η-ξ spacetime and thermo fields

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In this paper an intuitive picture of η-ξ spacetime is given. It is shown why η-ξ spacetime can be regarded as a geometric background of field theory at finite temperature. In particular, a new way is put forth. We can get the real-time thermal Green's function with a 2×2 matrix by rotating the imaginary-time thermal Green's function from the Euclidean section to the Lorentzian section in η-ξ spacetime.

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The geometry and topology of spacetime have a fundamental effect on quantum field theory. After Hawking's famous discovery, in recent studies of quantum fields in static spacetime with horizons it has been found that the natural vacuum states possess thermal properties. Standard results for the Schwarzschild and Rindler cases have been found by many authors. In the former, the Hartle-Hawking vacuum is a thermal state for a static observer in Schwarzschild spacetime. In the latter, the Minkowski vacuum agrees with a thermal state for an accelerated observer. In both cases, these observers are not inertial. Can we construct a new spacetime in which the vacuum is a usual thermal state for a Minkowski inertial observer so that we can find the direct connection between geometry and thermo fields?

The answer is yes. η-ξ spacetime [1] provides a geometrical background for thermo-field theory. Geometrically, η-ξ spacetime can be regarded as, as pointed out by Wald [2], a maximal analytical complex extension of $S^1\times R^3$. This is why η-ξ spacetime relates closely to thermo-field theory. By the geometrical background of thermo fields we mean that η-ξ spacetime should possess the following properties. (a) The vacuum state for quantum fields in η-ξ spacetime is a thermal state for an inertial observer in Minkowski spacetime. (b) Every finite-temperature field theory method, such as the imaginary- and real-time thermal Green's function methods, thermo-field dynamics (TFD), and the black-hole radiation theory, can find its position in the η-ξ formalism. That is, η-ξ spacetime should provide a unified geometrical background for the above thermo-field methods. (c) The vacuum Green's functions on the Euclidean section in η-ξ spacetime are equal to the imaginary-time thermal Green's functions in Minkowski spacetime, and the vacuum Green's functions on the Lorentzian section in η-ξ spacetime correspond to the real-time thermal Green's functions in Minkowski spacetime. We can get the real-time thermal Green's functions with a 2×2 matrix from the imaginary-time thermal Green's functions by analytically continuing from the Euclidean section to the Lorentzian section in η-ξ spacetime. In this Brief Report we will give a geometrical picture of the above conclusions to show the connection between quantum fields in η-ξ spacetime and other thermo-field methods.

It is interesting to sketch an intuitive picture to show how η-ξ spacetime naturally provides a periodicity for the imaginary time τ. It is well known that the thermal Green's functions possess the imaginary-time periodicity

$$G_\beta(t,\mathbf{x})=G_\beta(t+iN\beta,\mathbf{x}) \quad \text{for all integers } N .$$

They can be regarded as defined in flat Euclidean space:

$$ds^2=d\tau^2+dx^2+dy^2+dz^2 , \tag{1}$$

where the τ coordinate is periodic with $\tau\sim\tau+\beta$. The manifold is a hypercylinder with topology $S^1\times R^3$ (see Fig. 1). Under the topological transformation

$$\sigma=\frac{1}{\alpha}e^{\alpha x}\sin\alpha\tau,\quad \xi=\frac{1}{\alpha}e^{\alpha x}\cos\alpha\tau , \tag{2}$$

the hypercylinder is opened up and flattened onto a hyperplane, still retaining its periodicity for τ (with period $\beta=2\pi/\alpha$) and its topology. This hyperplane is nothing but the Euclidean section in η-ξ spacetime with the metric

$$ds^2=\frac{1}{\alpha^2(\xi^2+\sigma^2)}(d\sigma^2+d\xi^2)+dy^2+dz^2 ; \tag{3}$$

i.e., the hypercylinder (1) is mapped onto the Euclidean hyperplane (3). On the σ-ξ plane, $\theta=\alpha\tau$ plays the role of a polar angle with period 2π. The fields defined in η-ξ

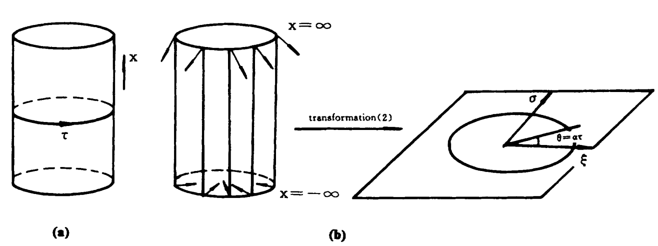

FIG. 1. (a) A hypercylinder of radius $\beta/2\pi$. Two coordinates (y,z) have been suppressed. (b) Under transformation (2) the cylinder is expanded out at the top, contracted at the bottom, and flattened onto the σ-ξ plane. On the σ-ξ plane, $\theta=\alpha\tau$ plays the role of a polar angle with period 2π.

