Effective action at finite temperature

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We consider the problem of generalizing the usual effective potential calculations by computing the kinetic terms in the one-loop effective action. Results for the leading-order gradient terms in the effective action for scalar fields valid for both zero and finite temperatures are given when quantum corrections arise from scalar, spinor, or gauge fields. In the gauge field calculation we present a generalization of the 't Hooft gauge-fixing condition which removes cross terms in the scalar and gauge fields, resulting in considerable simplification of the fluctuation operators. Our results imply that the consistency of the one-loop effective action requires at least three families of fermions.

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I. INTRODUCTION

The behavior of quantum field theory at finite temperature has received attention because of the existence of phase transitions and because it has important applications such as to the very early Universe [1].

In circumstances where local thermodynamic equilibrium is established the state of a physical system lies at the minimum of the free energy. In the case of quantum field theory, the free energy F can be identified with the effective action Γ by the relationship $\Gamma = \beta F$, where β is the inverse temperature. If the ensemble averages of the matter fields are homogeneous and unchanging with time, then the free energy density is given by the finite-temperature effective potential introduced by Weinberg [2, 3]. We have extended techniques for the evaluation of the effective action as a perturbation series in the gradients of the ensemble averaged fields, which can be used where the usual effective potential approach proves inadequate.

One situation of interest to us is the electroweak phase transition in the early Universe. Inhomogeneous fluctuations in the Higgs or gauge fields during this transition are particularly important. In the first place, sphaleron fluctuations are present and these can change the baryon number of the Universe [4–7]. The rate of sphaleron fluctuations at a temperature T is approximately given by $T^4 \exp(-\Delta\Gamma)$, where $\Delta\Gamma$ is the increase in the effective action caused by a sphaleron, calculated at finite temperature. Some of the temperature dependence in the effective action can be obtained by replacing coupling constants with temperature-dependant variables [5, 6], but this does not reproduce all of the temperature-dependent terms that can arise.

Another reason for considering the rate of inhomogeneous fluctuations is that they may determine the nature of the electroweak phase transition. The effective potential of a component of the Higgs field has the form [8]

$$V(\phi) = \tfrac{1}{2}m^2(T)\phi^2 + \tfrac{1}{6}g(T)\phi^3 + \lambda(T)\phi^4. \qquad (1)$$

The temperature-dependent mass $m(T)$ passes through zero during the phase transition. At a temperature T_c the potential can have two separate and equal minima, and if the rate of transitions over the barrier between them is small then the Universe would supercool. The full effective action, as calculated here, is important in determining the rate of these transitions.

The terms in the effective action which are quadratic in scalar field gradients have previously been calculated at zero temperature for scalar and Dirac field theories [9, 10]. A variety of related expansion techniques have been used to calculate these terms [13, 17]. At finite temperatures, a derivative expansion similar to the one used here has been applied by Hu [11, 12] to the scalar loop case. The quadratic term for a gauge field loop is given explicitly, we believe for the first time, in Sec. IV of this paper.

The important new feature of our calculation is the introduction of a new gauge which generalizes the 't Hooft gauge [15] for expansion about a nonconstant background scalar field. The new gauge choice leads to separable fluctuation operators, and should be useful in other applications outside the present context. One such application would be to simplify the analysis of the classical stability of Yang-Mills Higgs soliton solutions.

We give the contributions to the quadratic scalar gradient terms resulting from interactions with a Dirac field loop in Sec. III and a gauge field loop in Sec. IV. In the latter case the finite-temperature correction has the opposite sign to the original gradient term. This can be interpreted as an instability of the classical vacuum caused by the gauge field loop, but stability is restored if there are several families of fermions.

II. THE FINITE-TEMPERATURE EFFECTIVE ACTION FROM SCALARS

We have already explained the reasons for being interested in terms in the effective action which contain derivatives of fields. This requires generalizing the calculations of Coleman and Weinberg [9] for effective potentials. Previous work includes Refs. [9, 10, 16–20, 11, 12] which involve evaluations of the effective action for nonconstant background fields. The method which we will use is based upon the local-momentum space method for curved spacetime introduced by Bunch and Parker [14].

A. Zero-temperature results

The one-loop correction to the effective action for real scalar fields is expressed as the functional integral

$$\Gamma^{(1)} = -\ln \int d\mu[\phi] \exp\left(-\tfrac{1}{2}\phi\Delta\phi\right). \tag{2}$$

We choose to work in a spacetime with a Euclidean metric. $d\mu[\phi]$ represents the functional measure for the scalar fields, and Δ is some second-order, self-adjoint differential operator. In quantum theory we are interested in

$$\Delta = -\nabla^2 + m^2 + V(\phi) \tag{3}$$

where m is a constant, and $V(\phi)$ is a function of the background scalar field. [For example, in $\lambda\phi^4$ theory, $V(\phi) = 12\lambda\hat{\phi}^2$, where $\hat{\phi}$ is a background field. See Jackiw [21] or Iliopoulos [10] for lucid discussions of the background field method.] From (2), we have

$$\Gamma^{(1)} = \tfrac{1}{2}\ln[\det\Delta] = \tfrac{1}{2}\mathrm{tr}\,\ln\Delta, \tag{4}$$

using the definition of Gaussian functional integration over boson fields. It should be noted that both the determinant and the trace in (4) are to be performed in function space as well as over any indices carried by Δ.

Our method of calculation involves an expansion of the Green function $G(x,x')$ for the operator Δ, defined by

$$\Delta G(x,x') = \delta(x,x'). \tag{5}$$

We can relate $\Gamma^{(1)}$ to G by differentiation of (4) with respect to m^2:

$$\begin{aligned}\frac{\partial\Gamma^{(1)}}{\partial m^2} &= \tfrac{1}{2}\mathrm{tr}\left[\Delta^{-1}\frac{\partial\Delta}{\partial m^2}\right]\\ &= \tfrac{1}{2}\mathrm{tr}[\Delta^{-1}]\\ &= \tfrac{1}{2}\int d^4x\,\mathrm{tr}G(x,x).\end{aligned} \tag{6}$$

The remaining trace in (6) is over field indices carried by the Green function. (Dimensional regularization can be adopted here by integrating over N dimensions and using analytic continuation to $N=4$ at the end.) This relates $\Gamma^{(1)}$ to the coincidence limit of the Green function.

