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Quantum mechanics of history: The decoherence functional in quantum mechanics

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We study a formulation of quantum mechanics in which the central notion is that of a quantum-mechanical history—a sequence of events at a succession of times. The primary aim is to identify sets of "decoherent" (or "consistent") histories for the system. These are quantum-mechanical histories suffering negligible interference with each other, and, therefore, to which probabilities may be assigned. These histories may be found for a given system using the so-called decoherence functional. When the decoherence functional is exactly diagonal, probabilities may be assigned to the histories, and all probability sum rules are satisfied exactly. We propose a condition for approximate decoherence, and argue that it implies that most probability sum rules will be satisfied to approximately the same degree. We also derive an inequality bounding the size of the off-diagonal terms of the decoherence functional. We calculate the decoherence functional for some simple one-dimensional systems, with a variety of initial states. For these systems, we explore the extent to which decoherence is produced using two different types of coarse graining. The first type of coarse graining involves imprecise specification of the particle's position. The second involves coupling the particle to a thermal bath of harmonic oscillators and ignoring the details of the bath (the Caldeira-Leggett model). We argue that both types of coarse graining are necessary in general. We explicitly exhibit the degree of decoherence as a function of the temperature of the bath, and of the width to within which the particle's position is specified. We study the diagonal elements of the decoherence functional, representing the probabilities for the possible histories of the system. To the extent that the histories decohere, we show that the probability distributions are peaked about the classical histories of the system, with the distribution of their initial positions and momenta given by a smeared version of the Wigner function. We discuss this result in connection with earlier uses of the Wigner function in this context. We find that there is a certain amount of tension between the demands of decoherence and peaking about classical paths.

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I. INTRODUCTION

Few would dispute that quantum mechanics is a very successful theory. Indeed, there is, at present, no discernible discrepancy between the predictions of quantum theory and the results of experiment. Yet the conventional interpretation of quantum mechanics, the Copenhagen interpretation, is felt to be inadequate: it rests on an *a priori* division of the world into a classical observing apparatus and quantum-mechanical observed system and places heavy emphasis on the process of measurement [1]. What place is there for such notions in a world thought to be fundamentally quantum mechanical in nature? Or in the very early Universe when observers or measuring apparatus could not have existed?

These questions are not of a purely academic nature. A variety of recent developments suggest that extrapolation of quantum mechanics to the macroscopic domain might not only be of interest, but could even be obligatory. The possibility afforded by superconducting quantum interference devices (SQUID's) of preparing systems in macroscopic quantum states has forced a revision of the notion that only microscopic systems can exhibit quantum effects [2]. And the emergence of the field of quantum cosmology [3,4], in which it is asserted that quantum mechanics may be applied to the entire Universe, has necessitated a reconsideration of the foundations on which the conventional interpretation of quantum mechanics is based.

Even on the familiar territory of the microscopic level, quantum mechanics continues to be a source of conceptual difficulty. Although mathematically consistent, and in full agreement with experiment, it displays a number of features which are difficult to reconcile with physical intuition and are sometimes described as paradoxical.

Resolution of these difficulties may emerge from the observation that there is considerable scope for formulating the theory in different ways while preserving its physical predictions. For example, nonrelativistic quantum mechanics may be formulated in the Schrödinger picture, the Heisenberg picture, or in terms of a sum over histories. The theory looks very different in each of these

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approaches, but they are mathematically equivalent and their physical predictions are exactly the same. Viewing the theory from the perspective of these different formulations not only sheds new light on conceptual aspects of the theory, but also points the way to possible generalizations.

The conventional formulation of quantum mechanics, especially in the Schrödinger picture, places heavy emphasis on the notion of an event at a single moment of time: the quantum state of a system, the Hilbert space to which it belongs, and the "collapse of the wave function" of conventional quantum measurement theory, all involve a single moment of time [5]. It is, however, possible to generalize the usual formulation of quantum mechanics so that such notions are deemphasized, and one focuses instead on the notion of a *quantum-mechanical history*. By this is meant, loosely speaking, a sequence of quantum-mechanical events at successive moments of time.

The object of this paper is to study such a formulation of quantum mechanics, developed over the last few years primarily by Griffiths [6], Omnès [7], and Gell-Mann and Hartle [3,8,9]. This formulation specifically concerns closed quantum-mechanical systems and is assumed to apply to microscopic and macroscopic systems alike, up to and including the entire Universe. Its most important feature is that it focuses on the possible histories of a system. The formulation is explicitly time symmetric. It may be used to assign probabilities to noncommuting observables at different times. It makes no reference to external observers, classical apparatus, wave-function collapse, or indeed any of the usual machinery of conventional quantum measurement theory. The physical process of measurement may, however, be examined from within the formulation, and under appropriate circumstances the familiar results of the orthodox approach are recovered.

The central goal of this formulation is to assign probabilities to families of histories of a closed system. However, as we shall see, interference is generally an obstruction to assigning probabilities to histories. Attention therefore centers around a set of "consistency conditions" which determine the sets of histories suffering negligible interference, and therefore, to which probabilities obeying the standard probability sum rules may be assigned. A set of histories satisfying the consistency conditions are referred to as "consistent" or "decoherent" histories. They have the same status as the histories of a classical statistical system, such as a stochastic process. One may think of a system described by a set of consistent histories as possessing definite properties, but for which there are only probabilities of finding the system to be following a particular history.

In brief, therefore, in this "decoherent histories" (or "consistent histories") formulation of quantum mechanics many of the difficulties of the orthodox approach, and, in particular, the difficulties associated with central role played by measurement and the presumed existence of a classical domain, are replaced by the issue of satisfying the consistency conditions. These conditions act as a regulatory principle, or sieve, systematically sorting out the statements that may be made about a system into meaningful and meaningless. They identify the properties of a closed quantum system which may be regarded as definite, in an objective sense, that makes no reference to measurement or external observers.

The authors who developed this generalization of conventional quantum mechanics appeared to have somewhat different aims. Griffiths emphasized the formulation's potential for shedding light on the conceptual difficulties of quantum mechanics [6]. Omnès was likewise concerned with quantum-mechanical paradoxes, but, additionally, emphasized the role of formal logic. He also showed that the consistent histories formulation is based on fewer axioms than the Copenhagen interpretation, and moreover, on a different set of axioms [7]. The most ambitious point of view is that taken by Gell-Mann and Hartle, who were concerned with quantum mechanics as it might apply to the Universe as a whole [3,8,9]. The motivations for the present work are perhaps closest to those of Gell-Mann and Hartle. They concern the issue of the emergence of classical behavior and the interpretation of quantum cosmology.

As mentioned at the beginning of this paper, the Copenhagen interpretation posits a classical domain and is not sufficiently general to explain it in terms of an underlying quantum theory. By contrast, the decoherent histories approach assumes no separation of classical and quantum domains, and is taken to have an unrestricted domain of validity. Consider then, the requirements a quantum system must satisfy if it is to be approximately classical [8]. The most fundamental requirement is that it should be described by a decoherent set of histories. For then the histories of the system may be assigned probabilities obeying the standard probability sum rules. Second, the decoherent histories should consist of largely the same variables at different times. In this paper, we shall assume this and not explore the manner in which it may fail to be true. Third, the values of the dynamical variables at different times should be correlated according to classical laws. This means that the probability distributions for the histories should be strongly peaked about classical histories. There is some uncertainty as to what other requirements should be imposed. A further requirement discussed in Ref. [8] is that the histories must be characterized as precisely as is consistent with decoherence. Here, we will focus on decoherence and classical correlations.

We feel that the decoherent histories approach is likely to be both useful and important in the development of quantum mechanics and especially in quantum cosmology. It is therefore of interest to explore its features in the context of some simple models. This is what we do in this paper. The purpose is to develop some intuitive feel for the formalism in familiar circumstances and to obtain a quantitative understanding of how the decoherence conditions may be satisfied and the extent to which classicality may emerge. Our work consists largely of calculations in nonrelativistic quantum mechanics. Although quantum cosmology is one of our motivations, we will make no reference to any of its technical aspects. Other studies of the decoherent histories approach include that of Albrecht [10], who considered spin systems, and Blencowe [11], who considered the generalization to field theories.

We begin in Sec. II by reviewing the decoherent histories approach. The formalism as it currently stands is largely concerned with histories which satisfy the consistency conditions exactly. However, for most cases of interest, one has at best approximate decoherence. In Sec. III, we therefore address this issue and propose a condition for approximate decoherence. We also derive some useful inequalities for both the density matrix and the decoherence functional. A particularly useful model with which to discuss decoherence is the Caldeira-Leggett model, a model for quantum Brownian motion. It consists of a distinguished particle coupled to a thermal bath of harmonic oscillators. We review this model in Sec. IV. In Secs. V–VIII, we calculate the decoherence functional for this model in a variety of different circumstances. We summarize and conclude in Sec. IX.

