Cosmic strings with curvature corrections

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(Received 12 February 1992)

A generic model of string described by a Lagrangian density that depends on the extrinsic curvature of the string worldsheet is studied. Using a system of coordinates adapted to the string world sheet the equation of motion and the energy-momentum tensor are derived for strings evolving in curved spacetime. We find that the curvature corrections may change the relation between the string energy density and the tension. It can also introduce heat propagation along the string. We also find for the Polyakov as well as Nambu strings with a topological term that the open string end points can travel with a speed less than the velocity of light.

PACS number(s): 11.17.+y, 11.10.Lm, 98.80.Cq

I. INTRODUCTION

The possibility of describing filament structures in a simple way using field-theoretical string models has gained increasing interest in relativistic astrophysics [1] and general relativity [2], as well as cosmology [3,4]. The cosmic strings are represented as vortices of a gauge-field model or as a Nambu geometric string. The relations between the gauge-field vortices and Nambu strings has been established by a number of authors [5]. In particular, Förster obtains the Nambu action in the limit where the fields are condensed along a geometric line [6].

Polyakov proposed a modification of the Nambu action in which, in addition to the usual term constructed with the surface of the string world sheet, a term built with the extrinsic curvature of the string world sheet is also considered [7]. Another possibility along the same line has been explored by Lindstrom, Rocek, and van Nieuwenhuizen [8]. From a field-theoretical viewpoint these curvature corrections can be obtained by considering that the string has a certain thickness [9]. The dynamics of the Nambu string is dramatically changed by the inclusion of curvature-correction terms that are highly nonlinear in the action [10,11]. The effects of these corrections in the string gravitational field are yet unknown.

It is a rather remarkable fact that there are properties of the strings that are not changed by the inclusion in the action of the curvature correction of the Polyakov type. An example is provided by the critical dimension in which a consistent quantum theory of bosonics strings exists [12] ($D=26$).

The curvature corrections to the Nambu-string action depend on the particular field theory to which the strings are associated. In this paper we study generic curvature corrections built with the extrinsic curvature. We shall study string dynamics and in a special way the energy-momentum tensor that is a principal ingredient in general relativistic considerations.

The physical system that we study is described in four-dimensional spacetime by a Dirac-distribution-value Lagrangian density that will yield fourth-order differential equations [10,11]. Because of this difficulty, we shall use a more geometrical approach than the traditional one used when dealing with Nambu strings.

A geometrical string is a hybrid system intermediately between a field and a particle, since the number of parameters used in its description is less than the number of spacetime coordinates. Usually, this object is considered more like a field than a particle. In this work we shall take the opposite viewpoint. We shall consider, as did Förster [6] and Maeda and Turok [9], a coordinate system attached to the string world sheet; i.e., we take the parameters of the world sheet as two coordinates, and we choose the other two pointing in a direction orthogonal to the world sheet. These last two parameters are null on the world sheet. This system is analogous to the Serret-Frenet frame of a particle in classical mechanics, the so-called comoving frame.

We use a variational principle similar to the Hamiltonian principle for particles. We perform a variation only on the coordinates which define the position of the string in the spacetime. The energy-momentum tensor is obtained by a variation of the other two parameters.

In Sec. II we introduce the coordinate system adapted to the world sheet of a string evolving in a curve spacetime. Since the variational principle as well as the coordinate system are rather involved, for methodological reasons we derive the equation of motion and energy-momentum tensor for the Nambu string. We present a different derivation of the end-point condition for open strings.

In Sec. III we study the equation of motion and energy-momentum tensor for a generic model of strings described by a Lagrangian density depending on the metric of the string world sheet and its extrinsic curvature. We find that the inclusion of the extrinsic curvature can change the string equation of state, i.e., the relation between the energy density and tension, and may introduce heat propagation along the string. Also, we have the presence of typical dipolar terms.

In the final section, we briefly compare our method with the usual one, and we discuss the equation of motion and energy-momentum tensor for the Polyakov string, and a model of strings that depends on the intrinsic curvature that has a particular case the Nambu string with a topological term. We find, in both cases, that the velocity of the open-string end points can be less than the speed of light.

II. WORLD SHEETS, COORDINATE SYSTEMS, AND NAMBU STRINGS

In this section we study the equation of motion and energy-momentum tensor for the usual geometric string described by the Nambu action in an intrinsic system of coordinates. Most of the results of this section are well known, and they are presented mainly to introduce the notation and different systems of coordinates as well as the methodology that will be used later in the more general case of strings described by actions depending on the extrinsic curvature of the world sheet. We consider that the strings evolve in a four-dimensional Riemannian manifold endowed with a metric $g_{\mu\nu}$ of signature $(+---)$; greek indices run from 0 to 3, with $\mu,\nu,\rho,\ldots$ for ordinary systems of coordinates and $\alpha,\beta,\gamma,\ldots$ for the special systems attached to two-dimensional surfaces. Also, we shall use uppercase latin indices $A,B,C,\ldots$ and lower case latin indices $a,b,c,\ldots$ running from 0 to 1 and from 2 to 3, respectively.

A string spans a two-dimensional surface S (world sheet), which can be parametrized as $x^{\mu}=X^{\mu}(\tau^A)$, where $\tau^A=(\tau^0,\tau^1)$; τ^0 and τ^1 are a time and a space parameter, respectively. We shall associate with each point of the string world sheet four linearly independent vectors (e_A,N_a) that form a basis of the four-dimensional spacetime. We shall take the first two as

$$e_A=X^{\mu}_{,A}\partial_{\mu}\ , \tag{2.1}$$

where the comma stands for the usual partial derivative, i.e., tangent vectors to the parametric curves of the string world sheet. The other two

$$N_a=N^{\mu}_a(\tau)\partial_{\mu} \tag{2.2}$$

will be taken as being orthogonal to the word sheet and orthonormal:

$$N_a\cdot e_A=g_{\mu\nu}N^{\mu}_a X^{\nu}_{,A}=0\ , \tag{2.3}$$

$$N_a\cdot N_b=g_{\mu\nu}N^{\mu}_{\alpha}N^{\nu}_{\beta}=-\delta_{ab}\ . \tag{2.4}$$

The usual sum convention will be used for all kind of equivalent indices. There is still one degree of freedom left in the choice of the vectors N_a.

