Perturbative evaluation of renormalization-group functions in massive three-dimensional ϕ^6 theory

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Dimensional regularization characterizes divergences using poles that occur in Feynman integrals when n (the number of dimensions in the regulated theory) equals D (the number of dimensions in the initial classical action). Since these poles are generated by gamma functions of the form $\Gamma(A-n/2)$, a divergence arises only if $A-n/2$ is a negative integer; consequently the need for renormalization arises only beyond one-loop order when n is odd. We illustrate this by computing the renormalization group functions to lowest order in a model for a massive scalar in three dimensions with four- and six-point couplings.

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I. INTRODUCTION

Dimensional regularization [1] is a particularly useful technique in perturbative calculations of radiative effects for two principal reasons. First of all, it respects any symmetry that does not depend on the dimensionality of the classical theory (such as gauge symmetry). Second, it characterizes all divergences arising in the course of evaluating Feynman integrals by poles occurring when n (the dimension of regulated theory), equals D (the dimension of the classical theory), making it possible to renormalize by using a "mass-independent" subtraction scheme [2].

A peculiarity of dimensional regularization is that in odd dimensions integrals which are divergent by naive power counting may be regulated to a finite value, with no poles occurring when $n=D$. To see this, let us examine the standard n-dimensional (Euclidean) integral

$$\int \frac{d^nk}{(2\pi)^n}\frac{(k^2)^a}{(k^2+m^2)^b} = \frac{1}{(4\pi)^{n/2}}(m^2)^{n/2+a-b}\frac{\Gamma\left[\frac{n}{2}+a\right]\Gamma\left[b-a-\frac{n}{2}\right]}{\Gamma\left[\frac{n}{2}\right]\Gamma(b)} . \tag{1}$$

Irrespective of what integer values a and b may take, no poles occur on the right-hand side of (1) when n approaches any odd integer.

Nevertheless, odd dimensional theories can be divergent beyond one-loop order as a and b in (1) themselves in principle can then depend on $n/2$. We illustrate how this happens in the three-dimensional theory whose Lagrangian is

$$\mathcal{L}=-\frac{1}{2}(\partial_\mu\phi)^2-\frac{1}{2}m^2\phi^2-\frac{1}{4!}\kappa\phi^4-\frac{\lambda}{6!}\phi^6 . \tag{2}$$

This is the most general model for a self-interacting scalar in three dimensions consistent with renormalizability and the symmetry $\phi\rightarrow-\phi$.

In the next section we perform an explicit calculation of the renormalization-group functions to second order in the coupling constants κ and λ, using the techniques of (1) and (2). (An analogous computation has been done in the four-dimensional ϕ^4 (ϕ_4^4) model in [3].)

II. THE RENORMALIZATION-GROUP FUNCTIONS

To second order in κ and λ, the 26 diagrams that contribute to the two-, four-, and six-point functions are illustrated in Fig. 1. It becomes apparent that the integration formula (1) leads to poles when $n=3$ only if the diagram being evaluated involves having two vertices connected by an odd number of internal lines. Consequently the only divergent diagrams are i, j, k, l, s, t, and z. The only Feynman integrals we need to compute are consequently

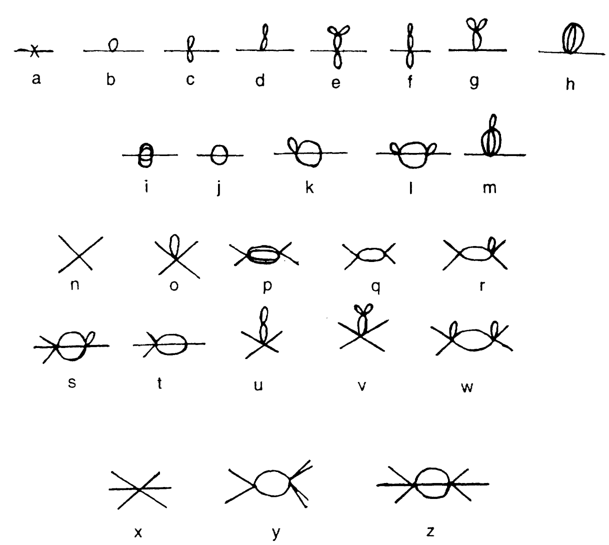

FIG. 1. Two-, four-, six-point functions.

