Quantum effects near a point mass in (2+1)-dimensional gravity

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We investigate the behavior of classical and quantum fields in the conical space-time associated with a point mass in $2+1$ dimensions. We show that the presence of conical boundary conditions alters the electrostatic field of a point charge leading to the presence of a finite self-force on the charge from the direction of the point mass exactly as if the point mass itself were charged. The conical space-time geometry also affects the zero-point fluctuations of a quantum scalar field leading to the existence of a vacuum polarization $\langle T_{\mu\nu}\rangle$ in the $(2+1)$-dimensional analogue of the Schwarzschild metric. The resulting linearized semiclassical Einstein equations $G_{\mu\nu}=8\pi G\langle T_{\mu\nu}\rangle$ possess a well-defined Newtonian limit, in marked contrast to the classical case for which no Newtonian limit is known to exist. An elegant reformulation of our results in terms of the method of images is also presented. Our analysis also covers the nonstatic de Sitter–Schwarzschild metric in $2+1$ dimensions, in which in addition to the vacuum polarization, a nonzero vacuum flux of energy $\langle T_{rt}\rangle$ is also found to exist. As part of this analysis, we evaluate the scalar field propagator in an n-dimensional de Sitter space; as a result some novel features of quantum field theory in odd dimensions are seen to emerge.

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I. INTRODUCTION

A well-established feature of Einstein gravity is that in a space-time of dimensionality $d<4$, it is devoid of any intrinsic dynamics. In three dimensions this result arises from the observation that both the Ricci and the Riemann tensors have an equal number of components $(=6)$. Consequently, the Riemann tensor can be expressed in terms of a combination of Ricci tensors:

$$R^{\mu\nu}_{\alpha\beta}=\epsilon^{\mu\nu\lambda}\epsilon_{\alpha\beta\gamma}(R^{\gamma}_{\lambda}-\tfrac{1}{2}\delta^{\gamma}_{\lambda}R)\ . \tag{1.1}$$

Clearly, if the Ricci tensor vanishes then so does the Riemann tensor, with the result that gravity does not propagate outside of matter sources. Since the Weyl tensor incorporates the internal degrees of freedom of the gravitational field, it follows from (1.1) that $C_{iklm}\equiv 0$ in a three-dimensional space-time. In a space-time of dimensionality $d>3$, the vanishing of the Weyl tensor (also called "the conformal tensor") is indicative of the fact that the space-time under consideration is conformally flat. This is not so in $d=3$, the issue of conformal flatness in this case being decided not by the Weyl tensor but by the symmetric, conserved, and traceless Cotton-York tensor (sometimes also known as the three-dimensional Weyl tensor),

$$C^{\alpha\beta}=\epsilon^{\alpha\gamma\delta}(R^{\beta}_{\gamma}-\tfrac{1}{4}\delta^{\beta}_{\gamma}R)_{;\delta}\ , \tag{1.2}$$

so that any three-dimensional space-time is conformally flat if and only if the Cotton-York tensor vanishes [1]. The Cotton-York tensor also features prominently in topologically massive gravity which has been the focus of considerable attention in recent years following the discovery by Deser that bosons and fermions can acquire exotic spin and statistics within the framework of this theory [2,3].

Topologically massive gravity is described by an action which is the sum of the standard Einstein action and a Chern-Simons term [3]:

$$I=I_E+\frac{1}{\mu}I_{\rm CS}\ , \tag{1.3a}$$

where

$$I_E=\int d^3x\sqrt{g}\,R \tag{1.3b}$$

and $I_{\rm CS}$ is the Chern-Simons action,

$$I_{\rm CS}=\frac{1}{2\mu}\int d^2x\sqrt{g}\,\epsilon^{\mu\alpha\nu}[\omega^a_\mu\partial_\alpha\omega_{\nu a}+\tfrac{1}{3}\omega^a_\mu\omega^b_\alpha\omega^c_\nu\epsilon_{abc}]\ , \tag{1.3c}$$

where ω^a_ν is the spin connection and μ is a constant having dimension of mass. Variation of I with respect to the metric results in the Einstein-Cotton equations [2,3]

$$R^{\mu\nu}+\frac{1}{\mu}C^{\mu\nu}=0\ , \tag{1.4}$$

$C^{\mu\nu}$ being the Cotton-York tensor.

The new equations of motion (1.4) do not constrain the curvature to vanish in the absence of sources, so that gravity has a nontrivial dynamics and can propagate. It is interesting to note that the external metric of a static point source is identical in both topologically massive gravity and Einstein gravity at large distances from the source ($\mu r>1$), and is given by [2–6]

$$ds^2=dt^2-dr^2-r^2d\varphi^2\ , \tag{1.5a}$$

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where

$$0 \le \varphi < \frac{2\pi}{p}, \quad p=(1-4G_2M)^{-1}, \quad (p \ge 1) ,$$

M being the mass of the point source and G_2 the gravitational constant in 2+1 dimensions. We note that the t=const two-hypersurface of this metric is a cone, and that the metric is flat everywhere except at the origin. In terms of a new polar coordinate $\tilde{\varphi}=p\varphi$ the metric takes the form

$$ds^2=dt^2-dr^2-\frac{r^2}{p^2}d\tilde{\varphi}^2 , \tag{1.5b}$$

with $\tilde{\varphi}$ extending over the entire range $0 \le \tilde{\varphi} < 2\pi$.

Metric (1.5) can be obtained from the well-known exterior metric of a straight cosmic string by suppressing the dimension along its length [7]. Recently many authors [8] have studied the scattering of point particles in the conical space-time metric (1.5). We shall follow an alternate approach and study the semiclassical one-loop quantum gravitational effects that arise in such a space-time due to its nontrivial topology. Such effects are also known to be associated with cosmic strings and have been extensively studied by a number of authors [9].

The outline of this paper is as follows.

In Sec. II we study the classical electrostatic field of a charged particle in the space-time of a point mass (1.5). We show that the existence of conical boundary conditions distorts the electrostatic field of the particle in a way that causes the particle to experience a repulsive self-force directed away from the point mass.

In Sec. III we examine quantum fluctuations of a massless scalar field in the conical space-time described by (1.5). We demonstrate the existence of a vacuum polarization characterized by a finite vacuum expectation value of the energy-momentum tensor $\langle T_{\mu\nu} \rangle$. We also show that the vacuum energy density $\langle T_{00} \rangle$ is negative for scalar fields coupling either conformally or minimally to gravity.

In Sec. IV we calculate the back reaction of one-loop quantum gravitational effects on the space-time geometry via the semiclassical Einstein equations $G_{\mu\nu} = 8\pi G_2 \langle T_{\mu\nu} \rangle$. We find that in the linearized approximation the semiclassical Einstein equations have a well-defined Newtonian limit, in marked contrast to the classical case where no such limit exists.

In Sec. V we extend our study to the Schwarzschild–de Sitter metric. We find in this case, in addition to the vacuum polarization, the presence of a vacuum energy flux $\langle T_{tr} \rangle$ directed radially away from the point source.

In Sec. VI we extend our analysis to include twisted scalar fields and evaluate $\langle \phi^2 \rangle^T$ and $\langle T_{\mu\nu} \rangle^T$ for twisted fields in the three-dimensional Schwarzschild metric. We find that $\langle \phi^2 \rangle^T$ and $\langle T_{\mu\nu} \rangle^T$ are generally of opposite sign to $\langle \phi^2 \rangle$ and $\langle T_{\mu\nu} \rangle$ for untwisted fields.

We end our paper with a discussion of our results in Sec. VII.

II. ELECTROSTATICS IN 2+1 DIMENSIONS

We begin our treatment of classical and quantum effects in conical space-times with a study of the classical Poisson equation in the conical background geometry (1.5). Since fields are generally sensitive to the global properties of a space-time, one would in general expect nontrivial modifications to arise for the standard electrostatic field of a point charge in (1.5).

A general solution to the Poisson equation

$$\Delta\varphi(\mathbf{x}) = -2\pi q \delta^2(\mathbf{x}-\mathbf{x}') \tag{2.1}$$

for a point charge located at $\mathbf{x}$ may be found by first constructing a Green's function satisfying

$$\Delta G(\mathbf{x},\mathbf{x}') = -2\pi\delta^2(\mathbf{x}-\mathbf{x}') . \tag{2.2}$$

The self-force on a test charge in the space-time (1.5) is then given by $\mathbf{F} = -\nabla U(\mathbf{x})$, where $U(\mathbf{x})$ is the electrostatic energy:

$$U(\mathbf{x}) = \tfrac{1}{2} q^2 G(\mathbf{x},\mathbf{x}) . \tag{2.3}$$

The symmetry of the problem makes it convenient to work in polar coordinates. The Poisson equation for the Green's function (2.2) then assumes the form

$$\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right] G_p(r,\theta;r',\theta') = -\frac{2\pi}{r}\delta(r-r')\delta(\theta-\theta') , \tag{2.4}$$

where $\delta(r-r')$ and $\delta(\theta-\theta')$ are one-dimensional δ functions. Since $\delta(\theta-\theta')=(p/2\pi)\sum_{m=-\infty}^{\infty} e^{ipm(\theta-\theta')}$, we shall use the polar coordinate expansion of the Green's function

$$G_p(r,\theta;r',\theta') = \frac{p}{2\pi} \sum_{m=-\infty}^{\infty} e^{ipm(\theta-\theta')} g_m(r,r') . \tag{2.5}$$

$g_m(r,r')$ then satisfies the radial differential equation

$$\left[\frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{p^2 m^2}{r}\right] g_m(r,r') = -2\pi\delta(r-r') . \tag{2.6}$$

Our one-dimensional Green's function can be written as

$$g_m(r,r') = \begin{cases} -\dfrac{1}{A} u_1(r) u_2(r'), & r < r', \\ -\dfrac{1}{A} u_1(r') u_2(r), & r > r', \end{cases} \tag{2.7}$$

where $u_{1,2}$ are solutions of the corresponding homogeneous equation: $u_1(r) \equiv r^{|pm|}$ and $u_2(r) \equiv r^{-|pm|}$.

The constant A may be determined from the Wronskian condition

$$u_1(r)\frac{d}{dr}u_2(r) - u_2(r)\frac{d}{dr}u_1(r) = \frac{A}{r} , \tag{2.8}$$

which gives $A = -2pm$, so that finally

$$g_m(r,r') = \frac{1}{2|pm|} X^{|pm|}, \quad (m \neq 0) , \tag{2.9a}$$

where

$$X=\frac{r}{r'} \quad \text{for } r'>r\,,$$
$$X=\frac{r'}{r} \quad \text{for } r'<r\,, \tag{2.9b}$$

and

$$g_0(r,r')=-\ln r', \quad 0\le r<r'\,,$$
$$g_0(r,r')=-\ln r, \quad 0\le r'<r\,. \tag{2.9c}$$

The two-dimensional Green's function $G_p(x,x')$ now assumes the form

$$G_p(r,\theta;r'\theta')=\frac{1}{2\pi}\sum_{m=1}^{\infty}\frac{X^{pm}}{m}\cos pm(\theta-\theta')-\frac{p}{2\pi}\ln r'\,, \tag{2.10}$$

where $X=r/r'<1$ is assumed.

Performing the summation in (2.10) we finally get [10]

$$G_p(r,\theta;r',\theta')=-\frac{1}{4\pi}\ln[r^{2p}+r'^{2p}-2(rr')^p\cos p(\theta-\theta')]\,, \tag{2.11}$$

which for $p=1$ reduces to

$$G_1(r,\theta;r',\theta')=-\frac{1}{2\pi}\ln|\mathbf{x}-\mathbf{x}'|\,, \tag{2.12}$$

the familiar form for the Green's function in (2+1)-dimensional Minkowski space. $G_p(\mathbf{x},\mathbf{x}')$ is formally divergent in the limiting case $\mathbf{x}\to\mathbf{x}'$, and must be regularized. Subtracting the flat-space contribution $G_1(\mathbf{x},\mathbf{x}')$ from $G_p(\mathbf{x},\mathbf{x}')$ and taking the limit $\mathbf{x}\to\mathbf{x}'$ we get

$$G_p^{\text{reg}}(\mathbf{x},\mathbf{x})=\lim_{\mathbf{x}\to\mathbf{x}'}[G_p(\mathbf{x},\mathbf{x}')-G_1(\mathbf{x},\mathbf{x}')]$$
$$=-\frac{1}{4\pi}\ln[p^2r^{2(p-1)}]\,, \tag{2.13}$$

which is finite.