The induced metric on the world sheet and its extrinsic curvature are, respectively,

$$\gamma_{AB}=e_A\cdot e_B=g_{\mu\nu}X^{\mu}_{,A}X^{\nu}_{,B}\ , \tag{2.5}$$

$$K_{aAB}=\nabla_A N_a\cdot e_B=g_{\mu\nu}\nabla_A N^{\mu}_a X^{\nu}_{,B}\ , \tag{2.6}$$

where ∇_A denotes the covariant derivative along the parametric curve τ_A. The curvature K_{aAB} is symmetric in A and B.

Now we shall define a coordinate system attached to the string world sheet, $\sigma^{\alpha}=(\tau^A,\rho^a)$, which generalizes to a Riemannian space the system introduced by Förster [6]. Let us consider at a point $P(\tau)$ of the world sheet the two-dimensional space spanned by the unit spacelike vectors N_a. All the vectors lying in this space are represented at $P(\tau)$ by $\lambda^a N_a$, where λ^2 and λ^3 are two parameters. The manifold of geodesics through P, tangent to this space, forms a geodesic surface. We shall assume that, at least in the neighborhood of the world sheet, we can always choose the two vectors $N_a(\tau)$ in such a way that the spacetime is foliated by the above-mentioned family of geodesic surfaces. Hence, locally, for any point Q near the world sheet, we can always define a unique geodesic (lying only in one of the surfaces of the family) that passes through Q and crosses the world sheet. Let s be the length of the segment of the geodesic between the world sheet and point Q. The point Q is determined on the geodesic surface by the two coordinates [13]

$$\rho^a=s\lambda^a\ ,\quad (\lambda^1)^2+(\lambda^2)^2=1\ . \tag{2.7}$$

The system of coordinates, $\sigma^{\alpha}=(\tau^A,\rho^a)$, with the quantities τ^A and ρ^a above defined, has a clear geometrical meaning and will be particularly useful in studying string dynamics. This system is similar to the Frenet-Serret system of coordinates used to study the Newtonian dynamics of a point particle. The spacetime metric in this new system of coordinates will be denoted by $G_{\alpha\beta}(\sigma)$.

By writing the geodesic equation for the spacelike geodesics, $\tau^A=$const, and using relation (2.7), it follows that [13]

$$(\Gamma^d_{ab})_0=0\ ,\quad (\partial_d G_{ab})_0=0\ . \tag{2.8}$$

The subscript zero indicates that the quantities are taken on the surface $\rho^a=0$.

A first-order series expansion of $G_{\alpha\beta}$, in terms of quantities defined on the world sheet, gives

$$G_{\alpha\beta}=(G_{\alpha\beta})_0+\rho^a(G_{\alpha\beta,a})_0+O(\rho^2)\ . \tag{2.9}$$

Also, from (2.8), the elementary properties of the holonomic basis ∂_{α} (that is, $\partial_{\alpha}\cdot\partial_{\alpha}=G_{\alpha\beta}$ and $\nabla_{\alpha}\partial_{\beta}=\Gamma^{\lambda}_{\alpha\beta}\partial_{\lambda}=\nabla_{\beta}\partial_{\alpha}$) and the fact that ∂_A and ∂_a on the world sheet reduce to e_A and N_a, we get

$$G_{AB}=\gamma_{AB}+2K_{aAB}\rho^a+O(\rho^2)\ , \tag{2.10a}$$

$$G_{AB}=\nabla_A N_a\cdot N_b\rho^a+O(\rho^2)\ , \tag{2.10b}$$

$$G_{ab}=-\delta_{ab}+O(\rho^2)\ . \tag{2.10c}$$

From (2.10) we verify that

$$(G_{Ab,a})_0=-(G_{Aa,b})_0\ . \tag{2.11}$$

A. Intrinsic motion equation of Nambu strings

The timelike world surface S describing the motion of a geometric string is given by the extremum of the Nambu action:

$$A=\mu\int\sqrt{-\gamma}\,d^2\tau\ , \tag{2.12}$$

where γ is the determinant of the induced metric γ_{AB} and μ is the geometric string constant.

In the coordinate system (τ^A,ρ^a), the equation of this surface is by definition $\rho^a=0$. Now we shall consider in the neighborhood of S another surface $\bar{S}$ represented by the equation $\rho^a=\delta\rho^a(\tau)$. We pass from the surface S to the surface $\bar{S}$ by a variation of ρ^a, $\delta\rho^a(\tau)$ at constant τ^A. This variation is independent of the particular parametrization of the surface S (parameters τ^A) and it separates the "true" or dynamical variation of the position of the string in spacetime from the variation of the parameterizations of the sheet.

Let us consider the variation of the action A:

$$\delta A=\tfrac{1}{2}\mu\int\sqrt{-\gamma}\gamma^{AB}\delta\gamma_{AB}\,d^2\tau\ . \tag{2.13}$$

And let $\bar{\gamma}_{AB}$ be the induced metric of $\bar{S}$. In order to obtain the variation

$$\delta\gamma_{AB}=\bar{\gamma}_{AB}-\gamma_{AB}\ , \tag{2.14}$$

we associate with $\bar{S}$ a similar system of coordinates $\bar{\sigma}'^{\alpha}=(\tau'^A,\rho'^a)$. We shall assume that the two system of coordinates are related by the infinitesimal transformation

$$\sigma'^{\alpha}=\sigma^{\alpha}+\xi^{\alpha}(\sigma)\ , \tag{2.15}$$

where the components of the vector $\xi^{\alpha}(\sigma)$ are first-order quantities. Therefore the transformation of the spacetime metric is given by

$$G'_{\alpha\beta}(\sigma)-G_{\alpha\beta}(\sigma)=-\pounds_{\xi}G_{\alpha\beta}\ , \tag{2.16a}$$

where $\pounds_{\xi}$ is the Lie derivative along the field ξ^{α}. Also,

$$\pounds_{\xi}G_{\alpha\beta}=\xi_{\alpha;\beta}+\xi_{\beta;\alpha}\ . \tag{2.16b}$$