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spacetime will automatically satisfy the periodicity for the imaginary time τ.

By analytically continuing σ to $\eta = i\sigma$ in Eq. (3), we obtain the Lorentzian section of η-ξ spacetime with the metric

$$ds^2 = \frac{1}{\alpha^2(\xi^2-\eta^2)}(-d\eta^2+d\xi^2)+dy^2+dz^2 \ . \tag{4}$$

The singularities at $\xi^2-\eta^2=0$ divide the Lorentzian section into four disjointed parts I, II, III, IV (Fig. 2), each of which is identified with a Minkowski spacetime. To see this, we introduce a coordinate transformation in regions I and II:

$$\begin{aligned} \eta &= \pm\frac{1}{\alpha}e^{\alpha x}\sinh\alpha t \quad \text{the upper sign for region I,} \\ \xi &= \pm\frac{1}{\alpha}e^{\alpha x}\cosh\alpha t \quad \text{the lower sign for region II;} \end{aligned} \tag{5}$$

then (4) becomes a Minkowski spacetime

$$ds^2 = -dt^2+dx^2+dy^2+dz^2 \ .$$

The singularities at $\xi^2-\eta^2=0$ can be formally regarded as the "horizons." The "causal" construction on the Lorentzian section, which is similar to one in black-hole spacetime, leads to direct application of Hawking's black-hole radiation theory. In Ref. [1] it has been shown that the vacuum defined on the whole Lorentzian section is just a thermal state for an integral observer in Minkowski spacetime.

The Killing fields on the Lorentzian section are defined by

$$\left(\frac{\partial}{\partial\lambda}\right)^a = \varepsilon\alpha(\xi\eta^a+\eta\xi^a) \ , \quad \varepsilon = \begin{cases} 1 & \text{in region I ,} \\ -1 & \text{in region II ,} \end{cases} \tag{6}$$

which are timelike in regions I and II. It is natural to choose the Killing parameters λ as the time coordinate, which coincides with the Minkowski times t and $-t$ in regions I and II, respectively. The Hamiltonian H^λ for the quantum fields on the Lorentzian section, which is defined on a spacelike surface (such as $\lambda=$const), can be written as

$$H^\lambda = H_{\rm I}^\lambda + H_{\rm II}^\lambda = H - \tilde{H} \ , \tag{7}$$

where H and $\tilde{H}$ are the Hamiltonians defined on the hypersurface $t=$const in Minkowski spacetime for regions I and II, respectively. The presence of the sign change of the second term is due to the opposite out-normal direction of the hypersurfaces $t=$const and $\lambda=$const. Equation (7) directly relates the quantum fields in η-ξ spacetime with the thermo-field dynamics of Takahashi and Umezawa [3]. This picture also provides an intuitive interpretation for $\tilde{H}$: it can be regarded as the field in the "mirror universe" (region II). By the above analysis, we find that thermo-field dynamics corresponds to the Hamiltonian formalism of quantum field theory in η-ξ spacetime.

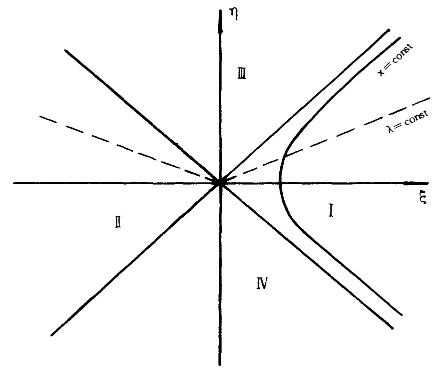

FIG. 2. The Lorentzian section in η-ξ spacetime. The "horizons" at $\xi^2-\eta^2=0$ divide it into four disjoint parts.