Because we only require the coincidence limit of the Green function, we may evaluate $G(x,x')$ for x in the neighborhood of x'. Let

$$x = x' + y. \tag{7}$$

(In curved spacetime this procedure would amount to the introduction of Riemann normal coordinates at x', or more generally, a synchronous frame in a vector bundle, and the calculation would parallel that of Bunch and Parker.) We allow $V(x)$ to be a general function of x and Taylor expand about $x = x'$:

$$V(x) = V(x') + \sum_{n=1}^{\infty}\frac{1}{n!}y^{\mu_1}\cdots y^{\mu_n}V_{\mu_1\cdots\mu_n} \tag{8}$$

where

$$V_{\mu_1\cdots\mu_n} = \left.\frac{\partial^n V(x)}{\partial x^{\mu_1}\cdots\partial x^{\mu_n}}\right|_{x=x'}. \tag{9}$$

Hence, (5) becomes

$$\left(-\nabla^2 + M^2 + \sum_{n=1}^{\infty}\frac{1}{n!}y^{\mu_1}\cdots y^{\mu_n}V_{\mu_1\cdots\mu_n}\right)G(x,x') = \delta(y) \tag{10}$$

where

$$M^2 = m^2 + V(x'). \tag{11}$$

In order to solve (10), Fourier transform $G(x,x')$ as

$$G(x,x') = \int\frac{d^Nk}{(2\pi)^N}e^{iky}G(k;x'), \tag{12}$$

substitute into (10), and integrate by parts to obtain

$$\left((k^2+M^2) + \sum_{n=1}^{\infty}\frac{i^n}{n!}V_{\mu_1\cdots\mu_n}\frac{\partial^n}{\partial k_{\mu_1}\cdots\partial k_{\mu_n}}\right)\times G(k;x') = 1. \tag{13}$$

We want terms in the one-loop effective action which involve kinetic terms in the background fields in V. Thus, we write

$$G(k;x') = \sum_{j=0}^{\infty}G_j(k;x') \tag{14}$$

where the subscript k counts the number of derivatives which occur in the expression G_j. Note that this procedure is different from Bunch and Parker [14] who solve (13) as an asymptotic series in k. The method which we adopt is more suitable for obtaining the kinetic part of the one-loop effective action. We may then solve (13) as an asymptotic series in the number of derivatives.

Equating terms in (13) on both sides with no derivatives leads to

$$G_0(k;x') = (k^2+M^2)^{-1}. \tag{15}$$

This will lead to the usual effective potential part in the one-loop effective action. The remainder of (13) is then

$$\sum_{j=1}^{\infty}G_j(k;x') + (k^2+M^2)^{-1}\times\sum_{n=1}^{\infty}\sum_{j=0}^{\infty}\frac{i^n}{n!}V_{\mu_1\cdots\mu_n}\frac{\partial^n G_j(k;x')}{\partial k_{\mu_1}\cdots\partial k_{\mu_n}} = 0. \tag{16}$$

The indices in the double summation may now be relabeled so that one of them counts the total number of derivatives:

$$\sum_{j=1}^{\infty}G_j(k;x') + (k^2+M^2)^{-1}\times\sum_{j=1}^{\infty}\sum_{n=1}^{j}\frac{i^n}{n!}V_{\mu_1\cdots\mu_n}\frac{\partial^n G_{j-n}(k;x')}{\partial k_{\mu_1}\cdots\partial k_{\mu_n}} = 0. \tag{17}$$

Thus

$$G_j(k;x') = -(k^2+M^2)^{-1}\sum_{n=1}^{j}\frac{i^n}{n!}V_{\mu_1\ldots\mu_n}\frac{\partial^n G_{j-n}(k;x')}{\partial k_{\mu_1}\ldots\partial k_{\mu_n}}. \tag{18}$$

This result fully determines all of the terms in the expansion (14) of the Green function in the number of derivatives, although the explicit expressions for G_j must be found by solving (18) iteratively. We have done this for terms which involve no more than four derivatives and find, after some straightforward calculation, the results summarized in Appendix A.

The desired expression for the one-loop effective action may now be obtained directly from (6), (12), (14), and the results in Appendix A. The term with no derivatives just gives the usual effective potential, and we shall not consider it further here. The terms with one and three derivatives are seen from (A1), (A3) to be odd in k, and therefore they will integrate to zero. The second derivative terms in $\Gamma^{(1)}$ are found, after an integration by parts, to be

$$\Gamma_2^{(1)} = \tfrac{1}{24}(4\pi)^{-2}\int d^4x\, M^{-2}V^\mu V_\mu. \tag{19}$$

[If there is more than one scalar field, then there will be a trace over group indices in (19).] The terms in $\Gamma^{(1)}$ which contain four derivatives are

$$\Gamma_4^{(1)} = \tfrac{1}{16\pi^2}\int d^4x\,[\,-\tfrac{1}{96}M^{-8}(V^\mu V_\mu)^2 + \tfrac{1}{60}M^{-6}V^\mu V^\nu V_{\mu\nu} - \tfrac{1}{180}M^{-4}V^{\mu\nu}V_{\mu\nu} + \tfrac{1}{720}M^{-4}(\nabla^2 V)^2 - \tfrac{1}{360}M^{-6}V^\mu V_\mu \nabla^2 V]. \tag{20}$$

In the case of massive $\lambda\phi^4$ theory, $V(x) = 12\lambda\hat{\phi}^2(x)$, and $M^2 = m^2 + 12\lambda\hat{\phi}^2(x)$, where $\hat{\phi}(x)$ is the background scalar field. The result in (19) agrees with the calculation of Iliopoulos [10], and the result in (20) may be shown to agree with a result of Fraser [13].

B. Finite-temperature results

The local momentum space method presented in the previous section may also be extended to calculations at finite temperature by replacing the momentum space measure in the zero-temperature results with [2, 3]

$$\sum_{n=-\infty}^{\infty}\frac{1}{\beta}\int\frac{d^{N-1}k}{(2\pi)^{N-1}}, \tag{21}$$

where $k^0 = 2\pi n/\beta$. The relevant integrals which can arise are summarized in Appendix B. Although it is possible to obtain results for arbitrary background fields, we will restrict attention to those which are time independent.

Using the result in (A2) and integrating by parts leads to the following expression for the second derivative terms in $\Gamma^{(1)}$:

$$\Gamma_2^{(1)} = \int d^4x[\tfrac{1}{6}I(3)V^\mu V_\mu - \tfrac{1}{2}I^{\mu\nu}(4)V_\mu V_\nu]. \tag{22}$$

The finite-temperature integrals are evaluated in Appendix B. We find that, at high temperature,

$$\Gamma_2^{(1)} = \frac{T}{384\pi}\int d^4x (M^2)^{-3/2}(V^iV_i) \tag{23}$$

where V_i denotes the spatial gradient only. For massive $\lambda\phi^4$ theory, we have

$$\Gamma_2^{(1)} = \frac{\lambda^2 T}{32\pi}\int d^4x(m^2+12\lambda\hat{\phi}^2)^{-3/2}\hat{\phi}^2\nabla_i\hat{\phi}\nabla^i\hat{\phi}. \tag{24}$$

This one-loop correction has the same sign as the kinetic term in the classical scalar field action.