The semicolon denotes a covariant derivative with the connection $\Gamma^{\alpha}_{\beta\gamma}$. On the surface $\bar{S}$, we have $\rho^a=\delta\rho^a(\tau)$ and $\rho'^a=0$; hence,

$$\xi^a(\tau,\delta\rho^a(\tau))=-\delta\rho^a(\tau)\ . \tag{2.17}$$

In order to have the same parameter τ^A on S and $\bar{S}$, we shall impose

$$\xi^A(\tau,\delta\rho^a(\tau))=0\ . \tag{2.18}$$

From (2.17), (2.18), and the transformation (2.15) at $\sigma^{\alpha}=(\tau^A,0)$, we get

$$\begin{aligned}G'_{AB}(\tau^A,0)-G_{AB}(\tau^A,0)&=\bar{\gamma}_{AB}-\gamma_{AB}\\&=\delta\rho^a(\tau)(G_{AB},a)_0\ ,\end{aligned} \tag{2.19}$$

and from (2.10) it follows that

$$\delta\gamma_{AB}=2K_{aAB}\delta\rho^a(\tau)\ . \tag{2.20}$$

Therefore the variation $S\rightarrow\bar{S}$ at τ^A=const gives

$$\delta A=\mu\int\sqrt{-\gamma}\gamma^{AB}K_{aAB}\delta\rho^a(\tau)d^2\tau=0\ . \tag{2.21}$$

Hence

$$K_a=\gamma^{AB}K_{aAB}=0\ . \tag{2.22}$$

The two equations (2.22) are the "intrinsic" equations of the strings; geometrically, they say that the mean curvature of the world sheet is null [14].

This variational principle does not give rise to the usual boundary conditions for the end points of open strings. These will appear naturally when we consider the string energy-momentum tensor.

It is easy to show that Eqs. (2.22) are equivalent to the usual motion equations of strings in the ordinary coordinate system (x^{μ}). First, we introduce the quantity

$$K^{\mu}_{AB}=X^{\mu}_{,AB}-\hat{\Gamma}^{\,C}_{AB}X^{\mu}_{,C}+\Gamma^{\mu}_{\lambda\nu}X^{\lambda}_{,A}X^{\nu}_{,B}\ , \tag{2.23}$$

where $\hat{\Gamma}^{\,C}_{AB}$ and $\Gamma^{\mu}_{\lambda\nu}$ are the connections associated with the world sheet of induced metric γ_{AB} and spacetime metric $g_{\mu\nu}$, respectively. By using the projection tensor

$$t^{\lambda\mu}=\gamma^{AB}X^{\lambda}_{,A}X^{\mu}_{,B}\ , \tag{2.24}$$

expression (2.23) can be written as

$$K^{\mu}_{AB}=(X^{\sigma}_{,AB}+\Gamma^{\sigma}_{\lambda\nu}X^{\lambda}_{,A}X^{\nu}_{,B})(\delta^{\mu}_{\sigma}-g_{\sigma\rho}t^{\rho\mu})\ . \tag{2.25}$$

Hence K^{μ}_{AB} is a world vector normal to the word sheet and can be written as

$$K^{\mu}_{AB}=\delta^{ab}K_{aAB}N^{\mu}_b\ . \tag{2.26}$$

Also,

$$K_{aAB}=-g_{\mu\nu}N^{\mu}_aK^{\nu}_{AB}\ . \tag{2.27}$$

By using (2.25) one can verify that (2.27) is equivalent to the extrinsic curvature (2.6). And from (2.23), (2.27), and (2.22), we get the string equation of motion written in the usual way:

$$\begin{aligned}&(1/\sqrt{-\gamma})\partial_A(\sqrt{-\gamma}\gamma^{AB}X^{\mu}_{,B})\\&\qquad+\Gamma^{\mu}_{\lambda\nu}X^{\lambda}_{,A}X^{\nu}_{,B}\gamma^{AB}=0\ .\end{aligned} \tag{2.28}$$

B. Energy-momentum tensor of Nambu strings

To obtain the symmetric energy-momentum tensor associated with the string, we shall use the usual method employed for matter fields in a Riemannian spacetime [15]; i.e., by assuming that the fields satisfy their equations of motion, the energy-momentum tensor is obtained as a consequence of the invariance of the action under infinitesimal coordinate transformations.

In the case of strings, it is sufficient to consider infinitesimal transformations of the particular form

$$\tau'^A=\tau^A+\xi^A(\tau)\ , \tag{2.29a}$$

$$\rho'^a=\rho^a+\xi^a(\tau)\ , \tag{2.29b}$$

where the fields $\xi^{\alpha}(\tau)=(\xi^A(\tau^B),\xi^a(\tau^B))$ depend only on the indicated variables. The transformation (2.29a) is a change of parameters on the string world sheet; the action (2.12) is invariant under this transformation. The transformation (2.29b) is the transformation that was used to derive the string equation of motion. Thus, assuming that Eq. (2.21) is verified, we have that the action is an extremum, $\delta A=0$.

From (2.13), (2.16), and (2.29), we get

$$\delta A=-\tfrac{1}{2}\mu\int\sqrt{-G}\,G^{AB}(\xi_{A;B}+\xi_{B;A})\delta^2(\rho)d^2\rho\,d^2\tau\ , \tag{2.30}$$

where G is the determinant of $G_{\alpha\beta}$ and $\delta^2(\rho)=\delta(\rho^2)\delta(\rho^3)$ is the usual Dirac distribution. To obtain this result, we have made use of the identities

$$G_{AB}\delta^2(\rho)=\gamma_{AB}\delta^2(\rho)\ , \tag{2.31}$$

$$G\delta^2(\rho)=\gamma\delta^2(\rho)\ , \tag{2.32}$$

$$\delta G_{AB}\delta^2(\rho)=\delta\gamma_{AB}\,\delta^2(\rho)\ . \tag{2.33}$$

Expression (2.30) can also be written as

$$\delta A=-\tfrac{1}{2}\mu\int\sqrt{-G}\,\mathcal{G}^{\alpha\beta}(\xi_{\alpha;\beta}+\xi_{\beta;\alpha})\delta^2(\rho)d^2\rho\,d^2\tau\ , \tag{2.34}$$

with

$$\mathcal{G}^{\alpha\beta}=\begin{bmatrix} G^{AB} & 0\\ 0 & 0\end{bmatrix}\ . \tag{2.35}$$

