Smeared Wigner functions and quantum-mechanical histories

J. J. Halliwell*

Center for Theoretical Physics, Laboratory for Nuclear Science, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

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We calculate the probability for a quantum-mechanical history consisting of imprecise samplings of position at two moments of time. In the limit of small time separation, this leads to an imprecise sampling of position together with a time-of-flight sampling of momentum. We also calculate the probability for the history consisting of direct momentum and position samplings a short time apart. In each case, we find that the resulting probability distribution on phase space is a smeared version of the Wigner function, and is positive. We show that these smearings belong to a class of smearings which make the Wigner function positive. In the case of the time-of-flight momentum sampling, it is more general than previously considered smearings, such as that of Husimi.

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I. INTRODUCTION

The Wigner function [1] has proved to be a useful mathematical and conceptual tool in a variety of circumstances [2]. For a system described by a density matrix $\rho(x,y)$, it is defined by

$$W(p,q)=\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}du\, e^{ipu/\hbar}\rho\left(q-\frac{u}{2},q+\frac{u}{2}\right) . \qquad (1.1)$$

For a pure state, $\rho(x,y)=\Psi(x)\Psi^*(y)$, it has the properties

$$\int dp\; W(p,q)=|\Psi(q)|^2 \qquad (1.2)$$

and

$$\int dq\; W(p,q)=|\tilde{\Psi}(p)|^2 , \qquad (1.3)$$

where $\tilde{\Psi}(p)$ is the Fourier transform of $\Psi(q)$. We will be concerned with systems described by a Hamiltonian of the form

$$H=\frac{p^2}{2m}+V(q) . \qquad (1.4)$$

The Wigner function for a pure state may then be shown to satisfy the Liouville-type equation

$$\frac{\partial W}{\partial t}+\frac{p}{m}\frac{\partial W}{\partial q}-\frac{dV}{dq}\frac{\partial W}{\partial p}=\frac{\hbar^2}{24}\frac{d^3V}{dq^3}\frac{\partial^3 W}{\partial p^3}+\cdots , \qquad (1.5)$$

where the ellipsis denotes terms of higher powers in $\hbar$ and higher derivatives in W and V. Hereafter we set $\hbar=1$.

These properties of the Wigner function show that it is very similar to a classical phase-space distribution and have prompted its use as a heuristic interpretational tool, especially in discussing the emergence of classical behavior in quantum cosmology [3–8]. However, there are a number of grounds on which an interpretational scheme based on the Wigner function must be regarded as problematic. The main objection is that it is not always positive. This may be seen as follows. Denote by $W_\psi(p,q)$ the Wigner function associated with the pure state ψ. Then a direct calculation shows that

$$\int dp\,dq\; W_\psi(p,q)W_\phi(p,q)=\frac{1}{2\pi}|(\psi,\phi)|^2 , \qquad (1.6)$$

where (ψ,ϕ) denotes the usual inner product between ψ and ϕ. By choosing ψ and ϕ to be orthogonal, we see that the integral of the product of their Wigner functions is zero, and therefore, since neither of the Wigner functions generally vanish identically, at least one of them must go negative somewhere. It seems to be the case, however, that the Wigner function is positive for many cases of interest, and only goes negative as a result of oscillations in small volumes of phase space. Indeed, it turns out that if the Wigner function is smeared over a suitable volume of phase space, a positive probability distribution can be obtained [2,9–11].

A second objection to this use of the Wigner function (and indeed, with any smeared version of it), is that there is no fundamental justification for its use. Its interpretation as a probability distribution in phase space is inspired by its properties outlined above, and by the fact that it gives the expected answers for a variety of simple cases. But predictions in quantum mechanics are made using a well-defined set of procedures involving projections operators, states, unitary evolution, etc., and the above use of the Wigner function makes no contact with these procedures.