The electrostatic energy of a charge distribution is

$$U=\tfrac{1}{2}\int\int\rho(\mathbf{x}')G_p^{\text{reg}}(\mathbf{x}',\mathbf{x}'')\rho(\mathbf{x}'')d^2x'\,d^2x''\,, \tag{2.14}$$

where $\rho(\mathbf{x})$ is the charge density. For a point charge located at $\mathbf{x}$, $\rho(\mathbf{x})=q\delta^2(\mathbf{x}'-\mathbf{x})$, so that

$$U(\mathbf{x})=\frac{q^2}{2}\int\int\delta^2(\mathbf{x}'-\mathbf{x})\delta^2(\mathbf{x}''-\mathbf{x})\times G_p^{\text{reg}}(\mathbf{x}',\mathbf{x}'')d^2x'\,d^2x''$$
$$=\frac{q^2}{2}G_p^{\text{reg}}(\mathbf{x},\mathbf{x})=\frac{-q^2}{8\pi}\ln[p^2r^{2(p-1)}]\,. \tag{2.15}$$

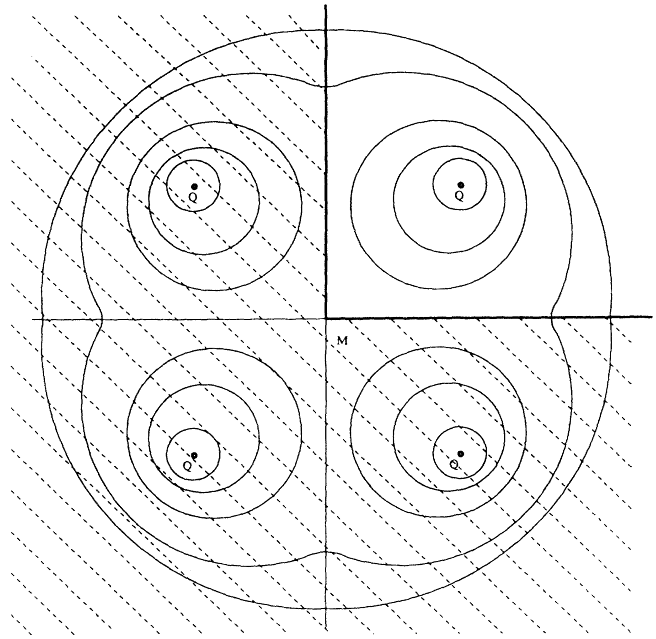

FIG. 1. The equipotential contours of the electrostatic force field of a point charge Q are shown for a space-time possessing a deficit angle $3\pi/2$ using the method of images (see Sec. II and Appendix A).

The self-force on the test charge is then

$$\mathbf{F}=-\nabla U=-\hat{\mathbf{r}}\frac{\partial U}{\partial r}=\hat{\mathbf{r}}\frac{(p-1)q^2}{4\pi r}\,. \tag{2.16}$$

We find that the self-force is repulsive and can be fairly large for $p\gg 1$, corresponding to large values of the deficit angle [11]. Interestingly, for $M\ll(4G_2)^{-1}$,

$$\mathbf{F}\simeq\frac{(4G_2Mq)q}{4\pi r}\hat{\mathbf{r}}\equiv\frac{Qq}{4\pi r}\hat{\mathbf{r}}\,; \tag{2.17}$$

i.e., the conical boundary conditions present in (1.5) have effectively induced a charge Q on the point source, proportional to its mass and of the same sign as the test charge q.

Our results can be elegantly rederived using the method of images [12], according to which, for integer p ($p\ge 1$),

$$G_p^{\text{reg}}(r,\theta;r',\theta')=\sum_{k=1}^{p-1}G_1\left[r,\theta;r',\theta'+\frac{2\pi k}{p}\right]=-\frac{1}{4\pi}\sum_{k=1}^{p-1}\ln\left[r^2+r'^2-2rr'\cos\left[\Delta\theta+\frac{2\pi k}{p}\right]\right]\,, \tag{2.18}$$

i.e., a test charge at (r,θ) *sees* $(p-1)$ images of itself located at $(r,\theta+2\pi k/p)$ $(k=1,2,\ldots,p-1)$. The proof of this assertion is straightforward and is given in Appendix A. (The method of images is illustrated in Fig. 1 for $p=4$.)

III. VACUUM POLARIZATION NEAR A POINT MASS IN 2+1 DIMENSIONS

It is well known that in a large variety of situations the imposition of nontrivial boundary conditions serves to alter the zero-point fluctuations of a quantum field leading to the existence of a vacuum polarization (DeWitt [13]). One might conjecture that the conical boundary conditions implicit in metric (1.5) will also lead to similar effects arising in the space-time of a massive point particle in 2+1 dimensions.

To investigate this possibility we shall consider a massless scalar field $\phi(x)$ propagating in the conical background geometry (1.5), and satisfying the field equation

$$\Box\phi(x)+\xi R=0 \tag{3.1}$$

(the curvature scalar R is taken to be equal to zero everywhere except at the location of the point mass; $\xi=\frac{1}{8}$ corresponds to conformal coupling in 2+1 dimensions [14]). Using conventional canonical quantization techniques, the field operator $\phi(x)$ may be expanded as a mode sum

$$\phi(x)=\sum_{\lambda}[a_\lambda u_\lambda(x)+a_\lambda^\dagger u_\lambda^*(x)]\ ,\quad x\equiv(t,r,\theta)\ , \tag{3.2}$$

where $a_\lambda, a_\lambda^\dagger$ are annihilation and creation operators satisfying the commutation relations $[a_\lambda,a_{\lambda'}^\dagger]=\delta_{\lambda\lambda'}$. The mode functions $u_\lambda(x)$ satisfy the differential equation (for $r>0$)

$$\left[\frac{\partial^2}{\partial t^2}-\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}-\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]u_\lambda(x)=0 \tag{3.3a}$$

and the boundary conditions

$$u_\lambda(r,\theta)|_{r=R}=0\ , \tag{3.3b}$$

$$u_\lambda(r,\theta)=u_\lambda\left[r,\theta+\frac{2\pi}{p}\right]\ . \tag{3.3c}$$

The essential features of the conical spatial geometry are incorporated in the boundary condition (3.3c). The boundary condition (3.3b) is imposed to facilitate normalization and mode counting. We shall take the $R\to\infty$ limit at a convenient point later on in our discussion. Equation (3.3a) can be solved exactly and its solution expressed as

$$u_\lambda\equiv u_{lm}(r,\theta,t)=N_{lm}J_{p|m|}(\omega_l r)e^{ipm\theta}e^{-i\omega_l t}\ , \tag{3.4}$$

where $\omega_l=\xi_l/R$, ξ_l being the lth zero of $J_{p|m|}(x)$, $m=0,\pm1,\ldots$, $l=1,2,\ldots$, and N_{lm} is a normalization constant whose value is fixed using the canonical equal-time commutation relation

$$[\phi(\mathbf{x},t),\pi(\mathbf{x}',t)]=i\delta^2(\mathbf{x}-\mathbf{x}')\ , \tag{3.5}$$

where the conjugate momentum $\pi(\mathbf{x},t)=\dot\phi(\mathbf{x},t)$. Equation (3.5) is equivalent to the condition

$$\int d^2x\,|u_{lm}|^2=(2\omega_l)^{-1}\ , \tag{3.6}$$

which yields a normalization constant N_{lm} given by

$$N_{lm}=\left[\frac{p}{2\pi\omega_l R^2}\right]^{1/2}[J_{p|m|+1}(\xi_l)]^{-1}\ . \tag{3.7}$$

The annihilation operator a_λ defined in (3.2) defines a vacuum $|0\rangle$ $(a_\lambda|0\rangle=0)$ in which the two-point (Wightman) function $D_p(x,x')$ can be expressed as a mode sum [14]:

$$\begin{aligned}D_p(x,x')&=\langle\phi(x)\phi(x')\rangle\\&=\sum_\lambda u_\lambda(x)u_\lambda^*(x')\\&=\sum_{l=1}^{\infty}\sum_{m=-\infty}^{\infty}N_{lm}^2u_{lm}(x)u_{lm}^*(x')\ .\end{aligned} \tag{3.8}$$

The dummy boundary $r=R$ is removed at this stage by taking $R\to\infty$ in (3.8) and by noting that

$$\begin{aligned}&\lim_{R\to\infty}\frac{1}{R}\sum_{l=1}^{\infty}u_{lm}\to\frac{1}{\pi}\int_0^\infty d\omega\,u_{lm}\ ,\\&\lim_{R\to\infty}J_{p|m|+1}^2(\xi_l)\to\frac{2}{\pi\xi_l}=\frac{2}{\pi\omega_l R}\ .\end{aligned} \tag{3.9}$$

The two-point function is obtained as an integral over ω and a summation over m:

$$D_p(x,x')=\frac{p}{4\pi}\sum_{m=-\infty}^{\infty}e^{imp(\theta-\theta')}\int_0^\infty d\omega\,e^{-i\omega(t-t')}J_{p|m|}(\omega r)J_{p|m|}(\omega r')\ . \tag{3.10}$$

Carrying out the integration over ω and setting

$$\cosh u_0=\frac{r^2+r'^2-(t-t')^2}{2rr'}\ , \tag{3.11a}$$

we obtain [10]

$$D_p(x,x')=\frac{p}{4\pi^2}\frac{1}{(2rr')^{1/2}}\int_{u_0}^\infty\frac{du}{(\cosh u-\cosh u_0)^{1/2}}\sum_{m=-\infty}^{\infty}\exp[imp(\theta-\theta')-|m|pu]\ . \tag{3.11b}$$

The summation over m can be obtained in a closed form, and the two-point function finally reduces to

$$D_p(x,x')=\frac{p}{4\pi^2}\frac{1}{(2rr')^{1/2}}\int_{u_0}^{\infty}\frac{du}{(\cosh u-\cosh u_0)^{1/2}}\frac{\sinh pu}{\cosh pu-\cos p(\theta-\theta')} . \tag{3.12}$$

It is straightforward to see that for $p=1$ one recovers the standard Minkowski space two-point function:

$$D_1(x,x')=\frac{1}{4\pi^2(2rr')^{1/2}}\int_{u_0}^{\infty}\frac{du}{(\cosh u-\cosh u_0)^{1/2}}\frac{\sinh u}{\cosh u-\cos(\theta-\theta')}=\frac{1}{4\pi\sigma},\quad \text{where } \sigma=|x-x'| . \tag{3.13}$$

The two-point function $D_p(x,x')$ obtained in (3.12) must be renormalized by subtracting out the Minkowski space contribution from it [13,14], so that

$$\begin{aligned}D_p(x,x')_{\rm ren}&=D_p(x,x')-D_1(x,x')\\&=\frac{1}{4\pi^2}\frac{1}{(2rr')^{1/2}}\int_{u_0}^{\infty}\frac{du}{(\cosh u-\cosh u_0)^{1/2}}\left[\frac{p\sinh pu}{\cosh pu-\cos p(\theta-\theta')}-\frac{\sinh u}{\cosh u-\cos(\theta-\theta')}\right] .\end{aligned} \tag{3.14}$$

At this stage one is in a position to evaluate the renormalized vacuum expectation values of the zero-point fluctuations of the field $\langle\phi^2(x)\rangle$, and the energy-momentum tensor $\langle T^\mu_\nu(x)\rangle$.

Given the propagator on an arbitrary (2+1)-dimensional manifold, the vacuum energy-momentum tensor may be determined by [14]

$$\begin{aligned}\langle T^\mu_\nu(x)\rangle=\lim_{x'\to x}&[(1-2\xi)\nabla^\mu\nabla'_\nu-(\tfrac{1}{2}-2\xi)g^\mu_\nu\nabla_\lambda\nabla'^\lambda-2\xi\nabla^\mu\nabla_\nu+\tfrac{2}{3}\xi g^\mu_\nu\nabla_\lambda\nabla^\lambda\\&-\xi(R^\mu_\nu-\tfrac{1}{2}Rg^\mu_\nu+\tfrac{4}{3}\xi Rg^\mu_\nu)+(\tfrac{1}{2}-\tfrac{4}{3}\xi)m^2g^\mu_\nu]D_p(x,x')_{\rm ren} .\end{aligned} \tag{3.15}$$

For a massless field in flat space-time (3.15) reduces to

$$\langle T^\mu_\nu(x)\rangle=\lim_{x'\to x}[(1-2\xi)g^{\mu\lambda}\partial_\lambda\partial'_\nu-(\tfrac{1}{2}-2\xi)g^\mu_\nu g^{\lambda\alpha}\partial_\alpha\partial'_\lambda-2\xi g^{\mu\lambda}\nabla_\lambda\partial_\nu]D_p(x,x')_{\rm ren} . \tag{3.16}$$

The coincidence limits of the various derivatives involved in (3.16) can all be related to the two quantities $\lim_{\theta'\to\theta}(\partial^2/\partial\theta^2)D_p(\theta,\theta')_{\rm ren}$ and $\lim_{\theta'\to\theta}D_p(\theta,\theta')$ in the following manner [15]:

$$\lim_{\theta'\to\theta}\frac{1}{r^2}\frac{\partial^2}{\partial\theta\,\partial\theta'}D_p(\theta,\theta')_{\rm ren}=\lim_{\theta'\to\theta}-\frac{1}{r^2}\frac{\partial^2}{\partial\theta\,\partial\theta}D_p(\theta,\theta')_{\rm ren} ,$$

$$\lim_{x'\to x}\frac{\partial^2}{\partial r\,\partial r'}D_p(x,x')_{\rm ren}=\lim_{\theta'\to\theta}\frac{1}{2r^2}\left[\frac{3}{4}+\frac{\partial^2}{\partial\theta^2}\right]D_p(\theta,\theta')_{\rm ren} ,$$

$$\lim_{x'\to x}\frac{\partial^2}{\partial r^2}D_p(x,x')_{\rm ren}=\lim_{\theta'\to\theta}\frac{1}{2r^2}\left[\frac{5}{4}-\frac{\partial^2}{\partial\theta^2}\right]D_p(\theta,\theta')_{\rm ren} ,$$

$$\lim_{x'\to x}\frac{\partial^2}{\partial t^2}D_p(x,x')_{\rm ren}=\lim_{x'\to x}-\frac{\partial^2}{\partial t\,\partial t'}D_p(x,x')_{\rm ren}=\lim_{\theta'\to\theta}\frac{1}{2r^2}\left[\frac{1}{4}+\frac{\partial^2}{\partial\theta^2}\right]D_p(\theta,\theta')_{\rm ren} ,$$

and, using $\Box D(x,x')_{\rm ren}=0$,

$$\lim_{x'\to x}\frac{1}{r}\frac{\partial}{\partial r}D_p(x,x')_{\rm ren}=\lim_{\theta'\to\theta}-\frac{1}{2r^2}D_p(\theta,\theta')_{\rm ren} . \tag{3.17}$$