II. THE QUANTUM MECHANICS OF HISTORY

We have described in the Introduction the motivations for studying a formulation of quantum mechanics based on history. We now describe the formalism for handling quantum-mechanical histories. This section is largely a review, with elaborations, of the material of Refs. [3,6–9]. A *history* is a sequence of events at a succession of times. Let us therefore first describe what we mean by an event in quantum mechanics.

A. Projection operators and events

In classical mechanics, systems are regarded as having definite properties and statements such as, "the position of the particle is x," are deemed to have an unambiguous meaning. In quantum mechanics, by contrast, although a system may have definite properties if its state is an eigenstate of some observable, it generally will not. We might be interested, for example, in knowing whether or not we can say of the system, at some moment of time, "the position of the particle lies in the range Δ," or "the momentum is p," or "the spin is up." Formally, possession of certain properties or the occurrence of events may be tested using projection operators. A projection operator associated with some event (or with some "proposition") is a Hermitian operator P satisfying $P^2=P$. The event is said to occur in quantum mechanics if $P|\Psi\rangle=|\Psi\rangle$, and not occur if $P|\Psi\rangle=0$. Since any state $|\Psi\rangle$ may be written as a superposition $|\Psi\rangle=P|\Psi\rangle+(1-P)|\Psi\rangle$, events cannot, in general, be said to definitely occur or definitely not occur, and one can at best assign a probability to each possibility. The probability of occurrence, for example, is $\langle\Psi|P|\Psi\rangle$.

A simple example is provided by the two-dimensional Hilbert space of spin states in a particular direction, $\{|\uparrow\rangle,|\downarrow\rangle\}$. The projection corresponding to the proposition, "the spin is up," is $P_\uparrow=|\uparrow\rangle\langle\uparrow|$, for which one clearly has $P_\uparrow|\uparrow\rangle=|\uparrow\rangle$, and $P_\uparrow|\downarrow\rangle=0$.

Relevant to the rest of this paper are propositions about a particle's position. The proposition, "the position of the particle is x," is implemented through the projection operator

$$P_x=|x\rangle\langle x| \; . \tag{2.1}$$

This corresponds to an infinitely precise specification of the particle's position. Of greater interest is the proposition, "the position of the particle lies in the range Δ," which is implemented through the projection operator

$$P_\Delta=\int_\Delta dx\,|x\rangle\langle x| \; . \tag{2.2}$$

If the particle is described by the state $|\Psi\rangle$, then its position definitely lies in the range Δ if $P_\Delta|\Psi\rangle=|\Psi\rangle$, and it definitely lies outside that range if $P_\Delta|\Psi\rangle=0$. The projection operators (2.2) actually turn out to be rather cumbersome to use in practice, and it is somewhat easier to use so-called "Gaussian slits." This involves using, instead of (2.2), the (approximate) projectors

$$P_\sigma(\bar{x})=\frac{1}{\pi^{1/2}\sigma}\int_{-\infty}^{\infty}dx\,\exp\left[-\frac{(x-\bar{x})^2}{\sigma^2}\right]|x\rangle\langle x| \; . \tag{2.3}$$

Generally, we will consider a set of projection operators P_α corresponding to a set of alternatives labeled by α, where α runs over some (possibly infinite and/or continuous range). The set of alternatives should be exhaustive, which means that

$$\sum_\alpha P_\alpha=1 \; , \tag{2.4a}$$

and mutually exclusive, meaning

$$P_\alpha P_\beta=\delta_{\alpha\beta}P_\beta \; . \tag{2.4b}$$

For the case of the projection operators (2.2), we write $P_\alpha=P_{\Delta_\alpha}$, and the set of alternatives is the set of intervals, $\{\Delta_\alpha\}$. This set of intervals must constitute a partition of the real line into nonoverlapping sets; i.e., $\cup_\alpha\Delta_\alpha=\mathbb{R}$ and $\Delta_\alpha\cap\Delta_\beta=\varnothing$ if $\alpha\neq\beta$. The Gaussian projectors (2.3) are exhaustive but satisfy the mutually exclusive condition only approximately, since one has

$$P_\sigma(\bar{x}_1)P_\sigma(\bar{x}_2)=\frac{1}{(2\pi\sigma^2)^{1/2}}\exp\left[-\frac{(\bar{x}_1-\bar{x}_2)^2}{2\sigma^2}\right]\times P_{\sigma/\sqrt{2}}\left[\frac{\bar{x}_1+\bar{x}_2}{2}\right] . \tag{2.5}$$

They are therefore exclusive only to the extent that the exponential on the right-hand side of (2.5) is zero for $\bar{x}_1\neq\bar{x}_2$. This means that the Gaussian projectors (2.3) affect a partition of the real line into regions with size of order σ centered around $\bar{x}$, and $\bar{x}$ has meaning only up to order σ. Quite how many times σ one takes each region to be in size depends on how well one needs the exclusivity condition to be satisfied, which, in turn, depends on the situation. We will return to this question in Sec. VII.

For a system in state $|\Psi\rangle$ at some moment of time, the probability of the occurrence of the event specified by the alternative α is

$$p(\alpha)=\langle\Psi|P_\alpha P_\alpha|\Psi\rangle\ . \tag{2.6}$$

A trivial rewriting of this, relevant to what follows, is

$$\mathrm{Tr}[P_\alpha|\Psi\rangle\langle\Psi|P_{\alpha'}]=p(\alpha)\delta_{\alpha\alpha'}\ , \tag{2.7}$$

where the trace is over a complete set of states.

A projection is said to be completely *fine-grained* if it corresponds to precise specification of a complete set of commuting observables. That is, the projectors are of the form

$$P_\alpha=|\alpha\rangle\langle\alpha|\ , \tag{2.8}$$

where the states $\{|\alpha\rangle\}$ are complete. For a particle moving in one dimension with position x, (2.1) would be an example of a fine-grained projection. A projection is said to be *coarse-grained* if it corresponds to imprecise specification of a complete set of commuting observables, precise specification of an incomplete set, or both. An example of the first possibility is Eq. (2.2) or (2.3). An example of the second (considered in the following sections) is provided by a composite system consisting of a distinguished subsystem with single coordinate x and an "environment" with a set of coordinates R_k. The Hilbert space for the total system is spanned by the states $\{|x,R_k\rangle\}$ and a coarse-grained projection corresponding to precise specification of an incomplete set of observables is

$$\bar{P}_x=\int d\mathbf{R}|x,R_k\rangle\langle x,R_k|\ . \tag{2.9}$$

Most generally, a coarse-grained projection is one of the form

$$\bar{P}_{\bar{\alpha}}=\sum_{\alpha\in\bar{\alpha}}P_\alpha\ , \tag{2.10}$$

where P_α is a fine-grained projector, and the sum is over all α not fixed by $\bar{\alpha}$.

B. Quantum-mechanical histories and interference

Turn now to the description of histories. As stated above, a history is a sequence of events at successive moments of time. A *quantum-mechanical history* is therefore characterized by a sequence of projection operators at a succession of times. The goal of quantum mechanics is to determine the probabilities for certain events, or sequences of events; thus, through the use of projection operators at a succession of times one might hope to assign probabilities to the possible histories of a system, in a manner analogous to Eqs. (2.6) and (2.7). However, interference generally forbids the assignment of probabilities to histories in quantum mechanics. To see why this is so, consider the following example.

Consider a system with Hamiltonian H which, at time t_0, is in a state $|\Psi\rangle$. At time t_1, it will be in the state

$$e^{-iH(t_1-t_0)}|\Psi\rangle\ . \tag{2.11}$$

Suppose at this time we ask whether or not the event corresponding to some set of projection operators P_{α_1} occurs. We therefore consider the object

$$P_{\alpha_1}e^{-iH(t_1-t_0)}|\Psi\rangle\ . \tag{2.12}$$

This will, of course, be zero if the event does not occur, equal to (2.11) if it does; but generally it will be nonzero and different from (2.11). Now suppose we evolve further to time t_2 and ask about the event corresponding to projectors P_{α_2}. We thus obtain the "path-projected state:"

$$|\alpha_2t_2,\alpha_1t_1,\Psi\rangle=P_{\alpha_2}e^{-iH(t_2-t_1)}P_{\alpha_1}e^{-iH(t_1-t_0)}|\Psi\rangle\ . \tag{2.13}$$

This state is the evolved state projected onto a sequence of alternatives at successive moments of time. It is the state for the *history* $(\Psi,t_0)\rightarrow(\alpha_1,t_1)\rightarrow(\alpha_2,t_2)$.