The purpose of this paper is to give a brief, direct, and elementary justification for the use of a certain class of smeared Wigner functions as interpretational tools. This we do by applying the standard machinery of quantum mechanics to calculate the probability of finding a particle with a sequence of properties at a succession of times, for a given initial state, i.e., the probability for a *quantum-mechanical history*. By focusing on the case of a history consisting of imprecise samplings of position at

*Electronic address: halliwell@mitlns.bitnet. Address after September 1, 1992: Blackett Laboratory, Imperial College of Science and Technology, South Kensington, London SW7 2BZ, UK.

two closely separated moments of time, we obtain a probability distribution for a position sampling together with a time-of-flight momentum sampling at approximately the same time. We also consider a history consisting of a direct (but imprecise) momentum sampling and an imprecise position sampling a short time interval apart. In each case, we find that the resultant probability distributions on phase space are smeared Wigner functions. In the first case, the smearing is slightly more general than those previously considered [2,9–11], and we investigate some of its properties.

This paper was inspired by the observation of Gell-Mann and Hartle that the Wigner function naturally arises in a formalism of quantum mechanics that assigns probabilities to histories [12]. It is also an offshoot of another paper on closely related matters by the present author in collaboration with H. F. Dowker [13].

II. SMEARED WIGNER FUNCTIONS

We begin by describing a general class of smeared Wigner functions. Define the smeared Wigner function

$$\overline{W}(\overline{p},\overline{q})=N\int dp\,dq\,W(p,q)\times\exp[-a(p-\overline{p})^2-b(q-\overline{q})^2-c(p-\overline{p})(q-\overline{q})]\ , \tag{2.1}$$

where $N=\pi^{-1/2}(ab-c^2/4)^{1/2}$. Smeared Wigner functions have been considered before in the literature, of the form (2.1), but with $c=0$ [2,9–11]. For convergence we must demand that $ab-c^2/4>0$. Let us ask for what values of a, b, and c it is positive. The simplest proof of positivity involves an elementary generalization to the case $c\neq 0$ of the argument given in Ref. [11] for the case $c=0$. More enlightening, however, is a generalization of the proof given in Ref. [10], and we take this approach here.

The basic idea is to show that the smearing function in (2.1) is itself a Wigner function, for certain values of a, b, and c. Positivity then follows from (1.6), suitably generalized.

Let us begin by obtaining the appropriate generalization of (1.6). Denote by W_ρ the Wigner function of the density matrix ρ. Then a direct calculation shows that

$$\int dp\,dq\,W_{\rho_1}(p,q)W_{\rho_2}(p,q)=\frac{1}{2\pi}\int dx\,dy\,\rho_1(x,y)\rho_2(y,x)=\frac{1}{2\pi}\mathrm{Tr}(\rho_1\rho_2)\ . \tag{2.2}$$

Now ρ_1 and ρ_2 are positive Hermitian operators. There therefore exist operators S_1 and S_2 such that $\rho_1=S_1S_1^\dagger$ and $\rho_2=S_2S_2^\dagger$, where a dagger denotes Hermitian conjugation. It follows that

$$\mathrm{Tr}(\rho_1\rho_2)=\mathrm{Tr}[(S_1^\dagger S_2)(S_1^\dagger S_2)^\dagger]\geq 0\ ; \tag{2.3}$$

hence, the desired result is achieved.

Now consider an arbitrary Gaussian density matrix. Let

$$\rho(x,y)=A\exp(-\alpha x^2-\beta y^2+\gamma xy+\mu x+\nu y)\ . \tag{2.4}$$

Hermiticity, $\rho(x,y)=\rho^*(y,x)$, implies that $\alpha=\beta^*$, $\gamma=\gamma^*$, and $\mu=\nu^*$. Normalizability requires $\alpha+\beta\geq\gamma$. The normalization $\mathrm{Tr}\rho=1$ fixes the values of A. Now consider positivity. It implies that

$$\int dx\,dy\,f(x)\rho(x,y)f^*(y)\geq 0 \tag{2.5}$$

for any normalizable function f. Let

$$g(x)=f(x)\exp(-\alpha x^2+\mu x)\ . \tag{2.6}$$

Then (2.5) becomes

$$\int dx\,dy\,g(x)g^*(y)e^{\gamma xy}\geq 0\ . \tag{2.7}$$

Expanding the exponential in a Taylor series, one has

$$\sum_{n=0}^{\infty}\frac{\gamma^n}{n!}|g_n|^2\geq 0\ , \tag{2.8}$$

where

$$g_n=\int dx\,x^n g(x)\ . \tag{2.9}$$

Positivity is therefore satisfied if and only if $\gamma>0$.