$$I_a=\int \frac{d^nk}{(2\pi)^n}\frac{1}{k^2+m^2} , \tag{3a}$$

$$I_b=\int \frac{d^nk}{(2\pi)^n}\frac{d^nq}{(2\pi)^n}\frac{1}{k^2+m^2}\frac{1}{q^2+m^2}\frac{1}{(p+k+q)^2+m^2} , \tag{3b}$$

and

$$I_c=\int \frac{d^nk}{(2\pi)^n}\frac{d^nq}{(2\pi)^n}\frac{d^nr}{(2\pi)^n}\frac{d^ns}{(2\pi)^n}\frac{1}{k^2+m^2}\frac{1}{q^2+m^2}\frac{1}{(k+r)^2+m^2}\frac{1}{(q+s)^2+m^2}\frac{1}{(p+r+s)^2+m^2} . \tag{3c}$$

In the Appendix we show that the relevant contributions to I_a, I_b, and I_c are given by ($\varepsilon\equiv 3-n$)

$$I_a=\frac{-m}{4\pi} , \tag{4a}$$

$$I_b=\frac{1}{32\pi^2\varepsilon} , \tag{4b}$$

$$I_c=\frac{-1}{2^{11}\pi^4\varepsilon}\left[\frac{p^2}{3}-5m^2\right] . \tag{4c}$$

We now apply renormalization theory to our perturbative expansion along the lines of [2] and [3]. Initially, the action of (2) is split into two parts,

$$\mathcal{L}^{(0)}=-\tfrac{1}{2}(\partial_\mu\phi)^2-\tfrac{1}{2}m_R^2\phi^2 \tag{5a}$$

and

$$\mathcal{L}^{(1)}=-\frac{1}{2}(m_B^2-m_R^2)\phi^2-\frac{1}{4!}\kappa_B\phi^4-\frac{1}{6!}\lambda_B\phi^6 \tag{5b}$$

so that the contributions of diagrams a, n, and x are

$$(a)=-(m_B^2-m_R^2) , \tag{6a}$$

$$(n)=-\kappa_B , \tag{6b}$$

and

$$(x)=-\lambda_B . \tag{6c}$$

The bare and renormalized quantities are now related by the equations [2]

$$\lambda_B=\mu^{2\varepsilon}\left[\lambda_R+\sum_{\nu=1}^{\infty}\frac{a_\nu(\lambda_R)}{\varepsilon^\nu}\right] , \tag{7a}$$

$$\kappa_B=\mu^{\varepsilon}\left[\kappa_R+\sum_{\nu=1}^{\infty}\frac{\bar a_\nu^{\kappa}(\lambda_R)\kappa_R+\bar a_\nu^{m}(\lambda_R)m_R}{\varepsilon^\nu}\right] , \tag{7b}$$

and

$$m_B^2=\left[m_R^2+\sum_{\nu=1}^{\infty}\frac{b_\nu^{m^2}(\lambda_R)m_R^2+b_\nu^{m\kappa}(\lambda_R)m_R\kappa_R+b_\nu^{\kappa^2}(\lambda_R)\kappa_R^2}{\varepsilon^\nu}\right] , \tag{7c}$$

with the wave-function renormalization being given by

$$Z=1+\sum_{\nu=1}^{\infty}\frac{c_\nu(\lambda_R)}{\varepsilon^\nu} . \tag{7d}$$

Here μ is the usual radiatively induced scale parameter.