We shall see later that the particular form of the matrix $\mathcal{G}$ is rather unique. Finally, by the usual calculation, we obtain

$$\delta A=-\mu\int\partial_\beta[\sqrt{-G}\,\mathcal{G}^{\alpha\beta}\xi_\alpha\delta^2(\rho)]d^2\rho\,d^2\tau + \int\sqrt{-G}\,T^{\alpha\beta}_{\ ;\beta}\xi_\alpha d^2\rho\,d^2\tau\ , \tag{2.36}$$

where

$$T^{\alpha\beta}=\mu\mathcal{G}^{\alpha\beta}\delta^2(\rho)\ . \tag{2.37}$$

In the case of open strings, in order to cancel the divergence term in (2.36), we must impose the boundary conditions

$$\sqrt{-\gamma}\gamma^{1B}|_{\tau^1=0,l}=0\ , \tag{2.38}$$

where $\tau^1=0,l$ are the open-string end points. This condition implements the physical requirement that no canonical momentum escapes for the end points and implies that these points move at the speed of light [16].

From (2.36), (2.38), and $\delta A=0$, we get, for all kind of strings,

$$\int\left[\int\sqrt{-G}\,T^{\beta}_{\alpha;\beta}d^2\rho\right]\xi^\alpha(\tau)d^2\tau=0\ . \tag{2.39}$$

Hence

$$T^{\alpha\beta}{}_{;\beta}=0\ . \tag{2.40}$$

The quantity $T^{\alpha\beta}$ is interpreted as the string energy-momentum tensor; its components are

$$T^{AB}=\mu\gamma^{AB}\delta^2(\rho)\ ,\quad T^{Ab}=0\ ,\quad T^{ab}=0\ . \tag{2.41}$$

In ordinary coordinates (x^μ),

$$T^{\mu\nu}=\mu X^\mu_{,A}X^\nu_{,B}\gamma^{AB}\delta^2(\rho)\ , \tag{2.42}$$

By using the identity

$$\delta^4(x^\mu,X^\mu(\tau'))/\sqrt{-g}=\delta^2(\rho)\delta^2(\tau,\tau')/\sqrt{-G}\ , \tag{2.43}$$

where g is the determinant of $g_{\mu\nu}$, (2.42) can be cast into the more usual form

$$T^{\mu\nu}=\mu\int\sqrt{-\gamma}\gamma^{AB}X^\mu_{,A}X^\nu_{,B}\delta^4(x^\nu,X^\nu(\tau))/\sqrt{-g}\ d^2\tau\ . \tag{2.44}$$

III. STRINGS WITH CURVATURE CORRECTIONS

In this section we shall study strings described by Lagrangian densities built not only with the string-induced metric, but also with the extrinsic curvature. The action will be taken as

$$A=\int\sqrt{-\gamma}L(\gamma_{AB},K_{aAB})d^2\tau\ , \tag{3.1}$$

where L transforms as a scalar under a change of parameters τ^A. A particularly interesting example is the Polyakov string, whose Lagrangian is

$$L_P=\mu(1-\alpha\delta^{ab}K_{aAB}K_{bCD}\gamma^{AC}\gamma^{BD})\ , \tag{3.2}$$

where α is a coupling constant. Another example is provided by the Lindstrom–Rocek–van Niewenhuizen (LRN) string

$$L_{\rm LRN}=\mu[\det(\gamma_{AB}-\beta\delta^{ab}K_{aAC}K_{bBD}\gamma^{CD})]^{1/2}\ , \tag{3.3}$$

where β is a constant. When $\beta\ll 1$ the LRN string reduces to the Polyakov string with $\alpha=\beta/2$ and this reduces to the Nambu string when $\alpha=0$. In general, when one considers the derivation of the string action from a gauge-field theory, the extra terms built with the extrinsic curvature appear as second-order corrections [9]. Nevertheless, since the particular form of the action, in general, will depend on the particular field theory that gives origin to the string action, we shall consider a generic function L wherein the curvature terms are not assumed necessarily to be small corrections.

The variation of the action (3.1) can be written

$$\delta A=\int\sqrt{-\gamma}(F^{AB}\delta\gamma_{AB}+F^{aAB}\delta K_{aAB})d^2\tau\ , \tag{3.4}$$

where

$$F^{AB}(\tau)=\frac{1}{2}L\gamma^{AB}+\frac{\partial L}{\partial\gamma_{AB}}\ , \tag{3.5}$$

$$F^{aAB}=\frac{\partial L}{\partial K_{aAB}}\ . \tag{3.6}$$

To find the variations $\delta\gamma_{AB}$ and δK_{aAB}, we shall proceed as in the last section. We recall that the coordinates $\sigma^\alpha=(\tau^A,\rho^a)$ and $\sigma'^\alpha=(\tau'^A,\rho'^a)$ are attached to S and $\bar{S}$ and they are related by (2.15). The general variation of the spacetime metric is given by (2.16). On the other hand, from (2.10), this variation can be written as

$$G'_{AB}(\sigma)-G_{AB}(\sigma) =\bar{\gamma}_{AB}(\tau)-\gamma_{AB}(\tau) +2\rho^a(\bar{K}_{aAB}(\tau)-K_{aAB}(\tau))+O(\rho^2) . \quad (3.7)$$

Now we restrict (2.15) to the case (2.29), i.e., when ξ^a depends only on τ^A. Note that this restriction does not impose constraints on the type of surface $\bar{S}$ represented by the equations $\rho^a=-\xi^a(\tau)$. By using (2.10) we can expand the last member of (2.16) in powers of ρ^a. By comparing this expansion with (3.7), we obtain

$$-\delta\gamma_{AB}=\partial_B\xi^D\gamma_{DA}+\partial_A\xi^D\gamma_{DB}+\xi^D\partial_D\gamma_{AB}+2\xi^aK_{aAB} , \quad (3.8)$$

$$-\delta K_{aAB}=\partial_B\xi^DK_{aDA}+\partial_A\xi^DK_{aDB}+\xi^D\partial_DK_{aAB} +\tfrac{1}{2}[\partial_B\xi^b(G_{Ab,a})_0+\partial_A\xi^b(G_{Bb,a})_0 +\xi^b(G_{AB,ab})_0] . \quad (3.9)$$