Let us now consider the Wigner function of the density matrix (2.4). Carrying out the Wigner transform, one obtains an exponential smearing function equal to that in (2.1) if the coefficients in (2.4) are chosen to satisfy

$$a=B\ , \tag{2.10}$$

$$b=-(\gamma^2-4\alpha\beta)B\ , \tag{2.11}$$

$$c=-2i(\alpha-\beta)B\ , \tag{2.12}$$

$$2a\overline{p}+c\overline{q}=i(\nu-\mu)B\ , \tag{2.13}$$

$$c\overline{p}+2b\overline{q}=[(\alpha-\beta)(\nu-\mu)+B(\mu+\nu)]B\ , \tag{2.14}$$

$$a\overline{p}^2+b\overline{q}^2+c\overline{p}\overline{q}=-\tfrac{1}{4}(\nu-\mu)^2B\ , \tag{2.15}$$

where $B=(\alpha+\beta+\gamma)^{-1}$. Solving (2.10)–(2.12), one obtains

$$\alpha=\beta^*=\frac{1}{4a}\left[1+ab-\frac{c^2}{4}+ic\right]\ , \tag{2.16}$$

$$\gamma=\frac{1}{2a}\left[1-ab+\frac{c^2}{4}\right]\ . \tag{2.17}$$

Equations (2.13) and (2.14) may also be solved for μ and ν, but the solutions will not be needed. Equation (2.15) is satisfied identically by the solutions to (2.10)–(2.14).

The important point now is that the positivity condition $\gamma\geq 0$ implies the restriction $ab-c^2/4\leq 1$. Together with the normalizability condition, we therefore have

$$0<ab-\frac{c^2}{4}\leq 1\ . \tag{2.18}$$

This then, is the condition the coefficients must satisfy in the smeared Wigner function (2.1) in order that it be positive.

Note that the quantity $ab-c^2/4$ is essentially the inverse of the volume of phase space over which the Wigner function is smeared. Reintroducing Planck's constant (2.18) is then the condition that the Wigner function be smeared over a positive volume greater than $\hbar$.

In place of (1.2), (1.3), the smeared Wigner function (2.1) satisfies

$$\int d\bar{p}\, \overline{W}(\bar{p},\bar{q}) = \frac{1}{\pi^{1/2}\sigma_q} \int dq\, |\Psi(q)|^2 \exp\left[-\frac{(q-\bar{q})^2}{\sigma_q^2}\right], \tag{2.19}$$

$$\int d\bar{q}\, \overline{W}(\bar{p},\bar{q}) = \frac{1}{\pi^{1/2}\sigma_p} \int dp\, |\tilde{\Psi}(p)|^2 \exp\left[-\frac{(p-\bar{p})^2}{\sigma_p^2}\right], \tag{2.20}$$

where

$$\sigma_q^2 = \frac{a}{ab-c^2/4}, \quad \sigma_p^2 = \frac{b}{ab-c^2/4} . \tag{2.21}$$

It follows that $\sigma_q^2\sigma_p^2 \geq 1$, consistent with the uncertainty principle.

There are many more properties of the smeared Wigner function (2.1) that could be worked out, and presumably they are very similar to those for the case $c=0$ [2,10]. This would be an interesting topic for future investigation.