Now we wish to assign a probability to this history. The obvious candidate for the probability of this history is

$$p(\alpha_2t_2,\alpha_1t_1)=\langle\alpha_2t_2,\alpha_1t_1,\Psi|\alpha_2t_2,\alpha_1t_1,\Psi\rangle\ . \tag{2.14}$$

If this is to be a true probability, it must satisfy the axioms of probability theory. These are that the candidate probability must be non-negative, normalized, and, most importantly, must satisfy the "probability sum rules." These sum rules are that the probabilities must be additive on disjoint regions of sample space (e.g., the probability of A or B is the probability of A plus the probability of B, if A and B are mutually exclusive). The expression (2.14) is clearly non-negative. It is also readily shown to be normalized to one (when summed over α_1 and α_2). The important point, however, is that the probabilities (2.14) will generally not satisfy the probability sum rules.

To see this, consider another history, similar to the one above, but in which no projection is made at time t_1; that is, the history $(\Psi,t_0)\rightarrow(\alpha_2,t_2)$. It has the path projected state

$$\begin{aligned}|\alpha_2t_2,\Psi\rangle&=P_{\alpha_2}e^{-iH(t_2-t_0)}|\Psi\rangle\\&=\sum_{\alpha_1}|\alpha_2t_2,\alpha_1t_1,\Psi\rangle\ ,\end{aligned} \tag{2.15}$$

where the final equality follows from the property (2.4a) of the projection operators. The probability for this second history is

$$p(\alpha_2t_2)=\langle\alpha_2t_2,\Psi|\alpha_2t_2,\Psi\rangle\ . \tag{2.16}$$

Then an example of the probability sum rules that should be obeyed is

$$p(\alpha_2t_2)=\sum_{\alpha_1}p(\alpha_2t_2,\alpha_1t_1)\ . \tag{2.17}$$

But this is not the case: (2.17) is generally not satisfied by the probabilities (2.14) and (2.16) defined in terms of the path-projected states. This follows immediately from (2.15) from which one has

$$\langle \alpha_2 t_2,\Psi|\alpha_2 t_2,\Psi\rangle=\sum_{\alpha_1}\langle \alpha_2 t_2,\alpha_1 t_1,\Psi|\alpha_2 t_2,\alpha_1 t_1,\Psi\rangle+\sum_{\alpha_1\neq\alpha_1'}\langle \alpha_2 t_2,\alpha_1 t_1,\Psi|\alpha_2 t_2,\alpha_1' t_1,\Psi\rangle \ . \tag{2.18}$$

This differs from (2.17) by the presence of the term

$$\sum_{\alpha_1\neq\alpha_1'}\langle \alpha_2 t_2,\alpha_1 t_1,\Psi|\alpha_2 t_2,\alpha_1' t_1,\Psi\rangle \ , \tag{2.19}$$

which is generally nonzero and represents interference between different quantum-mechanical histories. It is in this sense that interference generally prevents probabilities from being assigned to histories in quantum mechanics.

We may, nevertheless, still attempt to identify those sets of histories which suffer negligible interference with each other, and therefore to which probabilities may be assigned. From the above, it is readily seen that these histories may be found by studying the object

$$\begin{aligned}D(\alpha_1,\alpha_2|\alpha_1',\alpha_2)&=\langle \alpha_2 t_2,\alpha_1 t_1,\Psi|\alpha_2 t_2,\alpha_1' t_1,\Psi\rangle\\ &=\mathrm{Tr}[P_{\alpha_2}e^{-iH(t_2-t_1)}P_{\alpha_1}e^{-iH(t_1-t_0)}|\Psi\rangle\langle\Psi|e^{iH(t_1-t_0)}P_{\alpha_1'}e^{iH(t_2-t_1)}P_{\alpha_2}] \ ,\end{aligned} \tag{2.20}$$

where the trace is over a complete set of states. If Eq. (2.20) is zero for $\alpha_1\neq\alpha_1'$, we say that the histories *decohere* and the probability sum rule (2.17) will be satisfied. Moreover, the probabilities themselves are given by (2.20) with $\alpha_1=\alpha_1'$. The main goal, therefore, when studying the quantum mechanics of history, is to study an expression of the form (2.20) and identify those sets of histories which decohere. This simple example illustrates the key issues arising in any attempt to build a quantum mechanics based on history, and we now describe the more general formalism.

C. The decoherence functional

Generally, the system is described by an initial density matrix ρ at initial time t_0, and one considers histories consisting of n projections at times $t_1<t_2<\cdots<t_n$. Expression (2.20), the object which tells us whether or not probabilities may be assigned to histories and what those probabilities are, is a special case of an object called the *decoherence functional* and is given by

$$D([\alpha],[\alpha'])=\mathrm{Tr}[P^n_{\alpha_n}(t_n)\cdots P^1_{\alpha_1}(t_1)\rho P^1_{\alpha_1'}(t_1)\cdots P^n_{\alpha_n'}(t_n)] \ . \tag{2.21}$$

It is a functional of the pair of histories $[\alpha],[\alpha']$, where $[\alpha]$ denotes the string of alternatives, $\alpha_1,\alpha_2,\ldots,\alpha_n$ at times $t_1<t_2<\cdots<t_n$. The trace is over a complete set of states for the entire system, and we have introduced

$$P^k_{\alpha_k}(t_k)=e^{i(t_k-t_0)H}P^k_{\alpha_k}e^{-i(t_k-t_0)H} \ . \tag{2.22}$$

The superscript k has been added to allow for the possibility to have different types of projections at different moments of time, e.g., a position projection at t_1, a momentum projection at t_2, etc.

A final density matrix ρ_f could also be included at the end of the string of projections in (2.21), and it would then be necessary to divide by a normalization factor, $\mathrm{Tr}(\rho_f\rho)$. This form emphasizes the time-symmetric nature of the formulation [12]. Here, we will generally take ρ_f to be proportional to the identity operator.

We note the following elementary properties of the decoherence functional:

$$D([\alpha],[\alpha'])=D^*([\alpha'],[\alpha]) \ , \tag{2.23}$$

$$\sum_{[\alpha]}\sum_{[\alpha']}D([\alpha],[\alpha'])=\mathrm{Tr}\rho=1 \ . \tag{2.24}$$

The diagonal elements of the decoherence functional satisfy

$$D([\alpha],[\alpha])\geq 0 \ , \tag{2.25a}$$

$$\sum_{[\alpha]}D([\alpha],[\alpha])=1 \ . \tag{2.25b}$$

The last property, (2.25b), follows from the cyclic property of the trace, and from summing out the projections, starting with the projection at time t_n and working inwards. The diagonal elements are the candidates for the probabilities for the histories $(\rho,t_0)\rightarrow(\alpha_1,t_1)\cdots\rightarrow(\alpha_n,t_n)$ and we denote them

$$p(\alpha_1,\alpha_2,\ldots,\alpha_n)=D(\alpha_1,\alpha_2,\ldots,\alpha_n|\alpha_1,\alpha_2,\ldots,\alpha_n) \ . \tag{2.26}$$

Equations (2.25a) and (2.25b) ensure that they are nonnegative and properly normalized.

Consider now the sum rules the probabilities should satisfy. For a given set of histories, characterized by a sequence of projections $P_{\alpha_1},\ldots,P_{\alpha_n}$, one may construct coarser-grained histories by summing over the finer-grained projections, as in Eq. (2.10) (although note that here the P_α's need not be completely fine-grained projections). The coarser-grained histories are therefore characterized by a sequence of coarser-grained projections, $\bar{P}_{\bar{\alpha}_1},\ldots,\bar{P}_{\bar{\alpha}_n}$. We will be more explicit about the coarse-graining process below. The probability sum rules to be satisfied are that the probability of each coarser-grained history should be the sum of the probabilities of the finer-grained histories of which it is comprised. This means that

$$p(\bar{\alpha}_1,\bar{\alpha}_2,\ldots,\bar{\alpha}_n)=\sum_{[\alpha]\in[\bar{\alpha}]}p(\alpha_1,\alpha_2,\ldots,\alpha_n) \ . \tag{2.27}$$

Here we have used the notation

$$\sum_{[\alpha]\in[\bar{\alpha}]} = \sum_{\alpha_1\in\bar{\alpha}_1}\sum_{\alpha_2\in\bar{\alpha}_2}\cdots\sum_{\alpha_n\in\bar{\alpha}_n}, \qquad (2.28)$$

where $\alpha_k \in \bar{\alpha}_k$ denotes the sum over the alternatives α_k not fixed by the coarse graining $\bar{\alpha}_k$, and the coarse graining may be different at each moment of time. Equation (2.27) should hold *for all* coarse grainings $[\bar{\alpha}]$ of the finer-grained set of histories.

