Quark mass matrices on a minimal parameter basis

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General features of three-family quark mass matrices M_u and M_d on a specific quark basis are discussed. This quark basis is defined as the basis on which a traceless matrix $iK \equiv M_u M_d - M_d M_u$ takes a diagonal form. In this mass matrix scheme, the number of independent parameters in M_u and M_d is the same as that of the observable quantities, i.e., ten parameters.

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I. INTRODUCTION

Enormous progress in the experimental study of Z and B decays during the past few years has brought a realistic study of the quark mass matrix model almost within our reach, at least as far as three family quarks are concerned. There will be a simpler and more beautiful description of the quark mass matrices behind such observed quark and lepton mass spectra and their mixings. If we can find it, it will offer a promising clue to the origin of families and the mass generation mechanism of quarks and leptons. By choosing a specific mass matrix frame, can we obtain such a beautiful description of the mass matrices? Some interesting quark mass matrix models with specific matrix forms have been proposed: for example, Fritzsch type [1], Stech type [2], radiative type [3], democratic type [4], and so on. However, they are not based on a general parametrization of quark mass matrices. Now, we want to make a phenomenological study based on a general parametrization of the mass matrices.

There is, however, an obstacle to our phenomenological search for quark mass matrix models. Suppose a set of up- and down-quark mass matrices M_u and M_d (M_u, M_d), which provide excellent predictions of the diagonalized quark mass matrices D_u and D_d and the Kobayashi-Maskawa (KM) [5] matrix V as

$$\begin{aligned} D_u &= U_u M_u U_u^\dagger \ , \\ D_d &= U_d M_d U_d^\dagger \ , \\ V &= U_u U_d^\dagger \ , \end{aligned} \tag{1.1}$$

where, for simplicity, we have considered a Hermitian quark mass matrix model. As is well known, a mass matrix model (M_u', M_d'), which is connected to (M_u, M_d) by the relations $M_u' = U_0 M_u U_0^\dagger$ and $M_d' = U_0 M_d U_0^\dagger$, where U_0 is an arbitrary unitary matrix, is equivalent to the model (M_u, M_d) as far as the physically observable quantities (i.e., D_u, D_d, and V) are concerned. Therefore, even if we find a set of quark mass matrices (M_u, M_d) which can provide predictions in excellent agreement with experiment, the excellent agreement does not always guarantee that this model is really true.

In general, in quark mass matrices (M_u, M_d) the number of independent mass matrix parameters is bigger than that of observable quantities. For example, in the case of a three-family Hermitian quark mass matrix model, we have, in general, 18 independent parameters in (M_u, M_d), while we have only ten observable quantities, i.e., six up- and down-quark masses $m(u_i) = (m_u, m_c, m_t)$ and $m(d_i) = (m_d, m_s, m_b)$ $(i = 1, 2, 3)$ and four independent parameters in a 3×3 KM matrix V. The remaining eight unobservable parameters come from arbitrariness of a choice of the unitary matrix U_0 in (1.1).

However, note that the number of the independent parameters in (M_u, M_d) depends on what quark basis is chosen. The number 18 stated above is the maximal number in the most general case of 3×3 Hermitian mass matrices. If we choose a special quark basis, we can decrease this number.

For example, if we choose a quark basis where up-quark mass matrix M_u takes the diagonal form D_u, we have seven independent parameters in M_d, so that we can possess the same number of the independent parameters as that of the observable quantities, i.e., three down-quark masses and four KM-matrix parameters. These seven independent parameters in M_d can easily be represented [6] by these seven observable quantities, i.e., three in D_d and four in V, because M_d is given by $M_d = V D_d V^\dagger$. If we put some ansatz on M_d, then we can obtain sum rules for down-quark masses $m(d_i)$ and KM matrix elements $|V_{ij}|$. For instance, as pointed out by Weinberg [7], if we put an ansatz $(M_d)_{11} = 0$, we get the well-known sum rule for the Cabibbo mixing $|V_{us}| \simeq \sqrt{m_d/m_s}$. Recently, Ma [8] has derived interesting sum rules on the basis of a model with $M_u = D_u$. More phenomenological characteristics of Ma's model have been studied by Lavoura [9]. (However, since M_d in his model is not Hermitian, our mass matrix description stated below does not include his model.)

Although we cannot rule out the possibility that nature chooses such a quark basis as $M_u = D_u$, such a model does not satisfy our present interest in the top quark mass, because such a model, in general, does not include up-quark mass parameters, especially top quark mass m_t (although the down-quark mass matrix M_d in Ma's model has included a parameter m_u/m_c). On the other hand, there is a traditional idea that M_u and M_d have the same structure, except that the values of the parameters in M_u are different from those in M_d in their magnitudes. In such a model, sum rules for KM matrix elements $|V_{ij}|$ will include up-quark masses $m(u_i)$ as well as down-quark masses $m(d_i)$.

The purpose of the present paper is to make a general study, from the phenomenological point of view, of 3 × 3 Hermitian quark mass matrices in a frame in which quark mass matrices have minimal parameters, i.e., ten parameters in (M_u, M_d), and in which M_u has the same matrix form with M_d except for values of mass matrix parameters, so that in our sum rules the mass parameters appear symmetric as $m(u_i) \leftrightarrow m(d_i)$.

As discussed in the next section, our quark mass matrix frame (quark basis) is defined as the frame in which a traceless matrix $iK \equiv M_uM_d - M_dM_u$ takes a diagonal form. In our formulation, one of the ten independent parameters, ε, is a parameter with an extremely small value, which is proportional to the rephasing-invariant quantity J [10]. Therefore, our mass matrix frame is convenient for studying the case of the limit $J \to 0$, i.e., the limit of no CP violation.

In Sec. II, the general formulation in our quark mass matrix frame is given. In Sec. III, sum rules for $|V_{us}|$, $|V_{cb}|$, and $|V_{ub}|$ in an interesting case of the parameters are discussed. Finally, Sec. IV is devoted to summary and discussion.

II. FORMULATION

In this section, we give a formulation for general 3 × 3 Hermitian mass matrices on our minimal parameter basis, on which a traceless matrix K defined by

$$iK \equiv M_uM_d - M_dM_u \tag{2.1}$$

takes a diagonal form D_K.