The lowest-order contributions to the functions in (7) are now determined by evaluating the appropriate Feynman diagrams depicted in Fig. 1. They are given by

$$(\mathrm{i})=\lambda_B^2(1)\left[\frac{1}{120}\right]\left[\frac{-1}{2^{11}\pi^4\varepsilon}\left[\frac{p^2}{3}-5m_R^2\right]\right] , \tag{8a}$$

$$(\mathrm{j})=\kappa_B^2(1)\left[\frac{1}{6}\right]\left[\frac{1}{32\pi^2\varepsilon}\right] , \tag{8b}$$

$$(\mathrm{k})=\lambda_B\kappa_B(2)\left[\frac{1}{12}\right]\left[\left[\frac{-m_R}{4\pi}\right]\left[\frac{1}{32\pi^2\varepsilon}\right]\right] , \tag{8c}$$

$$(\mathrm{l})=\lambda_B^2(1)\left[\frac{1}{24}\right]\left[\left[\frac{-m_R}{4\pi}\right]^2\left[\frac{1}{32\pi^2\varepsilon}\right]\right] , \tag{8d}$$

$$(\mathrm{s})=\lambda_B^2(4)\left[\frac{1}{12}\right]\left[\left[\frac{-m_R}{4\pi}\right]\left[\frac{1}{32\pi^2\varepsilon}\right]\right] , \tag{8e}$$

$$(\mathrm{t})=\lambda_B\kappa_B(4)\left[\frac{1}{6}\right]\left[\frac{1}{32\pi^2\varepsilon}\right] , \tag{8f}$$

$$(\mathrm{z})=\lambda_B^2(10)\left[\frac{1}{6}\right]\left[\frac{1}{32\pi^2\varepsilon}\right] . \tag{8g}$$

The first terms, on the right-hand sides of (8) are the number of diagrams, the second the so-called "symmetry factor," and the third the contribution of the Feynman integrals, as determined by (4).

By (6a) and (8a)–(8d) we see that the portion of the radiative correction to the two-point function that contributes to wave function and mass renormalization is such that

$$\Delta_F^{-1}=p^2+m_R^2-\Sigma=\left[1+\frac{\lambda_R^2}{120\times3\times2^{11}\pi^4\varepsilon}\right]\left[p^2+m_B^2+\frac{m_R^2}{\varepsilon}\left[\frac{\lambda_R^2}{120\times3\times2^{11}\pi^4}-5\frac{\lambda_R^2}{120\times2^{11}\pi^4}-\frac{\lambda_R^2}{24\times16\times32\pi^4}\right]\right.$$
$$\left.+\frac{\kappa_R^2}{\varepsilon}\left[\frac{-1}{6\times32\pi^2}\right]+\frac{m_R\kappa_R}{\varepsilon}\left[\frac{\lambda_R}{6\times4\times32\pi^3}\right]\right]. \tag{9}$$

Similarly, the four-point function following from (6b), (8e), and (8f) is

$$\Gamma^{(4)}=-\kappa_B+\frac{1}{\varepsilon}\left[\frac{2\lambda_R\kappa_R}{3\times32\pi^2}-\frac{\lambda_R^2 m_R}{3\times4\times32\pi^3}\right] \tag{10}$$

while the six-point function following from (6c) and (8g) is

$$\Gamma^{(6)}=-\lambda_B+\frac{1}{\varepsilon}\left[\frac{10}{6}\frac{\lambda_R^2}{32\pi^2}\right]. \tag{11}$$

By (9), (10), and (11) we see that finiteness is ensured to this order of perturbation theory for these diagrams if the functions in (7) are taken to be

$$a_1=\frac{10}{6}\frac{\lambda_R^2}{32\pi^2}, \tag{12a}$$

$$\bar a_1^{\kappa}=\frac{2}{3\times32}\frac{\lambda_R}{\pi^2}, \tag{12b}$$

$$\bar a_1^{m}=\frac{-\lambda_R^2}{3\times4\times32\pi^3}, \tag{12c}$$

$$b_1^{m^2}=\lambda_R^2\left[\frac{1}{120\times3\times2^{11}\pi^4}+\frac{5}{120\times2^{11}\pi^4}+\frac{1}{24\times16\times32\pi^4}\right], \tag{12d}$$

$$b_1^{m\kappa}=\frac{-\lambda_R}{6\times4\times32\pi^3}, \tag{12e}$$

$$b_1^{\kappa^2}=\frac{1}{6\times32\pi^2}, \tag{12f}$$

$$c_1=\frac{-\lambda_R^2}{120\times3\times2^{11}\pi^4}. \tag{12g}$$

The vertices of (6b) and (6c) by (7a) and (7b) now acquire contributions bilinear in the coupling constants and proportional to ε^{-1}; the extra pieces are