The subscript zero indicates that the quantities are taken at the value $\rho^a=0$. In deriving this last equation, we have made use of

$$G_{AB}=\gamma_{AB}+2\rho^aK_{aAB}+\tfrac{1}{2}(G_{AB,ab})_0\rho^a\rho^b+\mathcal{O}(\rho^3) . \quad (3.10)$$

In Minkowski space the geodesics $\tau^A=$const are straight lines and $(G_{AB,ab})_0$ can be written as

$$(G_{AB,ab})_0=N_{a,A}\cdot N_{b,A}+N_{b,A}\cdot N_{a,B} . \quad (3.11)$$

As we mentioned before, the Lagrangian L is an invariant under a transformation of the parameters τ^A. This condition of invariance will be translated in constraint equations for L. In order to find these constraints, we shall specialize the variations (3.8) and (3.9) to the case $\xi^a(\tau)=0$; we find

$$-\delta_\tau\gamma_{AB}=\partial_B\xi^D\gamma_{DA}+\partial_A\xi^D\gamma_{DB}+\xi^D\partial_D\gamma_{AB} \quad (3.12)$$

and

$$-\delta_\tau K_{aAB}=\partial_B\xi^DK_{aDA}+\partial_A\xi^DK_{aDB}+\xi^D\partial_DK_{aAB} . \quad (3.13)$$

From (3.4), (3.12), and (3.13), we get

$$\delta_\tau A=-\int\sqrt{-\gamma}\{F^{AB}\partial_D\gamma_{AB}+F^{aAB}\partial_DK_{aAB}-(2/\sqrt{-\gamma})\partial_B[\sqrt{-\gamma}(F^{AB}\gamma_{DA}+F^{aAB}K_{aDA})]\}\xi^Dd^2\tau -2\int\partial_B[\sqrt{-\gamma}(\gamma_{DA}F^{AB}+F^{aAB}K_{aDA})\xi^D]d^2\tau . \quad (3.14)$$

From the variational principle, we find

$$F^{AB}\partial_D\gamma_{AB}+F^{aAB}\partial_DK_{aAB} -(2/\sqrt{-\gamma})\partial_B[\sqrt{-\gamma}(F^{AB}\gamma_{DA}+F^{aAB}K_{aDA})]=0 . \quad (3.15)$$

In the case of open strings, in order to cancel the divergence term in (3.14), we must also require

$$\sqrt{-\gamma}(F^{A1}\gamma_{AD}+F^{aA1}K_{aAD})|_{\tau^1=0,l}=0 . \quad (3.16)$$

Note that we are not allowed to consider $\xi^D|_{\tau^1=0,l}=0$, since these conditions severely restrict the type of transformation. We shall come back to this point later. Equation (3.15) can be cast in a manifestly covariant form with respect to reparametrizations of the world sheet as

$$\hat{\nabla}_B(F^B{}_A+F^{aBD}K_{aDA})-\tfrac{1}{2}F^{aBD}\hat{\nabla}_AK_{aBD}=0 . \quad (3.17)$$

A. Equation of motion

To obtain the string equation of motion, we must perform a variation of the world sheet at constant τ^A. In this case the transformation (3.8) reduces to (2.20) with $\xi^a=-\delta\rho^a$ and (3.9) to

$$\delta_\rho K_{aAB}=-\tfrac{1}{2}[\partial_B\xi^b(G_{Ab,a})_0+\partial_A\xi^b(G_{Bb,a})_0 +\xi^b(G_{AB,ab})_0] . \quad (3.18)$$

We find that the variation (3.4) when we pass from S to the neighboring $\bar{S}$ at constant τ^A is

$$\delta_\rho A=-\int\{\sqrt{-\gamma}[2F^{AB}K_{bAB}+\tfrac{1}{2}F^{aAB}(G_{AB,ab})_0] -\partial_B[\sqrt{-\gamma}F^{aAB}(G_{Ab,a})_0]\}\xi^bd^2\tau -\int\partial_B[\sqrt{-\gamma}F^{aAB}(G_{AB,a})_0\xi^b]d^2\tau . \quad (3.19)$$

The divergence term can be canceled by assuming a variation such that the boundary of S coincides with the boundary of $\bar{S}$, that is,

$$\xi^a(-\infty,\tau^1)=\xi^a(+\infty,\tau^1)=\xi^a(\tau^0,0)=\xi^a(\tau^0,l)=0 . \quad (3.20)$$

From the variational principle, we obtain the equations of motion,

$$2F^{AB}K_{bAB}+\tfrac{1}{2}F^{aAB}(G_{AB,ab})_0 -(1/\sqrt{-\gamma})\partial_B[\sqrt{-\gamma}F^{aAB}(G_{Ab,a})_0]=0 . \quad (3.21)$$

Note that this equation is covariant with respect to a general transformation of the parametrization of the string world sheet and as Eq. (3.17) they are written in the very particular system of spacetime coordinates adapted to the string world sheet. Also, for open strings, this equation of motion must be considered together with the end-point condition (3.16). This last condition can be implemented by considering either

$$\sqrt{-\gamma}\,|_{\tau^1=0,l}=0 \tag{3.22}$$

or

$$(F^{A1}\gamma_{AD}+F^{aA1}K_{aAD})|_{\tau^1=0,l}=0 \ . \tag{3.23}$$

The first tells us that the end points travel with the speed of light as in the case of Nambu strings. The second restriction, in general, cannot be implemented since it tells us that each end point must describe two parametric curves that, in the generic case, will not be the same. We shall return to this point later.

B. Energy-momentum tensor

As in the case of Nambu strings, we shall obtain the energy-momentum tensor from the variation (2.29) of the action together with the assumption that the equations of motion are satisfied.

