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Matrix elements of pseudoscalar $Q\overline{q}$ mesons including relativistic and asymptotic-freedom effects

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We present a detailed analysis of effects on the decay constants f_P and mixing parameters B of the pseudoscalar $Q\overline{q}$ mesons, coming from two fundamental dynamical properties of this system: relativistic dynamics and an asymptotically free Coulomb interaction. Our results derive entirely from the short-distance properties of a Dirac wave function for the light quark in a pure leading-log Coulomb potential. We obtain an analytical expression for the asymptotic behavior ($r \to 0$) of this wave function. We also make use of a QCD effective action model for $Q\overline{q}$ mesons, in order to demonstrate that our analysis does not depend on any assumptions about the long-distance properties of this system. We show that the short-distance effects of the relativistic wave function with asymptotic freedom play an important role in the evolution of f_P and B with meson mass M_P. These corrections to f_P, which vanish as $\alpha_s(M_P)$ for $M_P \to \infty$, are to be superimposed on the leading short-distance correction due to Voloshin and Shifman and Politzer and Wise, and are found to be quantitatively more significant than the latter in the phenomenologically important region $M_D \lesssim M_P \lesssim M_B$.

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I. INTRODUCTION AND GENERAL FRAMEWORK

The decay constants f_P and mixing parameters B of the pseudoscalar $Q\overline{q}$ mesons have been subject to considerable analysis in recent years (see, e.g., Refs. [1–11]). f_P gives the matrix element for decay of a pseudoscalar meson state $|M(K)\rangle$, of definite four-momentum K, through the axial-vector current A^5_μ:

$$\langle 0\,|A^5_\mu(X)|\,M(K)\rangle = iK_\mu f_P e^{-iK\cdot X}. \tag{1}$$

The matrix element B for neutral-meson mixing (e.g., B_d-$\bar{B}_d$) is conventionally normalized in terms of f_P^2:

$$\langle \overline{M}|(V_\mu - A^5_\mu)^2|M\rangle \equiv \tfrac{4}{3} f_P^2 M_P B, \tag{2}$$

where V_μ is the vector current.

Phenomenologically, the interest in these matrix elements stems from the fact that they appear in many processes from which we can extract quantities of fundamental importance to the standard model, such as the quark mixing matrix, CP violation, the mass of the top quark, and so on. Unfortunately, the available experimental information is scarce. $f_\pi = 132$ MeV and $f_K = 168$ MeV are well-known, while a Mark III experiment provides the upper bound $f_D < 290$ MeV [12], and a recent ARGUS analysis [13] gives an average over f_{D_s} and $f_{D_s^*}$ of (267 ± 28) MeV$\times[2.7\%/B(D_s^+ \to \phi\pi^+)]^{1/2}$. Moreover, there are large discrepancies between the various model calculations of the matrix elements for heavy $Q\overline{q}$ mesons (e.g., f_D, f_B). This situation has been a major impediment to attempts to extract reliable estimates of standard model parameters from experiment.

A scaling law for f_P that is derived from the nonrelativistic (NR) quark model is frequently used as a benchmark in theoretical calculations. For families of $Q\overline{q}$ mesons with the same flavor of light quark,

$$f_{\rm NR}(m_Q \to \infty) \propto \sqrt{\frac{1}{M_P}}, \tag{3}$$

where m_Q is the mass of the heavy quark and M_P is the meson mass. The factor of $1/\sqrt{M_P}$ has a kinematical origin, coming from a density of states when a Fourier transform is applied to Eq. (1). The mixing parameter is identically equal to one in any nonrelativistic constituent quark model [2]:

$$B_{\rm NR}(m_Q) \equiv 1. \tag{4}$$

Vacuum saturation of a sum over intermediate states inserted between the operators in Eq. (2) also gives $B \equiv 1$.

The question of the mass scale at which the NR scaling law for f_P becomes reliable has been hotly contested in the literature (for a recent review see Ref. [1]). For example, a variety of quark-model calculations (e.g., Ref. [3]) suggest that $f_K > f_D > f_B$, while recent QCD sum rule [4] and lattice [5] calculations suggest that f_P is roughly constant between the K and B mesons. The disagreement on this basic property of the decay constant underscores the unsatisfactory state of our understanding of the physics of f_P. Estimates of a given decay constant sometimes differ by factors of 3 or 4 from one model to another. Similarly, predictions for the neutral-meson mixing parameters cover a wide range of values—for example, calculations of B_K range at least from 1/3 to 1 (theoretical estimates of B_K are summarized in Ref. [6]).

In this paper we consider corrections to the NR scaling laws for f_P and B that are induced by two fundamental dynamical properties of the $Q\overline{q}$ system: relativistic dynamics for the light quark and an asymptotically-free Coulomb interaction. A very simple approach (which we first proposed in Ref. [14]) is used to draw conclusions that are as model independent as possible about the effects of these dynamics. Our basic premise is that for sufficiently large heavy quark masses m_Q, the light quark wave function at short distances is adequately described by a Dirac equation in a QCD leading-log-corrected Coulomb potential.

This picture is in several ways a natural, and conceptually simple, extension of the NR quark-model description which leads to Eqs. (3) and (4). In particular, for *very* large m_Q, our relativistic wave function leads to the scaling limit Eq. (3) for f_P, and exhibits the limit $B(m_Q \to \infty) = 1$ [cf. Eq. (4)]. However, it will be seen that relativistic dynamics and the short-distance QCD running coupling lead to substantial corrections to these scaling laws (which cannot be mocked up in NR quark models) in the phenomenologically important intermediate regime $M_D \lesssim M_P \lesssim M_B$. We believe that these effects on the decay constants and mixing parameters have either not been properly included, or understood, in previous model calculations.

We note that Voloshin and Shifman (VS) [9] and Politzer and Wise (PW) [10] have discovered an important (leading) short-distance correction to the NR scaling law, Eq. (3), for f_P. Their correction (which does not modify the NR scaling law for the B parameter) is due to a finite renormalization of the matrix element Eq. (1) which defines f_P, from the large momentum scale $\sim m_Q$ at which the matrix element is defined, down to the soft momentum scale $\sim R^{-1}$ which characterizes the $Q\overline{q}$ wave function. We will frequently refer to the nonrelativistic scaling law of Eq. (3) with the understanding that we actually include the correction of VS-PW when making explicit calculations.

We are concerned here with a subleading (and therefore different type of) short-distance correction to the NR scaling laws. We point out in this connection that VS assumed a nonrelativistic quark model wave function when estimating their effects on the decay constants. In their heavy quark effective theory, PW obtain the same result without referring explicitly to the quark wave function (see also Ref. [11]), as, to leading order, the wave function drops out when comparing f_P for two states with different M_P. The effects on f_P analyzed in this paper, coming from a relativistic wave function with an asymptotically free short-distance interaction, are in this sense of subleading order, and should be superimposed on the leading correction due to VS-PW. We note that the VS-PW correction to f_P dominates in the limit $M_P \to \infty$, with our effects vanishing as $\alpha_s(M_P)$ in this limit. However, we find that the effects due to the relativistic wave function are quantitatively more significant in the phenomenologically important region $M_D \lesssim M_P \lesssim M_B$.

Our analysis is based on the following three general dynamical ingredients.

A. The light quark wave function is calculated in a Born-Oppenheimer approximation, the recoil of the heavy quark and effects due to its color-magnetic moment being neglected in first approximation. According to perturbative QCD, the light quark therefore experiences a leading-logarithm-corrected Coulomb potential $V_{\rm Coul}$ in the vicinity of the heavy quark

$$V_{\rm Coul}(r) = -\frac{\overline{\alpha}(r)}{r}, \qquad (5)$$

where

$$\overline{\alpha}(r) \equiv \tfrac{4}{3}\alpha_s(r) = -\frac{1}{3\pi b_0 \ln(r\Lambda_{\rm pot})}. \qquad (6)$$

b_0 is the one-loop β-function coefficient,

$$b_0 = \frac{1}{8\pi^2}(11 - \tfrac{2}{3}N_f), \qquad (7)$$

and $\Lambda_{\rm pot}$ is the mass scale appropriate to the $Q\overline{q}$ potential. We derive a connection between $\Lambda_{\rm pot}$ and the QCD scale parameter $\Lambda_{\overline{\rm MS}}$ in Sec. IIIB, where $\overline{\rm MS}$ denotes the modified minimal subtraction scheme.