A. Expression of M_u and M_d

Since the determinant of K and the trace of K^2 are given by

$$\begin{aligned}\det K &\simeq 2u_3^2d_3^2u_2d_2J \\ &\simeq 2u_3^2d_3^2u_2d_2|V_{us}||V_{cb}||V_{ub}|\sin\delta_{13} ,\end{aligned} \tag{2.2}$$

$$\operatorname{tr} K^2 \simeq 2u_3^2d_3^2|V_{cb}|^2 , \tag{2.3}$$

where u_i and d_i denote $m(u_i)$ and $m(d_i)$, respectively, J is the rephasing-invariant quantity as a measure of CP violation [10], and δ_{13} is a CP-violation phase parameter in the standard parametrization [11] of the KM matrix V. [The derivations of (2.2) and (2.3) are given in Appendix A.] The fact that the present experiments give a very small value (of the order of 10^{-4}) for the ratio $\det K/(\operatorname{tr} K^2)^{3/2}$ suggests that one of the eigenvalues of the traceless matrix iK is extremely small compared to the other two, i.e., the diagonalized matrix of K, D_K, takes the form

$$iD_K = ik\left\{\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix} + \varepsilon\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}\right\} , \tag{2.4}$$

where ε is a parameter with a small value, which is given by

$$\varepsilon \simeq \sqrt{2}\frac{\det K}{(\operatorname{tr} K^2)^{3/2}} \simeq \frac{u_2}{u_3}\frac{d_2}{d_3}\frac{|V_{us}||V_{ub}|}{|V_{cb}|^2}\sin\delta_{13} . \tag{2.5}$$

Since $\operatorname{tr}(M_uK) = 0$ and $\operatorname{tr}(M_dK) = 0$, the quark mass matrices M_q ($q = u$ and d) on the $K = D_K$ basis must take the form

$$\begin{aligned}M_q = \tfrac{1}{2}m_1^q\begin{pmatrix}1-\varepsilon & \sqrt{1-\varepsilon^2}a_q & \sqrt{2\varepsilon}\sqrt{1+\varepsilon}c_q^* \\ \sqrt{1-\varepsilon^2}a_q^* & 1+\varepsilon & \sqrt{2\varepsilon}\sqrt{1-\varepsilon}b_q \\ \sqrt{2\varepsilon}\sqrt{1+\varepsilon}c_q & \sqrt{2\varepsilon}\sqrt{1-\varepsilon}b_q^* & 0\end{pmatrix} \\ +m_0^q\,\mathbb{1} ,\end{aligned} \tag{2.6}$$

where the complex parameters a, b, and c must satisfy the subsidiary conditions from the diagonalization condition of $(M_uM_d - M_dM_u)$

$$a_u - a_d = b_u^*c_d^* - b_d^*c_u^* , \tag{2.7}$$

$$b_u - b_d = a_u^*c_d^* - a_d^*c_u^* , \tag{2.8}$$

$$c_u - c_d = a_u^*b_d^* - a_d^*b_u^* . \tag{2.9}$$

The mass matrices M_u and M_d in (2.6) have included four real parameters m_1^q and m_0^q ($q = u, d$), six complex parameters a_q, b_q, and c_q ($q = u, d$), and one real parameter ε. Of the six phase parameters $\alpha_q \equiv \arg a_q$, $\beta_q \equiv \arg b_q$, and $\gamma_q \equiv \arg c_q$, however, only four parameters,

$$\alpha \equiv \alpha_u - \alpha_d , \quad \beta \equiv \beta_u - \beta_d , \quad \gamma \equiv \gamma_u - \gamma_d , \tag{2.10}$$

and

$$\phi \equiv \tfrac{1}{2}(\alpha_u + \alpha_d + \beta_u + \beta_d + \gamma_u + \gamma_d) , \tag{2.11}$$

play a substantial role in the predictions of the observable quantities $m(u_i)$, $m(d_i)$, and $|V_{ij}|$ $(i, j = 1, 2, 3)$. Therefore, we have 15 parameters in M_u and M_d. On the other hand, of the subsidiary conditions (2.7)–(2.9), there are five independent subsidiary conditions (in terms of real parameters). In conclusion, we have ten independent parameters in M_u and M_d on our quark basis.

B. Subsidiary conditions

Putting (2.8) and (2.9) into (2.7), we obtain

$$|a_u - a_d|^2 = |b_u - b_d|^2 - |c_u - c_d|^2 . \tag{2.12}$$

From (2.8)×$(b_u^* + b_d^*)$−(2.9)×$(c_u^* + c_d^*)$, we obtain

$$\begin{aligned}(a_u - a_d)(a_u^* + a_d^*) = {}& (b_u - b_d)(b_u^* + b_d^*) \\ &-(c_u - c_d)(c_u^* + c_d^*) .\end{aligned} \tag{2.13}$$

Eliminating $(a_u - a_d)$, $(b_u - b_d)$, and $(c_u - c_d)$ from (2.7)–(2.9), we obtain

$$|a_u + a_d|^2 - 4 = |b_u + b_d|^2 - |c_u + c_d|^2 . \tag{2.14}$$

Hereafter, we indicate $|a_q|$, $|b_q|$, and $|c_q|$ simply by a_q, b_q, and c_q. Relations (2.13) and (2.14) lead to the following subsidiary conditions in terms of real parameters:

$$a_u^2 - 1 = b_u^2 - c_u^2 \ , \tag{2.15}$$

$$a_d^2 - 1 = b_d^2 - c_d^2 \ , \tag{2.16}$$

$$a_u a_d \cos\alpha - 1 = b_u b_d \cos\beta - c_u c_d \cos\gamma \ , \tag{2.17}$$

$$a_u a_d \sin\alpha = b_u b_d \sin\beta - c_u c_d \sin\gamma \ . \tag{2.18}$$

The remaining subsidiary condition which is independent of (2.15)–(2.18) is obtained from (2.7): i.e.,

$$(a_u - a_d)\cos\frac{\alpha}{2}\cos\phi - (a_u + a_d)\sin\frac{\alpha}{2}\sin\phi = (b_u c_d - b_d c_u)\cos\frac{\beta-\gamma}{2} \ , \tag{2.19}$$

or

$$(a_u - a_d)\cos\frac{\alpha}{2}\sin\phi + (a_u + a_d)\sin\frac{\alpha}{2}\cos\phi = -(b_u c_d + b_d c_u)\sin\frac{\beta-\gamma}{2} \ . \tag{2.20}$$

Note that only one of the conditions (2.19) and (2.20) is independent of (2.15)–(2.18), because $(2.17)^2 + (2.18)^2$ and $(2.19)^2 + (2.20)^2$ lead to the same relation:

$$(a_u - a_d)^2 + 4 a_u a_d \sin^2\frac{\alpha}{2} = (b_u c_d - b_d c_u)^2 + 4 b_u b_d c_u c_d \sin^2\frac{\beta-\gamma}{2} \ . \tag{2.21}$$

Then, the matrix iD_K is given by (2.4), where

$$k = \tfrac{1}{2} m_1^u m_1^d \left[a_u a_d \sin\alpha - \varepsilon(b_u b_d \sin\beta + c_u c_d \sin\gamma)\right] \ . \tag{2.22}$$

Now we must represent the observable quantities $m(u_i)$, $m(d_i)$, and $|V_{ij}|$ in terms of these mass matrix parameters under the subsidiary conditions (2.15)–(2.20).