From (3.4), (2.31)–(2.33), and the relation obtained by deriving (3.7) with respect to ρ^a,

$$\delta^2(\rho)\delta K_{aAB}=\tfrac{1}{2}\delta^2(\rho)(\delta G_{AB})_{,a} \ . \tag{3.24}$$

We find

$$\delta A=\int\sqrt{-G}\,[F^{AB}\delta G_{AB} + \tfrac{1}{2}F^{aAB}(\delta G_{AB})_{,a}]\delta^2(\rho)d^2\rho\, d^2\tau \ . \tag{3.25}$$

Using the relations

$$(1/\sqrt{-G}\,)\partial_A\sqrt{-G}=\tfrac{1}{2}G^{MN}\partial_A G_{MN}+O(\rho^2) \ , \tag{3.26a}$$

$$(1/\sqrt{-G}\,)\partial_a\sqrt{-G}=\tfrac{1}{2}G^{MN}\partial_a G_{MN}+O(\rho) \ , \tag{3.26b}$$

and the usual properties of the δ function, we can write (3.25) as

$$\delta A=\int\sqrt{-G}\,\{[F^{AB}-\tfrac{1}{4}G^{MN}G_{MN,a}F^{aAB}]\delta^2(\rho) -\tfrac{1}{2}F^{aAB}\partial_a\delta^2(\rho)\}\delta G_{AB}d^2\rho\, d^2\tau \tag{3.27}$$

or

$$\delta A=-\tfrac{1}{2}\int\sqrt{-G}\,T^{\alpha\beta}(\xi_{\alpha;\beta}+\xi_{\beta;\alpha})d^2\rho\, d^2\tau \ , \tag{3.28}$$

with

$$T^{\alpha\beta}=\begin{bmatrix} T^{AB} & 0\\ 0 & 0\end{bmatrix} , \tag{3.29a}$$

$$T^{AB}=(2F^{AB}-\gamma^{MN}K_{aMN}F^{aAB})\delta^2(\rho)-F^{aAB}\partial_a\delta^2(\rho) \ . \tag{3.29b}$$

The introduction of the quantity $T^{\alpha\beta}$ as a tensor needs to be justified; we shall come back to this point later. The usual argument gives

$$\delta A=\int\sqrt{-G}\,T^{\alpha\beta}{}_{;\beta}\xi_\alpha d^2\rho\, d^2\tau - \int\partial_\beta(\sqrt{-G}\,T^{\alpha\beta}\xi_\alpha)d^2\rho\, d^2\tau \ . \tag{3.30}$$

The second integral of (3.30) can be written as

$$-2\int\partial_B\{\sqrt{-\gamma}[F^{AB}\gamma_{AB}+F^{aAB}K_{aAD}]\xi^D\}d^2\tau \ . \tag{3.31}$$

Thus we recover the term in (3.14) that gives the origin to the open-string end-point constraints [cf. Eq. (3.16)]. Therefore the integral (3.31) is null.

The variational principle gives

$$\int\sqrt{-G}\,T_\alpha{}^\beta{}_{;\beta}\xi^\alpha(\tau)d^2\rho\, d^2\tau=0 \ , \tag{3.32}$$

for all $\xi^\alpha(\tau)$. Therefore, formally, we can consider $T^{\alpha\beta}$ as the energy-momentum tensor. Note that we have chosen in (3.32) the mixed components $T_\alpha{}^\beta$ in order to have the contravariant components $\xi^\alpha(\tau)$, which are a function of τ^A only. Since $T_\alpha{}^\beta$ is a distribution in the variables ρ^a, we must be careful in the verification of the "conservation laws" $T^{\alpha\beta}{}_{;\beta}=0$. In order to be more specific, we shall introduce the distribution notation

$$\langle\sqrt{-G}\,,T_\alpha{}^\beta{}_{;\beta}\rangle=\int\sqrt{-G}\,T_\alpha{}^\beta{}_{;\beta}d^2\rho \ . \tag{3.33}$$

Equation (3.32) can be cast in the form

$$\int\langle\sqrt{-G}\,,T_\alpha{}^\beta{}_{;\beta}\rangle\xi^\alpha(\tau)d^2\tau=0 \ , \tag{3.34}$$

for all $\xi^\alpha(\tau)$. Hence, instead of $T^{\alpha\beta}{}_{;\beta}=0$, we shall consider the conservation laws

$$\langle\sqrt{-G}\,,T_\alpha{}^\beta{}_{;\beta}\rangle=0 \ . \tag{3.35}$$

In order to better understand the meaning of this last equation, we shall compute the explicit form of $T_\alpha{}^\beta{}_{;\beta}$; we find

$$T_\alpha{}^\beta{}_{;\beta}=Y_\alpha\delta^2(\rho)+Y_\alpha{}^d\partial_d\delta^2(\rho) \ , \tag{3.36}$$

where

$$Y_D(\tau)=\gamma_{DB}\hat{\nabla}_A(2F^{AB}-K_aF^{aAB}) \ , \tag{3.37a}$$

$$\begin{aligned} Y_a(\tau)=&(1/\sqrt{-\gamma}\,)(\partial_A\sqrt{-\gamma}\,)(G_{aB,b})_0F^{AB} +(G_{aB,b})_0\partial_AF^{bAB}-2K_{aAB}F^{AB}\\ &+K_{aAB}K_dF^{dAB}+F^{dAB}(G_{Ba,dA})_0 -\tfrac{1}{2}(G_{AB,ad})_0F^{dAB} \ , \end{aligned} \tag{3.37b}$$

$$-Y_D^d(\tau,\rho)=\tfrac{1}{2}G_{DB}(G^{MN}\partial_AG_{MN}F^{dBA}+\partial_AF^{dBA}) +\tfrac{1}{2}(2G_{AD,E}-G_{AE,D})F^{dAE} \ , \tag{3.37c}$$

$$Y_a^d(\tau)=K_{aAB}F^{dAB} \ . \tag{3.37d}$$

From (3.37) we get

$$\begin{aligned}\langle\sqrt{-G}\,,T_D{}^\beta{}_{;\beta}\rangle &=2\sqrt{-\gamma}[-\tfrac{1}{2}F^{aAB}\hat{\nabla}_DK_{aAB} +\hat{\nabla}_B(F^{AB}\gamma_{DA}+F^{aAB}K_{aDA})] \ ,\end{aligned} \tag{3.38a}$$

$$\begin{aligned}\langle\sqrt{-G},T_a{}^\beta{}_{;\beta}\rangle &=-\sqrt{-\gamma}\{2F^{AB}K_{aAB}+\tfrac{1}{2}F^{dAB}(G_{AB,da}) -(1/\sqrt{-\gamma}\,)\partial_B[\sqrt{-\gamma}F^{dAB}(G_{Aa,d})_0]\} \ .\end{aligned} \tag{3.38b}$$

These equations tell us that the conservation law (3.35) is a consequence of the requirement of covariance of action under reparametrization of the string world sheet [cf. Eq. (3.17)] and the string equation of motion [17] [Eq. (3.21)].