B. The light quark is a Dirac fermion with a small current mass m. We expect its wave function in the ground state to have the familiar form (as appropriate for a central $Q\overline{q}$ potential)

$$\psi(\mathbf{r}) = \mathcal{N}\begin{bmatrix} -i\chi(r) \\ \boldsymbol{\sigma}\cdot\hat{\mathbf{r}}\phi(r)\end{bmatrix}, \qquad (8)$$

where $\mathcal{N}$ is a normalization constant.

C. Quantum-mechanical fluctuations in the position of the heavy quark are accounted for in the overlap of the $Q\overline{q}$ pair, and are assumed to be described by a nonrelativistic wave function $\Psi_Q(\mathbf{r})$, which is "spread" over a distance r_Q at least of the order of the Compton wavelength $1/m_Q$. Following Donoghue and Johnson [15], the decay constant is then given by the overlap integral

$$f_{\rm rel}^2(m_Q) = \frac{12}{M_P}\mathcal{N}^2 \int d^3\mathbf{r}\, |\Psi_Q(\mathbf{r})|^2\, \chi^2(r), \qquad (9)$$

where 12 is a color-spin-flavor coefficient. We use the notation $f_{\rm rel}$ to emphasize the relativistic dynamics.

$f_{\rm rel}$ depends on the "spread" r_Q in the heavy quark wave function, but in the applications which we consider in this paper, we find that it turns out to be essentially independent of the exact shape of $\Psi_Q(\mathbf{r})$. Then Eq. (9) for $f_{\rm rel}$ reduces to the (physically reasonable) expression

$$f_{\rm rel}(m_Q) = \sqrt{\frac{12}{M_P}}\mathcal{N}\chi(r_Q), \quad r_Q = O(1/m_Q). \qquad (10)$$

We have verified that using a given value for r_Q in Eq. (10) gives essentially the same value for $f_{\rm rel}$ as is obtained from the overlap integral in Eq. (9), using a variety of possible forms for the wave function $\Psi_Q(\mathbf{r})$ (e.g., Gaussian and step function), when r_Q is taken as the rms radius of the heavy quark probability distribution.

The mixing parameter B can likewise be expressed in terms of an overlap integral [16]

$$B_{\rm rel}(m_Q) = \frac{12}{f_P^2 M_P}\mathcal{N}^2 \int d^3\mathbf{r}\, |\Psi_Q(\mathbf{r})|^2\, [\chi^2(r) - \phi^2(r)], \tag{11}$$

where the notation $B_{\rm rel}$ again emphasizes the relativistic dynamics. Notice that the normalization $\mathcal{N}$ of the light quark wave function cancels in $B_{\rm rel}$. Approximating the overlap by the distance r_Q which characterizes the "spread" of the heavy quark wave function, we obtain

$$B_{\rm rel}(m_Q) = 1 - \left[\frac{\phi(r_Q)}{\chi(r_Q)}\right]^2, \quad r_Q = O(1/m_Q). \tag{12}$$

We observe that Eqs. (9) or (10) for $f_{\rm rel}$ imply (for $m_Q \to \infty$) the NR scaling law Eq. (3) for the decay constant, up to the VS-PW correction, *provided* that the relativistic wave function at the origin is finite, $\chi(r \to 0) =$ finite. We will show that this is indeed the case for a $Q\overline{q}$ pair interacting through the asymptotically free Coulomb potential. Moreover, we show that $\phi(r \to 0) = 0$ in this potential, so that $B_{\rm rel}$ also satisfies the NR scaling law, Eq. (4), in the limit $m_Q \to \infty$. On the other hand, we find that the detailed behavior of the relativistic wave function at short distances leads to significant scaling violations at intermediate masses.

The main results presented in this paper all follow from the dynamical ingredients A, B, and C. We note that while quantum-mechanical fluctuations in the position of the heavy quark will be accounted for when we compute the $Q\overline{q}$ overlap (ingredient C), we will compute the light quark wave function $\psi(\mathbf{r})$ with $m_Q = \infty$ (ingredient A). We expect that finite m_Q effects on the wave function (in particular, the recoil and color-magnetic moment of the heavy quark), when treated as first-order perturbations, will mainly result in changes to the normalization $\mathcal{N}$, due to changes in the radii of the $Q\overline{q}$ triplet and singlet states. We show in Sec. IV that neglecting these finite m_Q effects only tends to suppress the corrections to the NR scaling law for f_P that are obtained from the wave function with $m_Q = \infty$ (the mixing parameter B is independent of the wave-function normalization).

We also point out that the overlap radius $r_Q = O(1/m_Q)$ in Eqs. (10) and (12) can, in principle, be determined from a more complete model calculation of the dynamics of the heavy quark motion than we will attempt in this paper.[1] Instead, we here simply use the fact that on physical grounds r_Q cannot be much smaller than $1/m_Q$, and we show in detail in Sec. IV that our underestimate of r_Q again only tends to suppress the relativistic corrections to the NR scaling laws.

The rest of this paper is organized as follows. In Sec. II, we analyze the Dirac equation for $\psi(\mathbf{r})$ in a *pure* leading-log-corrected Coulomb potential, and we obtain an analytical expression for the asymptotic behavior ($\mathbf{r} \to 0$) of the wave function. The short-distance limit of this wave function enables us, in principle, to make model-independent calculations for the effects of relativistic dynamics and asymptotic freedom on the evolution of $f_P(m_Q)$ and $B(m_Q)$ for sufficiently large m_Q [corresponding to distances $r = O(1/m_Q)$ where $V_{\rm Coul}(r)$ should provide a reliable approximation]. However, we must be sure to isolate genuine short-distance effects from possible spurious effects due to the failure of the leading-log potential at long distances. In particular, as we go below $M_P \approx 1$ GeV, we get dangerously close to the scale at which the leading-log coupling blows up, and $V_{\rm Coul}$ becomes unphysical.

To demonstrate that our results are in fact independent of the long-distance properties of the leading-log Coulomb wave function, we also make use of a relativistic model for the $Q\overline{q}$ system, based on a one-loop approximation to the effective action for QCD [17–24]. This effective action has been shown to provide a description of mesons containing one or two heavy quarks ($Q\overline{q}$ and QQ mesons) which has many of the qualitative features expected of exact QCD. In particular, this effective theory naturally includes both asymptotic freedom and linear confinement [18, 23].

In Sec. III we apply this QCD effective action to $Q\overline{q}$ systems with a light fermionic quark [25]. In particular, we show that the light quark wave function in this model has *exactly* the same short-distance behavior as in the pure leading-log Coulomb potential. This model is also shown to provide a well-behaved, and qualitatively correct, description of the long-distance physics of the $Q\overline{q}$ system. We emphasize that although this model could be used to make a complete analysis of the $Q\overline{q}$ system (including the spectrum and transition matrix elements), the QCD effective action is used in this paper only in order to establish that our analysis does not depend on assumptions about the long-distance properties of the $Q\overline{q}$ system, and to extract results that are due explicitly to the short-distance dynamical effects of interest.

In Sec. IV we compute the relativistic decay constant $f_{\rm rel}$ using the light quark wave function in both the pure leading-log Coulomb potential, and in the QCD effective action model (EAM). We find that these two descriptions give essentially the same results for the evolution of $f_{\rm rel}(m_Q)$ with m_Q. Given that the pure leading-log Coulomb wave function exhibits an unphysical singularity at long-distances (compared to a well-behaved limit in the EAM), we conclude from this comparison that the evolution of $f_{\rm rel}$ that is obtained from these wave functions is determined by their properties at short distances, even for m_Q as small as $\approx 5\Lambda_{\overline{\rm MS}}$. We clearly correlate our results for the evolution of $f_{\rm rel}$ with the detailed short-distance behavior of the asymptotically free relativistic wave function.

In Sec. V we estimate the contribution of these short-distance effects to the ratio f_B/f_D, which is less model dependent than the value of the individual decay constants. We find a sizable correction to the ratio that would be obtained from the NR scaling law. We also make estimates of these effects on the mixing parame-

[1] In Born-Oppenheimer approximation, one would account for the recoil of the heavy quark by solving a Schrödinger equation for Ψ_Q in the potential due to the color-charge cloud of the light quark. We thank R. L. Jaffe for several conversations in this connection.

ters B in the region $M_P \gtrsim M_D$. In Sec. VI we consider a possible extrapolation of our calculations down to $M_P \approx M_K$. To the extent that the application of our methods in this regime can be trusted, we can use the experimental value for f_K as input to obtain lower bounds for f_D and f_B individually. Our results obtained in this way are consistent with most recent QCD sum rule [4] and lattice [5] calculations, and with experimental bounds obtained by Mark III [12] and ARGUS [13]. Finally, in Sec. VII we summarize our main results, and describe some improvements that can be made to the present analysis.