$$(n)'=\frac{1}{\varepsilon}\left[-\frac{2\lambda_R\kappa_R}{3\times32\pi^2}+\frac{\lambda_R^2 m_R}{3\times4\times32\pi^3}\right] \tag{13a}$$

and

$$(x)'=-\frac{1}{\varepsilon}\frac{10}{6}\frac{\lambda_R^2}{32\pi^2}, \tag{13b}$$

respectively. With this in mind, we see that in Fig. 1, we must now include the diagrams (b), (c), and (o). These extra contributions are

$$(b)=(1)\left[\frac{1}{2}\right]\left[\frac{1}{\varepsilon}\left[\frac{-2\lambda_R\kappa_R}{3\times32\pi^2}+\frac{\lambda_R^2 m_R}{3\times4\times32\pi^3}\right]\left[\frac{-m_R}{4\pi}\right]\right], \tag{14a}$$

$$(c)=(1)\left[\frac{1}{8}\right]\left[\frac{1}{\varepsilon}\left[-\frac{10}{6}\frac{\lambda_R^2}{32\pi^2}\right]\left[\frac{-m_R}{4\pi}\right]^2\right], \tag{14b}$$

$$(o)=(1)\left[\frac{1}{2}\right]\left[\frac{1}{\varepsilon}\left[-\frac{10}{6}\frac{\lambda_R^2}{32\pi^2}\right]\left[\frac{-m_R}{4\pi}\right]\right], \tag{14c}$$

so that it is necessary to supplement the result of (12) by

$$\bar a_1^{m}=\frac{1}{2}\left[-\frac{10}{6}\frac{\lambda_R^2}{32\pi^2}\right]\left[-\frac{1}{4\pi}\right], \tag{15a}$$

$$b_1^{m\kappa}=\frac{1}{2}\left[\frac{-2\lambda_R}{3\times32\pi^2}\right]\left[-\frac{1}{4\pi}\right], \tag{15b}$$

and

$$b_1^{m^2}=\frac{1}{2}\left[\frac{\lambda_R^2}{3\times4\times32\pi^3}\right]\left[-\frac{1}{4\pi}\right]+\frac{1}{8}\left[-\frac{10}{6}\frac{\lambda_R^2}{32\pi^2}\right]\left[-\frac{1}{4\pi}\right]^2. \tag{15c}$$

A final contribution now arises due to the correction (15a) induces in (n) of Fig. 1:

$$(n)''=-\frac{1}{2}\left[-\frac{10}{6}\frac{\lambda_R^2}{32\pi^2}\right]\left[-\frac{m_R}{4\pi}\right]\frac{1}{\varepsilon}; \tag{16}$$

this means that (b) of Fig. 1 now receives the contribution

$$(b)'=(1)\left[\frac{1}{2}\right]\left[-\left[\frac{1}{2}\right]\left[-\frac{10}{6}\frac{\lambda_R^2}{32\pi^3}\right]\left[-\frac{m_R}{4\pi}\right]\left[\frac{1}{\varepsilon}\right]\left[-\frac{m_R}{4\pi}\right]\right] . \tag{17}$$

From (17), we find that the final contribution to the functions in (7) is

$$b_1^{m^2}=\frac{1}{2}\left[-\frac{1}{2}\right]\left[-\frac{10}{6}\frac{\lambda_R^2}{32\pi^2}\right]\left[-\frac{1}{4\pi}\right]^2 . \tag{18}$$