C. Quark masses

Quark masses $q_i \equiv m(q_i)$ $(q = u, d;\ i = 1, 2, 3)$ are obtained from the calculation of $\mathrm{tr}\, M_q$, $\mathrm{tr}\, M_q^2$, and $\det M_q$.

It is convenient to introduce the following matrices M_q^0 and D_q^0:

$$\begin{aligned} M_q &= M_q^0 + m_0^q\, \mathbb{1} \ , \\ D_q &= D_q^0 + m_0^q\, \mathbb{1} \ . \end{aligned} \tag{2.23}$$

We can easily calculate the case of $m_0^q = 0$. Then, results in the case of $m_0^q \neq 0$ are obtained from the replacement

$$q_i\ (\equiv q_i^0) \ \to q_i - m_0^q \ , \tag{2.24}$$

for the results in the case of $m_0^q = 0$, where q_i^0 are q_i in the limit of $m_0^q = 0$.

The sum rules for the quark masses q_i^0 are

$$q_3^0 + q_2^0 + q_1^0 = \mathrm{tr}\, M_q^0 = m_1^q \ , \tag{2.25}$$

$$\frac{q_3^0 q_2^0 + q_3^0 q_1^0 + q_2^0 q_1^0}{(q_3^0 + q_2^0 + q_1^0)^2} = \frac{\mathrm{csi}\, M_q^0}{(\mathrm{tr}\, M_q^0)^2} = \frac{1}{4}\left[(1 - 3\varepsilon^2)(1 - a_q^2) - 2\varepsilon(b_q^2 + c_q^2)\right] \ , \tag{2.26}$$

$$\frac{q_3^0 q_2^0 q_1^0}{(q_3^0 + q_2^0 + q_1^0)^3} = \frac{\det M_q^0}{(\mathrm{tr}\, M_q^0)^3} = -\tfrac{1}{4}\varepsilon\left[\,(1+\varepsilon^2)(b_q^2 + c_q^2) - 2(1-\varepsilon^2) a_q b_q c_q \cos\psi_q + 2\varepsilon(1 - a_q^2)\right] \ , \tag{2.27}$$

where the notation $\mathrm{csi}\, A$ is a function of the matrix A which is defined by Lavoura [12] as

$$\mathrm{csi}\, A \equiv \tfrac{1}{2}[(\mathrm{tr}\, A)^2 - \mathrm{tr}\, A^2] \tag{2.28}$$

and the phase parameters ψ_q denote

$$\begin{aligned} \psi_u &\equiv \alpha_u + \beta_u + \gamma_u = \phi + \tfrac{1}{2}(\alpha + \beta + \gamma) \ , \\ \psi_d &\equiv \alpha_d + \beta_d + \gamma_d = \phi - \tfrac{1}{2}(\alpha + \beta + \gamma) \ . \end{aligned} \tag{2.29}$$

D. KM matrix elements

The magnitudes of KM matrix elements $|V_{ij}|$ are obtained from the calculation of $\mathrm{tr}\,(M_u M_d)$, $\mathrm{tr}\,(M_u^2 M_d)$, $\mathrm{tr}\,(M_u M_d^2)$, and $\mathrm{tr}\,(M_u^2 M_d^2)$ as follows:

$$|V_{ij}|^2 - \delta_{ij} = \frac{J^{22} + u_i J^{12} + d_j J^{21} + u_i d_j J^{11}}{(u_i - u_k)(u_i - u_l)(d_j - d_m)(d_j - d_n)} \quad (i \neq k \neq l \neq i \text{ and } j \neq m \neq n \neq j) \ , \tag{2.30}$$

where

$$\begin{aligned} J^{11} &\equiv I^{11} \ , \\ J^{12} &\equiv I^{12} - \mathrm{tr}\,(D_d) I^{11} \ , \\ J^{21} &\equiv I^{21} - \mathrm{tr}\,(D_u) I^{11} \ , \\ J^{22} &\equiv I^{22} - \mathrm{tr}\,(D_u) I^{12} - \mathrm{tr}\,(D_d) I^{21} \\ &\quad + \mathrm{tr}\,(D_u)\mathrm{tr}\,(D_d) I^{11} \ , \end{aligned} \tag{2.31}$$

$$I^{mn} \equiv \mathrm{tr}\,(M_u^m M_d^n) - \mathrm{tr}\,(D_u^m D_d^n) \ . \tag{2.32}$$

The derivation of the formulas (2.30) is given in Appendix B. The importance of denoting $|V_{ij}|^2$ in terms of $\mathrm{tr}\,(M_u^m M_d^n)$ has been stressed by Hamzaoui [13]. From (2.30), we can readily express $|V_{ij}|^2$ in terms of quark masses and $\mathrm{tr}\,(M_u^m M_d^n)$ $(m, n = 1, 2)$. However, since $\mathrm{tr}\,(M_u^m M_d^n) \sim u_3^m d_3^n$, the numerical values of $|V_{ij}|^2$ are sensitive to the deviations of $\mathrm{tr}\,(M_u^m M_d^n)$ from $\mathrm{tr}\,(D_u^m D_d^n)$. Therefore, the expression (2.30) in terms of I^{mn} will be convenient for numerical study.

If we define a parameter ω, which gives deviation from the symmetric KM matrix [14], i.e., V with $|V_{ij}| = |V_{ji}|$, as

$$\omega \equiv |V_{21}|^2 - |V_{12}|^2 = |V_{32}|^2 - |V_{23}|^2 = |V_{13}|^2 - |V_{31}|^2 \ , \quad (2.33)$$

then any matrix elements V_{ij} are given in terms of four independent parameters $|V_{12}|$, $|V_{23}|$, $|V_{13}|$, and ω [15]. Therefore, four expressions of $|V_{12}|^2$, $|V_{13}|^2$, $|V_{21}|^2$, and $|V_{23}|^2$ obtained from (2.30) are sufficient to calculate every $|V_{ij}|^2$.