II. DIRAC EQUATION IN A "PURE" LEADING-LOG COULOMB POTENTIAL

According to our dynamical ingredients A and B (see the Introduction), the behavior of the light quark wave function $\psi(\mathbf{r})$ at short distances is given by the solution to the Dirac equation in the leading-log Coulomb potential $V_{\rm Coul}$ [see Eq. (5)]:

$$[\boldsymbol{\alpha}\cdot(-i\boldsymbol{\nabla}) + V_{\rm Coul}(r) + \beta m]\,\psi_{\rm Coul}(\mathbf{r}) = \xi\psi_{\rm Coul}(\mathbf{r}). \tag{13}$$

m is the current mass of the quark, and ξ is the energy eigenvalue of the stationary state $\psi(\mathbf{r},t) = e^{-i\xi t}\psi(\mathbf{r})$. Note that we have used $V_{\rm Coul}$ as the zeroth component of a vector potential in Eq. (13), as given by perturbative QCD. We use the notation $\psi_{\rm Coul}$ to emphasize that this wave function is calculated in the approximation that the $Q\overline{q}$ pair interacts via a "pure" leading-log Coulomb potential. The form of the ground state solution to Eq. (13) is given in Eq. (8).

Since the normalization $\mathcal{N}$ and energy eigenvalue ξ depend on the long-distance properties of the wave function, they cannot be determined from the short-distance potential $V_{\rm Coul}$. On the other hand, we show below that the short-distance limit of $\psi_{\rm Coul}(\mathbf{r})$, relevant to the overlap of the $Q\overline{q}$ pair, is independent of ξ and m, and we will be able to draw significant conclusions about the behavior of the matrix elements without knowing the wave function normalization.

It is therefore of interest to study the Dirac equation with ξ and m set to zero:

$$\left[\boldsymbol{\alpha}\cdot(-i\boldsymbol{\nabla}) - \frac{\overline{\alpha}(r\Lambda_{\rm pot})}{r}\right]\begin{pmatrix} -i\chi_{\rm Coul}(r) \\ \boldsymbol{\sigma}\cdot\hat{\mathbf{r}}\phi_{\rm Coul}(r)\end{pmatrix} \equiv 0. \tag{14}$$

Note that the scale for $\psi_{\rm Coul}(\mathbf{r})$ is set *explicitly* by the QCD scale parameter, since $\Lambda_{\rm pot} \propto \Lambda_{\overline{\rm MS}}$ [see Sec. III B below] is the only scale in Eq. (14). This is to be compared with the situation in many QCD-inspired models, where a connection between the scale for hadronic matrix elements and the QCD scale parameter is missing. Equation (14) implies that the scale for f_P in our calculation is correlated with other fundamental quantities of hadronic physics, such as the observed slope of light meson Regge trajectories [24].

To solve Eq. (14), we proceed in the usual way by separating it into two coupled, first-order radial equations, which are identical in form to the usual set of radial Dirac equations, except that ξ is here set to zero:

$$\begin{aligned} \frac{d}{dr}\chi_{\rm Coul} &= -\frac{\overline{\alpha}(r)}{r}\phi_{\rm Coul}, \\ \frac{d}{dr}\phi_{\rm Coul} &= -\frac{2}{r}\phi_{\rm Coul} + \frac{\overline{\alpha}(r)}{r}\chi_{\rm Coul}. \end{aligned} \tag{15}$$

The solution to Eq. (15) can be obtained in the form of an (asymptotic) series expansion in powers of the running coupling. This is very different from the usual series expansion for the Dirac equation in a Coulomb potential of fixed strength (in that case, the expansion is in powers of r, with fixed α appearing in the series coefficients). The change of variables

$$y(r\Lambda_{\rm pot}) \equiv \overline{\alpha}(r) = -\frac{\alpha_0}{\ln(r\Lambda_{\rm pot})}, \quad \alpha_0 \equiv \frac{1}{3\pi b_0}, \tag{16}$$

eliminates the explicit factors of r in Eq. (15), resulting in equations in terms of y only:

$$\begin{aligned} \frac{d}{dy}\chi_{\rm Coul} &= -\frac{\alpha_0}{y}\phi_{\rm Coul}, \\ \frac{d}{dy}\phi_{\rm Coul} &= -\frac{2\alpha_0}{y^2}\phi_{\rm Coul} + \frac{\alpha_0}{y}\chi_{\rm Coul}. \end{aligned} \tag{17}$$

Note that y, $\chi_{\rm Coul}$, and $\phi_{\rm Coul}$ are all dimensionless functions of $r\Lambda_{\rm pot}$. These differential equations can be solved by the (asymptotic) series expansions

$$\chi_{\rm Coul}(y) = \sum_{n=0}^{\infty} g_n y^n, \quad \phi_{\rm Coul}(y) = \sum_{n=0}^{\infty} f_n y^n. \tag{18}$$

For the solution that is regular at the origin, the coefficients are obtained from the recursion relations

$$\begin{aligned} g_0 &= \text{arbitrary}, \quad f_0 = 0, \\ f_n &= \frac{1}{2}\left[\frac{1-n}{\alpha_0}f_{n-1} + g_{n-1}\right], \\ g_n &= -\frac{\alpha_0}{n}f_n. \end{aligned} \tag{19}$$

These series expansions only converge asymptotically. However, the first few terms in the series are sufficient to compute the functions to arbitrarily high accuracy for sufficiently small $r\Lambda_{\rm pot}$ [in any event, the long-distance properties of the solution to Eq. (13) are not physically meaningful].

It is important to note that our dynamical ingredients A and B imply that the "true" light quark wave function, with the correct long-distance physics (not contained in $V_{\rm Coul}$), will have exactly the same leading short-distance behavior as the solution to Eq. (13). We therefore truncate the asymptotic series to obtain the following model-independent result for the short-distance behavior of the true wave function [26]:

$$\chi(r \to 0) = \chi(0)\left[1 + \frac{1}{18\pi^2 b_0^2}\frac{1}{\ln(r\Lambda_\chi)}\right],$$
$$\phi(r \to 0) = -\chi(0)\frac{1}{6\pi b_0}\frac{1}{\ln(r\Lambda_\phi)}, \quad (20)$$

where $\chi(0)$ is finite. Equation (20) can be written schematically as $\chi(r \to 0) = \chi(0)[1 + O(\alpha_s(r))] =$ finite, and $\phi(r \to 0) = O(\alpha_s(r)\chi(r)) \to 0$.

The leading corrections in ξ and m to Eq. (20) are of order $(\xi, m)r/\ln r$. Expressions for Λ_χ and Λ_ϕ are obtained by absorbing the $1/\ln^2$ terms in Eq. (18) into the scale factor $\Lambda_{\rm pot}$ in the leading logarithms [up to truly higher order remainders of $O(1/\ln^3)$]

$$\Lambda_\chi = \Lambda_{\rm pot}\exp[-(\alpha_0^2 + 1)/4],$$
$$\Lambda_\phi = \Lambda_{\rm pot}\exp[-(\alpha_0^2 + 1)/2]. \quad (21)$$

We stress that *the Dirac wave function at the origin in the leading-log Coulomb potential is finite*, unlike the more familiar case of the wave function in a Coulomb potential of fixed strength α, which is singular [e.g., for fixed $\alpha \ll 1$, we would have $\phi(r \to 0) \propto \alpha\chi(r \to 0) \sim \alpha r^{-\alpha^2/2} \to \infty$, to be compared with the above limits]. This result has apparently not been realized in earlier studies. The finiteness of the wave function in the leading-log potential is evidently due to the fact that the running coupling constant $\alpha_s(r)$ tends to zero sufficiently rapidly as $r \to 0$ [22]. We have also found that the Klein-Gordon wave function in the leading-log Coulomb potential is finite at the origin [23]. This differs from the singular behavior found by Durand for a nonlocal relativistic spin-0 equation in the same potential [27] (the spin-0 equation analyzed by Durand was used to calculate f_P in Ref. [28]).