Now we will return to the definition of the energy-momentum tensor (3.29). On the string world sheet, some of the quantities $\xi_{\alpha;\beta}+\xi_{\beta;\alpha}$ are null: namely,

$$(\xi_{a;b}+\xi_{b;a})_0=0 . \tag{3.39}$$

Therefore Eq. (3.28) shows that the tensor $T^{\alpha\beta}$ is not completely defined. It may also have components of the form

$$T^{ab}=S^{ab}(\tau)\delta^2(\rho) . \tag{3.40}$$

This apparent indetermination of $T^{\alpha\beta}$ comes from the fact that the transformations (2.29) are not a general change of coordinates as is needed to determine the energy-momentum tensor [15]. Nevertheless, the verification of the conservation laws, taking into account (3.40), shows that S^{ab} is identically null.

Finally, we want to say a few words about the physical meaning of the energy-momentum tensor (3.29). First, we note that in (3.29), in addition to the usual Dirac distribution, we also have the appearance of $\partial_a\delta^2(\rho)$, i.e., a typical dipolar distribution that describes in a first approximation the string width. We also have that now the tensor $T^A{}_B$ is no longer diagonal as in the case of the usual Nambu string, a fact that will produce a richer thermodynamics. Hence, depending on nature of the roots of the secular equation

$$\det(T^A{}_B)-\lambda\,\mathrm{Tr}(T^A{}_B)+\lambda^2=0 , \tag{3.41}$$

we can have three different situations that will give origin to the three canonical forms [18] of T_{AB}:

$$\begin{bmatrix}\lambda_0 & 0\\ 0 & -\lambda_1\end{bmatrix} , \quad \begin{bmatrix}\mathrm{Re}\lambda & \mathrm{Im}\lambda\\ \mathrm{Im}\lambda & -\mathrm{Re}\lambda\end{bmatrix} , \quad \begin{bmatrix}\lambda+\kappa & -\kappa\\ -\kappa & -\lambda+\kappa\end{bmatrix} . \tag{3.42}$$

The first two cases correspond to different roots of (3.41) and the last to equal roots $[T_{00}+T_{01}=-(T_{11}+T_{01})=\lambda$ and $T_{01}=-\kappa]$. In the case of different real roots, we will have, in general, that the string tension will not be the same as the string energy density; strings with this kind of equation of state have already appeared in the literature [19]. The case of complex roots is particularly interesting because we will have the same relation between the energy density and tension as in the case of Nambu strings; but now we will also have a heat flow along the string. In the last case, the equal-root case, we will also have a heat flow along the string, but now the tension is not the same as the energy density. In order to have tension, in other words, a string, we need $\lambda-\kappa>0$. Note that to have a well-defined secular equation, we need to consider (3.41) as an algebraic equation in the sense of the distribution theory. Also, since the coefficients of (3.41) depend on τ^A, we can have that the three listed possibilities may occur in the evolution of the same string.

IV. DISCUSSION

The method used to derive the equation of motion and energy-momentum tensor of strings described by Lagrangians that depend on the extrinsic curvature of the string world sheet presents some very special features. The equation of motion split in two groups. The first has no dynamical content; it is the geometric requirement of the invariance of the action under a change of coordinates. The second group defines the frame wherein the string evolves. The strings are described always by the same relation $\rho^a=0$, but the frame in which this equation has a meaning changes in each particular case. The actual finding of this frame requires the solution of a quite complicated system of equations. On the other hand, the computation of the energy-momentum tensor can be performed in a rather natural way using these intrinsic systems of coordinates. Our computation shows that the energy-momentum tensor has a nontrivial structure that will be difficult to obtain in usual coordinates. Also, in the case of open strings, the intrinsic coordinates are more suitable to describe the end-point condition. To be more specific, let us write the action in the usual coordinates X^μ:

$$A=\int \mathcal{L}\, d^2\tau , \tag{4.1}$$

where for strings with extrinsic curvature corrections evolving in a curved spacetime,

$$\mathcal{L}=\mathcal{L}(X^\mu,\partial_A X^\mu,\partial_A\partial_B X^\mu) . \tag{4.2}$$

The usual variational principle gives us the string equation of motion,

$$\frac{\partial^2}{\partial\tau^A\partial\tau^B}\frac{\partial\mathcal{L}}{\partial\,\partial_A\partial_B X^\mu}-\frac{\partial}{\partial\tau^A}\frac{\partial\mathcal{L}}{\partial\,\partial_A X^\mu}+\frac{\partial\mathcal{L}}{\partial X^\mu}=0 , \tag{4.3}$$

and for open strings the end-point condition

$$\left.\frac{\partial}{\partial\tau^B}\frac{\partial\mathcal{L}}{\partial\,\partial_1\partial_B X^\mu}-\frac{\partial\mathcal{L}}{\partial\partial_1 X^\mu}\right|_{\tau^1=0,l}=0 , \tag{4.4a}$$

$$\left.\frac{\partial\mathcal{L}}{\partial\,\partial_1\partial_B X^\mu}\right|_{\tau^1=0,l}=0 . \tag{4.4b}$$

In principle, the dependence of $\mathcal{L}$ on the indicated variables is through the spacetime metric tensor $g_{\mu\nu}$ and its first and second derivatives, the metric of the string world sheet γ_{AB}, and the symbol K^μ_{AB} or, better, on invariants built with these quantities.

The motion equation (4.3) gives directly the string world sheet. But when written in terms of $g_{\mu\nu}$, γ_{AB}, and K^μ_{AB} it is quite complicated. The only particular case already known is the simpler case of the Polyakov string in flat spacetime. The motion equation in this case is rather formidable [11]. In the energy-momentum tensor (3.29), derivatives appear in directions perpendicular to the world sheet. Note that to express this fact in the usual coordinates is not a simple matter. Also, condition (3.16) is simpler than (4.4). In order to understand better the meaning of the result obtained in the last section, we shall briefly discuss some particular cases of strings with curvature corrections evolving in flat spacetime.