As in the simple example discussed above, however, the probability sum rule (2.27) will generally not be satisfied by the diagonal elements of the decoherence functional, and one cannot assign probabilities to histories in the manner (2.26). Summing over the finer-grained projections, one obtains the decoherence functional for the coarser-grained histories:

$$D([\bar{\alpha}],[\bar{\alpha}']) = \sum_{[\alpha]\in[\bar{\alpha}]}\sum_{[\alpha']\in[\bar{\alpha}']} D([\alpha],[\alpha']) . \qquad (2.29)$$

From this it follows that

$$D([\bar{\alpha}],[\bar{\alpha}]) = \sum_{[\alpha]\in[\bar{\alpha}]} D([\alpha],[\alpha]) + \sum_{\substack{[\alpha]\neq[\alpha'] \\ [\alpha],[\alpha']\in[\bar{\alpha}]}} D([\alpha],[\alpha']) . \qquad (2.30)$$

Here $[\alpha]\neq[\alpha']$ means all pairs of histories $[\alpha],[\alpha']$ for which $\alpha_k \neq \alpha_k'$ for at least one value of k. In analogy with (2.18), therefore, the presence of the sum over off-diagonal terms generally prevents one from identifying the on-diagonal terms with the probabilities, (2.26).

For the probability sum rules to be obeyed, it is necessary that the sum over off-diagonal terms vanishes in (2.30). From the Hermiticity property, (2.23), it follows that only the real part of the decoherence functional contributes to the interference term in (2.30). A sufficient condition for decoherence, therefore, is

$$\mathrm{Re}[D(\alpha_1,\alpha_2,\ldots,\alpha_n|\alpha_1',\alpha_2',\ldots,\alpha_n')] = 0 \qquad (2.31)$$

except when $\alpha_k = \alpha_k'$ for all k. This is also a necessary condition because the sum over off-diagonal terms must vanish for *all possible* coarser grainings of the histories; i.e., all possible sums of the off-diagonal terms must vanish. The fundamental formula for the quantum mechanics of history may therefore be written

$$\mathrm{Re}[D(\alpha_1,\alpha_2,\ldots,\alpha_n|\alpha_1',\alpha_2',\ldots,\alpha_n')] = p(\alpha_1,\alpha_2,\ldots,\alpha_n)\delta_{\alpha_1\alpha_1'}\cdots\delta_{\alpha_n\alpha_n'} . \qquad (2.32)$$

This is both the condition for decoherence and the rule for the assignment of probabilities to decoherent histories.

Sets of histories that decohere are the *only* histories that are regarded as having meaning in this framework and constitute the predictive output of the theory. Sets of histories which do not decohere cannot be assigned probabilities. They are regarded as devoid of meaning and have no predictive content.

In all cases we are aware of, the real and imaginary parts of the decoherence functional generally vanish together (or are small, see below), and it is often convenient to work with the slightly stronger condition obtained by omitting the real part condition in (2.31). It would, however, be of interest to find examples for which one cannot do this.

Note that it is essential that the complete *set* of histories decoheres. That is, the decoherence condition must be satisfied for all possible values of the alternatives $[\alpha]$. It might be possible, for example, to find a particular pair of distinct histories $[\alpha],[\alpha']$ (i.e., particular values of $[\alpha],[\alpha']$) for which the decoherence condition (2.31) is satisfied, but not, in general, for all other pairs of values. It would not be correct, however, to say that this particular pair decoheres. The crucial point is that the probability sum rules must be satisfied and these sum rules involve a sum over *all* alternatives, i.e., over *all* possible values of α_k for each k The decoherence condition must therefore be satisfied for all possible pairs of histories in the set.

D. Coarse graining and decoherence

Turn now to the question of how to achieve decoherence. First we note a simple but very important case. The decoherence functional (2.21) is always diagonal in the final projection $D([\alpha],[\alpha'])\propto\delta_{\alpha_n\alpha_n'}$ by virtue of the cyclic property of the trace. In particular, suppose that we consider histories characterized by a single event at a single moment of time. Such histories *always* decohere, for one has

$$\mathrm{Tr}[P_{\alpha_1}(t_1)\rho P_{\alpha_1'}(t_1)] = \mathrm{Tr}[\rho P_{\alpha_1}(t_1)]\delta_{\alpha_1\alpha_1'} . \qquad (2.33)$$

It is perhaps for this reason that the need for decoherence is not apparent in conventional quantum mechanics, which largely focuses on events at a single moment of time. Let us go on, therefore, to study more general histories consisting of events at more than one moment of time.

The most refined description of history it is possible to give is a completely fine-grained history. This is one characterized by a set of fine-grained projections at every moment of time, i.e., one in which one precisely specifies a complete set of commuting observables at every moment of time. With the exception of some special cases, fine-grained histories do not decohere. To see this, insert into (2.21) the fine-grained projections

$$P^k_{\alpha_k} = |\alpha_k\rangle\langle\alpha_k| \qquad (2.34)$$

and for the moment let the projections be at a discrete, finite set of times. Then the decoherence functional has the form

$$D([\alpha],[\alpha'])=\delta_{\alpha_n\alpha'_n}\langle\alpha_n,t_n|\alpha_{n-1},t_{n-1}\rangle\langle\alpha'_{n-1},t_n|\alpha'_n,t_{n-1}\rangle \times\cdots \times\langle\alpha_2,t_2|\alpha_1,t_1\rangle\langle\alpha'_1,t_1|\alpha'_2,t_1\rangle\langle\alpha_1|\rho(t_1)|\alpha'\rangle\ . \tag{2.35}$$

Even before taking the limit that the projections are continuous in time, it is evident that a decoherence functional which has the product form (2.35) will generally not be diagonal. This will also be clear from the path-integral form below. As indicated above, however, there are some exceptions. For example, suppose that all the projections commute with each other and with the Hamiltonian (as would be the case with momentum projections for the free particle). Then it is not difficult to see that the decoherence functional will be diagonal for any initial state. Another special case is that of a pure initial state $|\Psi\rangle$, with the projections at the times t_k taken to be the state unitarily evolved to that time, $P^k_{\alpha_k}=|\Psi(t_k)\rangle\langle\Psi(t_k)|$ (together with its complement, $1-P^k_{\alpha_1}$). It is not difficult to show that these histories decohere.

To achieve decoherence, it is generally necessary to consider coarse-grained histories. There are three principle methods of coarse-graining histories. The first is to make projections at not every moment of time. Typically this involves making projections at discrete moments of time, but it could also involve making projections in a discrete set of continuous ranges of time. At the moments of time when the projections are made, one can then give imprecise specification of a complete set of commuting variables, or precise specification of an incomplete set, or both. This, of course, corresponds to making coarse-grained projections at those moments of time, as discussed earlier.

It is an important issue for investigation to determine the extent to which these coarse grainings lead to decoherence. This will be the topic of much of the remainder of this paper. We remark that it is immediately clear that the first of the three methods does not seem to be particularly relevant. Specifying a set of fine-grained projections at not every moment of time is a coarse graining, but as we saw above, it alone will generally not lead to decoherence. On the other hand, there is no reason why coarse-grained projections continuous in time should not lead to decoherence. We will concentrate on the second two methods in the following sections.

E. Path-integral form of the decoherence functional

The decoherence functional is very conveniently written in terms of a path integral, a form we will exploit in the following sections. Suppose the system is described by a set of configuration space variables $q^i(t)$. From the expression (2.21) for $D([\alpha,],[\alpha'])$, one may derive the path-integral expression

$$D([\alpha],[\alpha'])=\int_{[\alpha]}\mathcal{D}q^i\int_{[\alpha']}\mathcal{D}q^{i'}\exp(iS[q^i]-iS[q^{i'}])\delta(q^i_f-q^{i'}_f)\rho(q^i_0,q^{i'}_0)\ . \tag{2.36}$$

Here, $S[q^i]$ is the action for the system. The sum is over two sets of paths $q^i(t)$, $q^{i'}(t)$, which begin at q^i_0, $q^{i'}_0$, at $t=t_0$, weighted by the initial density matrix. They end at $t=t_f$ at a common point $q^i_f=q^{i'}_f$, which is summed over, and the result is independent of t_f [this follows from the trace form of the decoherence functional, (2.21)]. The paths also satisfy restrictions at times $t_1\cdots t_n$ corresponding to the projections $P^k_{\alpha_k}(t_k)$. The path-integral form is most useful when the projections are onto position. In this case, the paths are restricted to pass through certain ranges (i.e., pass through gates) on the time slices $t_1\cdots t_n$, but are otherwise free. Projections onto momenta are possible in a phase-space path-integral version of (2.36) [9].

The path-integral form of the decoherence functional provides an alternative way of seeing that completely fine-grained histories do not decohere. Suppose we project onto precise values of the coordinates at every moment of time; e.g., project q^i onto some value Q^i, say. Formally, this involves inserting into the path integral at every time t a δ function $\delta(q^i(t)-Q^i(t))$. It is not difficult to see that the decoherence functional then takes the form

$$D([\alpha],[\alpha'])=\exp(iS[Q^i]-iS[Q^{i'}]) \times\delta(Q^i_f-Q^{i'}_f)\rho(Q^i_0,Q^{i'}_0)\ . \tag{2.37}$$

This expression is the decoherence functional for completely fine-grained configuration space histories, $Q^i(t)$, $Q^{i'}(t)$. It is clearly not, in general, small for distinct histories.