Together, (12), (15), and (18) yield

$$a_1=\frac{5}{6}\frac{\lambda_R^2}{(4\pi)^2} , \tag{19a}$$

$$\bar{a}_1^{\kappa}=\frac{1}{3}\frac{\lambda_R}{(4\pi)^2} , \tag{19b}$$

$$\bar{a}_1^{m}=\frac{1}{4}\frac{\lambda_R^2}{(4\pi)^3} , \tag{19c}$$

$$b_1^{m^2}=\frac{17}{360}\frac{\lambda_R^2}{(4\pi)^4} , \tag{19d}$$

$$b_1^{m\kappa}=\frac{1}{12}\frac{\lambda_R}{(4\pi)^3} , \tag{19e}$$

$$b_1^{\kappa^2}=\frac{1}{12}\frac{1}{(4\pi)^2} , \tag{19f}$$

and

$$c_1=-\frac{\lambda_R^2}{2880(4\pi)^4} . \tag{19g}$$

The renormalization-group functions can now be expressed in terms of the quantities in (19). To show this, we follow [2]. Upon setting

$$\mu'=\mu(1+\rho) , \tag{20}$$

(7a) and (7b) become

$$\lambda_B=(\mu')^{2\varepsilon}\left[\lambda_R(1-2\varepsilon\rho)-2\rho a_1(\lambda_R)+\sum_{\nu=1}^{\infty}\frac{a_\nu(\lambda_R)-2\rho a_{\nu+1}(\lambda_R)}{\varepsilon^\nu}\right] , \tag{21a}$$

$$\kappa_B=(\mu')^{\varepsilon}\left\{\kappa_R(1-\varepsilon\rho)-\rho[\bar{a}_1^{\kappa}(\lambda_R)\kappa_R+\bar{a}_1^{m}(\lambda_R)m_R] + \sum_{\nu=1}^{\infty}\frac{[\bar{a}_\nu^{\kappa}(\lambda_R)-\rho\bar{a}_{\nu+1}^{\kappa}(\lambda_R)]\kappa_R+[\bar{a}_\nu^{m}(\lambda_R)-\rho\bar{a}_{\nu+1}^{m}(\lambda_R)]m_R}{\varepsilon^\nu}\right\} . \tag{21b}$$

Terms linear in ε in (21) can be eliminated by defining

$$\tilde{\lambda}_R=\lambda_R(1-2\varepsilon\rho) , \tag{22a}$$

$$\tilde{\kappa}_R=\kappa_R(1-\varepsilon\rho) , \tag{22b}$$

$$\tilde{m}_R=m_R . \tag{22c}$$

Equation (22) is now used to express λ_B, κ_B, and m_B^2 in terms of $\tilde{\lambda}_R$, $\tilde{\kappa}_R$ and $\tilde{m}_R^2$. It is now possible to identify λ'_R, κ'_R, and m'^2_R with the terms in the resulting relations that are independent of poles in ε; we find that

$$\lambda'_R=\tilde{\lambda}_R-2\rho[a_1(\tilde{\lambda}_R)-\tilde{\lambda}_R a'_1(\tilde{\lambda}_R)] , \tag{23a}$$

$$\kappa'_R=\tilde{\kappa}_R-\rho[\bar{a}_1^{m}(\tilde{\lambda}_R)\tilde{m}_R-2\tilde{\lambda}_R\bar{a}_1^{m'}(\tilde{\lambda}_R)\tilde{m}_R -2\tilde{\lambda}_R\bar{a}_1^{\kappa'}(\tilde{\lambda}_R)\tilde{\kappa}_R] , \tag{23b}$$

$$m'^2_R=\tilde{m}_R^2-\rho[-2b_1^{\kappa^2}(\tilde{\lambda}_R)\tilde{\kappa}_R^2-2\tilde{\lambda}_R b_1^{\kappa^{2\prime}}(\tilde{\lambda}_R)\tilde{\kappa}_R^2 -b_1^{m\kappa}(\tilde{\lambda}_R)\tilde{\kappa}_R\tilde{m}_R-2\tilde{\lambda}_R b_1^{m\kappa'}(\tilde{\lambda}_R)\tilde{\kappa}_R\tilde{m}_R -2\tilde{\lambda}_R b_1^{m^{2\prime}}(\tilde{\lambda}_R)\tilde{m}_R^2] . \tag{23c}$$