For the terms in the finite-temperature effective action which involve four derivatives, we find

$$\Gamma_4^{(1)} = \int d^4x\,[\,-\tfrac{5}{24}I(6)(V^iV_i)^2 - \tfrac{1}{30}I(5)V^iV_i\nabla^2V + \tfrac{1}{5}I(5)V^iV^jV_{ij} - \tfrac{1}{30}I(4)V^{ij}V_{ij} + \tfrac{1}{120}I(4)(\nabla^2V)^2]. \tag{25}$$

Here ∇^2 is the spatial Laplacian. Use of the high temperature expansions of Appendix B leads to

$$\Gamma_4^{(1)} = \int d^4x\Big\{-\tfrac{35T}{24576\pi}(M^2)^{-9/2}(V^iV_i)^2 - \tfrac{T}{3072\pi}(M^2)^{-7/2}V^iV_i\nabla^2V + \tfrac{T}{512\pi}(M^2)^{-7/2}V^iV^jV_{ij} - \tfrac{T}{1920\pi}(M^2)^{-5/2}V^{ij}V_{ij} + \tfrac{T}{7680\pi}(M^2)^{-5/2}(\nabla^2V)^2\Big\}. \tag{26}$$

Unlike the case for the second derivative at finite temperature, the fourth derivative terms have an indefinite sign.

III. THE FINITE-TEMPERATURE EFFECTIVE ACTION FROM SPINORS

The classical action for a Dirac spinor field with a nonconstant mass $M(x)$ in a spacetime with Euclidean signature is

$$\Gamma^{(0)} = -i\int d^Nx\,\bar{\psi}(x)\,[\gamma\cdot\nabla - M(x)]\,\psi(x). \tag{27}$$

The gamma matrices satisfy

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\delta^{\mu\nu}. \tag{28}$$

The nonconstant mass term here may come from a Yukawa coupling to a scalar field, for example.

The one-loop effective action arising from integration over the fermion field is

$$\begin{aligned}\Gamma^{(1)} &= -\ln \int d\mu[\bar{\psi},\psi]e^{-\Gamma^{(0)}} \\ &= -\ln \det\left[-i(\gamma\cdot\nabla - M)\right] \\ &= -\mathrm{tr}\ln\left[-i(\gamma\cdot\nabla - M)\right].\end{aligned} \tag{29}$$

As for the scalar field, we will relate $\Gamma^{(1)}$ to a Green function $S(x,x')$ for the operator occurring in (29). Define

$$-i\left[\gamma\cdot\nabla - M(x)\right]S(x,x') = \delta(x,x') \tag{30}$$

and assume that

$$M(x) = m + \bar{M}(x) \tag{31}$$

for some $\bar{M}(x)$ which does not depend on m. Then

$$\frac{\partial\Gamma^{(1)}}{\partial m} = -i\int d^N x\,\mathrm{tr}S(x,x) \tag{32}$$

where the functional trace has been performed leaving only the trace over any other indices carried by the spinor fields.

The local-momentum expansion may be used in (30). Write

$$M(x) = M(x') + \sum_{n=1}^{\infty}\frac{1}{n!}y^{\mu_1}\cdots y^{\mu_n}M_{\mu_1\cdots\mu_n}, \tag{33}$$

where

$$M_{\mu_1\cdots\mu_n} = \left.\frac{\partial^n M(x)}{\partial x^{\mu_1}\cdots\partial x^{\mu_n}}\right|_{x=x'}. \tag{34}$$

Define a Fourier transform of $S(x,x')$ by

$$S(x,x') = \int\frac{d^N k}{(2\pi)^N}e^{iky}S(k;x') \tag{35}$$

and substitute into (30). After integrating by parts,

$$\left((\gamma\cdot k + iM) + i\sum_{n=1}^{\infty}\frac{i^n}{n!}M_{\mu_1\cdots\mu_n}\frac{\partial^n}{\partial k_{\mu_1}\cdots\partial k_{\mu_n}}\right)\times S(k;x') = 1. \tag{36}$$

Write, analogously to (14),

$$S(k;x') = \sum_{j=0}^{\infty}S_j(k;x'), \tag{37}$$

where the subscript counts the number of derivatives. The coefficients with $j\le 2$ are given in Appendix A.

Since the calculation here is principally intended to illustrate our method, we will make the simplifying assumption that M in (36) is a multiple of the identity. The S_0 term in (37) again leads to the usual effective potential part of $\Gamma^{(1)}$ which we will not consider further. The S_1 term given in (A7) gives a vanishing contribution to the effective action. After some Dirac algebra, integrations by parts, and the use of zero-temperature results in Appendix B, the part of $\Gamma^{(1)}$ which contains two derivatives is found to be

$$\Gamma_2^{(1)} = \tfrac{1}{48\pi^2}(N-1)\Gamma(2-N/2)\times\int d^4x\left(\frac{M^2}{4\pi\mu^2}\right)^{N/2-2}M^\mu M_\mu \tag{38}$$

where μ is the renormalization scale. The result contains a pole term at $N=4$. For a Yukawa coupling we would have $M(x) = m + g\hat{\phi}(x)$, and the pole term would be absorbed by a field renormalization for the scalar field $\hat{\phi}(x)$. If we also perform a finite renormalization, so that $\Gamma_2^{(1)}$ begins at quadratic order in $\hat{\phi}$, then the renormalized $\Gamma_2^{(1)}$ is

$$\left(\Gamma_2^{(1)}\right)_{\mathrm{ren}} = -(4\pi)^{-2}\int d^4x\,\ln\left(\frac{M^2}{m^2}\right)M^\mu M_\mu. \tag{39}$$

The presence of the logarithmic term is expected on the basis of renormalization group considerations, and will be present whenever it is necessary to perform a field renormalisation for the scalar field. If the scalar field is massless, then (39) becomes (since $M = g\hat{\phi}$)

$$\left(\Gamma_2^{(1)}\right)_{\mathrm{ren}} = -g^2(4\pi)^{-2}\int d^4x\,\ln\left(\frac{\hat{\phi}^2(x)}{\mu^2}\right)\nabla^\mu\hat{\phi}\nabla_\mu\hat{\phi}. \tag{40}$$

We will see a similar result in Sec. IV for gauge field loops.