In the path-integral form of the decoherence functional, the two most important coarse grainings involve specifying not all of the q^i but only some of them, and specifying the q^i only imprecisely by projecting them onto some range. The sum over histories also affords the possibility of coarse grainings more general than those that can be implemented by projection operators in the trace form of the decoherence functional. The underlying notion that permits this generalization is that of a *partition* of the paths. Projection operators partition the paths according to their properties at a particular moment of time, e.g., the particle either does or does not pass through the region of configuration space Δ at time t. In the sum over histories, they can be partitioned without reference to time. For example, one can partition the paths into those that do or do not pass through the region Δ at *any* moment of time. Such a partition cannot be effected by a single chain of projection operators at fixed moments of time, yet it can in the sum over histories, and is sometimes a useful and interesting one to consider [13]. The sum-over-histories version of the decoherence functional is therefore more general than the trace form (2.21) in that it permits these more general coarse grainings, but it is also less general in that it is restricted to coarse grainings involving only positions and momenta.

As an aside, we note that the path-integral form of the decoherence functional may also be written

$$D([\alpha],[\alpha'])=\int_{[\alpha],[\alpha']}\mathcal{D}q^i\exp(iS[q^i])\rho(q_0^i,q_0^{i'}) \ . \qquad (2.38)$$

Here, the sum is over all paths $q^i(t)$ beginning at q_0^i at $t=t_0$, moving forwards in time to $t=t_f$ passing through the gates $[\alpha]$, and then moving backwards in time passing through the gates $[\alpha']$, ending at $q_0^{i'}$ at $t=t_0$.

F. Decoherent histories and quantum measurement theory

Much of the formalism described in this section bears close resemblance, at least mathematically, to the familiar machinery of standard quantum measurement theory. Indeed, the diagonal part of Eq. (2.21),

$$p(\alpha_1,\ldots,\alpha_n)=\mathrm{Tr}[P^n_{\alpha_n}(t_n)\cdots P^1_{\alpha_1}(t_1)\rho P^1_{\alpha_1}(t_2)\cdots P^n_{\alpha_n}(t_n)] \ , \qquad (2.39)$$

is a familiar formula of quantum measurement theory. It is the probability of a sequence of *measured* alternatives with an initial state ρ and with unitary evolution between measurements. It is, however, important to understand the distinction between this approach and the decoherent histories approach.

The decoherent histories approach concerns closed quantum systems. It makes no reference to the process of measurement or to collapse of the wave function. The projection operators are not models of measurement by an external agency or of interactions with other systems—they cannot be because the system is closed. The projections serve only to characterize the possible histories of the closed system. These histories are assigned candidate probabilities via the formula (2.39). The candidate probabilities are only true probabilities if they satisfy the consistency conditions. The sole predictive output of the theory consists of the probabilities for a set of consistent histories.

By contrast, conventional quantum measurement theory concerns quantum systems that are not genuinely closed. They are not closed because they are occasionally subjected to the influence of an external, classical, agency that performs a measurement. The state of the system evolves according to two laws of evolution: unitary evolution, when isolated, and nonunitary evolution (collapse of the wave function) when a measurement takes place. This measurement process is modeled by projection operators. Using this formalism one can construct the probability for a sequence of measurements: it is given by (2.39). In this case there is no obligation to show that the probabilities (2.39) obey probability sum rules, and, in general, they will not. This is because the probabilities for different sequences of measured alternatives correspond to different physical situations, and there is no reason in general why these probabilities should be related. For example, a sequence of measurements at times t_1, t_2, and t_3 is a quite different physical situation to the sequence in which the measurement at t_2 is omitted because of the physical disturbance the measurement at t_2 necessarily produces.

Of course, in standard quantum measurement theory, there *are* special measurements, quantum nondemolition measurements, which do not physically disturb the system. A sequence of such measurements will lead to a probability (2.39) which does, in fact, satisfy the probability sum rules. Also, it should be noted that expression (2.39) satisfies a limited set of probability sum rules for any set of measurements, namely, the rules

$$\sum_{\alpha_n}p(\alpha_1,\ldots,\alpha_{n-1},\alpha_n)=p(\alpha_1,\ldots,\alpha_{n-1}) \qquad (2.40)$$

for all n. This follows from the exhaustive property of the projections and the cyclic property of the trace. But neither of these features should detract from the fact that the interpretation of the mathematical formalism in quantum measurement theory is very different to its interpretation in the decoherent histories approach considered in this paper.

These, then, are the differences between the two approaches, but the connections between them should also be stressed. As indicated in the Introduction, an analysis of the quantum measurement process may be carried out from within the framework of the decoherent histories approach. Because the decoherent histories approach applies to genuinely closed quantum systems, the system carrying out the measurements must be included in the total closed system. One could, for example, consider a closed system consisting of two interacting subsystems, observer and observed, and study the correlations between them. In this way is may be shown that the Copenhagen view of quantum measurement theory outlined above emerges as a special case of the more general decoherent histories approach. The precise conditions under which this approximation is appropriate are described in Ref. [3].

This completes our survey of the general formalism of the quantum mechanics of history. As stated in the Introduction, we feel that this approach to quantum mechanics has considerable potential, on the one hand, for clarifying many conceptual issues, and on the other, as a possible tool with which to do quantum cosmology. It therefore becomes an interesting issue to calculate the decoherence functional for various models. Not only will this allow us to develop some feeling for how the formalism works in the context of simple examples, but, also, it will allow us to obtain a quantitative idea of the effectiveness of the coarse grainings discussed above. In particular, in the following sections, we wish to exhibit the decoherence explicitly and quantitatively as a function of the coarse graining.

III. APPROXIMATE DECOHERENCE AND SOME INEQUALITIES

In Sec. II, we described the formalism of the quantum mechanics of histories and gave the condition, Eq. (2.32), that must be satisfied if probabilities are to be assigned to sets of histories. This condition is the condition for *exact* decoherence, i.e., for the probability sum rules for histories to be satisfied exactly. While it is sometimes possible to exhibit histories which decohere exactly, it seems reasonable to expect that, in general, decoherence will not be exact, but will be approximate. This is the case, for example, for the models considered in this paper. It therefore becomes an interesting and important question to understand what is meant by approximate decoherence. This question is the topic of the present section.

A. Approximate decoherence

Recall that the probability sum rules to be satisfied are Eq. (2.27), i.e., that the probability of a coarser-grained history must be the sum of the probabilities for its constituent finer-grained histories, and that this must be true for *all* coarser-grained histories. The natural generalization of this is to demand that the probability sum rules are satisfied to order ϵ, for some constant $\epsilon<1$. By this we mean that the interference terms do not have to be exactly zero, but only suppressed by a factor ϵ; i.e.,

$$\Bigg| \sum_{\substack{[\alpha]\neq[\alpha'] \\ [\alpha],[\alpha']\in[\bar\alpha]}} \mathrm{Re}D([\alpha],[\alpha']) \Bigg| < \epsilon \sum_{[\alpha]\in[\bar\alpha]} D([\alpha],[\alpha]) \tag{3.1}$$

for all possible coarser grainings $[\bar\alpha]$ of the alternatives $[\alpha]$.

In the case of exact decoherence, $\epsilon=0$, condition (3.1) is equivalent to the much simpler condition (2.31), that the real parts of all the off-diagonal terms of the decoherence functional vanish. This enormously simplifies the problem of checking the probability sum rules. For the case of approximate decoherence considered here, however, in the worst possible case, we might have to check the probability sum rules for all possible choices of coarser-grained histories. It could be, for example, that the degree to which the sum rules are satisfied depends on the particular sum rule in question. Let us therefore ask, is there a particular sum rule which, if satisfied to order ϵ, will imply that all other sum rules are satisfied to the same order or better?

To address these issues, consider the finest coarser graining possible, in which two alternatives at time t_k are combined:

$$P_{\bar\alpha_k} = P_{\alpha_k} + P_{\alpha'_k} \ . \tag{3.2}$$

This means that the alternative $\bar\alpha_k$ consists of α_k or α'_k. Let us then demand that the probability sum rule for this coarser graining is satisfied to order ϵ. It is simple to show that this means

$$|\mathrm{Re}D(\cdots\alpha_k\cdots|\cdots\alpha'_k\cdots)| < \epsilon[p(\cdots\alpha_k\cdots)+p(\cdots\alpha'_k\cdots)] \ . \tag{3.3}$$

and taking the case in which $[\alpha]$ and $[\alpha']$ differ only in the values of the alternatives at time t_k and no other.