As described in the Introduction, the analysis leading to Eq. (20) enables us, in principle, to determine the influence of relativistic and leading-log Coulomb dynamics on the decay constants and mixing parameters. However, we must be sure to isolate genuine short-distance effects from possible spurious effects due to the failure of the leading-log potential at long distances. Note in particular that the solution to Eq. (13) will exhibit a form of Klein paradox as r approaches $\Lambda_{\rm pot}^{-1}$, where the potential $V_{\rm Coul}$ blows up, causing unphysical pair creation effects. On the other hand, the rapid changes in $\chi(r)$ and $\phi(r)$ as $r \to 0$, due to the logarithms in Eq. (20), are genuine physical effects, and will induce significant logarithmic departures from the NR scaling laws for f_P and B.

III. THE $Q\bar{q}$ SYSTEM IN A QCD EFFECTIVE ACTION APPROXIMATION

A. Formalism

We now describe a QCD effective action approximation for the $Q\bar{q}$ system that exhibits the correct short-distance limit for the light quark wave function, Eq. (20), and which also provides a well-behaved description of the system at long distances. The application of the EAM to a system containing a dynamical fermionic quark, summarized here, is very similar to that for a system containing a dynamical scalar quark, which we analyzed in detail in Refs. [23] and [24].

The effective action model (EAM) we use is defined by an effective Lagrangian density for the gauge fields, $\mathcal{L}_{\rm eff}^{\rm gauge}$, which includes leading renormalization-group corrections to the Lagrangian of classical chromodynamics [18, 23]:

$$\mathcal{L}_{\rm eff}^{\rm gauge}(\mu, -\tfrac{1}{2}F^{a\mu\nu}F^a_{\mu\nu}) = -\tfrac{1}{4}\bar{g}^{-2}(\mu, -\tfrac{1}{2}F^{a\mu\nu}F^a_{\mu\nu})F^{a\mu\nu}F^a_{\mu\nu}, \quad (22)$$

where

$$F^{a\mu\nu} = \partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} + f^{abc}A^{b\mu}A^{c\nu}, \quad (23)$$

and where the running coupling $\bar{g}$ here runs with the field strengths. μ is the subtraction point. To leading-logarithm order

$$\bar{g}^{-2}(\mu, -\tfrac{1}{2}F^{a\mu\nu}F^a_{\mu\nu}) = g^{-2}[1 + \tfrac{1}{4}b_0 g^2 \ln(-\tfrac{1}{2}F^{a\mu\nu}F^a_{\mu\nu}/\mu^4)], \quad (24)$$

where $g \equiv \bar{g}(\mu, \mu^4)$ and b_0 is the one-loop β function coefficient [Eq. (7)].

The total effective Lagrangian density $\mathcal{L}^{Q\bar{q}}$ for the $Q\bar{q}$ system is the sum of the effective Lagrangian for the gauge fields plus a fermionic Lagrangian $\mathcal{L}^F$:

$$\mathcal{L}^{Q\bar{q}} = \mathcal{L}_{\rm eff}^{\rm gauge} + \mathcal{L}^F. \quad (25)$$

We take $\mathcal{L}^F$ to have the usual form

$$\mathcal{L}^F = i\bar{\psi}\gamma^\mu(\partial_\mu - i\tfrac{1}{2}\lambda^a A^a_\mu)\psi + \bar{\psi}_M\gamma^\mu\tfrac{1}{2}\lambda^a A^a_\mu\psi_M - m\bar{\psi}\psi - M'\bar{\psi}_M\psi_M, \quad (26)$$

ignoring the kinetic term for the heavy quark (consistent with our dynamical ingredient A). ψ_M is a Dirac wave function for the heavy quark of (bare) mass M', and ψ is the wave function for the light quark of current mass m.

We assume that the action corresponding to $\mathcal{L}^{Q\bar{q}}$ is minimized by an Abelian configuration of fields and currents [18–24]. This amounts to dropping the color indices in Eqs. (22)–(26) and assigning equal and opposite Abelian charges of magnitude $\sqrt{4/3}$ to the quarks. The equations of motion which minimize the resulting Abelian action are similar to the usual Maxwell equations for (Abelianized) color-electric and -magnetic fields $\mathbf{E}$ and $\mathbf{B}$ plus an Abelian Dirac equation for the light quark (the heavy quark is treated as an external source). However, the fields and sources appear to be embedded in a "medium," characterized by a nonlinear dielectric $\epsilon(\mathcal{F})$, which measures the response of the QCD vacuum to an applied field $\mathcal{F}$ [18, 23]. In the leading-log model for $\mathcal{L}_{\rm eff}^{\rm gauge}$, Eqs. (22)–(24), we have

$$\epsilon(\mathcal{F}) = \tfrac{1}{4}b_0 \ln\left|\frac{\mathcal{F}}{\kappa^2}\right|, \quad \mathcal{F} \equiv \mathbf{E}^2 - \mathbf{B}^2. \quad (27)$$

The scale parameter $\kappa^{1/2}$ is related to the subtraction point μ [18]. The zero of the dielectric $\epsilon(\mathcal{F}=\kappa^2)=0$ implies that a vacuum electric field (condensate) $E^2_{\rm vac}=\kappa^2$ is spontaneously generated by one-loop radiative corrections [17, 18].

Because of the nonlinear dielectric, the equations of motion are highly coupled and nonlinear. A considerable simplification is obtained by neglecting $|\mathbf{B}|$ compared to $|\mathbf{E}|$. The effective action is then extremized by the solution to a set of nonlinear Maxwell equations for the (Abelianized) color-electric field $\mathbf{E}$,

$$\nabla\cdot\left(\epsilon(E^2)\mathbf{E}\right)=\sqrt{\frac{4}{3}}\left[\psi^\dagger(\mathbf{r})\psi(\mathbf{r})-\delta^3(\mathbf{r})\right],$$
$$\mathbf{E}(\mathbf{r})\equiv-\nabla A^0(\mathbf{r}), \tag{28}$$

and a nonlinear Dirac equation for the light quark wave function,

$$\left[\boldsymbol{\alpha}\cdot(-i\nabla)+\sqrt{\frac{4}{3}}A^0(\mathbf{r})+\beta m\right]\psi(\mathbf{r})=\xi\psi(\mathbf{r}). \tag{29}$$

The coupled system, Eqs. (28) and (29), is solved exactly.[2] The form of the ground state wave function is given in Eq. (8).

In order to uniquely determine the solution to Eqs. (28) and (29) we must impose suitable boundary conditions. We find that a solitonlike solution exists, in which all fields take on their vacuum values outside a sphere of radius R surrounding the heavy quark:

$$\psi(r>R)=0,\quad E(r\geq R)=\kappa. \tag{30}$$

The wave function is normalized according to $\int_{r\leq R}d^3\mathbf{r}\,\psi^\dagger\psi=1$.

Notice that the light quark wave function is identically zero outside R. This avoids a Klein paradox that would otherwise be induced by the vacuum electric field, since the size of the soliton acts as an effective infrared cutoff. A similar situation occurs in our model for the $Q\overline{q}$ system with a dynamical scalar quark [23], and in the model for the $Q\overline{Q}$ system [18].

The boundary conditions on the light quark wave function at $r=R$ are not unique, as expected from the chiral invariance of the effective action (not including the quark mass term). Following the usual MIT bag model [29], we use the boundary condition

$$\bar{\psi}(\mathbf{r})\psi(\mathbf{r})=0 \text{ at } r=R, \tag{31}$$

which breaks chiral symmetry explicitly. Equation (31) guarantees that the light quark current normal to the soliton surface vanishes. Finally, the radius R at which Eqs. (30)–(31) are imposed is determined by conservation of the energy-momentum tensor at R (cf. Ref. [23])

$$\frac{\partial}{\partial r}\bar{\psi}(\mathbf{r})\psi(\mathbf{r})=0 \text{ at } r=R. \tag{32}$$

While our boundary conditions, Eqs. (30)–(32), are similar to those in the usual MIT bag model [29], we emphasize that confinement in the EAM is due entirely to the nontrivial electric field generated in the one-loop approximation to the QCD vacuum, $E(r\geq R)=\kappa$. See Ref. [23] for further analysis of boundary conditions for relativistic matter fields in the EAM.