The result for finite temperatures may be found in a similar way to the result for scalar fields, with the momentum space measure (21). At the end of the calculation,

$$\Gamma_2^{(1)} = \frac{T}{12\pi}\int d^4x\,(M^2)^{-1/2}\nabla^i M\nabla_i M. \tag{41}$$

We have again considered only the case of spatially dependent background scalar fields. This finite-temperature radiative correction to the classical scalar field kinetic term is found to have the same sign as that obtained from scalar field loops in (23), in contrast with the contribution in the vacuum energy which has the opposite sign. It therefore appears that fermion contributions to the effective action stabilize the classical vacuum, even at high temperatures.

IV. THE FINITE-TEMPERATURE EFFECTIVE ACTION FROM GAUGE FIELDS

Consider the interactions between massless scalar meson fields Φ and gauge bosons A_μ described by the action

$$\Gamma^{(0)} = \int d^4x\left[\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + (D^\mu\Phi)^\dagger(D_\mu\Phi)\right], \tag{42}$$

where $D_\mu = \nabla_\mu + ieA_\mu$ is the gauge-covariant derivative and $F_{\mu\nu}$ is the corresponding field strength tensor. We shall take the gauge group to be U(1), but the generalization to arbitrary gauge groups can be carried out without difficulty, and some results for other gauge groups are given below.

The first step in constructing the effective action is to expand the action about a background field:

$$\Phi = \tfrac{1}{\sqrt{2}}\hat{\phi} + \tfrac{1}{\sqrt{2}}(\phi_1 + i\phi_2). \tag{43}$$

For one-loop order we can neglect terms which are not quadratic in ϕ_1, ϕ_2, or A_μ. When this is done it immediately becomes apparent that the scalar and vector fields are coupled by cross terms. These should be eliminated in the gauge-fixing process, otherwise they would considerably complicate the analysis.

We will choose the gauge-fixing functional

$$\mathcal{F}[A,\phi] = \nabla^\mu A_\mu + \alpha e\hat{\phi}\phi_2 + 2U^\mu A_\mu. \tag{44}$$

where α is the gauge parameter and

$$U_\mu = \hat{\phi}^{-1}\nabla_\mu\hat{\phi}. \tag{45}$$

For constant background fields, U_μ vanishes and our gauge reduces to the 't Hooft gauge [15].

In the generalization to Yang-Mills theory with a background Yang-Mills field A_μ, the gauge-fixing condition is defined by the replacement of $\nabla^\mu A_\mu$ with $D^\mu A_\mu$. We would also require U^μ to be a solution of $(D^\mu - U^\mu)\hat{\phi} = 0$.

Integrating over the gauge degrees of freedom in the path integral gives a ghost contribution $\det\Delta_g^{-1}$ and a gauge-fixing term $\mathcal{F}^2/2\alpha$ in the action. Since the gauge-fixing functional has been chosen to eliminate any cross terms, we get a quadratic contribution to the action of the form

$$\tfrac{1}{2}\int d^4x \, (A_\mu \Delta_v^{\mu\nu} A_\nu + \phi_1\Delta_1\phi_1 + \phi_2\Delta_2\phi_2), \tag{46}$$

where

$$\begin{aligned}\Delta_v^{\mu\nu} &= -\delta^{\mu\nu}\nabla^2 + (1-\alpha^{-1})\nabla^\mu\nabla^\nu + 4\alpha^{-1}U^\mu U^\nu \\ &\quad - 2\alpha^{-1}\nabla^\nu U^\mu + 2\alpha^{-1}(U^\mu\nabla^\nu - U^\nu\nabla^\mu) + e^2\hat{\phi}^2,\end{aligned} \tag{47}$$

$$\Delta_g = -\nabla^2 - 2U^\mu\nabla_\mu + \alpha e^2\hat{\phi}^2, \tag{48}$$

$$\Delta_1 = -\nabla^2, \tag{49}$$

$$\Delta_2 = -\nabla^2 + \alpha e^2\hat{\phi}^2. \tag{50}$$

In order to perform the path integration, the operators in (4.5) have to be self-adjoint, but the ghost operator Δ_g, which already appears as a determinant, must be left in its non-self-adjoint form. The one-loop effective action so obtained from the path integral is then

$$\Gamma^{(1)} = \Gamma^{(0)} + \tfrac{1}{2}\text{tr}\,\ln\Delta_v - \text{tr}\,\ln\Delta_g + \tfrac{1}{2}\text{tr}\,\ln\Delta_2 + \tfrac{1}{2}\text{tr}\,\ln\Delta_1. \tag{51}$$

We have the same problem expanding these logarithms as in the previous sections. This time, for variety, we will adapt the method used by Aitchison and Fraser [16], as described in Appendix C. (The relevant expressions for the local coordinate space expansion are contained in Appendix A.) First of all we set

$$M^2 = e^2\hat{\phi}^2 = m^2 + \bar{M}^2 \tag{52}$$

where m^2 is a constant. We then expand the logarithms in (4.10) as powers of $\bar{M}^2$. For the gauge loop example we also have to expand the logarithms in powers of U_μ, since this is also of first order in derivatives of $\hat{\phi}$.

The expansions are simplest with the choice $\alpha = 1$ for the gauge parameter. With this choice of gauge parameter we get the second derivative terms given below:

$$\begin{aligned}(\text{tr}\,\ln\Delta_v)_2 =& -\tfrac{1}{2}\text{tr}\left\{(-\nabla^2+m^2)^{-4}\left[(-\nabla^2+m^2)\bar{M}^{2\mu}_{\mu}\bar{M}^2 + 4\nabla_\mu\nabla_\nu\bar{M}^{2\mu\nu}\bar{M}^2\right]\right\} \\ &+\text{tr}\left[(-\nabla^2+m^2)^{-2}\left(4m^2U^2 - 4\nabla^\mu\nabla^\nu U_\mu U_\nu + 2\bar{M}^2 U_\mu{}^\mu\right)\right], \\ (\text{tr}\,\ln\Delta_g)_2 =& -\tfrac{1}{2}\text{tr}\left\{(-\nabla^2+m^2)^{-4}\left[(-\nabla^2+m^2)\bar{M}^{2\mu}_{\mu}\bar{M}^2 + 4\nabla_\mu\nabla_\nu\bar{M}^{2\mu\nu}\bar{M}^2\right]\right\} \\ &-\text{tr}\left[(-\nabla^2+m^2)^{-2}\left(2\nabla^\mu\nabla^\nu U_\mu U_\nu + \bar{M}^2 U_\mu{}^\mu\right)\right], \\ (\text{tr}\,\ln\Delta_2)_2 =& -\tfrac{1}{2}\text{tr}\left\{(-\nabla^2+m^2)^{-4}\left[(-\nabla^2+m^2)\bar{M}^{2\mu}_{\mu}\bar{M}^2 + 4\nabla_\mu\nabla_\nu\bar{M}^{2\mu\nu}\bar{M}^2\right]\right\}.\end{aligned} \tag{53}$$

The first trace in each equation has the same form as the scalar field case given in Appendix C, while the remaining terms come from the contributions to the operators depending on U_μ. The traces are over spacetime and vector indices and can be evaluated in momentum space, all of the relevant sums and integrals being given in Appendix B.