The path-integral formulation in quantum field theory is essentially related to geometry and topology. Using it, we now will show that the (vacuum) Green's functions on the Euclidean section in η-ξ spacetime are equal to the imaginary-time (Matsubara) thermal Green's functions, and the (vacuum) Green's functions on the Lorentzian section agree with the real-time thermal Green's functions. Consider a scalar field in η-ξ spacetime with an action

$$S^\eta = \int d\eta\, d\xi\, dx_\perp e^{-2\alpha x}\left\{\frac{1}{2}e^{2\alpha x}\left[\left(\frac{\partial\varphi}{\partial\eta}\right)^2-\left(\frac{\partial\varphi}{\partial\xi}\right)^2\right] - \frac{1}{2}(\nabla_\perp\varphi)^2-\frac{1}{2}m^2\varphi^2-V(\varphi)\right\} , \tag{8}$$

where $e^{2\alpha x}=\alpha^2|\xi^2-\eta^2|$. The Euclidean generating functional can be written as

$$Z_E[j] = N\int[d\varphi]\exp\left[-\int d\sigma\, d\xi\, dx_\perp(\mathcal{L}^\sigma - e^{-2\alpha x}j\varphi)\right] . \tag{9}$$

We now perform the change of variables (2) from (σ,ξ) to (τ,x). If we want the transformation to be single valued, we must have $0\le\alpha\tau\le 2\pi$, or equivalently $0\le\tau\le 2\pi/\alpha\equiv\beta$, and then $\varphi(\tau=0)=\varphi(\tau=\beta)$. We obtain

$$Z_E[0] = N\int_{\varphi(\tau=0)=\varphi(\tau=\beta)}[d\varphi]\exp\left\{-\int_0^\beta d\tau\int d^3x\left[\frac{1}{2}\left(\frac{\partial\varphi}{\partial\tau}\right)^2+\frac{1}{2}(\nabla\varphi)^2+\frac{1}{2}m^2\varphi^2+V(\varphi)\right]\right\} . \tag{10}$$

Equation (10) coincides with the path-integral form of the partition function ${\rm Tr}e^{-\beta H}$ in the temperature field theory [4]. And $Z_E[J]$ is the complete generating functional for the Euclidean source $J(\tau,\mathbf{x})$. All of the imaginary-time thermal Green's functions can be obtained by differentiating the generating functional $Z_E[J]$.

On the Lorentzian section, the generating functional in η-ξ spacetime has the form

$$Z[j] = N_1\int[d\varphi]\exp\left[i\int d\eta\, d\xi\, dx_\perp e^{-2\alpha x}\left\{\frac{1}{2}e^{2\alpha x}\left[\left(\frac{\partial\varphi}{\partial\eta}\right)^2-\left(\frac{\partial\varphi}{\partial\xi}\right)^2\right]-\frac{1}{2}(\nabla_\perp\varphi^2)-\frac{1}{2}m^2\varphi^2-V(\varphi)+j\varphi\right\}\right] . \tag{11}$$

Now we take the change of variables from (η,ξ) to (λ,X); here, λ is defined by (6) and X by

$$\left[\frac{\partial}{\partial X}\right]^a=\alpha(\eta\eta^a+\xi\xi^a) . \tag{12}$$

Then Eq. (11) can be written as

$$Z[J]=N_1\int[d\varphi]\exp\left[i\int d^4X[\mathcal{L}^\lambda(\varphi)+J\varphi]\right] , \tag{13}$$

where

$$d^4X=\varepsilon\, d\lambda\, dX\, dx_\perp,\quad J=e^{-2\alpha x}j ,$$
$$\mathcal{L}^\lambda(\varphi)=\tfrac{1}{2}\varphi(\Box_\lambda-m^2)\varphi-V(\varphi) , \tag{14}$$
$$\Box_\lambda=-\frac{\partial^2}{\partial\lambda^2}+\frac{\partial^2}{\partial X^2}+\nabla_\perp^2 ,$$

and the integral $\int d^4X$ in the exponential runs over the whole η-ξ spacetime, i.e., over all of regions I–IV.

In (13), the change of the field variables

$$\varphi(X)\to\varphi(X)+\int d^4Y\, D_\lambda(X-Y)J(Y) , \tag{15}$$

where

$$(\Box_\lambda-m^2)D_\lambda(X-Y)=\delta^4(X-Y) , \tag{16}$$

leads to

$$Z[J]=\exp\left[-i\int d^4X'\, V\left[\frac{1}{i}\frac{\delta}{\delta J(X')}\right]\right]\times\exp\left[-\frac{i}{2}\int d^4X\int d^4Y\, J(X)D_\lambda(X-Y)J(Y)\right] . \tag{17}$$