The two quantities $\lim_{\theta'\to\theta}D_p(\theta,\theta')_{\rm ren}$ and $\lim_{\theta'\to\theta}\partial^2D_p(\theta,\theta')_{\rm ren}/\partial\theta^2$ can be expressed in terms of finite integrals:

$$\begin{aligned}\lim_{\theta'\to\theta}D_p(\theta,\theta')_{\rm ren}&=\langle\phi^2(r)\rangle\\&=\frac{1}{4\pi^2r}\int_0^{\infty}\frac{du}{\sinh u}(p\coth pu-\coth u)\\&\equiv\frac{1}{8\pi r}s_1(p) ,\end{aligned} \tag{3.18}$$

$$\begin{aligned}\lim_{\theta'\to\theta}\frac{\partial^2}{\partial\theta^2}D_p(\theta,\theta')_{\rm ren}&=\frac{1}{8\pi^2r}\int_0^{\infty}\frac{du}{\sinh u}\left[\frac{\coth u}{\sinh^2u}-\frac{p^3\coth pu}{\sinh^2pu}\right]\\&\equiv\frac{1}{16\pi r}s(p) .\end{aligned} \tag{3.19}$$

Substituting relations (3.17) in (3.16) one obtains

$$\begin{aligned}\langle T^\mu_\nu(r)\rangle&=\frac{1}{2r^2}\left[\lim_{\theta'\to\theta}\frac{\partial^2}{\partial\theta^2}D_p(\theta,\theta')_{\rm ren}{\rm diag}(-1,-1,2)+(2\xi-\tfrac{1}{4})\lim_{\theta'\to\theta}D_p(\theta,\theta')_{\rm ren}{\rm diag}(-1,1,-2)\right]\\&=\frac{1}{32\pi r^3}[s(p){\rm diag}(-1,-1,2)+(4\xi-\tfrac{1}{2})s_1(p){\rm diag}(-1,1,-2)] .\end{aligned} \tag{3.20}$$

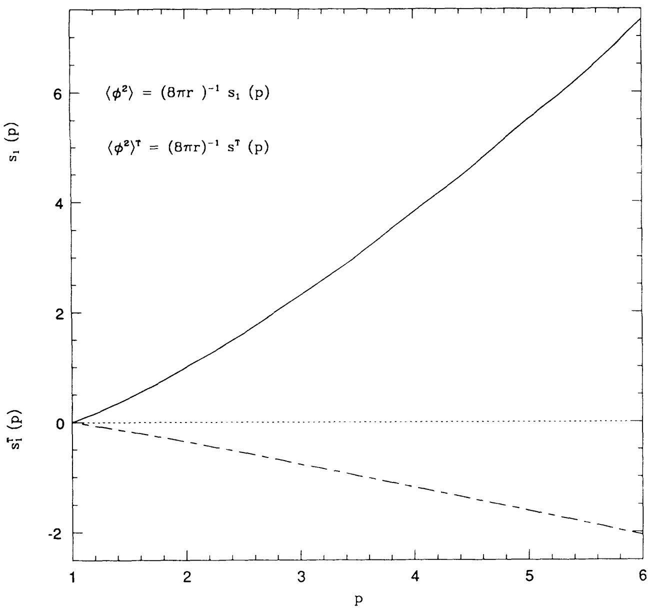

FIG. 2. The p dependence of $8\pi r\langle\phi^2(r)\rangle$ is shown for both twisted [$\equiv s_1^T(p)$] and untwisted fields [$\equiv s_1(p)$].

$s_1(p)=8\pi r\langle\phi^2\rangle$ and $s(p)=32\pi r^3\langle T_{00}\rangle_{\xi=1/8}$ are shown plotted against p in Figs. 2 and 3. The vacuum expectation value of the energy-momentum tensor so obtained satisfies the conservation equations $\langle T^{\mu}_{\nu}(x)\rangle_{;\mu}=0$ and is traceless for $\xi=\frac{1}{8}$.

It is interesting to note that as in the case of the electrostatic field, the method of images can once more be used for integer values of p to evaluate the two-point function $D_p(x,x')$. Using this approach, which is outlined in Appendix A, we find

$$D_p(x,x')_{\text{ren}}=\sum_{k=1}^{p-1} D_{\text{Mink}}(x,x_k') , \tag{3.21}$$

where $x\equiv(r,\theta,t)$, $x_k'\equiv(r',\theta'+2\pi k/p,t')$.

Using (3.16) we find that $\langle T^{\mu}_{\nu}(x)\rangle$ has precisely the same form as (3.20) with $s_1(p)$ and $s(p)$ now being the finite sums

$$s_1(p)=\sum_{k=1}^{p-1}\csc\frac{\pi k}{p} ,$$
$$s(p)=\sum_{k=1}^{p-1}\left[\csc^3\frac{\pi k}{p}-\frac{1}{2}\csc\frac{\pi k}{p}\right] . \tag{3.22}$$

The sums (3.22) can be obtained from the corresponding integrals for integer values of p by means of contour integration (see Appendix C).

It is also interesting to evaluate the total vacuum energy associated with a localized object of mass M:

$$\mathcal{E}=\int_0^{2\pi}\int_{M^{-1}}^{\infty}\langle T_{00}(r)\rangle r\,dr\,d\theta \tag{3.23a}$$

(where the lower limit M^{-1} has been imposed in order to make $\mathcal{E}$ a finite quantity); as a result,

$$\mathcal{E}=\frac{-M}{32\pi}[s(p)+(4\xi-\tfrac{1}{2})s_1(p)] , \tag{3.23b}$$

where $p=(1-4G_2M)^{-1}$. We find that for $\xi\geq 0,\mathcal{E}<0$. This is a consequence of the fact that for scalar fields with $\xi\geq 0$ the energy density $\langle T_{00}\rangle$ associated with the vacuum polarization is *always* negative.

IV. SEMICLASSICAL EINSTEIN GRAVITY IN 2+1 DIMENSIONS

Using the regularized vacuum expectation value for the energy-momentum tensor, obtained in the previous section, we can attempt to solve the semiclassical Einstein equations

$$G_{\mu\nu}=\kappa\langle T_{\mu\nu}\rangle \tag{4.1}$$

at a linearized level in order to obtain the first-order metric perturbation associated with the back reaction of the vacuum polarization $\langle T_{\mu\nu}\rangle$ on the space-time geometry [16,17]. We shall look for static solutions to (4.1).

As demonstrated in the previous section $\langle T^{\nu}_{\mu}\rangle$ has the form

$$\kappa\langle T^{\nu}_{\mu}(r)\rangle=\frac{A}{r^3}\text{diag}(-1,-1,2)+\frac{B}{r^3}\text{diag}(-1,1,-2) , \tag{4.2}$$

with $\kappa=2\pi G_2/c^4$, $A=(l_P/32\pi)s(p)$, $B=(l_P/32\pi)\times(4\xi-\frac{1}{2})s_1(p)$ ($l_P=G_2\hbar/c^3$ is the Planck length in 2+1 dimensions). Since $\kappa\langle T^{\nu}_{\mu}(r)\rangle$ is a function of r alone, one would expect the geometry of the perturbed metric to

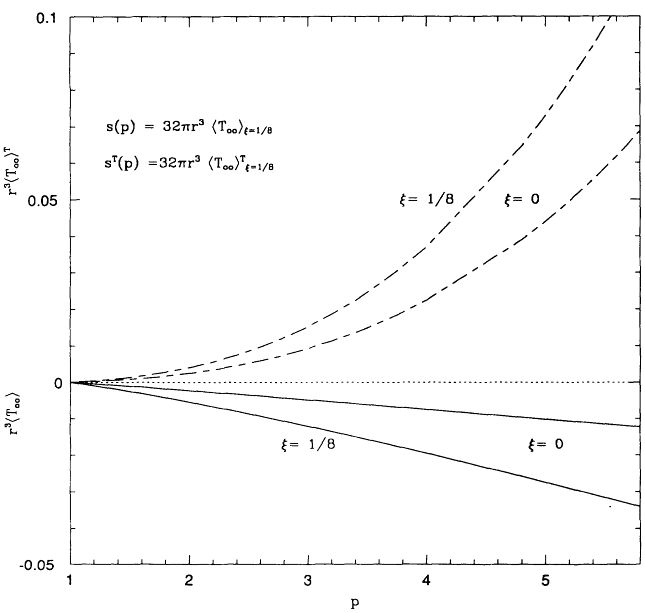

FIG. 3. The p dependence of the vacuum energy densities $r^3\langle T_{00}\rangle$ and $r^3\langle T_{00}\rangle^T$ [see (3.21) and (6.9)] is shown for untwisted (solid lines) and twisted scalar fields (dashed lines). Two values of the coupling parameter ξ are considered: minimal coupling ($\xi=0$) and conformal coupling ($\xi=\frac{1}{8}$).

respect axial symmetry. The most general form for such a metric is [1]

$$ds^2=e^{2\Phi(r)}(dt^2-dr^2)-e^{2\Psi(r)}d\theta^2 . \quad (4.3)$$

In the perturbative approach which we adopt, we shall expand the metric about the flat-space solution

$$ds^2=dt^2-dr^2-\left[\frac{r}{p}\right]^2 d\theta^2 , \quad (4.4)$$

so that each of $\Phi(r)$ and $\Psi(r)$ may be written as

$$\Phi(r)=\Phi_c+\phi(r)=\phi(r) ,$$
$$\Psi(r)=\Psi_c(r)+\psi(r)=\ln\frac{r}{p}+\psi(r) , \quad (4.5)$$

where $\Phi_c=1$, $\Psi_c(r)=\ln(r/p)$ are the lowest-order terms corresponding to the classical metric (4.4), and $\phi(r)$ and $\psi(r)$ are the first-order corrections in the Planck length l_P.

The Einstein equations (4.1), with $\langle T_{\mu\nu}\rangle$ given by (4.2), when linearized in $\phi(r)$ and $\psi(r)$ yield

$$\frac{d^2\psi}{dr^2}-\frac{1}{r}\frac{d\phi}{dr}+\frac{2}{r}\frac{d\psi}{dr}=\frac{2\pi}{r^3}(A+B) , \quad (4.6a)$$

$$\frac{1}{r}\frac{d\phi}{dr}=\frac{2\pi}{r^3}(A-B) , \quad (4.6b)$$

$$\frac{d^2\phi}{dr^2}=-\frac{4\pi}{r^3}(A-B) . \quad (4.6c)$$

(We shall now adopt the natural units $G_2=c=\hbar=1$; consequently, all length scales will be measured in units of l_P, the Planck length in 2+1 dimensions.) The functions $\phi(r)$ and $\psi(r)$ can be obtained by integrating (4.6), giving

$$\phi(r)=\frac{-2\pi}{r}(A-B)+k_1 , \quad (4.7a)$$

$$\psi(r)=\frac{-4\pi A}{r}\ln(r+1)+\frac{k_2}{r}+k_3 . \quad (4.7b)$$

In the above equations the constants of integration k_1 and k_3 must be set to zero since it is not possible to have them linear in l_P and dimensionless too. k_2 can also be set to zero since it reflects a scaling $r\rightarrow\alpha r$.

The line element of the metric (4.3) to first order in l_P now reads

$$ds^2=\left[1-\frac{2\pi}{r}(A-B)\right](dt^2-dr^2) -\frac{r^2}{p^2}\left[1-\frac{4\pi A}{r}\ln r\right]d\theta^2 . \quad (4.8)$$

The above approximation to the metric is valid so long as first-order corrections are small, i.e., when both $2\pi(A-B)/r$ and $4\pi A/r$ are small compared to unity.