At zero temperature, the second order term in derivatives in $\Gamma^{(1)}$ is

$$\Gamma_2^{(1)} = \int d^4x \, \frac{e^2}{8\pi^2}\left(\ln\frac{\hat{\phi}^2}{\mu^2}\right)\nabla^\mu\hat{\phi}\nabla_\mu\hat{\phi}. \tag{54}$$

This term could be deduced from the wave function renormalization. For a gauge-fixing term α, this implies that the numerical coefficient of $\ln\mu$ is [22] $(\alpha - 3)e^2/32\pi^2$. The remaining dependence on $\hat{\phi}$ follows by dimensional analysis.

We shall only give high temperature results in the case that $\hat{\phi}$ is independent of time. Then

$$\Gamma_2^{(1)} = -\int d^4x \, \frac{7e^2}{32\pi}\frac{T}{e\hat{\phi}}\nabla^i\hat{\phi}\nabla_i\hat{\phi}. \tag{55}$$

The singularity at $\hat{\phi} = 0$ could be removed, as with similar terms in the effective potential [3], by summing subsets of higher-loop graphs.

The result (4.16) can also be obtained with the gauge parameter choice $\alpha = 0$. This indicates that the leading term in the high temperature expansion is independent of gauge fixing, as is the case for the leading term in the effective potential. Higher-order terms will not be independent of the gauge-fixing in general. If a gauge–fixing independent result is required for all temperatures, a natural formalism to adopt is the Vilkovisky and DeWitt procedure [24, 25], although physical predictions should not depend on using the formalism. The higher order terms are currently under investigation.

A simple generalization of our results can be made to larger gauge groups:

O(N):

$$\Gamma_2^{(1)} = -\int d^4x \, \frac{7g^2}{32\pi}(N-1)\frac{T}{g\hat{\phi}}\nabla^i\hat{\phi}\nabla_i\hat{\phi}, \tag{56}$$

SU(N):

$$\Gamma_2^{(1)} = -\int d^4x \, \frac{7g^2}{32\pi}[N(N-R)R]^{1/2}\frac{T}{g\hat{\phi}}\nabla^i\hat{\phi}\nabla_i\hat{\phi}. \tag{57}$$

In the case on O(N), the background field was taken to be $\hat{\phi}E$, for a unit N vector E. For SU(N), the background field was $\hat{\phi}\lambda_1/\sqrt{2}$, where λ_1 is the group generator which breaks the SU(N) symmetry down to SU($N-R$)×SU(R)×U(1).

The sign of these results is the opposite of the sign of the corresponding classical terms, indicating possible instability of the classical vacuum. Stability can be restored by the inclusion of the scalar or fermion fields to ensure that the overall sign of the quadratic terms is positive. For theories in which the gauge and fermion couplings are equal (such as supersymmetric models), comparison of the gauge and fermion results indicate that at least three families of fermions are required.

V. DISCUSSION AND CONCLUSIONS

In the above, we have shown how to calculate in a systematic manner kinetic terms in the one-loop effective action which arise whenever there are nonconstant background scalar fields. We gave results valid for both zero and finite temperatures. In the scalar loop case we have given results up to fourth order in the number of derivatives and we have set up most of the expansions that are needed to calculate the same order terms with gauge loops.

The logarithmic terms which arise, for example in the gauge loop at zero temperature, are related to the renormalization of the four-dimensional theory. On the other hand, the finite high temperature terms come from quantum fluctuations which are constant in the time direction. These terms are finite corrections to a three-dimensional quantum theory and in the case of background scalar fields cannot be obtained by simple renormalization group arguments.

In applications to the early Universe it would be of interest to extend our analysis to include background gauge fields, along the lines indicated in Sec. IV. Shaposhnikov [5] has argued that the leading terms can be obtained by replacing the gauge coupling g with a temperature-corrected function $g(T)$. However, a complete analysis of the gradient terms in the effective action remains to be done.

Although we have limited our considerations to one-loop effects, the extension to multiloop processes is certainly possible. The local-momentum expansion of the Green functions could be used in this context. For example, at two loops the effective action involves products of two or three Green functions. It would be a straightforward calculation to substitute in our Green function expansions and then collect terms with the same number of derivatives of the background fields.

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APPENDIX A: COEFFICIENTS IN THE LOCAL MOMENTUM SPACE EXPANSION

After solving (18) iteratively, we find

$$G_1(k;x) = 2iD^{-3}k^\mu V_\mu,$$

$$G_2(k;x) = 2D^{-4}V^\mu V_\mu - 12D^{-5}k^{\mu_1}k^{\mu_2}V_{\mu_1}V_{\mu_2} - D^{-3}\nabla^2 V + 4D^{-4}k^{\mu_1}k^{\mu_2}V_{\mu\nu},$$

$$\begin{aligned} G_3(k;x) = {} & 120iD^{-7}k^{\mu_1}k^{\mu_2}k^{\mu_3}V_{\mu_1}V_{\mu_2}V_{\mu_3} + 40iD^{-6}k^{\mu_1}V_{\mu_1}V^{\mu_2}V_{\mu_2} \\ & -12iD^{-5}k^{\mu_1}V_{\mu_1}\nabla^2 V + 80iD^{-6}k^{\mu_1}k^{\mu_2}k^{\mu_3}V_{\mu_1}V_{\mu_2\mu_3} - 20iD^{-5}k^{\mu_1}V^{\mu_2}V_{\mu_1\mu_2} \\ & +4iD^{-4}k^{\mu_1}\nabla^2 V_\mu - 8iD^{-5}k^{\mu_1}k^{\mu_2}k^{\mu_3}V_{\mu_1\mu_2\mu_3}, \end{aligned} \tag{A1}$$