Exact expressions of $\mathrm{tr}\,(M_u^m M_d^n)$ $(m,n = 1,2)$ are given in Appendix C. As seen in Appendix C, the exact expression of J^{22} consists of somewhat complicated terms. Therefore, for calculation of $|V_{us}|$ and $|V_{cb}|$, it is convenient to use the approximations

$$\begin{aligned} J^{21} &= u_3(u_2-u_1)(d_2-d_1)|V_{12}|^2 + u_1(u_3-u_2)(d_3-d_2)|V_{23}|^2 \\ &\quad + u_2(u_3-u_1)(d_3-d_1)|V_{13}|^2 - u_1(u_3-u_2)(d_2-d_1)\omega \\ &\simeq u_3 u_2 d_2 |V_{12}|^2 \end{aligned} \quad (2.34)$$

and

$$\begin{aligned} J^{11} &= -(u_2-u_1)(d_2-d_1)|V_{12}|^2 - (u_3-u_2)(d_3-d_2)|V_{23}|^2 \\ &\quad -(u_3-u_1)(d_3-d_1)|V_{13}|^2 + (u_3-u_2)(d_2-d_1)\omega \\ &\simeq -u_3 u_2 |V_{23}|^2 \end{aligned} \quad (2.35)$$

rather than to use the exact expression (2.30) which includes J^{22}. However, if we can calculate J^{22} with a good approximation, the use of the relation

$$\begin{aligned} J^{22} &= -u_3 d_3(u_2-u_1)(d_2-d_1)|V_{12}|^2 - u_1 d_1(u_3-u_2)(d_3-d_2)|V_{23}|^2 \\ &\quad -u_2 d_2(u_3-u_1)(d_3-d_1)|V_{13}|^2 + u_1 d_3(u_3-u_2)(d_2-d_1)\omega \\ &\simeq -u_3 d_3 u_2 d_2 |V_{12}|^2 \end{aligned} \quad (2.36)$$

is also useful. [For (2.34)–(2.36), see Appendix B.]

Finally, we would like to note that $|V_{ij}|$ are independent of a choice of m_0^u and m_0^d.

III. THE $m_0^u = m_0^d = 0$ CASE

In this section, we investigate an interesting case

$$m_0^u = m_0^d = 0 \ . \quad (3.1)$$

A. Rough estimates of a_u, b_u, and c_u

In the case (3.1), from (2.26) and (2.27), we obtain

$$\frac{q_2}{q_3} \simeq \tfrac{1}{4}(1-a_q^2) \quad (3.2)$$

and

$$\frac{q_1 q_2}{q_3^2} \simeq -\tfrac{1}{4}\varepsilon(b_q^2 + c_q^2 - 2a_q b_q c_q \cos\psi_q) \ , \quad (3.3)$$

respectively. Here, since experimental values [16,17] of u_2/u_3 and d_2/d_3 [see Appendix A, (A8)] give $|1-a_u^2| \simeq 0.016$ and $|1-a_d^2| \simeq 0.13$, and the value of ε, which is given by (2.5), is smaller than the order of 10^{-4}, we have assumed that $|1-a_q^2| \gg |\varepsilon|(b_q^2+c_q^2)$.

We can show that in (3.3) the factor $(b_u^2 + c_u^2 - 2a_u b_u c_u \cos\psi_u)$ must be larger than $|1-a_u^2|\sqrt{|1-a_u^2\cos^2\psi_u|}$ for any values of b_u and c_u under the condition (2.15), so that we obtain the restriction

$$\left|\frac{u_1}{u_3}\right| > |\varepsilon|\sqrt{|1-a_u^2\cos^2\psi_u|} \quad (3.4)$$

from (3.3). The restriction (3.4) suggests that $\psi_u \simeq 0$ since $|u_1/u_3| \sim 2\times 10^{-5}$, while $|\varepsilon| \sim 10^{-4} \times |\sin\delta_{13}|$ (we consider $|\sin\delta_{13}| \sim 1$). (For ψ_d, such a restriction is not obtained.) Then, (3.3) leads to

$$\frac{u_1 u_2}{u_3^2} \simeq -\tfrac{1}{4}\varepsilon(b_u - c_u)^2 \ , \quad (3.5)$$

so that we can roughly estimate

$$b_u \simeq \sqrt{-\varepsilon\frac{u_2}{u_1}}\left(1 + \frac{4u_1u_2}{\varepsilon u_3^2}\right) \ , \quad (3.6a)$$

$$c_u \simeq \sqrt{-\varepsilon\frac{u_2}{u_1}}\left(1 - \frac{4u_1u_2}{\varepsilon u_3^2}\right) \ , \quad (3.6b)$$

where $\sqrt{-\varepsilon u_2/u_1} \sim 0.1$ and $|4u_1u_2/\varepsilon u_3^2| \sim 0.6\times 10^{-3}$ for $\sin\delta_{13} \simeq 1$.

B. Sum rule for $|V_{us}|^2$

In order to estimate $|V_{us}|$ by using the relation (2.34), we calculate J^{21}. From (2.26) and (2.27), we get the relation

$$\frac{u_3u_2+u_3u_1+u_2u_1}{(u_3+u_2+u_1)^2}-\frac{u_3u_2u_1}{(u_3+u_2+u_1)^3}=\tfrac{1}{4}(1-\varepsilon^2)[1-a_u^2-\varepsilon(b_u^2+c_u^2)-2\varepsilon a_ub_uc_u\cos\psi_u]\ . \tag{3.7}$$

By using (C1) and (C2) in Appendix C and (3.7), we obtain

$$\begin{aligned}\mathrm{tr}\,(M_u^2M_d)-\mathrm{tr}\,M_u\mathrm{tr}\,(M_uM_d)&=-\frac{1}{4}(m_1^u)^2m_1^d\{(1-\varepsilon^2)[1-a_u^2-\varepsilon(b_u^2+c_u^2)]\\&\quad+2\varepsilon(1+\varepsilon^2)(b_ub_d\cos\beta+c_uc_d\cos\gamma)+4\varepsilon(1-a_ua_d\cos\alpha)\\&\quad-2\varepsilon(1-\varepsilon^2)[a_ub_uc_d\cos(\psi_u-\gamma)\\&\quad+a_ub_dc_u\cos(\psi_u-\beta)+a_db_uc_u\cos(\psi_u-\alpha)]\}\\&=-\left(u_3u_2+u_3u_1+u_2u_1-\frac{u_3u_2u_1}{u_3+u_2+u_1}\right)(d_3+d_2+d_1)-(m_1^u)^2m_1^d\Delta^{21}\ ,\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}\Delta^{21}=&\ \tfrac{1}{2}\varepsilon(1+\varepsilon^2)(b_ub_d\cos\beta+c_uc_d\cos\gamma)+\varepsilon^2(1-a_ua_d\cos\alpha)\\&+\tfrac{1}{2}\varepsilon(1-\varepsilon^2)\{[a_u\cos\psi_u-a_d\cos(\psi_u-\alpha)]b_uc_u-a_u[b_uc_d\cos(\psi_u-\gamma)+b_dc_u\cos(\psi_u-\beta)]\}\ .\end{aligned} \tag{3.9}$$

On the other hand, $\mathrm{tr}\,(D_u^2D_d)-\mathrm{tr}\,D_u\mathrm{tr}\,(D_uD_d)$ is given by

$$\mathrm{tr}\,(D_u^2D_d)-\mathrm{tr}\,D_u\mathrm{tr}\,(D_uD_d)=-u_3u_2(d_3+d_2)-u_3u_2(d_3+d_1)-u_2u_1(d_2+d_1)\ , \tag{3.10}$$

so that we obtain

$$J^{21}=-u_3(u_2d_1+u_1d_2)+\frac{u_2u_1}{u_3+u_2+u_1}[u_3(d_2+d_1)-(u_2+u_1)d_3]-(m_1^u)^2m_1^d\Delta^{21}\ . \tag{3.11}$$