Let us define a new radial coordinate $R(r)$ such that

$$dR=\left[1-\frac{2\pi}{r}(A-B)\right]^{1/2}dr\simeq\left[1-\frac{\pi}{r}(A-B)\right]dr . \quad (4.9)$$

Then at large distances from the point mass ($R\gg1$) the line element (4.8) can be rewritten as

$$ds^2=\left[1-\frac{2\pi(A-B)}{R}\right]dt^2-dR^2 -\frac{R^2}{p^2}\left[1-\frac{2\pi(A+B)}{R}\ln R\right]d\theta^2 . \quad (4.10)$$

One finds that, although the first-order metric (4.10) is no longer locally flat, its R-θ section is still conical, the deficit angle now depending upon R, the proper radial distance from the point mass. To quantitatively describe this behavior it is convenient to introduce the deficit angle $\Delta\theta=2\pi-C/R$, where C is the circumference of a circle centered around the point mass at a fixed proper radius R from it. Then, for the metric (4.10),

$$\Delta\theta(R)=2\pi\left[1-\frac{1}{p}\left[1-\frac{2\pi(A+B)}{R}\ln R\right]\right] , \quad (4.11)$$

or, in terms of the classical deficit angle $\Delta\theta_{\rm class}=2\pi-2\pi/p$,

$$\Delta\theta(R)=\Delta\theta_{\rm class}+\frac{2\pi}{p}\frac{A+B}{R}\ln R . \quad (4.12)$$

One finds that for negative values of the energy density ($A+B>0$), the deficit angle *increases* as the point mass is approached. For positive values of the energy density, however, the deficit angle is seen to *decrease* with R. At large distances from the point mass $\Delta\theta(R)\simeq\Delta\theta_{\rm class}$, and the local geometry of the space-time approaches the asymptotic form described by the classical metric (4.4).

An important consequence of the linearized metric (4.10) is the existence of a well-defined Newtonian limit to the semiclassical Einstein equations (4.1) in 2+1 dimensions. A given space-time geometry is usually said to admit a Newtonian limit if the time-time component of its metric tensor has the form [18]

$$g_{00}=\left[1+\frac{2\Phi(r)}{c^2}\right] . \quad (4.13)$$

$\Phi(r)$ then plays the role of the Newtonian potential and, in the slow motion limit, the acceleration of a test particle is determined by $d^2\mathbf{x}/dt^2=-\nabla\Phi$. The Einstein equations in the same limit assume the form

$$R_0^0=\Delta\Phi=4\pi G(T_0^0-\tfrac{1}{2}T) \quad (4.14)$$

($T_\mu^\nu\equiv\langle T_\mu^\nu\rangle$ in our case). From (4.10) and (4.13) we find that

$$\Phi(R)=-G_3\frac{\pi(A-B)}{R} , \quad (4.15)$$

where G_3 ($\equiv G_2 l_P$) is the Newtonian gravitational constant *in 3+1 dimensions.* For a conformally coupled field $B=0$ and $\Phi(R)=-G_3\pi A/R$. A plot (Fig. 4) of A against M, the mass of the point particle, shows that in a broad range of parameter space, A is approximately proportional to M.

The above results prompt us to define a *gravitating*

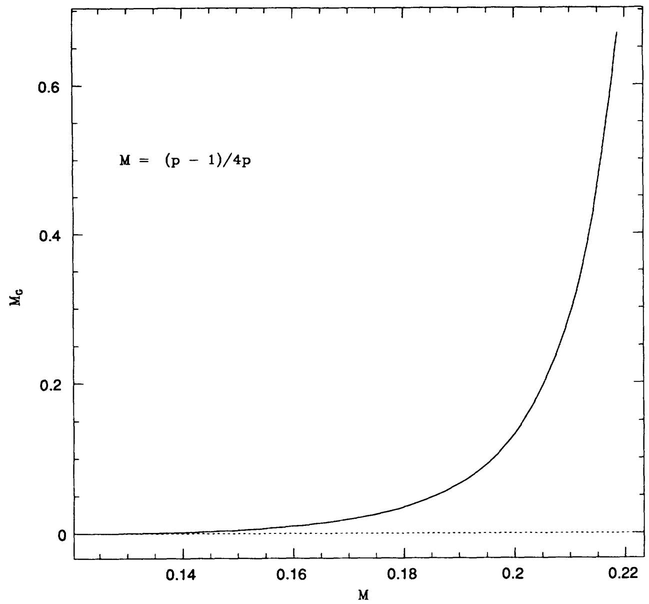

FIG. 4. The dependence of the gravitating mass M_G, defined in (4.16) for the conformally coupled case $\xi=\frac{1}{8}$, is plotted against the mass M of the point source. M is related to the deficit angle of the space-time through $\Delta\varphi_{\rm def}=8\pi M$. ($M_G$ and M are both expressed in units of the Planck mass.)

mass M_G:

$$M_G=\pi(A-B)=\frac{m_P}{32}[s(p)-(4\xi-\tfrac{1}{2})s_1(p)]\ , \qquad (4.16)$$

where $p=(1-4G_2M)^{-1}$, so that $\Phi=G_3M_G/R$ ($M_G>0$ for $\frac{1}{8}\geq\xi>0$).

Most of the results of the preceding analysis can be easily extended to other massless conformally coupled fields such as massless spinors and vectors. The conservation equation $\langle T^\mu_\nu\rangle_{;\mu}=0$ and the trace-free condition $\langle T^\mu_\mu\rangle=0$ in this case guarantee that the regularized vacuum expectation value of the energy-momentum tensor has the form

$$\langle T^\nu_\mu\rangle=\frac{A_\alpha(p)}{r^3}\text{diag}(-1,-1,2)\ . \qquad (4.17)$$

The numerical value of the constant A_α depends upon the contribution to the vacuum polarization from a quantum field having spin α. For conformally coupled scalars $A_0=s(p)/32\pi$, as demonstrated in the previous section. In general, when considering the net contribution to the vacuum polarization from fields with different spin, one would expect $A_\alpha(p)$ to be replaced by $\overline{A}(p)=\sum_\alpha n_\alpha A_\alpha(p)$, n_α being the number of spin-α fields present in nature. The Newtonian limit will then assume the somewhat more general form $\Phi(R)=-\pi G_3\overline{A}(p)/R$.

We should note that, strictly speaking, the potential $\Phi\sim R^{-1}$ corresponds to the Newtonian potential in 3+1 dimensions and not in 2+1 dimensions where the Newtonian potential has the form $\Phi\sim\ln r$. Thus a test particle near a point mass in 2+1 dimensions behaves just as if it were in the neighborhood of a gravitating mass in 3+1 dimensions—its motion is confined to a two-dimensional section of the 3D space.

The above results acquire a special significance in view of the fact that general relativity does not possess a Newtonian limit in 2+1 dimensions (a consequence of the lack of propagating modes in Einstein gravity in dimensions lower than 4). The attempt to construct alternate theories of gravitation which might have a well-defined Newtonian limit in lower dimensions has also proved to be very elusive [2–4,19]. For instance, the Einstein-Cotton equations (1.4), describing topologically massive gravity, do endow the gravitational field with a nontrivial dynamics but not with a Newtonian limit.

V. QUANTUM EFFECTS IN THE (2+1)-DIMENSIONAL DE SITTER–SCHWARZSCHILD METRIC

In this section we extend our previous analysis to nonstatic conical space-times such as the de Sitter–Schwarzschild metric, which describes the space-time of a point mass in the presence of a homogeneous cosmological constant Λ. The line element for this space-time has the form

$$ds^2=dt^2-e^{2Ht}\left[dr^2+\frac{r^2}{p^2}d\theta^2\right]\ , \qquad (5.1)$$

where $p=(1-4M)^{-1}$, $0\leq\theta<2\pi$, and $H=\sqrt{\Lambda/2}$.

For integer values of p the two-point function in this space can be expressed as a finite sum over the Green's function in de Sitter space $G(x,x')$, using the method of images described in Appendix A. As a result we get

$$G_p(x,x')=\sum_{k=0}^{p-1}G(x,x'_k)\ , \qquad (5.2)$$

where $x\equiv(r,t,\eta)$ and $x'_k\equiv(r',\theta'+2\pi k/p,\eta')$, η being the conformal time coordinate $\eta=\int dt\, e^{-Ht}$.

The de Sitter space propagator $G(\mathbf{x},\mathbf{x}')$ has been obtained in Appendix B for the general case of an n-dimensional space-time. In our treatment we shall set $n=3$ and regard the mass of the scalar field to be small ($m/H<1$). In terms of the conformal time, the scale factor assumes the form $a=-1/H\eta$; consequently, the proper distance to the point mass in (5.1) is given by $R=ar=-r/H\eta$. The $k=0$ term in (5.2) when suitably regularized and differentiated gives the one-loop vacuum energy-momentum tensor in (2+1)-dimensional de Sitter space. The additional contribution to the de Sitter propagator (B19) due to the conical nature of the space-time (5.1) is given by

$$G_p(x,x')_{\rm cone}=\frac{-H}{4\pi}\nu\csc\pi\nu\sum_{k=1}^{p-1}F(1+\nu,1-\nu;\tfrac{3}{2};\omega_k)\ , \qquad (5.3)$$

where

$$\nu=\left[1-\frac{m^2}{H^2}-6\xi\right]^{1/2}$$

and

$$\omega_k = 1 - \frac{r^2 + r'^2 - 2rr'\cos(\theta - \theta' + 2\pi k/p) - \Delta\eta^2}{4\eta\eta'} .$$

In obtaining (5.3) we have substituted the value of the de Sitter propagator (B19) into (5.2). Just as in Sec. III, we compute the expectation values $\langle \phi^2(x) \rangle$ and $\langle T^\mu_\nu(x) \rangle$ from the coincidence limits of $G_p(x,x')$ and its various derivatives. As a result

$$\langle \phi^2(x) \rangle_{\rm cone} = \lim_{x' \to x} G_p(x,x')_{\rm cone} = \frac{-H}{4\pi} \nu \csc\pi\nu \sum_{k=1}^{p-1} F \left[1+\nu, 1-\nu; \frac{3}{2}; 1 - \frac{r^2 \sin^2(\pi k/p)}{\eta^2} \right] . \tag{5.4a}$$

For conformally coupled massless scalar fields $\nu = \frac{1}{2}$, and we find that $\langle \phi^2(x) \rangle_{\rm cone}$ is conformally related to the flat space-time result obtained in (3.18):

$$\langle \phi^2(x) \rangle_{\rm cone} = -H\eta \langle \phi^2(x) \rangle_{\rm flat} = \frac{H}{8\pi R} s_1(p) , \tag{5.4b}$$

where $s_1(p) = \sum_{k=1}^{p-1} \csc(\pi k/p)$.

The vacuum expectation value of the energy-momentum tensor $\langle T^\mu_\nu(x) \rangle$ is obtained by means of the general relationship (3.15). After some lengthy calculations we obtain an expression for $\langle T^\mu_\nu(x) \rangle$ in terms of the coincidence limit of the hypergeometric function $F(1-\nu, 1+\nu; \frac{3}{2}, \omega_k)$ and its first and second derivatives with respect to ω_k. The derivatives of a hypergeometric function may be evaluated using [20]

$$\frac{d}{dz} F(a,b;c;z) = \left[\frac{ab}{c} \right] F(a+1, b+1; c+1; z) .$$

We find that for the case of a conformally coupled massless scalar field, $\langle T^\mu_\nu \rangle_{\rm cone}$ is conformally related to the flat space-time result:

$$\langle T^\mu_\nu(x) \rangle_{\rm cone} = (-H\eta)^3 \langle T^\mu_\nu(x) \rangle_{\rm flat} , \tag{5.5}$$

with $\langle T^\mu_\nu(x) \rangle_{\rm flat}$ given in (3.20).

For massless conformal fields $\langle T^\mu_\nu(x) \rangle_{\rm cone}$ as evaluated above provides the *entire* contribution to the vacuum expectation value of the energy-momentum tensor in the de Sitter–Schwarzschild metric. This is due to the fact that the $k=0$ term in (5.2) [which we had dropped while calculating $\langle T^\mu_\nu(x) \rangle_{\rm cone}$] when suitably differentiated and regularized gives the one-loop vacuum expectation value of the energy-momentum tensor in (2+1)-dimensional de Sitter space. Since the only maximally form-invariant rank-2 tensor under the de Sitter group is $g_{\mu\nu}$, the entire vacuum energy-momentum tensor in de Sitter space can be constructed out of its trace: $\langle T_{\mu\nu}(x) \rangle = g_{\mu\nu} \langle T \rangle / n$ (n being the dimensionality of the space-time). For massless conformally coupled fields in odd dimensions $\langle T \rangle = 0$ so that $\langle T^\mu_\nu(x) \rangle = 0$ in any odd-dimensional de Sitter space (Ref. [14], pp. 177 and 191). Consequently, while evaluating the energy-momentum tensor for conformally coupled fields, the $k=0$ term in (5.2) will not contribute and $\langle T^\mu_\nu(x) \rangle_{\rm ren} = \langle T^\mu_\nu(x) \rangle_{\rm cone}$.

For general values of m and ξ, $\langle \phi^2(x) \rangle$ and $\langle T^\mu_\nu(x) \rangle$ can be expressed in terms of elementary functions in the asymptotic regimes $R \ll H^{-1}$ and $R \gg H^{-1}$ (equivalently $r/\eta \ll 1$ and $r/\eta \gg 1$) using the well-known linear transformation formulas for the hypergeometric function [20].