B. Light quark wave function

The EAM for the $Q\overline{q}$ system exhibits asymptotic freedom at short distances, due to the properties of the dielectric $\epsilon(E)$ in strong fields [18, 23]. In particular, the $Q\overline{q}$ vector potential $V_{\rm EAM}$ [cf. Eqs. (13) and (29)]

$$V_{\rm EAM}(r)\equiv\sqrt{\frac{4}{3}}A^0(r), \tag{33}$$

has the correct short-distance limit

$$V_{\rm EAM}(r\ll\Lambda_{\rm pot}^{-1})=\frac{1}{3\pi b_0 r\ln(r\Lambda_{\rm pot})}, \tag{34}$$

as given by perturbative QCD. We derived the mass scale $\Lambda_{\rm pot}$ in terms of $\kappa^{1/2}$ in Ref. [23], $\Lambda_{\rm pot}=e(\sqrt{3}\pi b_0\kappa)^{1/2}$. Adler has determined the connection between $\kappa^{1/2}$ and the QCD scale parameter $\Lambda_{\overline{\rm MS}}$ [18, 19]. For three light quark flavors he found $\Lambda_{\overline{\rm MS}}=0.959\kappa^{1/2}$. We therefore obtain

$$\Lambda_{\rm pot}=2.23\Lambda_{\overline{\rm MS}}\quad[N_f=3]. \tag{35}$$

This is close to the analogous result for the mass scale in the quarkonium ($Q\overline{Q}$) potential, which equals $2.63\Lambda_{\overline{\rm MS}}$ for $N_f=3$ (see Ref. [19] and references therein).

We conclude from Eq. (34) that our EAM gives a model-independent description of the light quark wave function at short distances, since the model has the correct perturbative QCD result for the short-distance $Q\overline{q}$ interaction. The wave function at short distances is therefore identical in the EAM and in the pure leading-log Coulomb potential, and is given by our result in Eq. (20).

Our EAM also gives a well-behaved (and qualitatively correct) description of the long-distance properties of the $Q\overline{q}$ system. In particular, the model exhibits linear confinement, due to the nontrivial vacuum electric field. For example, excitations of the $Q\overline{q}$ system are found to lie on linear Regge trajectories [23, 24]. Linear confinement is also exhibited in the long-distance limit of the $Q\overline{q}$ vector potential; we find $V_{\rm EAM}(r\to R)=\sigma r$, where $\sigma=(1.12\Lambda_{\overline{\rm MS}})^2$ for $N_f=3$. The EAM therefore naturally interpolates between asymptotic freedom at short distances and linear confinement at long distances. This is clearly illustrated by a plot of $V_{\rm EAM}(r)$, shown in Fig. 1.

---

[2]It can be shown that the approximation of neglecting $|\mathbf{B}(\mathbf{r})|$ compared to $|\mathbf{E}(\mathbf{r})|$ becomes exact for $\mathbf{r}$ sufficiently close to the point charge of the heavy quark [22]. This is precisely the region where we evaluate $\psi(\mathbf{r})$ in order to calculate the matrix elements of interest. We also note that this approximation leads to a qualitatively correct description of the long-distances properties of the $Q\overline{q}$ system.

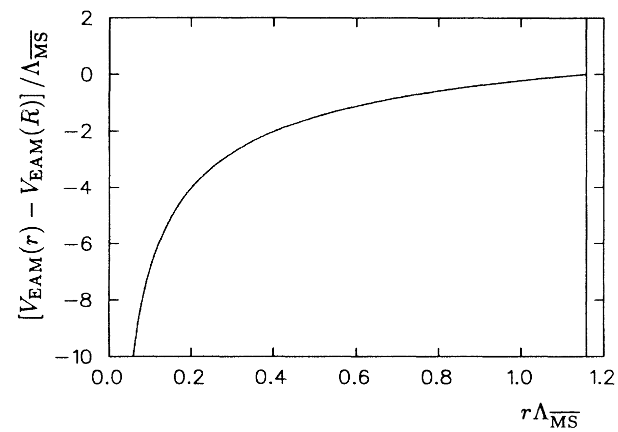

FIG. 1. $Q\overline{q}$ interaction potential $V_{\rm EAM}(r) \equiv \sqrt{4/3}A^0(r)$ in a one-loop QCD effective action model [Eqs. (28)–(35)], for $m = 0$ and $N_f = 3$. The light quark wave function vanishes outside the radius $R = 1.16\Lambda^{-1}_{\overline{\rm MS}}$, indicated by a vertical line in the figure.

We solve the system of coupled Maxwell and Dirac equations, Eqs. (28) and (29), subject to the boundary conditions Eqs. (30)–(32), using an iterative numerical method similar to the one we described in Ref. [23]. For $m = 0$ and $N_f = 3$, we find that the soliton radius is $R = 1.16\Lambda^{-1}_{\overline{\rm MS}}$. The wave-function components are plotted in Fig. 2, where the effect of the running coupling at short distances is clearly demonstrated. Note especially that for a fixed coupling α, we would have had $\phi(r \to 0) \propto \alpha\chi(r \to 0) \to \infty$. Instead we have $\chi \to$ finite and $\phi(r \to 0) \propto \alpha_s(r)\chi(r) \to 0$, as given in Eq. (20). The rapid changes in χ and ϕ at short distances, due to the logarithms in Eq. (20), are also evident in Fig. 2.

We show below that the rapid rise in $\chi(r \to 0)$ to a finite limit determines the main features of the evolution of $f_{\rm rel}(m_Q)$. It might be argued, however, that this behavior is some kind of a "residual" effect of the unphysical singularity that would be present if the running coupling $\overline{\alpha} \equiv \frac{4}{3}\alpha_s$ was nonzero at the origin. A strong indicator to the contrary is the fact that $\phi(r = 0) = 0$. To prove that there is no unphysical behavior, suppose that $\overline{\alpha}(r \to 0)$ decreases to a small nonzero value, instead of vanishing as $1/\ln(r\Lambda_{\rm pot})$. Then we would have $\chi(r \to 0) \sim (r\Lambda_{\overline{\rm MS}})^{-\overline{\alpha}^2(0)/2}$ ($\Lambda_{\overline{\rm MS}}$ being the only available scale). The singularity for $\overline{\alpha}(0) \neq 0$ would become important only at a distance $r\Lambda_{\overline{\rm MS}} \approx e^{-2/\overline{\alpha}^2(0)}$, where the wave function would become of order unity or greater [30]. Even if we take $\overline{\alpha}(0)$ as large as 0.5, this corresponds to a very short distance, $r\Lambda_{\overline{\rm MS}} \approx 10^{-3}$, which is much smaller than the radius $r_Q \approx 1/m_Q$ at which we evaluate the wave function in order to calculate the matrix elements of the D and B mesons, for example. We therefore conclude that the behavior of χ and ϕ at short distances in the leading-log Coulomb potential is a physical effect.

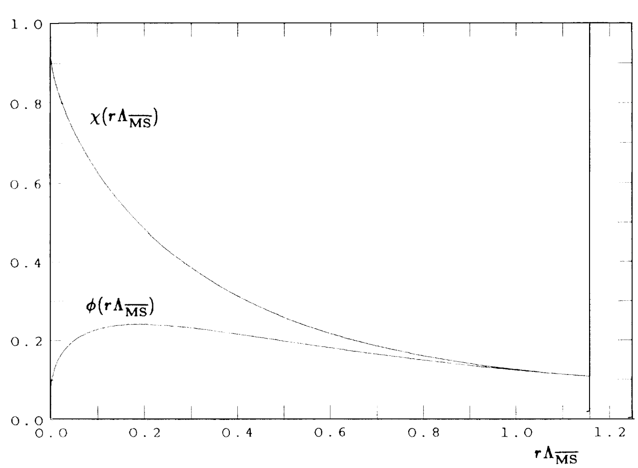

FIG. 2. Wave function components in the QCD effective action model, for $m = 0$ and $N_f = 3$. χ and ϕ are the upper and lower components, respectively [see Eqs. (8) and (29)].

IV. COMPARISON OF $f_{\rm rel}$ IN THE LEADING-LOG POTENTIAL AND IN THE EAM

We now have two relativistic descriptions of the short-distance physics of the $Q\overline{q}$ system: (i) the "pure" leading-log Coulomb potential, and (ii) the QCD effective action model (EAM). Both descriptions lead to the same short-distance limit for the light quark wave function, which we claim is an essentially model-independent property of the $Q\overline{q}$ system (implied by our dynamical ingredients A and B, see Sec. I).