$$
\begin{aligned}
G_4(k;x) = {} & 1680D^{-9}k^{\mu_1}k^{\mu_2}k^{\mu_3}k^{\mu_4}V_{\mu_1}V_{\mu_2}V_{\mu_3}V_{\mu_4} - 840D^{-8}k^{\mu_1}k^{\mu_2}V_{\mu_1}V_{\mu_2}V_{\mu_3}V^{\mu_3} \\
& +40D^{-7}V^{\mu_1}V_{\mu_1}V^{\mu_2}V_{\mu_2} + 180D^{-7}k^{\mu_1}k^{\mu_2}V_{\mu_1}V_{\mu_2}\nabla^2 V - 20D^{-6}V^{\mu}V_{\mu}\nabla^2 V \\
& -1680D^{-8}k^{\mu_1}k^{\mu_2}k^{\mu_3}k^{\mu_4}V_{\mu_1}V_{\mu_2}V_{\mu_3\mu_4} + 160D^{-7}k^{\mu_1}k^{\mu_2}V_{\mu_1\mu_2}V^{\mu_3}V_{\mu_3} \\
& +600D^{-7}k^{\mu_1}k^{\mu_2}V_{\mu_1}V_{\mu_2}V_{\mu_2\mu_3}V^{\mu_3} - 32D^{-6}V^{\mu_1}V^{\mu_2}V_{\mu_1\mu_2} - 80D^{-6}k^{\mu_1}k^{\mu_2}V_{\mu_1}\nabla^2 V_{\mu_2} \\
& +10D^{-5}V^{\mu}\nabla^2 V_{\mu} + 3D^{-5}(\nabla^2 V)^2 + 240D^{-7}k^{\mu_1}k^{\mu_2}k^{\mu_3}k^{\mu_4}V_{\mu_1}V_{\mu_2\mu_3\mu_4} \\
& -72D^{-6}k^{\mu_1}k^{\mu_2}V_{\mu_1\mu_2\mu_3}V^{\mu_3} - 40D^{-6}k^{\mu_1}k^{\mu_2}V_{\mu_1\mu_2}\nabla^2 V + 160D^{-7}k^{\mu_1}k^{\mu_2}k^{\mu_3}k^{\mu_4}V_{\mu_1\mu_2}V_{\mu_3\mu_4} \\
& -64D^{-6}k^{\mu_1}k^{\mu_2}V_{\mu_1\mu_3}V_{\mu_2}{}^{\mu_3} + 4D^{-5}V_{\mu_1\mu_2}V^{\mu_1\mu_2} \\
& +12D^{-5}k^{\mu_1}k^{\mu_2}\nabla^2 V_{\mu_1\mu_2} - 16D^{-6}k^{\mu_1}k^{\mu_2}k^{\mu_3}k^{\mu_4}V_{\mu_1\mu_2\mu_3\mu_4}.
\end{aligned}
$$

We have set

$$D = k^2 + M^2$$

in these expressions.

In the spinor field case of Sec. III, substitution of (3.11) into (3.10) results in the terms

$$
\begin{aligned}
S_0(k;x') &= \not{D}^{-1}, \\
S_1(k;x') &= -\not{D}^{-1}M_{\mu}\not{D}^{-1}\gamma^{\mu}\not{D}^{-1}, \qquad \text{(A2)} \\
S_2(k;x') &= \not{D}^{-1}M_{\mu_1}\not{D}^{-1}\gamma^{\mu_1}\not{D}^{-1}M_{\mu_2}\not{D}^{-1}\gamma^{\mu_2}\not{D}^{-1} + \not{D}^{-1}M_{\mu_1}\not{D}^{-1}M_{\mu_2}\not{D}^{-1}\gamma^{\mu_1}\not{D}^{-1}\gamma^{\mu_2}\not{D}^{-1} \\
&\quad +\not{D}^{-1}M_{\mu_1}\not{D}^{-1}M_{\mu_2}\not{D}^{-1}\gamma^{\mu_2}\not{D}^{-1}\gamma^{\mu_1}\not{D}^{-1} + i\not{D}^{-1}M_{\mu_1\mu_2}\not{D}^{-1}\gamma^{\mu_1}\not{D}^{-1}\gamma^{\mu_2}\not{D}^{-1},
\end{aligned}
$$

where we have set

$$\not{D} = \gamma\cdot k + iM. \qquad \text{(A3)}$$

For the gauge field case discussed in Sec. IV, we need the Green function for the operator Δ_v defined in (4.6), with $\alpha = 1$. Define

$$\Delta_v{}^{\mu}{}_{\nu}G^{\nu}{}_{\lambda}(x,x') = \delta^{\mu}_{\lambda}(x,x'). \qquad \text{(A4)}$$

Expanding the Green function in powers of derivatives as before leads to

$$
\begin{aligned}
G_0{}^{\mu}{}_{\nu}(k;x') &= \delta^{\mu}_{\nu}D^{-1}, \\
G_1{}^{\mu}{}_{\nu}(k;x') &= 2i\delta^{\mu}_{\nu}D^{-3}k^{\lambda}M^2_{,\lambda} - 2iD^{-2}(U^{\mu}k_{\nu} - U_{\nu}k^{\mu}), \qquad \text{(A5)} \\
G_2{}^{\mu}{}_{\nu}(k;x') &= -iD^{-1}M^2_{,\lambda}G_1{}^{\mu}{}_{\nu}{}^{,\lambda} + \tfrac{1}{2}D^{-1}M^2_{,\rho\sigma}D^{-1,\rho\sigma}\delta^{\mu}_{\nu} - 2iD^{-1}(U^{\mu}k_{\lambda} - U_{\lambda}k^{\mu})G_1{}^{\mu}{}_{\nu} \\
&\quad +2D^{-1}U^{\mu}{}_{,\sigma}(k_{\nu}D^{-1})^{,\sigma} - 2D^{-1}U_{\nu,\sigma}(k^{\mu}D^{-1})^{,\sigma} - (4U^{\mu}U_{\nu} - 2U^{\mu}{}_{,\nu})D^{-2},
\end{aligned}
$$

where μ denotes $\partial/\partial k_{\mu}$ when acting on G or D and $\partial/\partial x^{\mu}$ when acting on M^2 or U^{ν}. These expressions may be used to obtain the terms in the effective action which are quadratic in derivatives.