From (19) and (23) we obtain the renormalization-group functions so that, to second order in the coupling constants λ and κ,

$$\mu\frac{\partial\lambda}{\partial\mu}=\frac{10}{6}\frac{\lambda^2}{(4\pi)^2} , \tag{24a}$$

$$\mu\frac{\partial\kappa}{\partial\mu}=\frac{3}{4}\frac{\lambda^2 m}{(4\pi)^3}+\frac{2}{3}\frac{\lambda\kappa}{(4\pi)^2} , \tag{24b}$$

$$\mu\frac{\partial m^2}{\partial\mu}=\frac{1}{6}\frac{\kappa^2}{(4\pi)^2}+\frac{1}{4}\frac{\lambda\kappa m}{(4\pi)^3}+\frac{17}{90}\frac{\lambda^2 m^2}{(4\pi)^4} . \tag{24c}$$

III. DISCUSSION

We have seen how dimensional regularization can be used to compute radiative effects in odd dimensional theories. This is despite the fact that in some instances when power counting indicates that a divergence should occur, dimensional regularization does not give rise to a pole when $n=D$. For example, diagram (o) of Fig. 1 is linearly divergent when a cutoff is used, yet finite when dimensional regularization is employed. Once the cutoff is removed by renormalizing the coupling κ, the finite part left is identical to the result of dimensional regularization. This is consistent with the statement in [4] that in the limit $m^2=\kappa=0$, only λ and ϕ need to be renormalized if dimensional regularization is used.

It would be interesting to extend these considerations. For example, we could go beyond lowest order, as was done in [5] using a cutoff. Also, other theories, such as three-dimensional Chern-Simons theory and Yukawa theory with the renormalizable four-point interaction $\bar{\Psi}\Psi\phi^2$, can be treated using dimensional regularization. A further problem would be to see if the Bogolubov-Parasiuk-Hepp-Zimmermann (BPHZ) renormalization procedure can be applied in the context of dimensional regularization, using the approach of [6]. We intend to address these issues.

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APPENDIX

We consider in this appendix the integrals in (3). For I_a, we need only apply (1), so that

$$I_a=\frac{1}{(4\pi)^{n/2}}(m^2)^{n/2-1}\Gamma\left[1-\frac{n}{2}\right] \tag{A1}$$

which, because $\Gamma(\frac{1}{2})=\sqrt{\pi}$ and $\Gamma(x+1)=x\Gamma(x)$, becomes when $n=3$ the result of (4a). Next, repeatedly using the standard integral

$$\frac{1}{a^p b^q}=\frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}\int_0^1 dx\frac{x^{p-1}(1-x^{q-1}}{[xa+(1-x)b]^{p+q}} \tag{A2}$$

in conjunction with (1) we find that

$$\begin{aligned}I_b&=\int\frac{d^nk}{(2\pi)^n}\frac{1}{k^2+m^2}\int\frac{d^nq}{(2\pi)^n}\int_0^1 dx\frac{1}{[q^2+x(1-x)(p+k)^2+m^2]^2}\\&=\frac{\Gamma(3-n)}{(4\pi)^n}\int_0^1 dx\int_0^1 dy\, y^{1-n/2}[x(1-x)]^{n/2-2}\left[y(1-y)p^2+(1-y)m^2+\frac{ym^2}{x(1-x)}\right]^{n-3}.\end{aligned} \tag{A3}$$

Since near $\varepsilon=0$, $\Gamma(\varepsilon)\sim 1/\varepsilon$, (A3) reduces to (4b). Finally, by using (A2) to combine denominators in (3c) and integrating over k and q, then over r, and lastly over s, we arrive at the result

$$\begin{aligned}I_c=\int_0^1 dx\,dy\int_0^1 d\lambda\,d\sigma\frac{\Gamma(5-2n)}{(4\pi)^{2n}}[x(1-x)y(1-y)]^{n/2-2}[\lambda(1-\lambda)]^{n-3}\sigma^{3-3n/2}\\\times\left[\sigma(1-\sigma)p^2+(1-\sigma)m^2+\frac{\sigma m^2}{\lambda(1-\lambda)}\left[\frac{\lambda}{x(1-x)}+\frac{1-\lambda}{y(1-y)}\right]\right]^{2n-5}.\end{aligned} \tag{A4}$$

Equation (A2) can now be used to evaluate integrals over x, y, λ, and σ in (A4), leading to (4c).

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