On the other hand, these two descriptions are totally different at long distances. The EAM gives a well-behaved (and qualitatively correct) description of the long-distance physics, while the pure leading-log Coulomb potential is completely unphysical at long distances. Nevertheless, we find below that both the EAM and the pure Coulomb potential give essentially *identical* results for the evolution of $f_{\rm rel}(m_Q)$. Furthermore, it is easy to correlate the short-distance behavior of these wave functions [Eq. (20)] with the corrections that we find to the NR scaling law. We think that a comparison of the decay constant as computed in the "pure" leading-log Coulomb potential and in the EAM demonstrates that the short-distance effects of interest can be isolated in a model-independent way.

To make specific calculations, we must provide a definite connection between the overlap radius r_Q and the heavy quark mass m_Q. On physical grounds we do not expect r_Q to be much smaller than $1/m_Q$, $r_Q \gtrsim 1/m_Q$. Furthermore, since we do not know the connection between the quark mass m_Q and the meson mass M_P without a complete model calculation, we make the additional approximation $m_Q \approx M_P$ (which should hold in the limit of sufficiently large m_Q). We therefore take $r_Q \equiv 1/M_P$. Finally, since we have computed the wave function with $m_Q = \infty$, its normalization $\mathcal{N}$ is independent of m_Q, and does not contribute to the evolution of the decay constant. We show below that all of these approximations only *suppress* the effects we are interested in.

We therefore compare the M_P dependence of the "reduced" decay constants

$$\tilde{f}_{\rm EAM}(M_P) \propto \chi_{\rm EAM}\left(\frac{\Lambda_{\overline{\rm MS}}}{M_P}\right)\sqrt{\frac{1}{M_P}}\left(\frac{1}{\alpha_s(M_P)}\right)^{\gamma} \tag{36a}$$

and

$$\tilde{f}_{\rm Coul}(M_P) \propto \chi_{\rm Coul}\left(\frac{\Lambda_{\overline{\rm MS}}}{M_P}\right)\sqrt{\frac{1}{M_P}}\left(\frac{1}{\alpha_s(M_P)}\right)^{\gamma}. \tag{36b}$$

It is also instructive to compare these functions with the scaling law that is obtained in the extreme nonrelativistic quark-model picture (cf. Ref. [3]):

$$\tilde{f}_{\rm NR}(M_P) \propto \sqrt{\frac{1}{M_P}}\left(\frac{1}{\alpha_s(M_P)}\right)^{\gamma}. \tag{36c}$$

Notice that we have included in all three functions $\tilde{f}_{\rm EAM}$, $\tilde{f}_{\rm Coul}$, and $\tilde{f}_{\rm NR}$ a correction to the NR scaling law Eq. (3) due to a finite renormalization of the matrix element put forward by Voloshin and Shifman (VS) [9] and Politzer and Wise (PW) [10]. Their correction corresponds to the power of $\alpha_s(M_P)$ in Eqs. (36a)–(36c), where

$$\gamma \equiv \frac{1}{4\pi^2 b_0}. \tag{37}$$

We use $\chi_{\rm EAM}$ and $\chi_{\rm Coul}$ to denote the wave function as computed in the EAM and in the leading-log Coulomb potential, respectively. The scale for the evolution of these wave functions with meson mass M_P is set explicitly by the QCD scale parameter $\Lambda_{\overline{\rm MS}}$. We observe that both $\tilde{f}_{\rm EAM}$ and $\tilde{f}_{\rm Coul}$ approach the NR scaling law (including the VS-PW effect) Eq. (36c) for sufficiently large M_P, since our relativistic wave functions satisfy $\chi(1/M_P) \to 1$ for $M_P \gg \Lambda_{\overline{\rm MS}}$.

In order to study the m_Q evolution of the decay constant, we fix the light quark mass; we take $m = 0$ (cf. $m_{u,d} \ll \Lambda_{\overline{\rm MS}}$). We take $N_f = 3$ in the rest of this paper. Note that $\chi_{\rm Coul}$ is computed from Eq. (13) with ξ also set to zero (we cannot compute ξ and the wave-function normalization $\mathcal{N}$ in the pure leading-log Coulomb potential, since $\psi_{\rm Coul}$ is not normalizable at long distances).

To compute $\chi_{\rm EAM}$ we solve the coupled equations of motion in the EAM subject to the full set of boundary conditions [cf. Eqs. (30)–(32)]. In contrast, the wave function $\chi_{\rm Coul}$ in the leading-log Coulomb potential is obtained by integrating Eq. (13) with $\xi = 0$ outwards from the origin, subject only to the condition that the wave function is normalizable at the origin. Of course, this wave function develops a singularity as $r \to \Lambda_{\rm pot}^{-1}$. We use Eq. (35) to express the potential scale mass $\Lambda_{\rm pot}$ in terms of $\Lambda_{\overline{\rm MS}}$, when solving Eq. (13) for $\chi_{\rm Coul}$. We use $\Lambda_{\overline{\rm MS}}$ for the scale mass in the VS-PW correction [that is, as the scale for the running coupling $\alpha_s(M_P)$ in Eqs. (36a)–(36c)]. For reasonable values of $\Lambda_{\overline{\rm MS}}$, in the neighborhood of 200 MeV, this corresponds roughly to the scale mass used by VS in Ref. [9].

To compare the evolution of $\tilde{f}_{\rm EAM}$, $\tilde{f}_{\rm Coul}$, and $\tilde{f}_{\rm NR}$ it is appropriate to normalize them to have the same value at a large "reference" mass $M_{\rm ref}$:

$$\tilde{f}_{\rm EAM}(M_{\rm ref}) \equiv \tilde{f}_{\rm Coul}(M_{\rm ref}) \equiv \tilde{f}_{\rm NR}(M_{\rm ref}) \equiv 1, \quad M_{\rm ref} \gg \Lambda_{\overline{\rm MS}}. \tag{38}$$

The precise value of $M_{\rm ref}$ is unimportant. We take $M_{\rm ref}$ to be large since all calculations for the evolution of the decay constant should agree in the limit $m_Q \to \infty$, where the physics is well understood. The three "reduced" decay constants are plotted in Fig 3, with $M_{\rm ref} = 60\Lambda_{\overline{\rm MS}}$. The three functions approach the same asymptote at large meson masses M_P.

We see that the curves for $\tilde{f}_{\rm EAM}$ and $\tilde{f}_{\rm Coul}$ are very similar, all the way down to small masses $M_P \approx 5\Lambda_{\overline{\rm MS}}$. The outstanding feature of the relativistic curves is that they both possess a maximum. This maximum is due *entirely* to the relativistic dynamics of the light quark at short distances. These dynamics cause $\chi(r = 1/M_P)$ to *increase* as M_P increases, which "temporarily" overcomes the decrease in the kinematical factor $1/\sqrt{M_P}$. Thus, $f_{\rm rel}$ rises with M_P before it falls. The location of the maxima in $\tilde{f}_{\rm EAM}$ and $\tilde{f}_{\rm Coul}$ differs only by $\approx 15\%$. Although we must treat an extrapolation to small meson masses with caution, the occurrence of a maximum in the relativistic decay constant is a physical effect.

We conclude from this analysis that $\tilde{f}_{\rm EAM}$ and $\tilde{f}_{\rm Coul}$ are determined almost entirely by the short-distance properties of the relativistic wave function in the leading-log Coulomb potential, even at masses as low as $M_P \approx 5\Lambda_{\overline{\rm MS}}$. This conclusion is supported by the fact that $\chi_{\rm EAM}$ and $\chi_{\rm Coul}$ exhibit such different behaviors at long distances ($\chi_{\rm Coul}$ is completely unphysical, while $\chi_{\rm EAM}$ is well-behaved), and by the fact that the main features in the evolution of $\tilde{f}_{\rm EAM}$ and $\tilde{f}_{\rm Coul}$ are directly correlated with the evolution of $\chi(r)$ at short distances, due to the

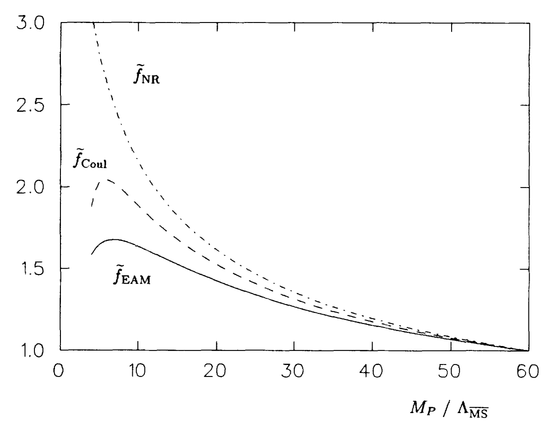

FIG. 3. Reduced decay constants in the effective action model $\tilde{f}_{\rm EAM}$ (solid line) and in the leading-log Coulomb potential $\tilde{f}_{\rm Coul}$ (dashed line) versus meson mass M_P. The extreme nonrelativistic scaling law $\tilde{f}_{\rm NR}$ is also illustrated (dashed-dot line). The functions are defined in Eqs. (36a)–(36c). All three curves include a correction due to Voloshin and Shifman, and Politzer and Wise. The curves are normalized to 1 at $M_P = 60\Lambda_{\overline{\rm MS}}$ [cf. Eq. (38)]. The value of $\Lambda_{\overline{\rm MS}}$ in these curves is arbitrary.

leading-log Coulomb interaction.