In order to estimate the Δ^{21} term exactly, we must assume an explicit model with specific values of the parameters. However, when we use (3.1)–(3.4), we can roughly estimate the Δ^{21} term. Under the approximation $\psi_u\simeq 0$, (3.9) becomes

$$\begin{aligned}\Delta^{21}&\simeq\tfrac{1}{2}\varepsilon\,[(b_u-c_u)(b_d\cos\beta-c_d\cos\gamma)+(a_u-a_d\cos\alpha)b_uc_u]\\&\simeq\tfrac{1}{2}\varepsilon\left[\sqrt{-\frac{4u_1u_2}{\varepsilon u_3^2}}\,\frac{1-a_ua_d\cos\alpha}{\sqrt{-\varepsilon u_2/u_1}}-(a_u-a_d\cos\alpha)\varepsilon\frac{u_2}{u_1}\right]\\&\simeq-2\frac{u_1d_2}{u_3d_3}\left(1+\tfrac{1}{2}\varepsilon^2\frac{u_2u_3}{u_1^2}\right)\ ,\end{aligned} \tag{3.12}$$

where we have used $\cos\alpha\simeq 1$ [see (3.24)] and $u_1<0$. The factor $\varepsilon^2u_2u_3/2u_1^2$ in (3.12) is the order of $u_2d_2^2/8u_1d_3^2\simeq 0.036$ for $\sin\delta_{13}\sim 1$. Therefore, the Δ^{21} term is negligibly small compared to $u_3u_2d_1$ in (3.11), but it cannot be neglected compared to $u_3u_1d_2$, i.e.,

$$J^{21}\simeq-u_3(u_2d_1+u_1d_2)+2u_3u_1d_2\ . \tag{3.13}$$

Then, comparing (3.11) with (2.34), we can obtain a sum rule

$$|V_{us}|\simeq\sqrt{-\left(\frac{d_1}{d_2}-\frac{u_1}{u_2}\right)}\simeq 0.22\ , \tag{3.14}$$

where we have used [16] $d_1\simeq -0.0089$ GeV, $d_2\simeq 0.175$ GeV, $u_1\simeq -0.0051$ GeV, and $u_2\simeq 1.35$ GeV.

C. Sum rule for $\omega\equiv|V_{21}|^2-|V_{12}|^2$

We adopt the parametrization [15] of the KM matrix in terms of the four independent parameters $|V_{us}|$, $|V_{cb}|$, $|V_{ub}|$, and $\omega\equiv|V_{cd}|^2-|V_{us}|^2$. Then, the rephasing-invariant J can be described by these four parameres [18], so that the CP-violation phase parameter δ_{13} is expressed by these four parameters. Since we have already known the values of $|V_{us}|$, $|V_{cb}|$, and $|V_{ub}|$, we now take a great interest in the value of the fourth parameter ω related to estimating the magnitude of CP nonconservation effects.

In a way similar to (3.11), we can obtain

$$J^{12}=-d_3(u_2d_1+u_1d_2)-\frac{d_2d_1}{d_3+d_2+d_1}[u_3(d_2+d_1)-(u_2+u_1)d_3]-m_1^u(m_1^d)^2\Delta^{12}\ , \tag{3.15}$$

where

$$\Delta^{12} = \tfrac{1}{2}\varepsilon(1+\varepsilon^2)(b_u b_d \cos\beta + c_u c_d \cos\gamma) + \varepsilon^2(1 - a_u a_d \cos\alpha)$$
$$+\tfrac{1}{2}\varepsilon(1-\varepsilon^2)\{[a_d\cos\psi_d - a_u\cos(\psi_d+\alpha)]b_d c_d - a_d[b_d c_u \cos(\psi_d+\gamma) + b_u c_d \cos(\psi_d+\beta)]\} \,. \tag{3.16}$$

From (3.11) and (3.16), we obtain

$$d_3 J^{21} - u_3 J^{12} = [u_3(d_2+d_1) - (u_2+u_1)d_3]\left(\frac{u_2 u_1 d_3}{u_3+u_2+u_1} + \frac{u_3 d_2 d_1}{d_3+d_2+d_1}\right)$$
$$-m_1^u m_1^d[(u_3+u_2+u_1)d_3\Delta^{21} - u_3(d_3+d_2+d_1)\Delta^{12}] \,. \tag{3.17}$$

On the other hand, since the left-hand side of (3.17) is approximately given by

$$d_3 J^{21} - u_3 J^{12} \simeq -u_3^2 d_3 d_2 \left(\frac{d_1}{d_2}|V_{23}|^2 + |V_{13}|^2 - \omega\right) \,, \tag{3.18}$$

comparing (3.17) with (3.18) we obtain a sum rule

$$\frac{d_1}{d_2}|V_{cb}|^2 + |V_{ub}|^2 - \omega \simeq -\frac{d_1}{d_2}\left(\frac{d_2}{d_3}\right)^2 \,, \tag{3.19}$$

or

$$-\omega \simeq |V_{us}|^2\left[|V_{cb}|^2 + \left(\frac{d_2}{d_3}\right)^2\right] - |V_{ub}|^2 \,. \tag{3.20}$$

Here, we have neglected the second term on the right-hand side of (3.17).

The first term in (3.17) is the order of $(u_3 d_2)(u_3 d_2 d_1/d_3) \sim u_3^2 d_3^2 \times 10^{-6}$, while the second term is of the order of $u_3^2 d_3^2(\Delta^{21} - \Delta^{12})$. If we estimate $(\Delta^{21} - \Delta^{12})$ optimistically, we get

$$(\Delta^{21} - \Delta^{12}) \sim \tfrac{1}{2}\varepsilon(a_d - a_u)(b_d - b_u)(c_d - c_u)$$
$$\sim \tfrac{1}{2}\varepsilon(a_d - a_u)^3 \sim 10^{-8} \,, \tag{3.21}$$

where we have used the relation $(a_d - a_u) \simeq 2d_2/d_3 \simeq 0.066$ from (3.2). However, the factor $(\Delta^{21} - \Delta^{12})$ can maximally be of the order of $|\varepsilon| \sim 10^{-4}$. Therefore, the numerical result from (3.17) should not be taken rigidly. The second term in (3.17) can, in general, contribute to estimates of such small quantities as ω.