As a result we find (for $R \ll H^{-1}$)

$$\langle \phi^2(x) \rangle_{\rm cone} = \frac{1}{4\pi R} s_1(p) . \tag{5.6}$$

The *diagonal* components of $\langle T^\mu_\nu(x) \rangle_{\rm cone}$ assume the form

$$\langle T^\mu_\nu(x) \rangle^{\rm diag}_{\rm cone} \approx \frac{1}{32\pi R^3} [s(p) {\rm diag}(-1,-1,2) + (4\xi - \tfrac{1}{2}) s_1(p) \times {\rm diag}(-1,1,-2)] , \tag{5.7}$$

where $s_1(p)$ and $s(p)$ are defined in (3.18) and (3.19). One immediately finds that $\langle \phi^2(x) \rangle_{\rm cone}$ and $\langle T^\mu_\nu(x) \rangle^{\rm diag}_{\rm cone}$ obtained in (5.6) and (5.7) are conformally related to the flat-space-time results (3.18) and (3.20) *for all values of* ξ. Both $\langle \phi^2 \rangle_{\rm cone}$ and $\langle T^\mu_\nu \rangle^{\rm diag}_{\rm cone}$ also seem to be independent of mass m in this limit. In addition to $\langle T^\mu_\nu(x) \rangle^{\rm diag}_{\rm cone}$ given by (5.7) there also exists an energy flux given by the off-diagonal component

$$\langle T^\eta_r \rangle_{\rm cone} \approx \frac{H}{16\pi R^2} (4\xi - \tfrac{1}{2}) s_1(p) . \tag{5.8}$$

The energy flux is a measure of the energy flowing away from the point source (it is absent in the case of a homogeneous and isotropic space-time such as the (2+1)-dimensional de Sitter metric). One can see from (5.8) that the direction of the energy flux is *outwards* from the origin for $\xi > \frac{1}{8}$ and *inwards* for $\xi < \frac{1}{8}$. It may be noted that the diagonal components $\langle T^\eta_r(x) \rangle_{\rm cone}$ are of one order higher in HR than $\langle T^\mu_\nu(x) \rangle^{\rm diag}_{\rm cone}$. [We feel that the results (5.6)–(5.8) will also remain valid for noninteger p if $s(p)$ and $s_1(p)$ are expressed as integrals using (3.21).]

In the other limiting case $R \gg H^{-1}$, $\langle T^\mu_\nu(x) \rangle_{\rm cone}$ can be obtained using the following transformation property of the hypergeometric function [20]:

$$F(a,b;c;\omega) = (1-\omega)^{-a} \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} \times F\left[a, c-b; a-b+1; \frac{1}{1-z} \right] + (1-\omega)^{-b} \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} \times F\left[b, c-a; b-a+1; \frac{1}{1-z} \right] . \tag{5.9}$$

We give below only the results for $\langle \phi^2 \rangle_{\rm cone}$ and $\langle T^\eta_r \rangle_{\rm cone}$ to leading orders in HR:

$$\langle \phi^2(x) \rangle_{\rm cone} = \frac{H^{2\nu-1}}{16\pi} \frac{2^{2\nu}}{\sin\pi\nu} R^{2\nu-2} \sum_{k=1}^{p-1} \sin^{2\nu-2}\frac{k\pi}{p} , \tag{5.10}$$

$$\langle T_r^\eta(x) \rangle_{\rm cone} = \frac{H^{2\nu}}{16\pi} \frac{2^{2\nu}(1-\nu)}{\sin\pi\nu} R^{2\nu-3} \times [(6\xi-1)-\nu(4\xi-1)] \times \sum_{k=1}^{p-1} \sin^{2\nu-2}\frac{\pi k}{p} . \tag{5.11}$$

The expressions in (5.10) and (5.11) are well defined only for $\nu<1$. For $\nu=1,2,\ldots,$ equations (5.10) and (5.11) display an infrared divergence which is similar to the infrared divergence found for the massless propagator in de Sitter space [21,22]. It may be noted that for $\nu<1$, $\langle T_r^\eta \rangle \to 0$ as $R\to\infty$, so that there is no radiation flux at infinity.

We would finally like to point out that although, as mentioned earlier in this section, vacuum polarization effects for conformally coupled scalar fields are absent in a (2+1)-dimensional de Sitter space-time, an Unruh-DeWitt particle detector nevertheless experiences a finite response. To see this we note that the response function of an Unruh-DeWitt particle detector is given by [23]

$$F(E) = \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' e^{-iE(\tau-\tau')} D(x(\tau),x(\tau')) , \tag{5.12}$$

where $D(x(\tau),x(\tau'))$ is the positive frequency Wightman Green's function evaluated for two points lying on the world line of the detector.

For (2+1)-dimensional de Sitter space the response rate for an inertial detector is given by

$$\frac{F(E)}{T} = \int_{-\infty}^{\infty} d(\Delta t) e^{-iE\Delta t} D(\Delta t) , \tag{5.13}$$

where, for a massless conformally coupled scalar field (see B17),

$$D(\Delta t) = \frac{H}{16\pi} \csc\frac{iH\Delta t}{2} . \tag{5.14a}$$

It is convenient to rewrite $D(\Delta t)$ as a sum [10]:

$$D(\Delta t) = \frac{H}{8\pi} \left[\frac{2}{iH\Delta t} + \frac{iH\Delta t}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{[(iH/2\pi)\Delta t + k][(iH/2\pi)\Delta t - k]} \right] . \tag{5.14b}$$

The integral in (5.13) can be evaluated by means of contour integration, so that finally

$$\frac{F(E)}{T} = \frac{1}{4} \frac{1}{e^{2\pi E/H}+1} , \tag{5.15}$$

which describes a thermal Fermi-Dirac distribution at the *de Sitter temperature* [14] $T=H/2\pi$.

Equation (5.15) points to a remarkable feature common to all odd-dimensional space-times—the inversion of statistics [24]. This can be viewed to be a consequence of the fact that the de Sitter Green's function in odd dimensions displays an antiperiodicity in imaginary time. One can see this in the general case by applying the transformation property of the hypergeometric function [20]

$$F(a,b;c;z) = (1-z)^{c-a-b} F(c-a,c-b;c;z) \tag{5.16}$$

to the n-dimensional Green's function (B15). As a result we obtain

$$D(\Delta t) = \csc^{n-2}\left[\frac{iH}{2}\Delta t\right] \frac{1}{(4\pi)^{n/2}} \left[\frac{H}{2}\right]^{n-2} \frac{\Gamma((n-1)/2+\nu)\Gamma((n-1)/2-\nu)}{\Gamma(n/2)} F\left[\frac{1}{2}-\nu, \frac{1}{2}+\nu; \frac{n}{2}; \cosh^2\frac{H\Delta t}{2}\right] , \tag{5.17}$$

where

$$\nu = \left[\frac{(n-1)^2}{4} - \frac{m^2}{H^2} - n(n-1)\xi\right]^{1/2} .$$

The above expression is manifestly antiperiodic in Δt with period $2\pi/H$ if n is odd, and periodic in Δt with the same period, if n is even. This leads us to conjecture that the response of a particle detector will be of the Fermi-Dirac type in odd space-time dimensions even for a massive, nonconformally coupled scalar field. (In an even-dimensional space-time $F(E)/T \propto (e^{2\pi E/H}-1)^{-1}$; for details regarding the behavior of particle detectors in space-times of arbitrary dimension see Takagi [24].)

The response of a particle detector placed at fixed distance R from the point mass in the conical space-time (5.1) will be modified to

$$\frac{F(E)}{T} = \frac{H}{8\pi} \int_{-\infty}^{\infty} d(\Delta t) e^{-iE\Delta t} \sum_{k=0}^{p-1} \frac{1}{[(1-H^2R^2)\sin^2(iH\Delta t/2) + H^2R^2\sin^2(\pi k/p)]^{1/2}} . \tag{5.18}$$

At distances close to the point mass (i.e., $R \ll H^{-1}$), (5.18) reduces to

$$\frac{F(E)}{T}=\frac{1}{4}\frac{p}{e^{2\pi E/H}+1}\ , \tag{5.19}$$

which is just the detector response in de Sitter space enhanced by a factor of p. This result could have been anticipated since the vacuum in de Sitter space has now effectively been compressed into a region of space p times smaller, due to the removal of the angular wedge from the space-time.

VI. TWISTED SCALAR FIELDS

As first pointed out by Isham [25] a twisted variety of scalar and spinor fields can be defined on a nonsimply connected manifold by considering antiperiodic boundary conditions along the identified coordinate (see also Ford [26]). We shall consider a massless, real, scalar field twisted around the mass point M located at $r=0$. The twisted field $\phi(x)$ obeys the same field equations as (3.1). However, the boundary conditions for the modes of ϕ are now different:

$$u_\lambda(r,\theta)=-u_\lambda\left[r,\theta+\frac{2\pi}{p}\right]. \tag{6.1}$$

Solving the field equation $\Box\phi=0$ with the new boundary conditions (6.1) we obtain

$$u_\lambda(x)=N_{lm}e^{ip(m+1/2)(\theta-\theta')}e^{-i\omega_l(t-t')}J_{p|m+1/2|}(\omega r)J_{p|m+1/2|}(\omega r')\ , \tag{6.2}$$

where N_{lm}, the normalization constant, is identical to the untwisted case (3.7). Using (3.8) and (3.9) to define the two-point function for the twisted field $D_p^T(x,x')$ we get

$$D_p^T(x,x')=\frac{p}{4\pi}\sum_{m=-\infty}^{\infty}e^{ip(m+1/2)(\theta-\theta')}\int_0^\infty d\omega\, e^{-i\omega_l(t-t')}J_{p|m+1/2|}(\omega r)J_{p|m+1/2|}(\omega r')\ . \tag{6.3}$$

The integration over ω is similar to the untwisted case (3.11a), and making identical substitutions as in that case we obtain

$$D_p^T(x,x')=\frac{p}{4\pi^2}\frac{1}{\sqrt{2rr'}}\int_{u_0}^\infty\frac{du}{(\cosh u-\cosh u_0)^{1/2}}\sum_{m=-\infty}^{\infty}\exp[ip(m+\tfrac{1}{2})(\theta-\theta')-p|m+\tfrac{1}{2}|u]\ . \tag{6.4}$$

The above summation can be rewritten in a closed form,

$$\sum_{m=-\infty}^{\infty}\exp[-p|m+\tfrac{1}{2}|u+ip(m+\tfrac{1}{2})(\theta-\theta')]=\frac{2\sinh(\tfrac{1}{2}pu)\cos\tfrac{1}{2}p(\theta-\theta')}{\cosh pu-\cos p(\theta-\theta')}\ ,$$

so that the two-point function $D_p^T(x,x')$ reduces to

$$D_p^T(x,x')=\frac{p}{2\pi^2}\frac{1}{\sqrt{2rr'}}\int_{u_0}^\infty\frac{du}{(\cosh u-\cosh u_0)^{1/2}}\frac{\sinh(\tfrac{1}{2}pu)\cos\tfrac{1}{2}p(\theta-\theta')}{\cosh pu-\cos p(\theta-\theta')}\ . \tag{6.5}$$

We now adopt the same procedure in renormalizing $D_p^T(x,x')$ as was used in Sec. III; namely, we subtract out the $p=1$ term in (6.5) so that the renormalized two-point function becomes

$$\begin{aligned}D_p^T(x,x')_{\rm ren}&=D_p^T(x,x')-D_1^T(x,x')\\&=\frac{1}{2\pi^2\sqrt{2rr'}}\int_{u_0}^\infty\frac{du}{(\cosh u-\cosh u_0)^{1/2}}\left[\frac{p\sinh\tfrac{1}{2}pu\,\cos\tfrac{1}{2}p\Delta\theta}{\cosh pu-\cos p\Delta\theta}-\frac{\sinh\tfrac{1}{2}u\cos\tfrac{1}{2}\Delta\theta}{\cosh u-\cos\Delta\theta}\right].\end{aligned} \tag{6.6}$$

As in the case of untwisted fields, the vacuum expectation values $\langle\phi^2(x)\rangle^T$ and $\langle T^\mu_\nu(x)\rangle^T$ can be obtained from the coincidence limit of the renormalized two-point function $D_p^T(x,x')$ and its second derivative with respect to θ. As a result

$$\langle\phi^2(x)\rangle^T=\lim_{\theta'\to\theta}D_p^T(\theta,\theta')_{\rm ren}=\frac{1}{8\pi r}s_1^T(p)\ , \tag{6.7a}$$

where

$$s_1^T(p)=\frac{2}{\pi}\int_0^\infty\frac{du}{\sinh u}(p\,\mathrm{csch}\,pu-\mathrm{csch}\,u) \tag{6.7b}$$

and

$$\lim_{\theta'\to\theta}\frac{\partial^2}{\partial\theta^2}D_p^T(\theta,\theta')_{\rm ren}=\frac{1}{8\pi r}s^T(p)\ , \tag{6.8a}$$

where

$$s^T(p)=\frac{2}{\pi}\int_0^\infty \frac{du}{\sinh u}\left[p^3\left[\operatorname{csch}^3 pu+\frac{\operatorname{csch} pu}{2}\right]-\left[\operatorname{csch}^3 u+\frac{\operatorname{csch} u}{2}\right]\right] . \tag{6.8b}$$

Using arguments akin to the ones used in Sec. III while evaluating $\langle T^\mu_\nu \rangle$ [see (3.16), (3.17), and (3.20)] we finally obtain

$$\begin{aligned}\langle T^\mu_\nu \rangle^T &= \frac{1}{2r^2}\left[\lim_{\theta'\to\theta}\frac{\partial^2}{\partial\theta^2}D^T_p(\theta,\theta')_{\rm ren}\mathrm{diag}(-1,-1,2)+(2\xi-\tfrac{1}{4})\lim_{\theta'\to\theta}D^T_p(\theta,\theta')_{\rm ren}\mathrm{diag}(-1,1,-2)\right]\\ &=\frac{1}{32\pi r^3}[s^T(p)\mathrm{diag}(-1,-1,2)+(4\xi-\tfrac{1}{2})s^T_1(p)\mathrm{diag}(-1,1,-2)] ,\end{aligned} \tag{6.9}$$

which is precisely (3.20) with $s(p)$ replaced by $s^T(p)$ and $s_1(p)$ replaced by $s^T_1(p)$. $s^T_1(p)=8\pi r\langle\phi^2\rangle^T$ and $s^T(p)=32\pi r^3\langle T_{00}\rangle_{\xi=1/8}$ are shown plotted against p in Figs. 2 and 3. It is interesting to note that $\langle T_{00}\rangle^T>0$ for massless conformally and minimally coupled twisted fields in contrast with the untwisted case.