III. PROBABILITIES FOR HISTORY

Now we show how smeared Wigner functions might arise naturally when one studies quantum-mechanical histories. An event in quantum mechanics at a fixed moment of time is characterized by a projection operator P_α, where α denotes the set of possible alternatives. The projections should be exhaustive and exclusive, which means, respectively, that

$$\sum_\alpha P_\alpha = 1, \quad P_\alpha P_\beta = \delta_{\alpha\beta} P_\alpha . \tag{3.1}$$

A projection corresponding to imprecise sampling of position, for example, is

$$P_{\bar{x}}^{\Delta} = \int_{\bar{x}-\Delta/2}^{\bar{x}+\Delta/2} |x\rangle\langle x| . \tag{3.2}$$

This projector asks whether the particle is in a region of size Δ around the point $\bar{x}$, and the variables $\bar{x}$ take a discrete set of values, $\bar{x}=0,\pm\Delta,\pm 2\Delta,\ldots$. Generally, the probability for the event in question is given by $\mathrm{Tr}[P_\alpha \rho]$, where the trace is over a complete set of states, and ρ is the density matrix.

A history is a sequence of events at a succession of times. A quantum-mechanical history is therefore characterized by a sequence of projections $P_{\alpha_1}, P_{\alpha_2}, \ldots, P_{\alpha_n}$ at a succession of times, $t_1, t_2, \ldots, t_n$. From the standard formalism of quantum mechanics, the probability for such a history may be shown to be

$$p(\alpha_1,\alpha_2,\ldots,\alpha_n) = \mathrm{Tr}[P_{\alpha_n}(t_n)\cdots P_{\alpha_1}(t_1)\rho P_{\alpha_1}(t_1)\cdots P_{\alpha_n}(t_n)] , \tag{3.3}$$

where ρ is the initial density matrix and $P_\alpha(t)$ denotes projection operators in the Heisenberg picture:

$$P_\alpha(t) = e^{iH(t-t_1)} P_\alpha e^{-iH(t-t_1)} . \tag{3.4}$$

Formula (3.3) is the central result for a formulation of quantum mechanics based on histories. We will exploit it in what follows.

There are two ways of regarding expression (3.3). One can think of it in the context of closed systems, such as the Universe as a whole [14]. In this case, in order to assign probabilities such as (3.3) to the histories, it is necessary to show that the probabilities obey a certain set of sum rules. This is equivalent to saying that sets of histories must decohere. The projection operators in this case are *not* to be thought of as measurements or interactions with another system. They are merely "samplings" determining the properties a system might be consistently ascribed. Alternatively, one can think of (3.3) in the context of open systems, as in the Copenhagen interpretation of quantum mechanics [15]. In this case, the projection operators *do* correspond to measurements performed by an external, classical measuring apparatus. The probabilities for different sets of histories need not obey the probability sum rules because the different sets correspond to distinct physical situations. Either point of view ultimately leads to the same expression (3.3) for the probabilities. The present work concerns only the mathematical properties of this expression, and is not specific to either of these interpretations of it.

A. Time-of-flight momentum sampling

Now consider the history consisting of an initial density matrix ρ at time t_1, a sampling of position at time t_1, evolution to time t_2, and a sampling of position at time t_2. The probability that these samplings will return the results $\bar{x}_1$ at t_1, and $\bar{x}_2$ at t_2 is

$$p(\bar{x}_1,t_1,\bar{x}_2,t_2) = \mathrm{Tr}\left[P_{\bar{x}_2} e^{-iH(t_2-t_1)} P_{\bar{x}_1} \rho P_{\bar{x}_1} e^{iH(t_2-t_1)}\right] . \tag{3.5}$$

Our goal is to evaluate this expression in the short time limit.