D. Sum rule for $|V_{cb}|^2$

For estimate of J^{11}, it is convenient to use the relation

$$\frac{q_3 q_2 + q_3 q_1 + q_2 q_1}{(q_3+q_2+q_1)^2} - 2\frac{q_3 q_2 q_1}{(q_3+q_2+q_1)^3}$$
$$= \tfrac{1}{4}(1+\varepsilon^2)(1-a_q^2) - \varepsilon(1-\varepsilon^2)a_q b_q c_q \cos\psi_q \,. \tag{3.22}$$

Then, we obtain

$$J^{11} = u_3(d_2+d_1) + (u_2+u_1)d_3 + u_2 d_1 + u_1 d_2$$
$$-\tfrac{1}{2}m_1^u m_1^d\left[(1-3\varepsilon^2)\left(1 - a_u a_d + 2a_u a_d \sin^2\frac{\alpha}{2}\right) - 2\varepsilon(b_u b_d\cos\beta + c_u c_d \cos\gamma)\right]$$
$$\simeq u_3 d_3\left[\left(\frac{u_2}{u_3} + \frac{d_2}{d_3}\right)\left(\frac{u_1}{u_3} + \frac{d_1}{d_3}\right) - \sin^2\frac{\alpha}{2} + \cdots\right] \,. \tag{3.23}$$

Therefore, we obtain

$$|V_{cb}|^2 \simeq \sin^2\frac{\alpha}{2} - \frac{d_2 d_1}{d_3^2} \,. \tag{3.24}$$

Since the present data show $|V_{cb}|^2 \gg |d_2 d_1/d_3^2|$, the dominant term in the right-hand side of (3.24) must be $\sin^2(\alpha/2)$.

The result (3.24), i.e., $|V_{cb}| \simeq |\sin(\alpha/2)|$, suggests that a model with $\alpha = 0$ leads to a prediction of $|V_{cb}|$ which is in poor agreement with experiment, so that the model is ruled out. A model with $\beta = \gamma = 0$ leads to $\alpha = 0$ by the subsidiary conditions (2.15)–(2.20), so that such a model is also ruled out.

E. Democratic-type matrix form

Recently, considerable interest in the democratic-type mass matrices has been taken. We would like to comment on a relation between our mass matrix expression (2.6) and a democratic type.

We consider a unitary matrix

$$U = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \end{pmatrix} \,, \tag{3.25}$$

which transforms democratic-type matrices X and Y as

$$UXU^\dagger = \tfrac{1}{3}Y + \frac{1}{3}\begin{pmatrix} 1-\sqrt{2} & 0 & 0 \\ 0 & 1+\sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \,, \tag{3.26}$$

$$UYU^\dagger = Y \,, \tag{3.27}$$

where the democratic-type matrices X and Y are defined by

$$X = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \,, \quad Y = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \,. \tag{3.28}$$

Then, the dominant term in our mass matrix (2.6) (i.e., the remaining term in the limit of $\varepsilon \to 0$) is transformed into a real matrix form as

$$U\begin{pmatrix} 1 & a_q e^{i\alpha_q} & 0 \\ a_q e^{-i\alpha_q} & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} U^\dagger = (1+a_q\cos\alpha_q)Y + (1-a_q\cos\alpha_q)\begin{pmatrix} 0&0&0\\0&0&0\\0&0&1\end{pmatrix} - \frac{1}{\sqrt{2}}a_q\sin\alpha_q\begin{pmatrix}0&0&1\\0&0&1\\1&1&0\end{pmatrix} . \quad (3.29)$$

It should be noticed that the $(M_q)_{ij}$ elements $(i,j=1,2)$ which include the phase factor α_q are transformed into real matrix elements. The imaginary parts come from the remaining terms $(M_q)_{3i}$ and $(M_q)_{i3}$, which include the phase factors β_q and γ_q.

If we put an ansatz that the left-hand side of (3.29) is given only by democratic-type matrices X and Y, we obtain the restriction

$$1-a_q\cos\alpha_q = -\frac{1}{\sqrt{2}}a_q\sin\alpha_q , \quad (3.30)$$

i.e.,

$$\tan\frac{\alpha_q}{2} = \frac{a_q}{\sqrt{2}(1+a_q)}\left(\sqrt{1-2\frac{1-a_q^2}{a_q^2}}-1\right) \simeq -\frac{1}{\sqrt{2}}(1-a_q) \simeq -\sqrt{2}\frac{q_2}{q_3} , \quad (3.31)$$

which leads to an excellent prediction

$$|V_{cb}| \simeq \left|\sin\frac{\alpha_u-\alpha_d}{2}\right| \simeq \sqrt{2}\left|\frac{d_2}{d_3}-\frac{u_2}{u_3}\right| \simeq 0.040 . \quad (3.32)$$

This sum rule (3.32) has been derived by Tanimoto [19] on the basis of the democratic-type mass matrix scheme

$$M_q = m_X^q X + m_Y^q Y + m_Z^q Z , \quad (3.33)$$

where the matrices X and Y are given by (3.28), the matrix Z is a constant traceless matrix, and their coefficients satisfy $m_X^q \gg m_Y^q \gg m_Z^q \simeq 0$. Our ansatz that the right-hand side of (3.29) should be expressed only in terms of the democratic-type matrices X and Y is essentially identical with the Tanimoto model, although $|m_X^q| = |1-a_q\cos\alpha_q| \ll |m_Y^q| = |1+a_q\cos\alpha_q|$ in our model, while $|m_X^q| \gg |m_Y^q|$ in the model of Tanimoto. Of course, this is not essential, because there is a unitary transformation which exchanges an X term for a Y term.

In addition, the successful derivation of the sum rule (3.32) seems to suggest that the following scenario is promising: the dominant terms, which provide $J=0$ and $u_1=d_1=0$, are given only by the democratic-type matrix X and the "partially" democratic-type matrix Y, and the effects of CP nonconservation $(J\neq 0)$ and nonvanishing first-family quark masses $(u_1\neq 0,\ d_1\neq 0)$ come from a third term with small parameter values and with a mass matrix form which violates democratic or partially democratic family mixing, for example the Z term in Tanimoto's model (3.33).

IV. SUMMARY AND DISCUSSION

We have studied 3×3 Hermitian quark mass matrices on a quark basis in which a traceless matrix $iK \equiv M_uM_d - M_dM_u$ takes a diagonal form and the number of independent parameters of M_u and M_d is the same as that of observable quantities.

One of the ten independent parameters, ε, is a parameter with an extremely small value, which is proportional to the rephasing-invariant quantity J, so that our mass matrix frame will be convenient for studying the case of the limit $J\to 0$, i.e., the limit of no CP violation.