The only significant qualitative difference between the two relativistic functions in the region $M_P \gtrsim 5\Lambda_{\overline{\rm MS}}$ is that the maximum of $\tilde{f}_{\rm Coul}$ is somewhat more pronounced. This is due to the increasing effects of the unphysical Klein paradox on $\chi_{\rm Coul}(1/M_P)$ as $M_P \to \Lambda_{\rm pot}$ (from above). As there is *no* Klein paradox in $\chi_{\rm EAM}$, we use $\tilde{f}_{\rm EAM}$ to extract quantitative results throughout this paper.

A practical use for Fig. 3 is that if one is given a value for f_P at a particular mass M_P, then one can read off the value for the decay constant that would be predicted by extrapolating to another meson mass. In this connection, we observe that the NR scaling law (including the correction due to VS-PW) substantially overestimates the decay constant at small M_P. Alternatively, if the NR scaling law is used to extrapolate from the known value of f_P for a light meson (such as f_K), then one will significantly underestimate the value of f_P for heavier states.

The reduced decay constant $\tilde{f}_{\rm EAM}$ gives the following *lower bound* on the meson mass $M_{\rm peak}$ at which f_P is maximal:

$$M_{\rm peak} \gtrsim 6.7\Lambda_{\overline{\rm MS}}. \tag{39}$$

The reason why we obtain a lower bound for $M_{\rm peak}$ is that we underestimated the overlap radius r_Q (i.e., we took $r_Q = 1/M_P$, while we expect on physical grounds that $r_Q > 1/m_Q > 1/M_P$). To see what effect this has on $f_{\rm rel}$, suppose for the sake of illustration that the correct overlap radius is given by $r_Q = \rho/M_P$, where we expect that $\rho > 1$. Then (neglecting the correction due to VS-PW for simplicity) if the "reduced" decay constant $\tilde{f}_{\rm EAM} \equiv \chi(1/M_P)/\sqrt{M_P}$ is maximal at $M_P = M_{\rm red}$, the "true" decay constant $f_{\rm rel} = \mathcal{N}\chi(r_Q)[12/M_P]^{1/2}$ has its maximum at $M_P = \rho M_{\rm red}$, which follows from a simple rescaling.

With the reasonable value[3] $\Lambda_{\overline{\rm MS}} \approx 220$ MeV, Eq. (39) gives $M_{\rm peak} \gtrsim 1.5$ GeV, near the D meson mass. Note that the development of a maximum in $f_{\rm rel}$ is evident at even larger meson masses. For example, our results show that $f_{\rm rel}(M_P)$ changes concavity at $M_P \gtrsim 12.9\Lambda_{\overline{\rm MS}}$ (=2.8 GeV for $\Lambda_{\overline{\rm MS}}$ =220 MeV).

V. COMPARISON WITH THE NR SCALING LAWS FOR THE D AND B MESONS

In order to minimize the model dependence inherent in a calculation of the decay constants of individual mesons (which requires a complete calculation of the $Q\bar{q}$ overlap, including the wave-function normalization and heavy quark recoil), it has become customary to compute ratios of decay constants (see e.g. Refs. [9–11]). Using the reduced decay constant $\tilde{f}_{\rm EAM}$, we place the following *lower bound* on the ratio between the decay constants of the B and D mesons[4]:

$$\frac{f_B}{f_D} \gtrsim \sqrt{\frac{M_D}{M_B}}\left(\frac{\alpha_s(M_D)}{\alpha_s(M_B)}\right)^{\gamma} \times 1.32 \approx 0.86 \qquad (\Lambda_{\overline{\rm MS}} = 220 \text{ MeV}), \tag{40}$$

where we now use $r_Q = 1/m_Q$ for the overlap radius (with $m_c = 1.6$ GeV and $m_b = 5.0$ GeV). The fact that we obtain a lower bound for this ratio is again due to a systematic underestimate of the overlap radius [see the paragraph following Eq. (39)]. This can be readily verified from the curve for χ in Fig. 2, by using $r_Q = \rho/m_Q$, with $\rho > 1$.

We have written our result in terms of a correction to the NR scaling law, with the renormalization correction due to Voloshin and Shifman (VS) and Politzer and Wise (PW) also factored out separately. We note that while the correction due to VS-PW dominates in the limit $M_P \to \infty$, the effects due to the relativistic wave function are quantitatively more significant in the subasymptotic region $M_P \lesssim M_B$. For example, their correction to the NR scaling law for f_B/f_D amounts to a factor ≈ 1.09 [9], while our effects give a correction of at least 1.32.

We also compare the evolution of the relativistic mixing parameter $B_{\rm rel}(m_Q)$, with the corresponding NR scaling law, $B_{\rm NR} \equiv 1$ [Eq. (4)]. It is important to note that, unlike the case of the decay constant, to leading order there is no renormalization of the matrix element which defines B [9, 10]. On the other hand, we find that the effects of a relativistic wave function with asymptotic freedom are quantitatively as important for B as for the decay constant.

Since the lower component of the Dirac wave function in the leading-log Coulomb potential goes at short distances like $\phi(r \to 0) \sim \alpha_s(r)$ [cf. Eq. (20)], we find that $B_{\rm rel}(m_Q) \to 1$ only for $m_Q \to \infty$ [cf. Eqs. (11) and (12)]. Figure 4 shows $B_{\rm rel}(m_Q)$ as computed in our EAM, using Eq. (12) to approximate the overlap of the $Q\bar{q}$ pair. We take $r_Q = 1/m_Q$ for the overlap radius. We see that our effects cause B to decrease extremely rapidly for $m_Q \lesssim 10\Lambda_{\overline{\rm MS}}$. Unlike the decay constant, the mixing parameter therefore depends strongly on the shape of the heavy quark wave function that is used in the overlap integral. We thus require a better treatment of the $Q\bar{q}$ overlap than we have used here. However, it is clear from Fig. 4 that relativistic effects are quantitatively significant, making a correction of about 20% to the NR scaling law at the D meson mass scale, for example.

[3]We obtained $\Lambda_{\overline{\rm MS}} \approx 240$ MeV in a fit to the slope of the light meson Regge trajectories [24], based on our leading-log EAM for $Q\bar{q}$ mesons with a dynamical scalar quark [23]. Adler and Piran obtained $\Lambda_{\overline{\rm MS}} \approx 220$ MeV in a fit to the quarkonium potential in their EAM for the $Q\bar{Q}$ system [19]. A somewhat smaller value of $\Lambda_{\overline{\rm MS}}$ in the same model was obtained in a fit to the spin-averaged quarkonium spectrum by Hiller [31].

[4]We note that the ratios of decay constants obtained from a two-loop approximation [19, 24] to the QCD effective action is roughly the same (to within $\approx 10\%$) as the ratios obtained in the one-loop approximation used in this paper, for a given value of $\Lambda_{\overline{\rm MS}}$.

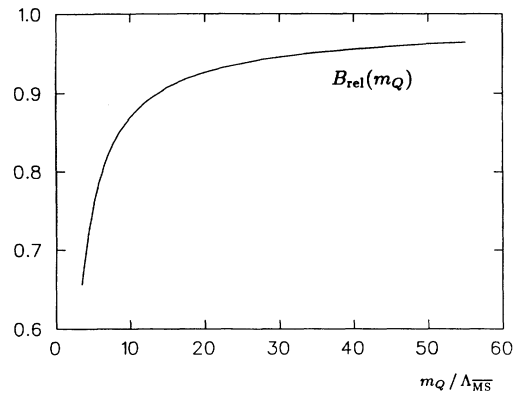

FIG. 4. Relativistic neutral-meson mixing parameter $B_{\rm rel}(m_Q)$ versus heavy quark mass m_Q. The curve is computed from Eq. (12) using an overlap radius $r_Q = 1/m_Q$ and the wave function in the effective action model. We observe that $B_{\rm rel}(m_Q \to \infty) \to 1$. The value of $\Lambda_{\overline{\rm MS}}$ in this curve is arbitrary.