One might contemplate generalizing this type of condition to the case in which the α_k's were different on each side of the decoherence functional for *all* values of k, not just one value, as in (3.3). The right-hand side might then involve some kind of arithmetic mean of the corresponding on-diagonal terms.

However, for reasons that will become clear below, it turns out that the condition (3.3), or its generalizations, are not the most appropriate ones. A condition that we have found instead to be more useful is

$$|R_{\alpha\alpha'}| < \epsilon[R_{\alpha\alpha}R_{\alpha'\alpha'}]^{1/2} \ , \tag{3.4}$$

where we have introduced the convenient notation

$$R_{\alpha\alpha'} = \mathrm{Re}D([\alpha],[\alpha']) \ . \tag{3.5}$$

We therefore take the geometric mean of the diagonal terms on the right-hand side rather than the arithmetic mean.

First of all, note that (3.4) implies (3.3). This follows (apart from a factor of 2) using the relation

$$\begin{aligned}[R_{\alpha\alpha}R_{\alpha'\alpha'}]^{1/2} &= \tfrac{1}{2}[(R_{\alpha\alpha}+R_{\alpha'\alpha'})^2-(R_{\alpha\alpha}-R_{\alpha'\alpha'})^2]^{1/2} \\ &\le \tfrac{1}{2}(R_{\alpha\alpha}+R_{\alpha'\alpha'})\end{aligned} \tag{3.6}$$

Now consider what condition (3.4) implies for the more general coarser grainings of the histories. Consider first the strict upper bound on the left-hand side of (3.1). One has

$$\begin{aligned}\Bigg| \sum_{\substack{\alpha\neq\alpha' \\ \alpha,\alpha'\in\bar\alpha}} R_{\alpha\alpha'} \Bigg| &\le \sum_{\substack{\alpha\neq\alpha' \\ \alpha,\alpha'\in\bar\alpha}} |R_{\alpha\alpha'}| \\ &< \epsilon \sum_{\substack{\alpha\neq\alpha' \\ \alpha,\alpha'\in\bar\alpha}} [R_{\alpha\alpha}R_{\alpha'\alpha'}]^{1/2} \ .\end{aligned} \tag{3.7}$$

To streamline the notation we temporarily drop the square brackets notation $[\alpha]$ in favor of a simple α.

We need an expression involving a sum over probabilities, as in the right-hand side of (3.1). We therefore write (3.7) as

$$\Bigg| \sum_{\substack{\alpha\neq\alpha' \\ \alpha,\alpha'\in\bar\alpha}} R_{\alpha\alpha'} \Bigg| < \Delta\epsilon \sum_{\alpha\in\bar\alpha} R_{\alpha\alpha} \ , \tag{3.8}$$

where

$$\Delta = \left[\sum_{\alpha \in \bar{\alpha}} R_{\alpha\alpha} \right]^{-1} \sum_{\substack{\alpha \neq \alpha' \\ \alpha, \alpha' \in \bar{\alpha}}} [R_{\alpha\alpha} R_{\alpha'\alpha'}]^{1/2} . \quad (3.9)$$

It is not difficult to see that the factor Δ will generally be much greater than 1, meaning that more general probability sum rules will not be satisfied to the same degrees as the basic condition, Eq. (3.4), but will be satisfied to degree $\Delta\epsilon$, a number generally much greater than ϵ.

The above analysis gives rigorous bounds on the probability sum rules, but these bounds are perhaps not the most relevant ones. The sum over off-diagonal terms on the left-hand side of (3.7) will typically involve a large number of positive and negative terms, and it is reasonable to assume that terms of a particular sign will not be favored. This means that there will be a considerable amount of cancellation, and the upper bound (3.8) is not representative of the typical value of the sum over off-diagonal terms. It is like a random walk in one dimension, with random step lengths and equal probabilities of stepping left or right. If the average step length is l, and the number of steps is N, the maximum distance one can walk is lN. However, if N is large, walks of such length are exceedingly rare, and it may be shown that by far the most probable walks have lengths of order $lN^{1/2}$ or less [14].

More generally, the central limit theorem permits an estimation of the typical value of the size of the left-hand side of (3.7). Consider the off-diagonal terms of $R_{\alpha\alpha'}$. If the α's run over N values, there are (N^2-N) off-diagonal terms in the decoherence functional. (We are therefore restricting attention to the case in which α is a discrete label, but N may be infinite.) A coarse graining $\bar{\alpha}$ of α corresponds to selecting a subset of, say, M α's from the N α's. There are therefore (M^2-M) terms in the sums in Eq. (3.7). We have very little information about $R_{\alpha\alpha'}$, but one thing we do know is that

$$\sum_{\substack{\alpha \neq \alpha' \\ \text{all } \alpha, \alpha'}} R_{\alpha\alpha'} = 0 , \quad (3.10)$$

where all α,α' means over all (N^2-N) values (in distinction to $\alpha,\alpha' \in \bar{\alpha}$). This follows from (2.30). (The only other thing of value we know about $R_{\alpha\alpha'}$ is the inequality derived below, but this will not be needed here.)

Without more detailed information about the $R_{\alpha\alpha'}$'s, a useful way to proceed is to assume that N is large and perform a statistical analysis. We are given a set of (N^2-N) $R_{\alpha\alpha'}$'s. Equation (3.10) implies that their mean value is zero. The quantity we are interested in is the restricted sum over $R_{\alpha\alpha'}$'s for a particular choice of coarse graining $\bar{\alpha}$, i.e., the quantity

$$(M^2-M) Y_{\bar{\alpha}} = \sum_{\substack{\alpha \neq \alpha' \\ \alpha, \alpha' \in \bar{\alpha}}} R_{\alpha\alpha'} . \quad (3.11)$$

The quantity $Y_{\bar{\alpha}}$ is therefore an approximate "measurement" of the exact mean value of the $R_{\alpha\alpha'}$'s. The question is, how close is this approximation ? The central limit theorem supplies the answer: for large M, the distribution of $Y_{\bar{\alpha}}$ approaches the normal form centered about the mean, with width $\sigma(M^2-M)^{-1/2}$, where σ is the standard deviation of the off-diagonal $R_{\alpha\alpha'}$'s [14]. This implies that "most" approximations to the mean $Y_{\bar{\alpha}}$ will lie within a few widths of it. For example, 98% of them will lie within four times the width. Up to factors of order unity, therefore, most $Y_{\bar{\alpha}}$'s will satisfy

$$|Y_{\bar{\alpha}}| < \sigma (M^2-M)^{-1/2} . \quad (3.12)$$

The standard deviation is given by

$$\sigma^2 = \frac{1}{N^2-N} \sum_{\substack{\alpha \neq \alpha' \\ \text{all } \alpha, \alpha'}} R^2_{\alpha\alpha'} . \quad (3.13a)$$

But since M is taken to be large, a reasonable approximation to (3.13a) is

$$\sigma^2 \approx \frac{1}{M^2-M} \sum_{\substack{\alpha \neq \alpha' \\ \alpha, \alpha' \in \bar{\alpha}}} R^2_{\alpha\alpha'} . \quad (3.13b)$$

Combining (3.11), (3.12), and (3.13b), the factors of M^2-M all drop out, and we are left with

$$\left| \sum_{\substack{\alpha \neq \alpha' \\ \alpha, \alpha' \in \bar{\alpha}}} R_{\alpha\alpha'} \right| < \left[\sum_{\substack{\alpha \neq \alpha' \\ \alpha, \alpha' \in \bar{\alpha}}} R^2_{\alpha\alpha'} \right]^{1/2} \quad (3.14)$$

as the bound on typical values of the left-hand side of (3.7).

Now we repeat the above analysis using (3.14). Using condition (3.4), it is straightforward to show that (3.14) implies

$$\left| \sum_{\substack{\alpha \neq \alpha' \\ \alpha, \alpha' \in \bar{\alpha}}} R_{\alpha\alpha'} \right| < \tilde{\Delta}\epsilon \sum_{\alpha \in \bar{\alpha}} R_{\alpha\alpha} , \quad (3.15)$$

where

$$\tilde{\Delta} = \left[\sum_{\alpha \in \bar{\alpha}} R_{\alpha\alpha} \right]^{-1} \left[\sum_{\substack{\alpha \neq \alpha' \\ \alpha, \alpha' \in \bar{\alpha}}} R_{\alpha\alpha} R_{\alpha'\alpha'} \right]^{1/2}$$
$$= \left[\sum_{\alpha \in \bar{\alpha}} R_{\alpha\alpha} \right]^{-1} \left[\left[\sum_{\alpha \in \bar{\alpha}} R_{\alpha\alpha} \right]^2 - \sum_{\alpha \in \bar{\alpha}} R^2_{\alpha\alpha} \right]^{1/2} . \quad (3.16)$$

It is readily seen that $\tilde{\Delta} \leq 1$. This is the main result of this section: given condition (3.4), most probability sum rules are satisfied to order ϵ or better, where "most" is understood in the sense explained above. We anticipate that the result will continue to hold, in some form, in the case where the α's are continuous labels, although we do not demonstrate this explicitly here.