As in the untwisted case, the vacuum energy-momentum tensor will in general back react on the space-time geometry via the semiclassical Einstein equations $G_{\mu\nu}=8\pi G_2\langle T_{\mu\nu}\rangle^T$, giving rise to the linearized metric

$$ds^2=\left[1-\frac{2\pi(A^T-B^T)}{R}\right]dt^2-dR^2-\frac{R^2}{p^2}\left[1-\frac{2\pi(A^T+B^T)}{R}\ln R\right]d\theta^2 , \tag{6.10}$$

where $A^T=(l_P/32\pi)s^T(p)$ and $B^T=(l_P/32\pi)(4\xi-\frac{1}{2})s^T_1(p)$ (l_P being the Planck length). From Fig. 3 we see that for $0\le\xi<\frac{1}{8}$, $A^T-B^T<0$ and $A^T+B^T<0$, so that the deficit angle in (6.10) *decreases* as the point mass is approached, in contrast with the untwisted case.

As in the case of untwisted fields the method of images can also be used to determine $D^T_p(x,x')$, with p now restricted to even integer values $p=2,4,6,\ldots$ (see Appendix A), so that

$$D^T_p(x,x')=\sum_{k=0}^{p-1}(-1)^k D_{\rm Mink}(x,x'_k) . \tag{6.11}$$

Proceeding as in Sec. III and regularizing (6.11) by subtracting out $D^T_1(x,x')$ we find that the final form of $\langle\phi^2(x)\rangle^T$ is given once more by (6.7a) with $s^T_1(p)$ now being the finite sum $s^T_1(p)=\sum_{k=0}^{p-1}(-1)^k\csc(\pi k/p)+2/\pi$.

The method of images can also be applied to obtain the corrections to the propagator for twisted fields in the de Sitter–Schwarzschild metric discussed in Sec. V. Following the procedure outlined in Appendix A we find

$$G^T_p(x,x')_{\rm cone}=\sum_{k=1}^{p-1}(-1)^k G(x,x'_k)=\frac{-H}{4\pi}\nu\csc\pi\nu\sum_{k=1}^{p-1}(-1)^k F\left[1+\nu,1-\nu;\frac{3}{2};1-\frac{\Delta x^2_\mu-\Delta\eta^2}{4\eta\eta'}\right] , \tag{6.12}$$

where $\Delta x^2_k=r^2+r'^2-2rr'\cos(\theta-\theta'+2\pi k/p)$, $H=\sqrt{\Lambda/2}$, and $\nu=(1-m^2/H^2-6\xi)^{1/2}$. For a massless, conformally coupled twisted scalar field $\nu=\frac{1}{2}$, and $\langle\phi^2(x)\rangle^T$ will simply be conformally related to the flat-space-time result (6.7) so that

$$\langle\phi^2(x)\rangle^T_{\rm cone}=\frac{1}{8\pi R}s^T_1(p) , \tag{6.13}$$

where $R=-r/H\eta$. This result is also true for light scalars ($m/H<1$) in the vicinity of the point mass ($R\ll H^{-1}$).

In addition, we also find, as in the untwisted case, the existence of an energy flux

$$\langle T^\eta_r(x)\rangle^T_{\rm cone}=\frac{H}{16\pi R^2}(4\xi-\tfrac{1}{2})s^T_1(p) , \tag{6.14}$$

which vanishes for conformally coupled fields ($\xi=\frac{1}{8}$). We notice that $\langle T^\eta_r\rangle_{\rm cone}$ has the opposite sign to (6.8), signifying that in an expanding universe, if the vacuum flux for untwisted fields is *outwards*, then the corresponding flux for twisted fields is *inwards* (and vice versa).

VII. CONCLUSIONS AND DISCUSSION

We have shown how nontrivial boundary conditions can affect the behavior of scalar fields at both the classical and quantum levels. At the classical level we find that the electric field associated with a point charge in a conical space-time is distorted, leading the charge to experience a self-force, in the complete absence of any other charges in the space-time. At a more fundamental level we find that the point mass, the source of the conical geometry, also induces a vacuum polarization in the surrounding space-time described by a finite vacuum expectation value of the energy-momentum tensor $\langle T_{\mu\nu}\rangle\propto 1/r^3$ (r being the distance to the point mass). Quantum effects of a similar nature are also known to arise in the space-time of a cosmic string, where the smallness of the string tension, $G\mu<10^{-6}$, prevents the effects from becoming large. No such constraint is present, however, in 2+1 dimensions, with the result that quantum effects can be significant in this case. We have also extended our analysis to an expanding universe by considering the behavior of $\langle\phi^2\rangle$ and $\langle T_{\mu\nu}\rangle$ in a conical de Sitter space-time. We find that in this case, in addition to the vacuum polarization, there exists a finite vacuum energy flux $\langle T_{rt} \rangle$ that describes a net flow of energy *away* from the point source.

As part of our analysis we also evaluate the scalar field propagator in an n-dimensional de Sitter space. Extending previous work by Takagi [24], we show that if n is even the propagator exhibits a periodicity in imaginary time with the period $T=2\pi/H$ (H being the Hubble constant in de Sitter space). On the other hand, for odd n the propagator displays an antiperiodicity in imaginary time. This leads us to conjecture that the vacuum for scalar fields in even- (odd-) dimensional de Sitter space resembles a thermal Bose (Fermi) distribution. We are also faced with the surprising result that an inertial particle detector registers a finite thermal response in de Sitter space of odd dimensions [24], even though the vacuum expectation value of the energy-momentum tensor for conformal fields vanishes identically in this case.

An important outcome of our analysis is that, at the semiclassical level, solutions to the Einstein equations $G_{\mu\nu}=(8\pi G_2/c^4)\langle T_{\mu\nu}\rangle$ possess a well-defined Newtonian limit. This result is significant since it is well known that the classical equations of general relativity do not have a Newtonian limit either in $2+1$ or in $1+1$ dimensions. We would like to point out that the existence of a Newtonian limit to the semiclassical Einstein equations is not a unique feature of $2+1$ dimensions, but extends to other space-time dimensions as well. For instance, it is well known that in $1+1$ dimensions the Einstein action is a topological invariant and consequently has no dynamical content. However, the semiclassical Einstein equations now give [27] $0=8\pi G_1(T+\langle T\rangle)$. For conformally invariant fields, $\langle T\rangle$ is given by the trace anomaly, which in $1+1$ dimensions is simply proportional to the Ricci scalar (Ref. [14], p. 178). As a result the semiclassical Einstein equations yield

$$R=8\pi\bar{G}_1 T\ , \tag{7.1}$$

which has both dynamical content and a Newtonian limit [28].

We would finally like to mention that although our analysis in this paper has regarded point sources to be fixed, it might be of interest to extend the present approach to moving sources as well. This case clearly bears a close resemblance to moving mirrors—in both cases boundary conditions, instead of remaining fixed, are functions of both space and time. Consequently, as in the case of moving mirrors one might expect particle creation effects to be present, modifying the motion of the point mass and leading to an increase in the entropy of the $(2+1)$-dimensional universe.

It would also be of interest to extend the treatment given in this paper to $(2+1)$-dimensional space-times containing several point masses. Such a space-time is well described by the static multicenter metric [29]

$$ds^2=dt^2-\left[\prod_i |r-r_i|^{-8Gm_i}\right](dr^2+r^2d\theta^2)\ , \tag{7.2}$$

in which particles of mass m_i are located at r_i. [For a single point mass (7.2) can easily be brought to the form (1.5) by a change of coordinates.] For two masses the problem reduces to one of finding the Casimir force between two cones. (In $3+1$ dimensions, the corresponding problem would be one of determining the Casimir force between two cosmic strings.) This problem bears a close affinity to that of the Casimir force between two wedges [30] and is presently being studied.

Note added in proof: It is interesting to note that the space-time associated with a plane domain wall has the form [7]

$$ds^2=(1-\kappa z)^2dt^2-dz^2 -(1-\kappa z)^2\exp(2\kappa t)(dx^2+dy^2)\ ,$$

where $\kappa=2\pi G\sigma$ is the surface tension of the wall.

In the plane of the wall ($z=0$), this metric reduces to that of a $(2+1)$-dimensional de Sitter space. It has been recently shown that a vacuum bubble after nucleation also has the internal geometry of a $(2+1)$-dimensional de Sitter space, and that perturbations on it can be described by a scalar field with a tachyonic mass [32]. Consequently the analysis of Sec. V, is relevant to the study of both classical and quantum fluctuations on domain walls and vacuum bubbles. [A conical $(2+1)$-dimensional de Sitter space, of the kind considered in Sec. V, would describe the metric on a domain wall pierced by a cosmic string.] From the form of the domain wall metric described above and the phenomenon of the inversion of statistics in odd-dimensional space-times discussed in Sec. V and [24], it follows that a comoving particle detector registering scalar particles and confined to move in the $z=0$ plane will register a thermal Fermi-Dirac response, characterized by a temperature $T=\kappa/2\pi$. On the other hand, since the (z,t) part of the above metric describes a $(1+1)$-dimensional Rindler space, a comoving particle detector outside of the wall will, for the same scalar particles, register a Bose-Einstein distribution at an identical temperature. Thus, depending upon its trajectory, a comoving particle detector in the space-time of a domain wall seems to register either a Fermi-Dirac or a Bose-Einstein distribution of particles.

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APPENDIX A: THE METHOD OF IMAGES

In Sec. II we made the assertion

$$G_p(r,\theta;r',\theta')=\sum_{k=0}^{p-1} G_1\left[r,\theta;r',\theta'+\frac{2\pi k}{p}\right] \tag{A1}$$

[see (2.10)], where

$$G_p(x,x')=\frac{1}{4\pi}\sum_{\substack{m=-\infty\\(m\neq 0)}}\frac{X^{p|m|}}{|m|}e^{ipm(\theta-\theta')}-\frac{p}{2\pi}\ln r' \tag{A2}$$

and $X=r/r'<1$.

In order to prove (A1) it is sufficient to establish that

$$\sum_{\substack{m=-\infty\\(m\neq 0)}} \frac{X^{p|m|}}{|m|} e^{ipm(\theta-\theta')} = -\sum_{k=0}^{p-1} \ln\left[1+X^2-2X\cos\left[\theta-\theta'+\frac{2\pi k}{p}\right]\right] . \tag{A3}$$

To do this we rewrite the right-hand side (RHS) of (A3) using the expansion [10]

$$-\ln\left[1+X^2-2X\cos\left[\Delta\theta+\frac{2\pi k}{p}\right]\right] = \sum_{\substack{m=-\infty\\m\neq 0}} \frac{X^{|m|}}{|m|}\exp\left[im\left[\Delta\theta+\frac{2\pi k}{p}\right]\right] . \tag{A4}$$

Then

$$-\sum_{k=0}^{p-1} \ln\left[1+X^2-2X\cos\left[\Delta\theta+\frac{2\pi k}{p}\right]\right] = \sum_{\substack{m=-\infty\\m\neq 0}} \frac{X^{|m|}}{|m|}\sum_{k=0}^{p-1}\exp\left[im\left[\Delta\theta+\frac{2\pi k}{p}\right]\right] . \tag{A5}$$

Since

$$\sum_{k=0}^{p-1} e^{im(2\pi k/p)} = p\sum_{n=-\infty}^{\infty} \delta_{m,np} , \tag{A6}$$

substituting (A6) in (A5) we get [$m\neq 0$ is assumed in (A7) and (A8)]

$$\sum_{m=-\infty}^{\infty} \frac{X^{|m|}}{|m|}\sum_{k=0}^{p-1}\exp\left[im\left[\Delta\theta+\frac{2\pi k}{p}\right]\right] = p\sum_{m=-\infty}^{\infty} \frac{X^{|m|}}{|m|} e^{im\Delta\theta}\sum_{n=-\infty}^{\infty}\delta_{m,np} \tag{A7}$$

$$= \sum_{n=-\infty}^{\infty} \frac{X^{p|n|}}{|n|} e^{ipn\Delta\theta} , \tag{A8}$$

which is the LHS of (A3).