APPENDIX B: MOMENTUM SPACE INTEGRALS

Define

$$I(p) = \int \frac{d^N k}{(2\pi)^N}(k^2 + M^2)^{-p}$$

where M^2 was defined in (11). The standard formula of dimensional regularization leads to ($p \geq 1$)

$$I(p) = (4\pi)^{N/2}\frac{\Gamma(p - N/2)}{\Gamma(p)}(M^2)^{N/2-p}. \qquad \text{(B1)}$$

It is easy to see from (B2), or directly from (B1), that

$$I(p) = -(p-1)^{-1}\frac{\partial}{\partial m^2}I(p-1) \qquad \text{(B2)}$$

for $p \geq 2$.

The other integrals which we require are

$$
\begin{aligned}
I^{\mu_1\mu_2}(p) &= \int \frac{d^N k}{(2\pi)^N}k^{\mu_1}k^{\mu_2}(k^2 + M^2)^{-p} \\
&= A(p)\delta^{\mu_1\mu_2}, \qquad \text{(B3)}
\end{aligned}
$$

$$
\begin{aligned}
&\int \frac{d^N k}{(2\pi)^N}k^{\mu_1}k^{\mu_2}k^{\mu_3}k^{\mu_4}(k^2 + M^2)^{-p} \\
&\qquad = B(p)(\delta^{\mu_1\mu_2}\delta^{\mu_3\mu_4} + \text{symmetries}). \qquad \text{(B4)}
\end{aligned}
$$

It is easy to show that, for $p \geq 2$,

$$A(p) = \tfrac{1}{2}(p-1)^{-1}I(p-1) \tag{B5}$$

and, for $p \geq 3$,

$$B(p) = \tfrac{1}{4}(p-1)^{-1}(p-2)^{-1}I(p-2). \tag{B6}$$

These results are sufficient to calculate the terms in the one-loop effective action at zero temperature which contain no more than four derivatives.

At finite temperature, the integral given in (B1) gets replaced with

$$I(p) = \sum_{n=-\infty}^{\infty} \frac{1}{\beta} \int \frac{d^{N-1}k}{(2\pi)^{N-1}} \left[k^2 + (2n\pi/\beta)^2 + M^2\right]^{-p}. \tag{B7}$$

The relation given in (B3) still holds, so that we only need to work out $I(1)$. Doing the integration before the summation leads to

$$I(1) = (4\pi)^{-(N-1)/2}\Gamma(\tfrac{3}{2} - \tfrac{N}{2})\beta^{-1}(2\pi\beta^{-1})^{N-3} \times\zeta(\tfrac{3}{2} - \tfrac{N}{2};\beta M/2\pi) \tag{B8}$$

where

$$\zeta(s;\nu) = \sum_{n=-\infty}^{\infty} (n^2+\nu^2)^{-s}. \tag{B9}$$

It may be shown [23] that $\zeta(s;\nu)$ is analytic for $\mathcal{R}(s) > \frac{1}{2}$ and has simple poles at $s = \frac{1}{2}, -\frac{1}{2}, \ldots$. In particular,

$$\zeta(\tfrac{3}{2} - \tfrac{N}{2};\nu) = -\nu^2(N-4)^{-1} + \zeta(1;\nu) + O(N-4) \tag{B10}$$

where for small ν,

$$\zeta(1;\nu) = -\tfrac{1}{6} + \nu + (\gamma-1)\nu^2 - \tfrac{1}{4}\zeta(3)\nu^4 + \tfrac{1}{8}\zeta(5)\nu^6 + \cdots. \tag{B11}$$

[The exact expression for $\zeta(1;\nu)$ may be found in Ref. [23].] The high temperature expansion for $I(1)$ is then found to be

$$I(1) = \tfrac{M^2}{8\pi^2}(N-4)^{-1} + \tfrac{M^2}{16\pi^2}\left[\ln(4\pi T^2) - \gamma\right] + \tfrac{1}{12}T^2 - \tfrac{T}{4\pi}(M^2)^{1/2} + O(M^4T^{-2}). \tag{B12}$$

Results for $I(p)$ with $p \geq 2$ are easily obtained by repeated differentiation of (B13) using (B3). In particular, for $p \geq 3$ there will be no poles at $N = 4$ and no logarithmic terms in the temperature. In fact, as long as the original integrals are convergent (which is the case for $p \geq 3$), the leading term in the high temperature expansion comes from the $n = 0$ term in the expansion. We have

$$I(2) = -\tfrac{1}{8\pi^2}(N-4)^{-1} - \tfrac{1}{16\pi^2}\left[\ln(4\pi T^2) - \gamma\right] \tag{B13}$$

$$+\tfrac{T}{8\pi}(M^2)^{-1/2} + \cdots, \tag{B14}$$

$$I(3) = \tfrac{T}{32\pi}(M^2)^{-3/2} + \cdots, \tag{B15}$$

$$I(4) = \tfrac{T}{64\pi}(M^2)^{-5/2} + \cdots, \tag{B16}$$

$$I(5) = \tfrac{5T}{512\pi}(M^2)^{-7/2} + \cdots, \tag{B17}$$

$$I(6) = \tfrac{7T}{1024\pi}(M^2)^{-9/2} + \cdots. \tag{B18}$$

The finite-temperature generalization of (B4) may be written as

$$\sum_{n=-\infty}^{\infty} \frac{1}{\beta} \int \frac{d^{N-1}k}{(2\pi)^{N-1}} k^\mu k^\nu (k^2 + M^2)^{-p} = A_1(p)\delta^{\mu\nu} + A_2(p)n^\mu n^\nu \tag{B19}$$

where n^μ is a unit vector in the time direction. We have

$$A_i(p) = -(p-1)^{-1}\frac{\partial}{\partial M^2}A_i(p-1) \tag{B20}$$

and

$$A_1(p) = \tfrac{1}{2}(p-1)^{-1}I(p-1) \tag{B21}$$

for $p \geq 2$. In the high temperature limit it may be shown that

$$A_1(1) = \tfrac{M^4}{32\pi^2}(N-4)^{-1} + \tfrac{M^4}{64\pi^2}\left[\ln(4\pi T^2) - \gamma\right] + \tfrac{1}{24}M^2T^2 - \tfrac{T}{12\pi}(M^2)^{3/2} + \cdots \tag{B22}$$

and

$$A_2(1) = -\tfrac{2\pi^2}{45}T^4 + \tfrac{1}{32\pi^2}M^4 + \tfrac{1}{12}M^2T^2 - \tfrac{T}{12\pi}(M^2)^{3/2} + \cdots. \tag{B23}$$