VI. EXTRAPOLATION TO THE K MESON AND ABSOLUTE NORMALIZATION FOR f_D AND f_B

In order to extract absolute values for the individual decay constants f_D and f_B with as little model dependence as possible, one may consider a possible extrapolation of our calculations to the K meson. This would enable us, in principle, to use the experimental value of f_K as a normalization.

We recognize however that our methods for treating finite m_Q effects are of uncertain reliability when applied to the s quark (although for the application to the K meson, we use a constituent mass $m_s \approx 500$ MeV). Furthermore, our model explicitly breaks chiral symmetry. On the other hand this extrapolation is, in our opinion, more justified in the context of a "one heavy quark" approximation that in purely NR models where the extrapolation from f_K is frequently made.

In this connection, it is instructive to compare the evolution of our relativistic decay constant with NR quark model calculations (e.g., Ref. [3]) which claim that the NR scaling law is already valid for meson masses as small as $M_P \approx M_K$. Indeed, we find

$$\frac{f_D}{f_K} \gtrsim \sqrt{\frac{M_K}{M_D}} \times 2.00 \quad (\Lambda_{\overline{\rm MS}} = 220 \text{ MeV}), \tag{41}$$

where the factor of 2.00 comes entirely from the relativistic wave function.

We do not include the Voloshin and Shifman, Politzer and Wise correction in our result for f_D/f_K, since their (perturbative) correction cannot be reliably estimated in the K to D region. We observe that the tendency of their effect in this region would be in the same direction as ours. We do however include their correction below in going from f_D to f_B, in a manner consistent with Sec. V.

We clearly see from Eq. (41) that if the NR scaling law is used to extrapolate from the value of f_P for a light meson, such as f_K, then one will significantly underestimate the value of f_P for heavier states. We point out that a constituent quark model based on the nonrelativistic (or quasirelativistic) Schrödinger equation cannot account for the significant relativistic effects described here, which derive from the short-distance behavior of the Dirac wave function.

We can make a more complete estimate of these ratios by accounting for the m_Q dependence of the light quark wave function normalization (recall that we computed the wave function in Born-Oppenheimer approximation, with $m_Q = \infty$). In typical potential model calculations, finite m_Q effects on the wave function (due in particular to the color-magnetic moment of the heavy quark) tend, for example, to make the radius of the D meson smaller than the radius of the K. As conservative estimates of the resulting corrections to the wave-function normalizations, we take

$$\begin{aligned} \mathcal{N}_D/\mathcal{N}_K &\approx (R_K/R_D)^{3/2} \approx (1.2)^{3/2}, \\ \mathcal{N}_B/\mathcal{N}_D &\approx (R_D/R_B)^{3/2} \approx 1. \end{aligned} \tag{42}$$

These values for the ratio of meson radii are typical of constituent quark model calculations that include the perturbation due to the heavy quark magnetic moment (see, e.g., Refs. [7, 8, 32]).[5]

If we now assume that we can apply our calculations to the K system, in order to use f_K as input to individually estimate f_D and f_B, we obtain the following lower bounds, which now include our estimates of the above finite m_Q effects:

$$\frac{f_D}{f_K} \gtrsim \sqrt{\frac{M_K}{M_D}} \times 2.00 \times 1.3 \;\Rightarrow\; f_D \gtrsim 225 \text{ MeV} \tag{43a}$$

and

$$\frac{f_B}{f_D} \gtrsim \sqrt{\frac{M_D}{M_B}} \left(\frac{\alpha_s(M_D)}{\alpha_s(M_B)}\right)^{\gamma} \times 1.32 \;\Rightarrow\; f_B \gtrsim 195 \text{ MeV}, \tag{43b}$$

where we again used $\Lambda_{\overline{\rm MS}} = 220$ MeV. If we use instead $\Lambda_{\overline{\rm MS}} \approx 180$ MeV, then these values are lowered by about 10%. The first numerical factor in each line of Eqs. (43a) and (43b) is the correction due to the relativistic wave function; the second numerical factor in Eq. (43a) comes from the wave-function normalization correction, Eq. (42).

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[5]We note that in a full calculation of finite m_Q effects in our relativistic EAM, we expect to obtain R_B somewhat smaller than R_D ($R_D/R_B \approx 1.10$), as a result of an additional perturbation due to the recoil of the heavy quark. This will tend to increase the value of f_B relative to f_D, further enhancing our relativistic correction to the NR scaling law in this case.

We must again stress that we consider results obtained with f_K as input to be rough, given the uncertainties involved in our treatment of finite m_Q effects when applied to the K meson. However, it is obvious from Eqs. (43a) and (43b) that the NR scaling law should not be used to extrapolate from f_K, due to substantial corrections from relativistic and leading-log Coulomb dynamics. Moreover, our estimate for the ratio f_B/f_D in Sec. V [Eq. (40)], which does not depend on f_K, demonstrates that the NR scaling law should not be used to extrapolate from f_D.

In this connection, we note that one measure of the reliability of our Born-Oppenheimer approximation for a heavy quark of given mass m_Q is the probability of finding the light quark inside the Compton wavelength $r_Q \approx 1/m_Q$. If the probability is very large, then we expect that finite m_Q effects on the wave function cannot be neglected. However, even inside a distance as large as $r \approx (2\Lambda_{\overline{\rm MS}})^{-1}$, we calculate (using our EAM) that the total probability of finding the light quark is only about 20%.

A phenomenlogically interesting check of the consistency of our extrapolation to the K meson is to consider a more complete model calculation of f_K within our QCD effective action approximation. The absolute value for f_K is obtained from the wave-function normalization in the EAM, $\mathcal{N} \propto (\Lambda_{\overline{\rm MS}})^{3/2}$ [cf. Eq. (10)]. We parametrize the overlap radius of the $Q\overline{q}$ pair using $r_Q = \lambda/m_Q$ in Eq. (10) and fit λ from the experimental value of f_K. Although such an application must again be viewed with caution, we think that this example lends support to our results—this fit to f_K is obtained with an overlap radius that is consistent with the Compton wavelength of the constituent s quark, $r_Q \approx 1.50/m_Q$ ($m_s \approx 500$ MeV), and with a reasonable value for the QCD scale parameter, $\Lambda_{\overline{\rm MS}} \approx 220$ MeV.

VII. SUMMARY AND OUTLOOK

We have found substantial corrections to the well-known nonrelativistic scaling laws for the decay constants and mixing parameters of the pseudoscalar $Q\overline{q}$ mesons, due to two fundamental dynamical properties of this system: relativistic dynamics for the light quark and an asymptotically free Coulomb interaction. Our analysis is based on very general, model-independent, features of these dynamics. An important element of our analysis is that the scale for the matrix elements is set explicitly by the QCD scale parameter. This implies a direct correlation between our calculations of the matrix elements and the values of other fundamental quantities of hadronic physics. Our quantitative estimates in the regime $M_K \lesssim M_P \lesssim M_B$ [cf. Eqs. (39)–(43)] have important consequences for phenomenology. We note that while the short-distance correction due to Voloshin and Shifman [9] and Politzer and Wise [10] (coming from a finite renormalization of the matrix element which defines f_P) dominates in the limit $M_P \to \infty$, we find that the effects due to the relativistic wave function are quantitatively more significant in the subasymptotic region $M_P \lesssim M_B$ [33].

The analysis presented in this paper can be made more complete by improving our treatment of finite m_Q effects in the $Q\overline{q}$ overlap integral, and in the normalization of the light quark wave function. We note, for example, that the corrections we found to the NR scaling law will be enhanced by using a realistic connection between the overlap radius r_Q and the heavy quark mass m_Q. The overlap radius would be uniquely determined by computing the wave function for the heavy quark in a full treatment of the Born-Oppenheimer approximation. It would also be interesting to analyze the influence of an asymptotically free relativistic wave function on other hadronic matrix elements of current interest, such as those required in setting limits on possible extensions to the standard model.

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