The position projections (3.2) are rather cumbersome to use mathematically, and it turns out to be more useful to use the Gaussian projectors instead:

$$P_{\bar{x}} = \frac{1}{\pi^{1/2}\sigma} \int_{-\infty}^{\infty} dx \exp\left[-\frac{(x-\bar{x})^2}{\sigma^2}\right] |x\rangle\langle x| . \tag{3.6}$$

These are exhaustive, but they are only approximately mutually exclusive. Equation (3.5) may now be written

$$p(\bar{x}_1,t_1,\bar{x}_2,t_2)=\int dx_2dx_1dy_1\langle x_2,t_2|x_1,t_1\rangle\rho(x_1,y_1)\langle y_1,t_1|x_2,t_2\rangle\exp\left[-\frac{(x_1-\bar{x}_1)^2}{\sigma_1^2}-\frac{(y_1-\bar{x}_1)^2}{\sigma_1^2}-\frac{(x_2-\bar{x}_2)^2}{\sigma_2^2}\right]. \tag{3.7}$$

For convenience we will generally omit prefactors. They may be recovered in the final result using normalization conditions. Next we write the density matrix in terms of its Wigner function

$$\rho(x_1,y_1)=\int dk\, W\left[k,\frac{x_1+y_1}{2}\right]e^{ik(x_1-y_1)}. \tag{3.8}$$

In the short time limit, the propagator is given by

$$\langle x_2,t_2|x_1,t_1\rangle=\left[\frac{m}{2\pi it}\right]^{1/2}\exp\left[\frac{im}{2t}(x_2-x_1)^2\right], \tag{3.9}$$

where we have introduced $t=t_2-t_1$. Introducing $\xi=x_1-y_1$ and $X=(x_1+y_1)/2$, Eq. (3.7) becomes

$$p(\bar{x}_1,t_1,\bar{x}_2,t_2)=\int dx_2d\xi\, dk\, dX\, W(k,X)\times\exp\left[i\xi\left[k+\frac{m}{t}(X-x_2)\right]-\frac{(x_2-\bar{x}_2)^2}{\sigma_2^2}-\frac{\xi^2}{2\sigma_1^2}-\frac{2}{\sigma_1^2}(X-\bar{x}_1)^2\right]. \tag{3.10}$$

The integrals over x_2 and ξ are readily carried out, and one obtains the result

$$p(\bar{x}_1,t_1,\bar{x}_2,t_2)=\int dk\, dX\, W(k,X)\times\exp\left[-\frac{\sigma_1^2}{2\alpha}\left[k-\frac{m}{t}(\bar{x}_2-X)\right]^2-\frac{2}{\sigma_1^2}(X-\bar{x}_1)^2\right], \tag{3.11}$$

where

$$\alpha=1+\frac{m^2\sigma_1^2\sigma_2^2}{2t^2}. \tag{3.12}$$

Finally, we can rewrite this as a probability distribution not for $\bar{x}_1$ and $\bar{x}_2$, but for $\bar{x}_1$ and the momentum $\bar{k}=(\bar{x}_2-\bar{x}_1)m/t$. The result is

$$p(\bar{x}_1,\bar{k},t)=\int dk\, dX\, W(k,X)\times\exp[-a(k-\bar{k})^2-b(X-\bar{x}_1)^2-c(k-\bar{k})(X-\bar{x})], \tag{3.13}$$

where

$$a=\frac{\sigma_1^2}{2\alpha},\quad b=\frac{m^2\sigma_1^2}{2\alpha t^2}+\frac{2}{\sigma_1^2},\quad c=-\frac{m\sigma_1^2}{\alpha t}. \tag{3.14}$$

Equation (3.14) is the first main result: a sampling of position, accompanied with a time-of-flight sampling of momentum at approximately the same time yield the results $\bar{x}_1$, $\bar{k}$ with a probability given by a smeared Wigner function of the form (2.1).

Equation (3.13) is positive by construction because expression (3.5) is positive. Still, it is gratifying to check that it falls into the class of smeared Wigner functions described in Sec. II. From (3.14), one has

$$ab-\frac{c^2}{4}=\left[1+\frac{m^2\sigma_1^2\sigma_2^2}{2t^2}\right]^{-1}\leq 1 \tag{3.15}$$

as expected.