For the case $1 \gg |1-a_q^2| \gg |\varepsilon|(b_q^2+c_q^2)$, (3.3), if we set $m_0^u = m_0^d = 0$, we can obtain an excellent sum rule for the Cabibbo mixing (3.14). This ansatz is substantially correspondent to the ansatz $(M_d)_{11}=0$ in another minimal parameter frame where $M_u = D_u$. The condition (3.3) for our parameters also leads to a sum rule (3.21) for small quantities such as $|V_{us}|^2|V_{cb}|^2$, $|V_{ub}|^2$, and ω. For a further detailed check on our sum rules, a numerical study by using a computer will be needed. Such a systematical search for possible numerical values of our parameters is a future task, because the purpose of the present paper is to give a general formulation of our mass matrix frame with $K=D_K$.

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APPENDIX A: CALCULATION OF det K AND TR K^2

The relation (2.2) is readily obtained from the definition of the rephasing-invariant quantity J [10]:

$$\det(M_uM_d - M_dM_u) \equiv i\det K \equiv 2i(u_3-u_2)(u_3-u_1)(u_2-u_1) \times(d_3-d_2)(d_3-d_1)(d_2-d_1)J. \quad (A1)$$

The derivation of (2.3) is somewhat intricate. From the general formula for arbitrary 3×3 matrices A and B,

$$\mathrm{tr}\,(A^2B^2) - \left[\mathrm{tr}\,A\,\mathrm{tr}\,(AB^2) + \mathrm{tr}\,B\,\mathrm{tr}\,(A^2B)\right] + \mathrm{tr}\,A\,\mathrm{tr}\,B\,\mathrm{tr}\,(AB) - \mathrm{csi}\,(AB) - \mathrm{csi}\,A\,\mathrm{csi}\,B = 0 , \quad (A2)$$

where $\mathrm{csi}\,A$ is Lavoura's function [12] for a matrix A defined by (2.28), we obtain

$$\mathrm{tr}\,(M_u^2M_d^2) - \left[\mathrm{tr}\,D_u\mathrm{tr}\,(M_uM_d^2) + \mathrm{tr}\,D_d\mathrm{tr}\,(M_u^2M_d)\right] + \mathrm{tr}\,D_u\mathrm{tr}\,D_d\mathrm{tr}\,(M_uM_d) - \mathrm{csi}\,(M_uM_d) - \mathrm{csi}\,D_u\mathrm{csi}\,D_d = 0 , \quad (A3)$$

which leads to

$$I^{22} - \left(I^{12}\mathrm{tr}\, D_u + I^{21}\mathrm{tr}\, D_d\right) + I^{11}\mathrm{tr}\, D_u \mathrm{tr}\, D_d + \tfrac{1}{2}\left(I^{22} - \tfrac{1}{2}\mathrm{tr}\, K^2\right) - \tfrac{1}{2}I^{11}\left[I^{11} + 2\,\mathrm{tr}\,(D_u D_d)\right] = 0 \ , \tag{A4}$$

where

$$\mathrm{tr}\, K^2 = -2\left[\mathrm{tr}\,(M_u M_d M_u M_d) - \mathrm{tr}\,(M_u^2 M_d^2)\right] \ , \tag{A5}$$

and I^{mn} $(m,n = 1,2)$ are defined by (2.32). Therefore, we obtain

$$\begin{aligned}\mathrm{tr}\, K^2 &= 6I^{22} - 4\left[I^{12}\mathrm{tr}\, D_u + I^{21}\mathrm{tr}\, D_d\right] \\ &\quad +4I^{11}\left[\mathrm{tr}\, D_u \mathrm{tr}\, D_d - \mathrm{tr}\,(D_u D_d)\right] - 2(I^{11})^2 \\ &\simeq 2u_3^2 d_3^2 |V_{cb}|^2 \ ,\end{aligned} \tag{A6}$$

where we have used the experimental facts

$$\begin{aligned}&|V_{us}|^2 = 0.0486 \ , \quad |V_{cb}|^2 \simeq 1.9\times 10^{-3} \ , \\ &|V_{ub}|^2 \simeq 2\times 10^{-5} \ [20],\end{aligned} \tag{A7}$$

$$\begin{aligned}&|u_1/u_2| \simeq 0.0038 \ [16], |u_2/u_3| \sim 0.004 \ [17], \\ &|d_1/d_2| \simeq 0.051 \ [16], \quad |d_2/d_3| \simeq 0.033 \ [16].\end{aligned} \tag{A8}$$

(For an expression of I^{mn} in terms of $|V_{ij}|$ and quark masses q_i, see Appendix B.)

Relation (2.5) is derived from the exact relations

$$\det D_K = 2\varepsilon(1-\varepsilon^2)k^3 \tag{A9}$$

and

$$\mathrm{tr}\, D_K^2 = 2(1+3\varepsilon^2)k^2 \ . \tag{A10}$$

APPENDIX B: GENERAL FORMULAS FOR $|V_{ij}|$

The general formulas (2.30) for $|V_{ij}|^2$ are derived as follows: The traces of $M_u^n M_d^m$ (n, m : integers) are given by

$$\mathrm{tr}\,(M_u^n M_d^m) = \mathrm{tr}\,(D_u^n V D_d^m V^\dagger) = \sum_{i,\,j} u_i^n d_j^m |V_{ij}|^2 \ . \tag{B1}$$

Since $|V_{ij}|^2$ are expressed in terms of the four independent KM matrix parameters

$$\begin{aligned}&\alpha^2 \equiv |V_{12}|^2 \ , \quad \beta^2 \equiv |V_{23}|^2 \ , \quad \gamma^2 \equiv |V_{13}| \ , \\ &\omega \equiv |V_{21}|^2 - |V_{12}|^2 \ ,\end{aligned} \tag{B2}$$

as

$$|V_{ij}|^2 = \begin{pmatrix} 1-\alpha^2-\gamma^2 & \alpha^2 & \gamma^2 \\ \alpha^2+\omega & 1-\alpha^2-\beta^2-\omega & \beta^2 \\ \gamma^2-\omega & \beta^2+\omega & 1-\beta^2-\gamma^2 \end{pmatrix} \ , \tag{B3}$$

we can write $\mathrm{tr}\,(M_u^n M_d^m)$ in terms of α^2, β^2, γ^2 and ω as

$$\begin{aligned}\mathrm{tr}\,(M_u^n M_d^m) &= u_3^n d_3^m + u_2^n d_2^m + u_1^n d_1^m - (u_2^n - u_1^n)(d_2^m - d_1^m)\alpha^2 \\ &\quad -(u_3^n - u_2^n)(d_3^m - d_2^m)\beta^2 - (u_3^n - u_1^n)(d_3^m - d_1^m)\gamma^2 \\ &\quad -(u_3^n - u_2^n)(d_2^m - d_1^m)\omega \ .\end{aligned} \tag{B4}$$

[Throughout this appendix we use α, β, and γ as those defined by (B2), but not as those defined by (2.10).]