A. Polyakov string

The condition (3.17) for the Lagrangian L_P [cf. Eq. (3.2)] is satisfied identically, and the equation of motion (3.21) in flat spacetime reduces to

$$(L_P\gamma^{AB}+4\alpha\mu\delta^{ac}K^A_{aC}K^{BC}_c)K_{bAB} +2\alpha\mu[\hat{\nabla}_B(K_a^{AB}\delta^{ac}N_{c,A}\cdot N_b)-K_a^{AB}\delta^{ac}N_{c,A}\cdot N_{b,B}]=0 . \quad (4.5)$$

and the open-string end-point equations to either (3.22) or

$$(1-\alpha\delta^{ab}K_{aAB}K_b^{AB})|_{\tau^1=0,l}=0 . \quad (4.6)$$

In this case we have that the two equations for each string end point (3.23) reduce to only one. Thus condition (4.6) in this case is a perfectly well-defined "boundary condition." The equations of motion for the Polyakov string in usual coordinates and in covariant form can be found in Ref. [11].

A class of solutions of the Polyakov-string equation of motion in Minkowski spacetime is the Nambu strings with the usual motion equation (2.28). The simplest solution of this last equation that describes an open string in Minkowski coordinates is

$$X^\mu=[\tau^0, b\cos\tau^1\cos(\omega\tau^0) , b\cos\tau^1\sin(\omega\tau^0), 0] , \quad (4.7)$$

i.e., a bar of length $2b$ rotating in the plane (x,y) with a constant angular velocity ω. The bar end points are represented by $\tau^1=0$ and $\tau^1=\pi$. For this particular solution, the boundary condition (3.22) tell us that $\omega b=1$, i.e., that the end points travel with the speed of light. The condition (4.6) reduces to

$$\omega b=[1+2\alpha\omega^2]^{1/2} . \quad (4.8)$$

When $-1<2\alpha\omega^2<0$ we also have a physically acceptable solution in which the interaction reduces the speed of the string end points. When $\alpha=0$ we recover $\omega b=1$.

The energy momentum (3.29) for the Polyakov string in flat spacetime can be cast as

$$T^{AB}=[L_P\gamma^{AB}+2\alpha\mu\delta^{ab}(K_aK_b^{AB}+2K^A_{aC}K_b^{BC})]\delta^2(\rho) +2\alpha\mu K_a^{AB}\delta^{ab}\partial_b\delta^2(\rho) . \quad (4.9)$$

For strings evolving on a plane, say, on the (x,y) plane, it is easy to show that the extrinsic curvature K_{aAB} has only one component along only one of the directions ∂_a. This property of the extrinsic curvature is a consequence of the fact that plane strings evolve effectively in three-dimensional Minkowski spacetime. Thus, for plane strings, we will have an energy-momentum tensor with a dipolar term that contains only one derivative of the Dirac distribution. This means that plane Polyakov strings will look like a ribbon with some extension along the above-mentioned direction ∂_a.

B. Strings with intrinsic curvature corrections

Another particularly interesting model of string is the one in which the dependence on the extrinsic curvature is through the combination

$$R=\delta^{ab}(K_aK_b-K_{aAB}K_b^{AB}) . \quad (4.10)$$

This last equation is the contracted Gauss formula that relates the extrinsic curvature of the world sheet with its Ricci-scalar curvature R. Note that (4.10) is only valid in flat spacetime; otherwise, we need to add corrections coming from the curvature of the spacetime [20]. The Lagrangian in this case will be written as $L_R=L_R(R)$.

The equation of motion with the respective end-point conditions (3.23) in the present case is

$$(L_R'R-L_R/2)K_b +\delta^{cd}N_b\cdot\hat{\nabla}_A[L_R'N_{d,B}(K_c\gamma^{AB}-K_c^{AB}K)]=0 , \quad (4.11)$$

$$L_R|_{\tau^1=0,l}=0 , \quad (4.12)$$

where $(\ )'=d/dR$. Thus, in this case, we can also have alternative boundary conditions to (3.22).

The energy-momentum tensor reduces to

$$T^{AB}=-2\{[(L_R'R-L_R/2)\gamma^{AB} +L_R'\delta^{ab}(K_aK_b\gamma^{AB}-K_aK_b^{AB})]\delta^2(\rho) +L_R'(K_a\gamma^{AB}-K_a^{AB})\delta^{ab}\partial_b\delta^2(\rho)\} . \quad (4.13)$$

There is a specialization of this last case that deserves particular attention, the case in which L_R is a linear function of R:

$$L_T=\mu(1+\alpha R) . \quad (4.14)$$

We have that in two dimensions $\sqrt{-\gamma}R$ is a total divergence. Therefore the equation of motion in this case is again the usual Nambu equation, but the addition of this topological term will change the boundary conditions for open strings and the energy-momentum tensor. In this case (4.13) reduces to

$$T^{AB}=\mu[(1-\alpha R)\gamma^{AB}+2\alpha K_a^{AB}\delta^{ab}\partial_b\delta^2(\rho)] . \quad (4.15)$$

For Nambu strings we have that

$$R=-\delta^{ab}K_{aAB}K_b^{AB} . \quad (4.16)$$

Therefore, in this case, for the end-point conditions, we can also have (4.6). The energy-momentum tensor (4.15) is equivalent to (4.9) with $K_a=0$. To show this last equivalence, we recall that in a two-dimensional manifold the Ricci tensor and Ricci scalar are related by $R_{AB}=\gamma_{AB}R/2$. The fact that the thermodynamics changes by the addition to the Lagrangian of total divergences has been studied by a variety of authors [21]. Note that the discussion concerning plane Polyakov strings and the boundary condition (4.8) for the string

(4.7) also applies in this case.

Finally, we want to add that the same methodology presented here can be used to study membranes with curvature corrections. A special case of a membrane with curvature corrections is studied in Ref. [22].

ACKNOWLEDGMENTS

We want to thank the hospitality of Université de Tours (P.S.L.) and Universidade Estadual de Campinas (B.B.).

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