There will, of course, be situations in which our statistical assumptions must fail. For example, one could choose coarse grainings in which all the M^2-M $R_{\alpha\alpha'}$'s summed over in (3.11) are of the same sign. This would therefore be a very bad approximation to the mean value, and the probability sum rules would then be satisfied only to the much poorer degree indicated by the strict bound (3.8). Also, we have implicitly assumed that the N^2-N $R_{\alpha\alpha'}$'s are distributed reasonably evenly, without particular bias towards positive or negative values. One could envisage decoherence functionals for which most of the $R_{\alpha\alpha'}$'s were positive, say, but the negative ones were sufficiently large for (3.10) to hold. Samplings of the $R_{\alpha\alpha'}$'s would therefore also possess this bias, and the statistical reasoning given above would be less accurate. But these situations are exceptional, and they may have some special significance which it would be of interest to investigate. Moreover, they do not detract from the above analysis, the point of which was to define what is meant by the "typical" case and explore its properties.

It may be enlightening to explain why condition (3.4) is more appropriate than (3.3). The main difficulty with (3.3) arises when the coarser graining $\bar{\alpha}$ involves a sum over an infinite number of α's. This happens in the models of this paper, for example. For, suppose one repeated the above analysis using (3.3) in place of (3.4), then in the expressions replacing (3.7) and (3.16), one would obtain expressions in which $R_{\alpha\alpha}$ are summed over both α and α' and would therefore diverge.

B. Some inequalities

We now derive some inequalities which will be useful and lend support to the approximate decoherence condition, (3.4). Consider the matrix elements of the density operator ρ is an arbitrary basis, $\{|A\rangle\}$. It is given by

$$\rho_{AB}=\langle A|\rho|B\rangle \ . \tag{3.17}$$

Now ρ is a non-negative Hermitian operator. This means that there exists some operator S such that $\rho=S^\dagger S$. It follows from the Cauchy-Schwarz inequality that

$$|\langle A|S^\dagger S|B\rangle|^2\leq\langle A|S^\dagger S|A\rangle\langle B|S^\dagger S|B\rangle \ . \tag{3.18}$$

We therefore have the inequality

$$|\rho_{AB}|^2\leq\rho_{AA}\rho_{BB} \tag{3.19}$$

for all $A\neq B$ with equality if and only if ρ is pure.

An analogous result also holds for the decoherence functional. Write the decoherence functional

$$D([\alpha],[\alpha'])=N\sum_\beta\langle\beta|\rho_f C_\alpha\rho_0 C^\dagger_{\alpha'}|\beta\rangle \ , \tag{3.20}$$

where we have explicitly written out the trace over a complete set of states, $\{|\beta\rangle\}$, and we use the notation

$$C_\alpha=P_{\alpha_n}(t_n)\cdots P_{\alpha_2}(t_2)P_{\alpha_1}(t_1) \ . \tag{3.21}$$

For generality, we have also included a final density matrix ρ_f. The normalization factor N is given by $N^{-1}=\mathrm{Tr}(\rho_f\rho)$.

For simplicity, consider first of all the case in which ρ_f is mixed but the initial state ρ_0 is pure, $\rho_0=|\Psi_0\rangle\langle\Psi_0|$. One then has

$$D([\alpha],[\alpha'])=N\langle\Psi_0|C^\dagger_{\alpha'}\rho_f C_\alpha|\Psi_0\rangle \ . \tag{3.22}$$

Since we may write $\rho_f=S_f^\dagger S_f$, it follows from the Cauchy-Schwarz inequality that

$$|D([\alpha],[\alpha'])|\leq[D([\alpha],[\alpha])D([\alpha'],[\alpha'])]^{1/2} \tag{3.23}$$

with equality if ρ_f is pure. The case of a pure ρ_f and mixed ρ_0 is essentially the same.

The case of general ρ_f and ρ_0 is a little more complicated. Write $\rho_f=S_f^\dagger S_f$ and $\rho_0=S_0S_0^\dagger$. Then the decoherence functional may be written

$$D([\alpha],[\alpha'])=N\sum_\beta\langle\beta|\mathcal{A}_\alpha\mathcal{A}^\dagger_{\alpha'}|\beta\rangle \ , \tag{3.24}$$

where we have introduced $\mathcal{A}_\alpha=S_fC_\alpha S_0$. One therefore has

$$\begin{aligned}|D(\alpha],[\alpha'])|&\leq N\sum_\beta|\langle\beta|A_\alpha A^\dagger_{\alpha'}|\beta\rangle|\\&\leq N\sum_\beta\langle\beta|\mathcal{A}_\alpha\mathcal{A}^\dagger_\alpha|\beta\rangle^{1/2}\langle\beta|\mathcal{A}_{\alpha'}\mathcal{A}^\dagger_{\alpha'}|\beta\rangle^{1/2} \ .\end{aligned} \tag{3.25}$$

For simplicity of notation, introduce

$$X_\beta=\langle\beta|\mathcal{A}_\alpha\mathcal{A}^\dagger_\alpha|\beta\rangle^{1/2} \ , \tag{3.26}$$

$$Y_\beta=\langle\beta|\mathcal{A}_{\alpha'}\mathcal{A}^\dagger_{\alpha'}|\beta\rangle^{1/2} \ . \tag{3.27}$$

Then (3.25) reads

$$|D([\alpha],[\alpha'])|\leq N\sum_\beta X_\beta Y_\beta \ . \tag{3.28}$$

Also,

$$D([\alpha],[\alpha])D([\alpha'],[\alpha'])=N^2\sum_{\beta,\gamma}X_\beta^2Y_\gamma^2 \ . \tag{3.29}$$

Now consider the inequality

$$\sum_{\beta,\gamma}(X_\beta Y_\gamma-X_\gamma Y_\beta)^2\geq 0 \ . \tag{3.30}$$

This implies that

$$\sum_{\beta,\gamma}X_\beta Y_\beta X_\gamma Y_\gamma\leq\sum_{\beta,\gamma}X_\beta^2Y_\gamma^2 \tag{3.31}$$

and hence that

$$\sum_\beta X_\beta Y_\beta\leq\left[\sum_{\beta,\gamma}X_\beta^2Y_\gamma^2\right]^{1/2} \ . \tag{3.32}$$

Comparing with (3.28) and (3.29), we therefore again obtain the inequality (3.23). This is the main result: the decoherence functional satisfies the inequality (3.23), with equality if the initial and final states are pure.

It is not true that equality is obtained only if the initial and final states are pure. It is not difficult to construct examples with a mixed initial state in which all but one of the probabilities for a set of histories are zero. But one then has equality in (3.23) because both sides are zero.

The inequality (3.23) lends support for the use of our approximate decoherence condition, (3.4). The degree of decoherence is basically the amount by which the left-hand side of (3.23) is less than the right-hand side. A search for other, more concrete measures of approximate decoherence would clearly be both useful and interesting.

IV. THE CALDEIRA-LEGGETT MODEL

An important class of systems in the study of decoherence are those in which there is a preferred split of the total system into a distinguished system, and the rest, summarily referred to as the environment. A natural coarse graining in such composite systems then consists of projecting onto the distinguished system only while tracing out over the environment. Models of this type have been considered extensively in the context of the reduced density matrix approach to decoherence [15]. Here, we will consider such a model in the context of the decoherence functional. The model is the Caldeira-Leggett model, originally proposed as a model of quantum Brownian motion [16]. This model is, in turn, based on earlier work of Feynman and Vernon [17].

The Caldeira-Leggett model is a comparatively simple model for decoherence in which the evolution of the reduced density matrix may be determined exactly. It consists of a distinguished system A with action

$$S_A[x]=\int_0^\tau dt[\tfrac{1}{2}M\dot{x}^2-\tfrac{1}{2}M\omega^2x^2] \tag{4.1}$$

coupled to a reservoir or environment B consisting of a large number of harmonic oscillators with coordinates R_k and action

$$S_B[\mathbf{R}]=\sum_k\int_0^\tau dt[\tfrac{1}{2}m\dot{R}_k^2-\tfrac{1}{2}m\omega_k^2R_k^2] . \tag{4.2}$$

The coupling is described by the action

$$S_I[x,\mathbf{R}]=-\sum_k\int_0^\tau dt\, C_kR_kx , \tag{4.3}$$

where the C_k's are coupling constants.