Setting

$$\begin{aligned}v_{12} &\equiv (u_2-u_1)(d_2-d_1)\alpha^2 \ , \\ v_{23} &\equiv (u_3-u_2)(d_3-d_2)\beta^2 \ , \\ v_{13} &\equiv (u_3-u_1)(d_3-d_1)\gamma^2 \ , \\ w &\equiv (u_3-u_2)(d_2-d_1)\omega \ ,\end{aligned} \tag{B5}$$

we can write (B4) explicitly as

$$I^{11} = -v_{12} - v_{23} - v_{13} + w \ , \tag{B6}$$

$$\begin{aligned}I^{21} &= -(u_2+u_1)v_{12} - (u_3+u_2)v_{23} \\ &\quad -(u_3+u_1)v_{13} + (u_3+u_2)w \ ,\end{aligned} \tag{B7}$$

$$\begin{aligned}I^{12} &= -(d_2+d_1)v_{12} - (d_3+d_2)v_{23} \\ &\quad -(d_3+d_1)v_{13} + (d_2+d_1)w \ ,\end{aligned} \tag{B8}$$

$$\begin{aligned}I^{22} &= -(u_2+u_1)(d_2+d_1)v_{12} - (u_3+u_2)(d_3+d_2)v_{23} \\ &\quad -(u_3+u_1)(d_3+d_1)v_{13} + (u_3+u_2)(d_2+d_1)w \ .\end{aligned} \tag{B9}$$

Then J^{nm}, which were defined by (2.31), are expressed as

$$\begin{pmatrix} J^{11} \\ J^{21} \\ J^{12} \\ J^{22} \end{pmatrix} = \begin{pmatrix} -1 & -1 & -1 & 1 \\ u_3 & u_1 & u_2 & -u_1 \\ d_3 & d_1 & d_2 & -d_3 \\ -u_3 d_3 & -u_1 d_1 & -u_2 d_2 & u_1 d_3 \end{pmatrix} \begin{pmatrix} v_{12} \\ v_{23} \\ v_{13} \\ w \end{pmatrix} \ . \tag{B10}$$

Therefore, by solving (B10) inversely, we can obtain the formulas (2.30).

APPENDIX C: FULL EXPRESSIONS OF TR $(M_u^m M_d^n)$

The full expressions of tr $(M_u^m M_d^n)$ for the mass matrix form (2.6) are given as

$$\frac{\mathrm{tr}\,(M_u M_d)}{m_1^u m_1^d} = 1 - \tfrac{1}{2}(1-3\varepsilon^2)\,(1 - a_u a_d \cos\alpha) + \varepsilon\,(b_u b_d \cos\beta + c_u c_d \cos\gamma)\ , \tag{C1}$$

$$\begin{aligned}\frac{\mathrm{tr}\,(M_u^2 M_d)}{(m_1^u)^2 m_1^d} = {} & 1 - \tfrac{1}{2}(1-\varepsilon^2)(1 - a_u a_d \cos\alpha) - \tfrac{1}{4}(1-\varepsilon^2)(1-a_u^2) \\ & +\tfrac{1}{4}\varepsilon(1-\varepsilon^2)\left[b_u^2 + c_u^2 + 2(b_u b_d \cos\beta + c_u c_d \cos\gamma)\right] \\ & +\tfrac{1}{2}\varepsilon(1-\varepsilon^2)\left[a_u b_u c_d \cos(\psi_u - \gamma) + a_u b_d c_u \cos(\psi_u - \beta) + a_d b_u c_u \cos(\psi_u - \alpha)\right]\ ,\end{aligned} \tag{C2}$$

$$\begin{aligned}\frac{\mathrm{tr}\,(M_u M_d^2)}{m_1^u (m_1^d)^2} = {} & 1 - \tfrac{1}{2}(1-\varepsilon^2)(1 - a_u a_d \cos\alpha) - \tfrac{1}{4}(1-\varepsilon^2)(1-a_d^2) \\ & +\tfrac{1}{4}\varepsilon(1-\varepsilon^2)\left[b_d^2 + c_d^2 + 2(b_u b_d \cos\beta + c_u c_d \cos\gamma)\right] \\ & +\tfrac{1}{2}\varepsilon(1-\varepsilon^2)\left[a_d b_d c_u \cos(\psi_d + \gamma) + a_d b_u c_d \cos(\psi_d + \beta) + a_u b_d c_d \cos(\psi_d + \alpha)\right]\ ,\end{aligned} \tag{C3}$$

$$\begin{aligned}\frac{\mathrm{tr}\,(M_u^2 M_d^2)}{(m_1^u)^2 (m_1^d)^2} = {} & 1 - \tfrac{1}{2}(1-\varepsilon^2)(1 - a_u a_d \cos\alpha) - \tfrac{1}{8}(1-\varepsilon^2)\left[(1+2\varepsilon^2)(2 - a_u^2 - a_d^2) + (1-3\varepsilon^2)(1 - a_u^2 a_d^2)\right] \\ & +\tfrac{1}{8}\varepsilon(1+\varepsilon)(1-\varepsilon^2)\left(b_u^2 + b_d^2 + 2b_u b_d \cos\beta\right) + \tfrac{1}{8}\varepsilon(1-\varepsilon)(1-\varepsilon^2)\left(c_u^2 + c_d^2 + 2c_u c_d \cos\gamma\right) \\ & +\tfrac{1}{8}\varepsilon(1-\varepsilon^2)\left[a_u^2(b_d^2 + c_d^2) + a_d^2(b_u^2 + c_u^2)\right] \\ & +\tfrac{1}{4}\varepsilon(1-\varepsilon^2)a_u a_d\left[(1-\varepsilon)b_u b_d \cos(\alpha+\beta) + (1+\varepsilon)c_u c_d \cos(\alpha+\gamma)\right] \\ & +\tfrac{1}{4}\varepsilon(1-\varepsilon^2)\{(1+\varepsilon)\left[a_u b_d c_u \cos(\psi_u - \beta) + a_d b_u c_d \cos(\psi_d + \beta)\right] \\ & \qquad +(1-\varepsilon)\left[a_u b_u c_d \cos(\psi_u - \gamma) + a_d b_d c_u \cos(\psi_d + \gamma)\right] \\ & \qquad +2\left[a_u b_d c_d \cos(\psi_d + \alpha) + a_d b_u c_u \cos(\psi_u - \alpha)\right]\} \\ & +\tfrac{1}{2}\varepsilon^2\left[(1-\varepsilon)^2 b_u^2 b_d^2 + (1+\varepsilon)^2 c_u^2 c_d^2\right] + \tfrac{1}{4}\varepsilon^2(1-\varepsilon^2)\left[b_u^2 c_d^2 + b_d^2 c_u^2 + 2b_u b_d c_u c_d \cos(\beta-\gamma)\right]\ .\end{aligned} \tag{C4}$$

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