For the particular values (3.14), one has

$$\sigma_q^2=\frac{\sigma_1^2}{2},\quad \sigma_p^2=\frac{2}{\sigma_1^2}\left[1+\frac{3m^2\sigma_1^2\sigma_2^2}{4t^2}\right]. \tag{3.16}$$

The uncertainty in momentum goes to infinity as $t\to 0$, while the uncertainty in position remains finite. a and c go to zero in this limit, and all dependence on $\bar{p}$ completely drops out. This is not surprising since a sampling of momentum by a time-of-flight method requires a nonzero amount of time to elapse. Also, note that the previously considered smearings of the Wigner function with $c=0$ are realized by the coefficients (3.14) only in the limit $t\to\infty$, a limit which can be taken in the calculation of this section only for the case of a free particle (for which the calculation is exact).

Another interesting case to consider is that in which one has a distinguished system coupled to an environment, and the projections in (3.5) refer only to the distinguished system. This arises, for example, in the model of Caldeira and Leggett [16], discussed in Refs. [12,13]. This coupling to an environment is generally necessary to produce decoherence of alternative histories for the system. A complicating factor is that the evolution of the density matrix is not unitary, and the expression (3.7) will no longer be valid. However, one can show that it is unitary in the small time limit [12,13], and then is described by (3.7). This means that the above result holds for the more general situations discussed in Refs. [12,13].

B. Direct momentum sampling

One may also consider histories in which there are projection operators which sample momentum directly. Let us therefore consider the history consisting of an initial density matrix ρ, a sampling of position at time t_1, evolution to time t_2, and a sampling of momentum at time t_2.

The probability that these samplings will return the results $\bar{x}$ at t_1 and $\bar{p}$ at t_2 is

$$p(\bar{x},t_1,\bar{p},t_2)=\mathrm{Tr}[P_{\bar{p}}e^{-iH(t_2-t_1)}P_{\bar{x}}\rho P_{\bar{x}}e^{iH(t_2-t_1)}] , \tag{3.17}$$

where

$$P_{\bar{x}}=\frac{1}{\pi^{1/2}\sigma_{\bar{x}}}\int_{-\infty}^{\infty}dx \exp\left[-\frac{(x-\bar{x})^2}{\sigma_{\bar{x}}^2}\right]|x\rangle\langle x| , \tag{3.18}$$

$$P_{\bar{p}}=\frac{1}{\pi^{1/2}\sigma_{\bar{p}}}\int_{-\infty}^{\infty}dp \exp\left[-\frac{(p-\bar{p})^2}{\sigma_{\bar{p}}^2}\right]|p\rangle\langle p| . \tag{3.19}$$

Equation (3.17) may thus be written

$$p(\bar{x},t_1,\bar{p},t_2)=\int dp\, dx\, dy\, \langle p,t_2|x,t_1\rangle\rho(x,y)\langle y,t_1|p,t_2\rangle \times\exp\left[-\frac{(x-\bar{x})^2}{\sigma_{\bar{x}}^2}-\frac{(y-\bar{x})^2}{\sigma_{\bar{x}}^2} -\frac{(p-\bar{p})^2}{\sigma_{\bar{p}}^2}\right] . \tag{3.20}$$

In the short time limit, one has

$$\langle p,t_2|x,t_1\rangle \approx \exp\left[-\frac{ip^2t}{2m}-ipx\right] , \tag{3.21}$$

where $t=t_2-t_1$.

Inserting (3.21), and also (3.8), one finds that all dependence of t drops out. Introducing $\xi=x-y$ and $X=(x+y)/2$, one obtains

$$p(\bar{x},t_1,\bar{p},t_2)=\int dp\, d\xi\, dX\, dk\, W(k,X) \times\exp\left[i(k-p)\xi-\frac{\xi^2}{2\sigma_{\bar{x}}^2} -2\frac{(X-\bar{x})^2}{\sigma_{\bar{x}}^2}-\frac{(p-\bar{p})^2}{\sigma_{\bar{p}}^2}\right] . \tag{3.22}$$

The integrals over p and ξ are readily carried out, with the result

$$p(\bar{x},t_1,\bar{p},t_2)=\int dX\, dk\, W(k,X) \times\exp[-a(k-\bar{p})^2-b(X-\bar{x})^2] , \tag{3.23}$$

where

$$a=\frac{\sigma_{\bar{x}}^2}{2[1+(1/2)\sigma_{\bar{x}}^2\sigma_{\bar{p}}^2]} , \quad b=\frac{2}{\sigma_{\bar{x}}^2} . \tag{3.24}$$

The result is therefore a smeared Wigner function of the form (2.1) with $c=0$.