Having established (A3), we note that $G_p(x,x')$ as defined in (A2) can be written as

$$G_p(r,\theta;r',\theta') = -\frac{1}{4\pi}\sum_{k=0}^{p-1} \ln\left[1+X^2-2X\cos\left[\Delta\theta+\frac{2\pi k}{p}\right]\right] - \frac{p}{2\pi}\ln r'$$
$$= -\frac{1}{2\pi}\sum_{k=0}^{p-1} \ln\left[r^2+r'^2-2rr'\cos\left[\Delta\theta+\frac{2\pi k}{p}\right]\right]^{1/2} = \sum_{k=0}^{p-1} G_1\left[r,\theta;r',\theta'+\frac{2\pi k}{p}\right] , \tag{A9}$$

which is what we set out to establish. [Equivalently $G_p^{\rm reg}(x,x')=G_p-G_1=\sum_{k=1}^{p-1}G_1(r,\theta,r',\theta'+2\pi k/p)$.]

Similarly the two-point functions of Secs. III and VI can also be obtained according to the method of images:

$$D_p(x,x') = \sum_{k=0}^{p-1} D_{\rm Mink}(x,x_k') \tag{A10}$$

($p=1,2,\ldots$) for a field satisfying periodic boundary conditions in θ, and

$$D_p^T(x,x') = \sum_{k=0}^{p-1} (-1)^k D_{\rm Mink}(x,x_k') \tag{A11}$$

($p=2,4,6,\ldots$) for a twisted field satisfying antiperiodic boundary conditions in θ. Here $x\equiv(t,r,\theta)$, $x_k'\equiv(t',r',\theta'+2\pi k/p)$, and for flat space,

$$D_{\rm Mink}(x,x_k') = \frac{1}{4\pi\sigma_k} , \tag{A12}$$

where

$$\sigma_k = |x-x_k'| = \left[r^2+r'^2-2rr'\cos\left[\Delta\theta+\frac{2\pi k}{p}\right]-\Delta t^2\right]^{1/2} .$$

To prove (A10) and (A11) we note that, from (3.10) and (6.3),

$$D_p(x,x') = \frac{p}{4\pi}\int_0^\infty d\omega\, e^{-i\omega(t-t')}\sum_{m=-\infty}^{\infty} e^{ipm(\theta-\theta')} J_{p|m|}(\omega r) J_{p|m|}(\omega r') \tag{A13}$$

and

$$D_p^T(x,x') = \frac{p}{4\pi}\int_0^\infty d\omega\, e^{-i\omega(t-t')}\sum_{m=-\infty}^{\infty} e^{ip(m+1/2)(\theta-\theta')} J_{p|m+1/2|}(\omega r) J_{p|m+1/2|}(\omega r') . \tag{A14}$$

Equations (A10) and (A11) are established immediately if we use the generalized laws of addition for Bessel functions (Davies and Sahni [9]):

$$p\sum_{m=-\infty}^{\infty} J_{p|m|}(\omega r)J_{p|m|}(\omega r')e^{ipm\Delta\theta}=\sum_{k=0}^{p-1} J_0(\omega r_k)\ ,$$
$$r_k=\left[r^2+r'^2-2rr'\cos\left[\theta-\theta'+\frac{2\pi k}{p}\right]\right]^{1/2} \tag{A15}$$

$(p=1,2,\ldots)$ for normal modes, and

$$p\sum_{m=-\infty}^{\infty} J_{p|m+1/2|}(\omega r)J_{p|m+1/2|}(\omega r')e^{ip(m+1/2)\Delta\theta}=\sum_{k=0}^{p-1}(-1)^k J_0(\omega r_k) \tag{A16}$$

$(p=2,4,6,\ldots)$ for twisted modes.

To establish (A15) we use the standard summation formula for Bessel functions [10]:

$$J_0(\omega r_0)=\sum_{l=-\infty}^{\infty} J_{|l|}(\omega r)J_{|l|}(\omega r')e^{il\Delta\theta}\ . \tag{A17}$$

Substituting (A17) into the right-hand side of (A15) we obtain

$$\sum_{k=0}^{p-1} J_0(\omega r_k)=\sum_{l=-\infty}^{\infty} J_{|l|}(\omega r)J_{|l|}(\omega r')e^{il\Delta\theta}\sum_{k=0}^{p-1} e^{il2\pi k/p}\ . \tag{A18}$$

Again, since

$$\sum_{k=0}^{p-1} e^{il2\pi k/p}=p\sum_{l=-\infty}^{\infty}\delta_{l,mp}\ , \tag{A19}$$

(A18) reduces to

$$\sum_{k=0}^{p-1} J_0(\omega r_k)=p\sum_{l=-\infty}^{\infty} J_{p|m|}(\omega r)J_{p|m|}(\omega r')e^{ipm\Delta\theta}\ , \tag{A20}$$

which is what we set out to establish.

Similarly substituting (A17) into the right-hand side of (A16) we get

$$\sum_{k=0}^{p-1}(-1)^k J_0(\omega r_k)=\sum_{l=-\infty}^{\infty} J_{|l|}(\omega r)J_{|l|}(\omega r')e^{il\Delta\theta}\times\sum_{k=0}^{p-1}(-1)^k e^{il2\pi k/p}\ . \tag{A21}$$

Again noting that, for $p=2,4,6,\ldots,$

$$\sum_{k=0}^{p-1} e^{i\pi k(1+2l/p)}=p\sum_{m=-\infty}^{\infty}\delta_{l,(m+1/2)p}\ , \tag{A22}$$

and substituting (A22) in (A21), we obtain (A16):

$$\sum_{k=0}^{p-1}(-1)^k J_0(\omega r_k)=p\sum_{m=-\infty}^{\infty} J_{p|m+1/2|}(\omega r)J_{p|m+1/2|}(\omega r')\times e^{ip(m+1/2)\Delta\theta}\ . \tag{A23}$$

Finally we would like to point out that the method of images retains its validity for nonstatic space-times possessing a symmetry axis, such as the metric

$$ds^2=a^2(\eta)\left[d\eta^2-dr^2-\frac{r^2}{p^2}d\theta^2\right]\ . \tag{A24}$$

The Green's function in (A24) has the form (for normal modes)

$$D_p(x,x')=\frac{p}{4\pi}\frac{1}{\sqrt{a(\eta)a(\eta')}}\int_0^{\infty}d\omega\,\chi_\omega(\eta)\chi_\omega^*(\eta')\sum_{m=-\infty}^{\infty}e^{ipm(\theta-\theta')}J_{p|m|}(\omega r)J_{p|m|}(\omega r')\ , \tag{A25}$$

where $\chi_\omega(\eta)$ satisfies the time component of the Klein-Gordon equation [14]

$$\frac{d^2\chi_\omega}{d\eta^2}+\{\omega^2+a^2(\eta)[m^2+(\xi-\tfrac{1}{8})R(\eta)]\}\chi_\omega=0 \tag{A26}$$

and the normalization condition $\chi_\omega\partial_\eta\chi_\omega^*-\chi_\omega^*\partial_\eta\chi_\omega=i$. $R(\eta)$ is the scalar curvature for the space-time (A23), and η is the conformal time $\eta=\int dt/a$.

Similarly, for twisted modes $(p=2,4,6,\ldots)$,

$$D_p^T(x,x')=\frac{p}{4\pi}\frac{1}{\sqrt{a(\eta)a(\eta')}}\int_0^{\infty}d\omega\,\chi_\omega(\eta)\chi_\omega^*(\eta')J_{p|m+1/2|}(\omega r)J_{p|m+1/2|}(\omega r')\sum_{m=-\infty}^{\infty}e^{ip(m+1/2)\Delta\theta}\ . \tag{A27}$$

Clearly the proof of the image formulas for the flat-space Green's function [(A13) and (A14)] can be extended to this case also, since the essential element in the proof, the generalized summation formulas for Bessel functions [(A15) and (A16)], can with equal validity be applied to (A25) and (A27), resulting in

$$D_p(x,x')=\frac{p}{4\pi}\frac{1}{\sqrt{a(\eta)a(\eta')}}\times\sum_{k=0}^{p-1}\int d\omega\,\chi_\omega(\eta)\chi_\omega^*(\eta')J_0(\omega r_k)=\sum_{k=0}^{p-1}D_1(x,x_k')\ , \tag{A28a}$$

and

$$D_p^T(x,x')=\frac{p}{4\pi}\frac{1}{\sqrt{a(\eta)a(\eta')}}\times\sum_{k=0}^{p-1}(-1)^k\int d\omega\,\chi_\omega(\eta)\chi_\omega^*(\eta')J_0(\omega r_k)=\sum_{k=0}^{p-1}(-1)^k D_1(x,x_k')\ . \tag{A28b}$$

APPENDIX B: THE TWO-POINT FUNCTION IN *n*-DIMENSIONAL DE SITTER SPACE

The n-dimensional spatially flat Robertson-Walker line element has the form

$$ds^2=dt^2-a^2(t)dl^2=a^2(\eta)(d\eta^2-dl^2)\ , \quad \text{(B1)}$$

where $dl^2=\sum_{i=1}^{n-1}dx_i^2$ and η is the conformal time coordinate $\eta=\int dt/a(t)$. In the case of de Sitter space $a(\eta)=-1/H\eta$, $-\infty\le\eta<0$, where H is the Hubble parameter, $H=\sqrt{\Lambda/(n-1)}$, and Λ is the n-dimensional cosmological constant. A massive, real scalar field in (B1) satisfies the n-dimensional Klein-Gordon equation

$$[\Box+m^2+\xi R(x)]\phi(x)=0\ , \quad \text{(B2)}$$

where $\Box=(1/\sqrt{-g})\partial_\mu(g^{\mu\nu}\sqrt{-g}\,\partial_\nu)$.

For purposes of quantization, the scalar field ϕ may be treated as an operator and decomposed into modes so that

$$\hat\phi(x)=\int d^{n-1}k[\hat a_k u_k(x)+\hat a_k^\dagger u_k^*(x)]\ , \quad \text{(B3)}$$

where $\hat a_k$ and $\hat a_k^\dagger$ are the $(n-1)$-dimensional annihilation and creation operators. $u_k(x)$ can be written as [14]

$$u_k(x)=(2\pi)^{(1-n)/2}a(\eta)^{(2-n)/2}e^{ik_ix^i}\chi_k(\eta)\ , \quad \text{(B4)}$$

where $k=(\sum_{i=1}^{n-1}k_i^2)^{1/2}$ and $\chi_k(\eta)$ satisfies

$$\frac{d^2}{d\eta^2}\chi_k(\eta)+(k^2+a^2(\eta)\{m^2+[\xi-\xi(n)]R(\eta)\})\times\chi_k(\eta)=0\ , \quad \text{(B5)}$$

where $\xi(n)=(n-2)/4(n-1)$ is the conformal coupling factor in n dimensions and $R(\eta)$ is the scalar curvature of the space-time.

In de Sitter space $R=n(n-1)H^2=\text{const}$, and (B5) may be solved exactly to obtain

$$\chi_k(\eta)=\tfrac{1}{2}(\pi\eta)^{1/2}H_\nu^{(2)}(k\eta)\ . \quad \text{(B6)}$$

$\chi_k(\eta)$ are positive frequency solutions of (A5) normalized according to the Wronskian condition

$$\chi_k(\eta)\frac{\partial}{\partial\eta}\chi_k^*(\eta)-\chi_k^*(\eta)\frac{\partial}{\partial\eta}\chi_k(\eta)=i\ , \quad \text{(B7)}$$

and $\nu=[(n-1)^2/4-m^2/H^2-n(n-1)\xi]^{1/2}$.

The complete mode functions are now

$$u_k(x)=2^{-3/2}(2\pi)^{(2-n)/2}(-H\eta)^{(n-2)/2}\times e^{ik_ix^i}H_\nu^{(2)}(k\eta)\ . \quad \text{(B8)}$$

The two-point function $G(x,x')$ can be obtained using a mode-sum approach [14,31]:

$$G(x,x')=\int d^{n-1}k\ u_k(x)u_k^*(x')\ , \quad \text{(B9)}$$

which in this case leads to

$$G(x,x')=\frac{1}{8}\left[-\frac{H}{2\pi}\right]^{n-2}(\eta\eta')^{(n-1)/2}\times\int d^{n-1}k\ H_\nu^{(1)}(k\eta)H_\nu^{(2)}(k\eta')e^{ik_i\Delta x^i}\ , \quad \text{(B10)}$$

where $\Delta x^i=x^i-x'^i$.

