uniquely determine the coefficient $\{\delta_i\}$ in the series $\alpha^*(R)=R(1+\delta_1R+\delta_2R^2+\cdots)$, for given ω and γ. In general, Eq. (7.8) has to be solved by iteration, except if $\gamma(R,\alpha)$ does not depend on α, i.e., if $\beta(R,\alpha)$ is linear in α. In this case Eq. (7.8) becomes

$$\frac{d\alpha^*}{dR}=\frac{\omega(\alpha^*)}{\gamma(R)} \quad (7.10)$$

which can be integrated in the standard way, with the boundary condition Eq. (7.9):

$$\delta_1=-\beta_1^*\left[\int^{\alpha^*}\frac{dx}{\omega(x)}-\int^{R}\frac{dx}{\gamma(x)}\right]. \quad (7.11)$$

One then deduces

$$\beta(R,\alpha)=[\alpha^*(R)-\alpha]\gamma(R) \quad (7.12)$$

which gives an improved form of β, useful for large $|\beta_2^*/\beta_1^*|$. I also note that if one defines a *running* coupling $\bar{\alpha}$ by the relation

$$\bar{\alpha}(R)|_{L=0}\equiv\alpha^*(R) \quad (7.13)$$

Eqs. (7.10) and (7.12) imply

$$\frac{d\bar{\alpha}}{d\ln Q^2}\equiv\bar{\beta}(\bar{\alpha})=(\bar{\alpha}-\alpha)\omega(\bar{\alpha}); \quad (7.14)$$

i.e., $\bar{\alpha}$ in fact coincides in the case where $\beta(R,\alpha)$ is linear in f with the universal coupling $\tilde{\alpha}$ of Eq. (6.22). It may be more convenient to predict $R(Q)$ using Eq. (7.11) together with Eq. (7.14) than from Eq. (7.12). These remarks apply in particular to the two-loop approximation, where $\omega(x)=\gamma(x)=-\beta_1^*x^2+\beta_{11}^*x^3$. Then Eq. (7.11) gives

$$\delta_1=\frac{\beta_2^*+\beta_{11}^*}{\beta_1^*}=\int_R^{\alpha^*}\frac{dx}{x^2-(\beta_{11}^*/\beta_1^*)x^3}. \quad (7.15)$$

Equation (7.14) (in the 2-loop approximation) and (7.15) are equivalent to Eqs. (6.23) and (6.24) [with the identification $\rho(R)\equiv\alpha^*(R)=\tilde{\alpha}(R)|_{L=0}$]. In fact, in the case where the β function is linear in f, the B, $\tilde{B}$, and γ functions all coincide, and consequently the improvements methods of Secs. VI and VII are identical in this case.

VIII. CONCLUSION

The three different methods which have been presented to ameliorate the perturbation expansion in case of large BLM coefficient r_1^* all rely on the following assumptions: (i) the $\bar{\beta}_i$ coefficients are "large," and (ii) they have weak f dependence. This means in particular these methods may be useful if one finds that $|\bar{\beta}_2(f)|\gg 1$, with $\bar{\beta}_2(f=0)\approx\bar{\beta}_2(f=f^*)$. I emphasize the assumption of weak f dependence of $\bar{\beta}_2$ is stronger than just assuming weak f dependence of r_2, as shown by Eqs. (4.11)–(4.14). It may also be worth pointing out the present methods exploit, and preserve, the polynomial f dependence of the r_i's in contrast with more standard procedures. The latter yield estimates such as $r_2=r_1^2+(\beta_1/\beta_0)r_1$ (if $\beta_2,\bar{\beta}_2$ are neglected), or $r_2=(\bar{\beta}_2-\beta_2)/\beta_0$ [if the so-called "fastest apparent convergence" (FAC) choice [7] of μ, $r_1(\mu)=0$, is adopted] which are singular in the limit $\beta_0\rightarrow 0$. Finally, I note all these methods are specific of QCD, and cannot be applied to QED since the relation (1.4) between r_1^* and the β functions does not hold in QED. In the latter case, however, all β function coefficients vanish in the $f\rightarrow 0$ limit (the QED analogue of the $f\rightarrow f^*$ limit), in such a way the ratios $\bar{\beta}_i/\beta_0$ and β_i/β_0 stay finite. It follows that the standard method of effective charges [7] is applicable to improve the QED perturbative series for large BLM coefficient r_1^* (and works even in the $f\rightarrow 0$ limit).

APPENDIX: PERTURBATIVE EXPANSION OF THE IR FIXED POINT AT $f \to f^*$

In QCD (and more generally non-Abelian gauge theories coupled to fermions) the β function has the property that the one-loop coefficient β_0 vanishes, while the higher-order coefficients β_i's stay finite, as f approach the value f^* where β_0 changes sign. It follows that for $f \to f^*$, $\beta(a)$ possesses a nontrivial IR fixed point [15] a^*, such that $\beta(a^*)=0$, perturbatively calculable in powers of $(f-f^*)$. In perturbation theory, the condition $\beta(a^*)=0$ becomes

$$1+\frac{\beta_1}{\beta_0}a^*+\frac{\beta_2}{\beta_0}a^{*2}+\frac{\beta_3}{\beta_0}a^{*3}+\cdots=0 . \tag{A1}$$

In leading order, keeping only the first two terms in Eq. (A1) (two-loop β function), one gets $a^*=-\beta_0/\beta_1^* \equiv \alpha$. In higher orders, a systematic expansion of a^* in powers of α may be derived from Eq. (A1). Setting

$$a^*=\alpha(1+a_1\alpha+a_2\alpha^2+a_3\alpha^3+\cdots) \tag{A2}$$

the a_i's can be computed by inserting Eq. (A2) into Eq. (A1) and requiring the resulting series vanish order by order, when reexpanded in powers of α [taking into account that the β_i's are polynomials in α, see Eqs. (3.2), (3.3), and (3.10)]. One gets

$$a_1=-\frac{\beta_2^*+\beta_{11}^*}{\beta_1^*} ,$$

$$a_2=-\left[\frac{\beta_3^*}{\beta_1^*}-2\left[\frac{\beta_2^*}{\beta_1^*}\right]^2-\left[\frac{\beta_{11}^*}{\beta_1^*}\right]^2+\frac{\beta_{21}^*}{\beta_1^*}-3\frac{\beta_2^*}{\beta_1^*}\frac{\beta_{11}^*}{\beta_1^*}\right] , \tag{A3}$$

$\vdots$

The critical exponent $\omega(\alpha)=\partial\beta/\partial a|_{a=a^*}$ can also be computed as a power series in α. From the definition of $\omega(\alpha)$, one has

$$\begin{aligned}\omega(\alpha)&=-(2\beta_0 a^*+3\beta_1 a^{*2}+4\beta_2 a^{*3}\\&\quad+5\beta_3 a^{*4}+6\beta_4 a^{*5}+\cdots)\\&\equiv+\omega_0\alpha^2+\omega_1\alpha^3+\omega_2\alpha^4+\omega_3\alpha^5+\cdots .\end{aligned} \tag{A4}$$

Inserting Eq. (A2) into Eq. (A4) yields the result for the ω_i's Eq. (3.24). Since ω and α are universal, RS-invariant quantities, so are the ω_i's as checked in Sec. III, where they are related to the universal invariants $\{I_i\}$.

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