Because t dropped out of the calculation, we may take it to zero and regard the position and momentum projections as occurring at the same time. It then becomes an issue of whether one would obtain the same result if the projections were taken in a different order. Repeating the calculation with a momentum projection at time t_1 and a position projection at time t_2, and then letting $t_2\to t_1$, one finds a distribution of the form (2.1) with $c=0$, and

$$a=\frac{2}{\sigma_{\bar{p}}^2} , \quad b=\frac{\sigma_{\bar{p}}^2}{2[1+(1/2)\sigma_{\bar{x}}^2\sigma_{\bar{p}}^2]} . \tag{3.25}$$

C. Coherent state projectors

Finally, it is pertinent to ask what other possibilities exist for sampling positions and momenta. A natural possibility to look at are coherent state projectors,

$$P_{\bar{p}\bar{q}}=|\bar{p},\bar{q}\rangle\langle\bar{p},\bar{q}| , \tag{3.26}$$

where $|\bar{p},\bar{q}\rangle$ are coherent states:

$$\langle x|\bar{p},\bar{q},\rangle=\frac{1}{(\pi\sigma)^{1/4}}\exp\left[-\frac{(x-\bar{q})^2}{2\sigma}+i\bar{p}x\right] . \tag{3.27}$$

The coherent state projectors affect simultaneous imprecise samplings of position and momentum. To find the connection with the Wigner function, it turns out not to be necessary to look at histories, but it is sufficient to look at the probability that, in a pure state $|\Psi\rangle$, a simultaneous imprecise sampling of position and momentum yields the approximate results $\bar{q},\bar{p}$. This is given by

$$P(\bar{p},\bar{q})=\mathrm{Tr}[P_{\bar{p},\bar{q}}\rho]=|\langle\bar{p},\bar{q}|\Psi\rangle|^2 . \tag{3.28}$$

But, through the relation (1.6), (3.28) is equal to

$$P(\bar{q},\bar{p})=\frac{1}{2\pi}\int dp\, dq\, W_\Psi(p,q)W_{\bar{p},\bar{q}}(p,q) . \tag{3.29}$$

Here, $W_{\bar{p},\bar{q}}(p,q)$ is the Wigner function for the coherent state (3.27), and is given by

$$W_{\bar{p},\bar{q}}(p,q)=\frac{1}{\pi}\exp\left[-\frac{(q-\bar{q})^2}{\sigma}-\sigma(p-\bar{p})^2\right] . \tag{3.30}$$

Equation (3.29) is therefore a smeared Wigner function of the form (2.1) with $c=0$ and $a=b^{-1}=\sigma$. This is not a new result [10], but it is perhaps useful to see how it fits into the general approach considered here.

IV. CONNECTIONS WITH EARLIER WORK

With the particular case of quantum cosmology in mind, it was proposed in Ref. [3] that one could use the usual, unsmeared Wigner function as an interpretational tool for identifying correlations in the wave function of an isolated system. The basic idea was to look for peaks in the Wigner function.

Anderson [7] and Habib and Laflamme [5] have correctly pointed out some of the deficiencies in this approach and suggested improvements. In addition to the problems discussed and dealt with in this paper, a problem these authors pointed out is that the peaks of the Wigner function obtained for certain examples in Ref. [3] were artifacts of the approximation methods used. More accurate treatments often lead to a wildly oscillatory

Wigner function possessing many peaks, and most of these peaks do not correspond to behavior of interest, e.g. to classical behavior. Habib and Laflamme suggest that these problems may be avoided by coarse graining [5]. They investigated a (rather complicated) form of coarse graining in which the system of interest is coupled to an environment, which is traced out of the density matrix. The overall effect is to smear the Wigner function and smooth out the rapid oscillations. In connection with the approach of this paper, we point out that it is most likely that adequate smoothing of the oscillations could also be achieved by smearing of the form (2.1). Here, the origin of this smearing is not coarse graining due to tracing out an environment, but coarse graining due to imprecise sampling of positions and momenta.