The $d^{n-1}k$ integration is carried out in polar coordinates using the following expression for the integration over the solid angle [10]:

$$\int d\Omega_{m-1}e^{i\mathbf{k}\cdot\Delta\mathbf{x}}=\frac{(2\pi)^{m/2}J_{(m-2)/2}(k\Delta x)}{(k\Delta x)^{(m-2)/2}}\ , \quad \text{(B11)}$$

where $k=|\mathbf{k}|$ and $\Delta x=|\Delta\mathbf{x}|$. Rewriting the Hankel functions $H_\nu(k\eta)$ in terms of the McDonald functions $K_\nu(k\eta)$,

$$H_\nu^{(2)}(k\eta)=\frac{2}{\pi}K_\nu(-ik\eta)\ ,\qquad H_\nu^{(2)*}(k\eta')=\frac{2}{\pi}K_\nu(ik\eta')\ , \quad \text{(B12)}$$

we obtain $G(x,x')$ as an integral over k $(=|\mathbf{k}|)$:

$$G(x,x')=\frac{(-H)^{n-2}}{2^{(n-1)/2}}\frac{1}{\pi^{(n+1)/2}}\frac{(\eta\eta')^{(n-1)/2}}{(\Delta x)^{(n-3)/2}}\times\int_0^\infty dk\ k^{(n-1)/2}K_\nu(-ik\eta)K_\nu(ik\eta')\times J_{(n-3)/2}(k\Delta x)\ . \quad \text{(B13)}$$

The above integral can be evaluated in terms of Legendre functions [10] so that, finally,

$$G(x,x')=\frac{(-H)^{n-2}}{2(2\pi)^{n/2}}\frac{\Gamma(\frac{1}{2}(n-1)+\nu)\Gamma(\frac{1}{2}(n-1)-\nu)}{(u^2-1)^{(n-2)/4}}\times P_{\nu-1/2}^{-(n-2)/2}(u)\ , \quad \text{(B14)}$$

where $u=(\Delta x^2-\eta^2-\eta'^2)/2\eta\eta'$. $G(x,x')$ in Eq. (B14) can be rewritten in terms of a hypergeometric function [20] as

$$G(x,x')=\frac{(-H)^{n-2}}{(4\pi)^{n/2}}\frac{\Gamma((n-1)/2+\nu)\Gamma((n-1)/2-\nu)}{\Gamma(n/2)}\times F\left[\frac{n-1}{2}+\nu,\frac{n-1}{2}-\nu;\frac{n}{2};\omega\right]\ , \quad \text{(B15)}$$

where $\omega=1-(\Delta x^2-\Delta\eta^2)/4\eta\eta'$.

One can easily verify that the propagator $D(x,x')$, for a massless and conformally coupled scalar field $[m=0,\xi=\xi(n)]$, scales conformally with the n-dimensional Minkowski space propagator $D_{\rm Mink}(x,x')$. Substituting $\nu=\frac{1}{2}$ [corresponding to $m=0$ and $\xi=\xi(n)$] in (B15) and using the relation [20]

$$F(a,b;b;z)=\frac{1}{(1-z)^a}\ , \quad \text{(B16)}$$

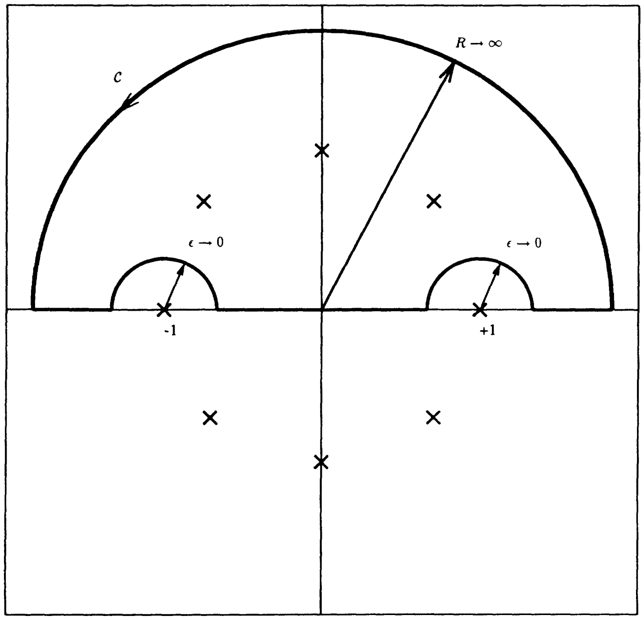

FIG. 5. The contour $\mathcal{C}$ and poles of (3.18), (6.7b), and (3.19), used to establish (3.22) and (6.12) by means of contour integration in Appendix C, are shown in the complex Z plane for $p=8$.

we obtain

$$D(x,x')=\frac{(-H\eta)^{(n-2)/2}(-H\eta')^{(n-2)/2}}{(4\pi)^{n/2}}\times\frac{\Gamma(\frac{1}{2}(n-2))}{(\Delta x^2-\Delta\eta^2)^{(n-2)/2}}=[a(\eta)a(\eta')]^{(2-n)/2}D_{\text{Mink}}(x,x') \ , \quad \text{(B17)}$$

as expected.

In four dimensions (B15) assumes the well-known form [21]

$$G(x,x')=\frac{H^2}{16\pi}(\tfrac{1}{4}-\nu^2)\sec\pi\nu F(\tfrac{3}{2}+\nu,\tfrac{3}{2}-\nu;2;\omega) \ . \quad \text{(B18)}$$

In three dimensions (B15) reads

$$G(x,x')=\frac{-H}{4\pi}\nu\csc\pi\nu F(1+\nu,1-\nu;\tfrac{3}{2};\omega) \ , \quad \text{(B19)}$$

which is used in Sec. V to construct the scalar field propagator in the conical de Sitter metric.

APPENDIX C

We outline a few steps involved in the evaluation of the integrals $s_1(p)$, $s(p)$, and $s_1^T(p)$ [(3.18), (3.19), (6.7b)] for integer p to get the summations (3.22) and (6.12). The substitution $z=e^u$ brings these integrals to a form suitable for contour integration over the contour $\mathcal{C}$ shown in Fig. 5. The contour integrals involved in $s_1(p)$, $s(p)$, and $s_1^T(p)$ are

$$I_1=\frac{2}{\pi}\oint_{\mathcal{C}}\frac{dz}{z^2-1}\left[p\frac{z^{2p}+1}{z^{2p}-1}-\frac{z^2+1}{z^2-1}\right] , \quad \text{(C1a)}$$

$$I_2=\frac{8}{\pi}\oint_{\mathcal{C}}\frac{dz}{z^2-1}\left[\frac{z^2+1}{(z^2-1)^3}-p^3\frac{z^{2p}+1}{(z^{2p}-1)^3}\right] , \quad \text{(C1b)}$$

$$I_3=\frac{4}{\pi}\oint_{\mathcal{C}}\frac{dz}{z^2-1}\frac{z^p}{z^{2p}-1} \quad (p \text{ even}) , \quad \text{(C1c)}$$

respectively. The poles of all the contour integrals (C1) are the $(2p)$th roots of unity, of which the poles $e^{ik\pi/p}$, $k=1,\dots,p-1$, lie within the contour $\mathcal{C}$ (Fig. 5). Evaluating (C1) using the residue theorem we obtain (omitting lengthy intermediate steps)

$$s_1(p)=\sum_{k=1}^{p-1}\csc\frac{k\pi}{p} \ , \quad \text{(C2a)}$$

$$s(p)=\sum_{k=1}^{p-1}\left[\csc^3\frac{k\pi}{p}-\frac{1}{2}\csc\frac{k\pi}{p}\right] , \quad \text{(C2b)}$$

$$s_1^T(p)=\sum_{k=1}^{p-1}(-1)^k\csc\left[\frac{k\pi}{p}\right]+\frac{2}{\pi} \ . \quad \text{(C2c)}$$

[1] D. Kramer, H. Stephani, E. Herlt, and M. MacCallum, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, England, 1980).

[2] S. Deser, in *Fields, Strings and Gravity,* Proceedings of the CCAST (World Laboratory) Symposium/Workshop, edited by H. Guo, Z. Qiu, and H. Tye (Gordon and Breach, New York, 1990), Vol. 6.

[3] S. Deser, R. Jackiw, and S. Templeton, Ann. Phys. (N.Y.) **140**, 372 (1982).

[4] M. E. Ortiz, Nucl. Phys. **B363**, 185 (1991).

[5] S. Habib, University of Maryland report, 1984 (unpublished).

[6] In Einstein gravity this form of the metric is valid everywhere outside the point source. It is also interesting to note that the exterior metric of an extended static source also has the asymptotically conical form (1.5), as recently demonstrated in N. J. Cornish and N. E. Frankel, Phys. Rev. D **43**, 2555 (1991).

[7] A. Vilenkin, Phys. Rep. **121**, 263 (1985).

[8] G. 't Hooft, Commun. Math. Phys. **117**, 685 (1988); S. Deser and R. Jackiw, *ibid.* **118**, 495 (1988); A. Cappelli, M. Ciafaloni, and P. Valtancoli, Nucl. Phys. **B369**, 669 (1992).

[9] Quantum effects near cosmic strings were investigated by T. M. Helliwell and D. A. Konkowski, Phys. Rev. D **34**, 1918 (1986); B. Linet, *ibid.* **33**, 1833 (1986); **35**, 536 (1987); A. G. Smith, Tufts University report, 1986 (unpublished); in *The Formation and Evolution of Cosmic Strings,* edited by G. W. Gibbons, S. W. Hawking, and T. Vachaspati (Cambridge University Press, Cambridge, England, 1990); J. S. Dowker, J. Phys. A **10**, 115 (1977); Class. Quantum

Grav. **4**, L157 (1987); Phys Rev. D **36**, 3095 (1987); in *The Formation and Evolution of Cosmic Strings,* edited by G. W. Gibbons, S. W. Hawking, and T. Vachaspati (Cambridge University Press, Cambridge, England, 1990); P. C. W. Davies and V. Sahni, Class. Quantum Grav. **5**, 1 (1987); V. P. Frolov and E. M. Serebriany, Phys. Rev. D **34**, 2840 (1986); L. Parker, Phys. Rev. Lett. **59**, 1369 (1987); V. Sahni, Mod. Phys. Lett. A **3**, 1425 (1988); W. A. Hiscock, Phys. Lett. B **188**, 317 (1987); I. H. Russell and D. J. Toms, Class. Quantum Grav. **6**, 1343 (1989); A. Sarmiento and S. Hacyan, Phys. Rev. D **38**, 1331 (1988).

[10] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic, New York, 1980).

[11] We refer the reader to the papers by Linet [9] and Smith [9] where a similar analysis has been carried out for cosmic strings.

[12] The method of images has also been discussed in the context of cosmic strings by Smith [9] and by Davies and Sahni [9].

[13] B. S. DeWitt, Phys. Rep. **19C**, 295 (1975); S. M. Christensen, Phys. Rev. D **14**, 2490 (1976).

[14] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).

[15] T. M. Helliwell and D. A. Konkowski [9] also obtain similar relationships between the various derivatives of the Green's function in the space-time of a cosmic string.

[16] Dimensionally $G_2 \equiv G_3 l^{-1}$, where G_3 is the gravitational constant in $3+1$ dimensions and l is a fundamental length scale—say the Planck length.

[17] Our treatment in this section closely follows that of Hiscock [9], who evaluated the semiclassical correction to the cosmic string metric.

[18] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, New York, 1975).

[19] J. D. Barrow, A. B. Burd, and D. Lancaster, Class. Quantum Grav. **3**, 551 (1986). See also N. J. Cornish and N. E. Frankel, Phys. Rev. D **43**, 2555 (1991).

[20] *Handbook of Mathematical Functions,* edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1970).

[21] T. S. Bunch and P. C. W. Davies, Proc. R. Soc. London **A360**, 117 (1978).

[22] B. Allen and A. Folacci, Phys. Rev. D **35**, 3771 (1987).

[23] W. G. Unruh, Phys. Rev. D **14**, 870 (1976); B. S. DeWitt, in *General Relativity,* edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979).

[24] S. Takagi, Prog. Theor. Phys. Suppl. **88**, 1 (1986).

[25] C. J. Isham, Proc. R. Soc. London **A362**, 383 (1978); **A364**, 591 (1978).

[26] L. H. Ford, Phys. Rev. D **21**, 949 (1980).

[27] T. S. Bunch and P. C. W. Davies, J. Phys. A **11**, 1315 (1978); N. Sanchez, Nucl. Phys. **B266**, 487 (1986).

[28] $\bar{G}_1$ should be viewed as an "induced" gravitational constant (in the manner of Sakharov) since its value is determined solely by one-loop quantum gravitational effects. Equation (7.1) can also be derived from a local action principle as demonstrated by C. Teitleboim, in *Quantum Theory of Gravity,* edited by S. Christensen (Hilger, Bristol, 1984), p. 327; R. Jackiw, *ibid.*, p. 403. A discussion on the Newtonian limit of (7.1) can also be found in N. J. Cornish and N. E. Frankel, Phys. Rev. D **43**, 2555 (1991); J. Gegenberg, P. F. Kelly, R. B. Mann, and D. Vincent, *ibid.* **37**, 3463 (1988).

[29] S. Deser, R. Jackiw, and G. 't Hooft, Ann. Phys. (N.Y.) **152**, 220 (1984).

[30] D. Deutch and P. Candelas, Phys. Rev. D **20**, 3063 (1979).

[31] An alternative derivation of the Green's function in n-dimensional de Sitter space is given in P. Candelas and D. J. Raine, Phys. Rev. D **12**, 965 (1975); J. S. Dowker and R. Critchley, *ibid.* **13**, 3224 (1976); B. Ratra, *ibid.* **31**, 1931 (1985).

[32] J. Garriga and A. Vilenkin, Phys. Rev. D **45**, 3469 (1992).