Note that regions III and IV are spacelike. When calculating the Green's functions for a physical field, we can drop them by letting $J(X)=0$, if $X\in$ III or IV. Equation (17) then becomes

$$Z[J]=\exp\left\{-i\int d^4x'\left[V\left[\frac{1}{i}\frac{\delta}{\delta J_1(x')}\right]-V\left[\frac{1}{i}\frac{\delta}{\delta J_2(x')}\right]\right]\right\}\exp\left[\frac{-i}{2}\int d^4x\int d^4y\, J_a(x)D_\lambda^{ab}(x-y)J_b(y)\right] , \tag{18}$$

where $a,b=1,2$ and

$$J_1(x)=J(X),\ X\in\text{I},\quad J_2(x)=-J(X),\ X\in\text{II} ,$$
$$D_\lambda^{11}(x-y)=D_\lambda(X-Y),\quad X,Y\in\text{I} ,$$
$$D_\lambda^{22}(x-y)=D_\lambda(X-Y),\quad X,Y\in\text{II} , \tag{19}$$
$$D_\lambda^{12>}(x-y)=-D_\lambda(X-Y),\quad X\in\text{I},\ Y\in\text{II} ,$$
$$D_\lambda^{21<}(x-y)=-D_\lambda(X-Y),\quad Y\in\text{I},\ X\in\text{II} .$$

Equation (18) is the generating functional for the real-time thermal Green's functions [5]. Using the Bogoliubov transformation between the η-ξ modes and the Minkowski modes in Ref. [1], we can get a matrix propagator in momentum space:

$$iD_\lambda^{ab}(k)=U(\beta,\omega)\times\begin{bmatrix}\dfrac{i}{k^2-m^2+i\varepsilon} & 0\\ 0 & \dfrac{-i}{k^2-m^2-i\varepsilon}\end{bmatrix}U(\beta,\omega) , \tag{20}$$

with

$$U(\beta,\omega)=\begin{bmatrix}\cosh\theta_\omega & \sinh\theta_\omega\\ \sinh\theta_\omega & \cosh\theta_\omega\end{bmatrix} ,$$

where

$$\tanh\theta_\omega=\exp(-\pi\omega/\alpha)=\exp(-\beta\omega/2) .$$

It is well known [6] that a doubling of the degrees of freedom is necessary in order to be able to calculate real-time thermal Green's functions. This doubling is absent in the Matsubara formalism. The η-ξ formulation provides a geometrical interpretation for this fact. On the Euclidean section in η-ξ spacetime, the imaginary-time periodicity of Matsubara Green's functions is provided by the periodicity of a polar-angle variable. When the field is rotated from the Euclidean section to the Lorentzian section in η-ξ spacetime, the appearance of "horizons" naturally leads to the doubling of the degrees of freedom, corresponding to the fields in regions I and II. It leads to the real-time thermal Green's functions.

It is helpful to compare the field theory in η-ξ spacetime with other calculation methods of thermal Green's functions. Dolan and Jackiw [5] performed the analytic continuation of the thermal free propagator from imaginary time τ to real time t, but they could get only a 1-1 component of the real-time thermal Green's function with a 2×2 matrix such as Eq. (20), which led to hopeless difficulties. By choosing a suitable time contour in a complex t plane which contains the real axis, Niemi and Semenoff [6] obtained the real-time thermal Green's functions with a 2×2 matrix. This method is called a nontrivial process of analytic continuation in Ref. [7]. Now it becomes clear that if we hope to get the real-time thermal Green's functions with a 2×2 matrix by "trivial" analytic continuation, we must rotate the imaginary-time thermal Green's functions from the Euclidean section to the Lorentzian section in η-ξ spacetime; i.e., simply replace σ by $-i\eta$. This is because only η-ξ spacetime, instead of Minkowski spacetime, can essentially represent the geometry of the field theory at finite temperature. As mentioned above, the Euclidean section of η-ξ spacetime intrinsically contains the periodicity for Minkowski imaginary time τ by transformation (2); the singularity at $\xi^2+\sigma^2=0$ indicates that the Euclidean section possesses the topology $S^1\times R^3$. When rotated to the Lorentzian section by $\sigma \to -i\eta$, the singularity at $\xi^2+\sigma^2=0$ corresponds to the "horizons" at $\xi^2-\eta^2=0$ on the Lorentzian section, and the sine functions in the transformation (2) become the hyperbolic functions in the transformation (5), which leads to the application of Hawking's radiation theory. All of these geometric and topologic characteristics lead to a very interesting connection between η-ξ spacetime and thermo fields.

So far in the η-ξ formulation we have, in a unified way, discussed the real and imaginary thermal Green's functions, thermo-field dynamics, and Hawking's radiation theory. We expect that quantum field theory in η-ξ spacetime may be regarded as a universal theory of field theory at finite temperature.

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