We also note that Anderson [7] has proposed the smeared Wigner function (3.29) as an interpretational tool, and as an improvement on the approach of Ref. [3]. Finally, after completion of this work, the author became aware that the appearance of the Wigner function in connection with joint probability distributions of the form (3.5) has previously been noted by Barchielli, Lanz, and Prosperi [17].

A useful approach to prediction in quantum mechanics, and, in particular, to prediction in quantum cosmology, is one in which probabilities are assigned to histories [14]. A more heuristic approach that has been used in the past assigns probabilities to histories through studying the Wigner function for the state of the system at a fixed moment of time [3–8]. The content of this paper has been to show that the latter can emerge from the former, although it is a class of smeared Wigner functions that naturally emerge, and not the Wigner function itself.

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[1] E. P. Wigner, Phys. Rev. **40**, 749 (1932).

[2] For extensive reviews of the Wigner function and related functions, see N. L. Balazs and B. K. Jennings, Phys. Rep. **104**, 347 (1984); M. Hillery, R. F. O'Connell, M. O. Scully, and E. P. Wigner, *ibid.* **106**, 121 (1984).

[3] J. J. Halliwell, Phys. Rev. D **36**, 3626 (1987).

[4] S. Habib, Phys. Rev. D **42**, 2566 (1990).

[5] S. Habib and R. Laflamme, Phys. Rev. D **42**, 4056 (1990).

[6] E. Calzetta, Phys. Rev. D **40**, 380 (1989).

[7] A. Anderson, Phys. Rev. D **42**, 585 (1990).

[8] T. P. Singh and T. Padmanabhan, Ann. Phys. (N.Y.) **196**, 296 (1989).

[9] K. Husimi, Proc. Phys. Math. Soc. Jpn. **22**, 264 (1940).

[10] R. F. O'Connell and E. P. Wigner, Phys. Lett. **85A**, 121 (1981).

[11] N. D. Cartwright, Physica **A83**, 210 (1976).

[12] M. Gell-Mann and J. B. Hartle, Santa Barbara Report No. UCSB-91-15, 1991 (unpublished).

[13] H. F. Dowker and J. J. Halliwell, preceding paper, Phys. Rev. D **46**, 1580 (1992).

[14] M. Gell-Mann and J. B. Hartle, in *Complexity, Entropy and the Physics of Information,* edited by W. Zurek, Santa Fe Institute Studies in Sciences of Complexity Vol. VIII (Addison-Wesley, Reading, 1990); and in *Proceedings of the Third International Symposium on the Foundations of Quantum Mechanics in the Light of New Technology,* edited by S. Kobayashi, H. Ezawa, Y. Murayama, and S. Nomura (Physical Society of Japan, Tokyo, 1990); J. B. Hartle, in *Quantum Cosmology and Baby Universes,* Proceedings of the 7th Jerusalem Winter School, Jerusalem, Israel, 1990, edited by S. Coleman, J. Hartle, T. Piran, and S. Weinberg (World Scientific, Singapore, 1991).

[15] See, for example, E. P. Wigner, Am. J. Phys. **31**, 6 (1963); W. Unruh, in *New Techniques and Ideas in Quantum Measurement Theory,* edited by D. Greenberger (New York Academy of Sciences, New York, 1986), Vol. 480.

[16] A. O. Caldeira and A. J. Leggett, Physica A **121**, 587 (1983).

[17] A. Barchielli, L. Lanz, and G. M. Prosperi, Nuovo Cimento **72B**, 79 (1982).