The result given in (B20) leads to

$$\begin{aligned} A_1(2) &= -\tfrac{M^2}{16\pi^2}(N-4)^{-1} - \tfrac{M^2}{32\pi^2}\left[\ln(4\pi T^2) - \gamma\right] + \tfrac{1}{24}T^2 - \tfrac{T}{8\pi}(M^2)^{1/2} + \cdots, \\ A_2(2) &= -\tfrac{M^2}{16\pi^2} - \tfrac{T^2}{12} + \tfrac{T}{8\pi}(M^2)^{1/2} + \cdots, \\ A_1(3) &= -\tfrac{1}{32\pi^2}(N-4)^{-1} - \tfrac{1}{64\pi^2}\left[\ln(4\pi T^2) - \gamma\right] + \tfrac{T}{32\pi}(M^2)^{-1/2} + \cdots, \\ A_2(3) &= \tfrac{1}{32\pi^2} - \tfrac{T}{32\pi}(M^2)^{-1/2} + \cdots. \end{aligned} \tag{B24}$$

For $p \geq 4$, when the integral in (B19) is convergent, we have

$$A_2(p) \sim A_1(p) \tag{B25}$$

in the high temperature limit. (This result follows from the fact that the dominant term in the limit comes from the $n = 0$ term in the summation.) Use of (B21) then gives the remaining results:

$$A_1(4) \sim -A_2(4) \sim \tfrac{T}{192\pi}(M^2)^{-3/2}, \tag{B26}$$

$$A_1(5) \sim -A_2(5) \sim \tfrac{T}{512\pi}(M^2)^{-5/2}, \tag{B27}$$

$$A_1(6) \sim -A_2(6) \sim \tfrac{T}{1024\pi}(M^2)^{-7/2}, \tag{B28}$$

$$A_1(7) \sim -A_2(7) \sim \tfrac{7T}{12288\pi}(M^2)^{-9/2}. \tag{B29}$$

Finally, we need the finite-temperature generalization of (B5):

$$\sum_{n=-\infty}^{\infty}\frac{1}{\beta}\int\frac{d^{N-1}k}{(2\pi)^{N-1}}k^{\mu_1}k^{\mu_2}k^{\mu_3}k^{\mu_4}(k^2+M^2)^{-p}$$

$$= B_1(p)(\delta^{\mu_1\mu_2}\delta^{\mu_3\mu_4} + \text{symmetries}) + B_2(p)(\delta^{\mu_1\mu_2}n^{\mu_3}n^{\mu_4} + \text{symmetries}) + B_3(p)n^{\mu_1}n^{\mu_2}n^{\mu_3}n^{\mu_4}. \quad \text{(B30)}$$

The $B_i(p)$ which we require have $p \geq 5$, and are all finite at $N = 4$. This means that the dominant term in the high temperature expansion comes from the $n = 0$ term in the summation. By taking $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0$, and $\mu_1 = \mu_2 = 0$, $\mu_3 = \mu_4 = 1$ we can relate $B_2(p)$ and $B_3(p)$ to $B_1(p)$:

$$B_2(p) \sim -B_1(p), \quad \text{(B31)}$$

$$B_3(p) \sim 3B_1(p). \quad \text{(B32)}$$

$B_1(p)$ may be found by taking $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1$,

$$B_1(p) = \tfrac{1}{4}(p-1)^{-1}(p-2)^{-1}I(p-2). \quad \text{(B33)}$$

The previous results found for $I(p-2)$ in the high temperature limit may now be used to obtain the desired results for $B_i(p)$.

APPENDIX C: OTHER METHODS OF EVALUATING THE EFFECTIVE ACTION

Another method of evaluating the effective action has been invented by Aitchison and Fraser [16]. This method can easily be adapted to high temperature calculations. Consider the one-loop contribution to the effective action,

$$\Gamma^{(1)} = \tfrac{1}{2}\text{tr}\,\ln\left[-\nabla^2 + M^2\right] \quad \text{(C1)}$$

where M^2 depends on the background field. The mass M^2 is split up into $m^2 + \bar{M}^2$, where m is a constant, and then $\Gamma^{(1)}$ expanded in powers of $\bar{M}$:

$$\Gamma^{(1)} = \tfrac{1}{2}\text{tr}\,\ln\left(-\nabla^2 + m^2\right) + \tfrac{1}{2}\text{tr}\left[(-\nabla^2+m^2)^{-1}\bar{M}^2\right] - \tfrac{1}{4}\text{tr}\left[(-\nabla^2+m^2)^{-1}\bar{M}^2\right]^2 + \cdots. \quad \text{(C2)}$$

The effective potential contribution to the effective action can be separated out by collecting all of the factors of $G = (-\nabla^2 + m^2)^{-1}$ together:

$$\Gamma^{(1)} = \tfrac{1}{2}\text{tr}\,\ln G^{-1} + \tfrac{1}{2}\text{tr}\left[G\bar{M}^2\right] - \tfrac{1}{4}\text{tr}\left[G^2\bar{M}^4\right] - \tfrac{1}{4}\text{tr}\left[G\bar{M}^2G\bar{M}^2\right] + \tfrac{1}{4}\text{tr}\left[G^2\bar{M}^4\right] + \cdots. \quad \text{(C3)}$$

The first three terms contain no derivatives of $\bar{M}$ and they reconstruct the effective potential as a function of M. The remaining terms can be simplified by grouping factors of G together using commutator identities:

$$GM^2GM^2 = G^2M^4 + G^3(LM^2)M^2 + G^4(L^2M^2)M^2 + \cdots \quad \text{(C4)}$$

where LM^2 denotes the commutator $[-\nabla^2, M^2]$. Expansion of the commutator yields

$$LM^2 = -2\nabla_\mu(M^2)^{,\mu} + (M^2)_{,\mu}^{\ \ \mu}, \quad \text{(C5)}$$

$$L^2M^2 = 4\nabla_\mu\nabla_\nu(M^2)^{,\mu\nu} + O((M^2)_{,\mu\nu\sigma}). \quad \text{(C6)}$$

Therefore the contribution to the second derivative terms from $\Gamma^{(1)}$ is

$$\Gamma_2^{(1)} = -\tfrac{1}{2}\text{tr}\left[G^3(\bar{M}^2)_{,\mu}^{\ ;\mu}\bar{M}^2 + 4G^4\nabla_\mu\nabla_\nu(\bar{M}^2)^{,\mu\nu}\bar{M}^2\right]. \quad \text{(C7)}$$

The trace can be evaluated by replacing ∇^μ with ik^μ. Using the results of Appendix B for the high temperature limits of the momentum integrals, one recovers the same result as (23). Higher order terms can be found by expanding the logarithm to higher orders and using further commutator identities.

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