The object is to study the quantum evolution of this system, but focusing on the system A only. At any time t, the most complete quantum description of A only is given by the reduced density matrix

$$\tilde{\rho}(x,y,t)=\int d\mathbf{R}\,d\mathbf{Q}\,\delta(\mathbf{R}-\mathbf{Q})\rho(x,\mathbf{R},y,\mathbf{Q},t) , \tag{4.4}$$

where $\rho(x,\mathbf{R},y,\mathbf{Q},t)$ is the density matrix of the combined system.

The evolution of a pure state would be given by the usual propagator for the total system, which may be expressed in path-integral form:

$$\langle x_f,\mathbf{R}_f,\tau|x_0,\mathbf{R}_0,0\rangle=\int\mathcal{D}x\,\mathcal{D}\mathbf{R}\exp(iS[x,\mathbf{R}]) . \tag{4.5}$$

Here, $S[x,\mathbf{R}]$ is the total action for the system

$$S[x,\mathbf{R}]=S_A[x]+S_B[\mathbf{R}]+S_I[x,\mathbf{R}] , \tag{4.6}$$

and the sum is over paths $(x(t),\mathbf{R}(t))$ satisfying the boundary conditions

$$x(0)=x_0,\quad x(\tau)=x_f,\quad \mathbf{R}(0)=\mathbf{R}_0,\quad \mathbf{R}(\tau)=\mathbf{R}_f . \tag{4.7}$$

The solution of the total density matrix is therefore given by

$$\begin{aligned}\rho(x_f,\mathbf{R}_f,y_f,\mathbf{Q}_f,\tau)=\int & dx_0dy_0d\mathbf{R}_0d\mathbf{Q}_0\\ &\times\langle x,\mathbf{R},\tau|x_0,\mathbf{R}_0,0\rangle\\ &\times\langle y,\mathbf{Q},\tau|y_0,\mathbf{Q}_0,0\rangle^*\\ &\times\rho(x_0,\mathbf{R}_0,y_0,\mathbf{Q}_0,0) .\end{aligned} \tag{4.8}$$

Using (4.5) and (4.8), we may therefore obtain a path-integral expression for the evolution of the reduced density matrix:

$$\begin{aligned}\tilde{\rho}(x_f,y_f,\tau)=\int & dx_0dy_0d\mathbf{R}_0d\mathbf{Q}_0d\mathbf{R}_fd\mathbf{Q}_f\mathcal{D}x\,\mathcal{D}y\,\mathcal{D}\mathbf{R}\,\mathcal{D}\mathbf{Q}\,\delta(\mathbf{R}_f-\mathbf{Q}_f)\\ &\times\exp(iS_A[x]-iS_A[y]+iS_B[\mathbf{R}]-iS_B[\mathbf{Q}]+iS_I[x,\mathbf{R}]-iS_I[y,\mathbf{Q}])\\ &\times\rho(x_0,\mathbf{R}_0,y_0,\mathbf{Q}_0,0) .\end{aligned} \tag{4.9}$$

Next, it is assumed that the initial density matrix for the total system has the form

$$\rho(x_0,\mathbf{R}_0,y_0,\mathbf{Q}_0,0)=\rho_A(x_0,y_0,0)\rho_B(\mathbf{R}_0,\mathbf{Q}_0,0) . \tag{4.10}$$

For then it is possible to completely integrate out the environment in the path integral (4.9). The resulting expression may then be written

$$\tilde{\rho}(x_f,y_f,\tau)=\int dx_0dy_0J(x_f,y_f,\tau|x_0,y_0,0)\rho_A(x_0,y_0,0) . \tag{4.11}$$

Here, we have introduced

$$J(x_f,y_f,\tau|x_0,y_0,0)=\int\mathcal{D}x\,\mathcal{D}y\,\exp(iS_A[x]-iS_A[y])\mathcal{F}[x,y;\tau] , \tag{4.12}$$

where $\mathcal{F}[x,y;\tau]$ is the influence functional:

$$\begin{aligned}\mathcal{F}[x,y;\tau]=\int & d\mathbf{R}_0d\mathbf{Q}_0d\mathbf{R}_fd\mathbf{Q}_f\delta(\mathbf{R}_f-\mathbf{Q}_f)\rho_B(\mathbf{R}_0,\mathbf{Q}_0,0)\\ &\times\int\mathcal{D}\mathbf{R}\,\mathcal{D}\mathbf{Q}\exp(iS_B[\mathbf{R}]-iS_B[\mathbf{Q}]\\ &\qquad+iS_I[x,\mathbf{R}]\\ &\qquad-iS_I[y,\mathbf{Q}]) .\end{aligned} \tag{4.13}$$

The quantity J defined in Eq. (4.12) is the central object of interest in the Caldeira-Leggett model because it describes the evolution of the reduced density from any initial total density matrix of the form (4.10). It will also turn out to play an important role in the decoherence functional described in the next section.

The influence functional (4.13) may be evaluated exactly given the initial density matrix of the environment B. A useful choice, taken by Caldeira and Leggett, is to take the environment to begin in thermal equilibrium at temperature T, with the density matrix

$$\rho_B(\mathbf{R},\mathbf{Q}) = \prod_k \frac{m\omega_k}{2\pi \sinh(\omega_k/kT)} \exp\left[-\frac{m\omega_k}{2\sinh(\omega_k/kT)}[(R_k^2+Q_k^2)\cosh(\omega_k/kT) - 2R_kQ_k] \right] . \quad (4.14)$$

The influence functional is then given by

$$\mathcal{F}[x,y;\tau] = \exp\left[-\int_0^\tau f(s) \right] , \quad (4.15)$$

where

$$f(s) = \int_0^2 ds'ds[x(s)-y(x)] \times \alpha_R(s-s')[x(s')-y(s')] + i\int_0^2 ds'ds[x(s)-y(s)] \times \alpha_I(s-s')[x(s')-y(s')] \quad (4.16)$$

and

$$\alpha_R(s-s') = \sum_k \frac{C_k^2}{2m\omega_k} \coth\left[\frac{\omega_k}{2kT}\right] \cos\omega_k(s-s') , \quad (4.17)$$

$$\alpha_I(s-s') = \sum_k \frac{C_k^2}{2m\omega_k} \sin\omega_k(s-s') . \quad (4.18)$$

Caldeira and Leggett next choose to take a continuum of oscillators in the environment, with density $\rho_D(\omega)$, which involves the replacements

$$\sum_k \to \int_0^\infty d\omega\, \rho_D(\omega), \quad C_k \to C(\omega) \quad (4.19)$$

in (4.17) and (4.18). Furthermore, a high-frequency cutoff in the sum over ω is taken of the form

$$\rho_D(\omega)C^2(\omega) = \begin{cases} \dfrac{4Mm\gamma\omega^2}{\pi} & \text{if } \omega < \Omega , \\ 0 & \text{if } \omega > \Omega . \end{cases} \quad (4.20)$$

The result has the general form

$$J(x_f,y_f,\tau|x_0,y_0,0) = \int \mathcal{D}x\, \mathcal{D}y\, \exp(i\tilde{S}[x,y] - \phi[x,y]) . \quad (4.21)$$

The effect of tracing out the environment leads, amongst other effects, to a renormalizing of the frequency of the distinguished oscillator from ω to ω_R. We will work in the Fokker-Planck limit, for which $kT \gg \Omega \gg \omega_R$. One then has

$$\tilde{S}[x,y] = \int_0^\tau dt[\tfrac{1}{2}M\dot{x}^2 - \tfrac{1}{2}M\dot{y}^2 - \tfrac{1}{2}M\omega_R^2x^2 + \tfrac{1}{2}M\omega_R^2y^2 - M\gamma(x-y)(\dot{x}+\dot{y})] \quad (4.22)$$

and

$$\phi[x,y] = 2M\gamma kT\int_0^\tau dt[x(t)-y(t)]^2 . \quad (4.23)$$

The environment therefore has three effects of significance: renormalization of the frequency ω, the introduction of dissipation characterized by γ, and the suppression of contributions from widely differing pairs of paths in (4.21) through (4.23). It is this latter effect that will lead to decoherence.

It is particularly useful to introduce the variables $X = x+y$, $\xi = x-y$. In terms of these variables, the above expressions are

$$\tilde{S}[X,\xi] = \int_0^\tau dt(\tfrac{1}{2}M\dot{X}\dot{\xi} - \tfrac{1}{2}M\omega_R^2X\xi - M\gamma\dot{X}\xi) \quad (4.24)$$

and

$$\phi[X,\xi] = 2M\gamma kT\int_0^\tau dt\, \xi^2 . \quad (4.25)$$

Two features that will be important in what follows are first that ϕ depends only on ξ, and second that X occurs linearly in $\tilde{S}[X,\xi]$.

Now we review the evaluation of J, (4.21). This will be useful for the next sections. It is convenient to expand about the extremum of $\tilde{S}$. The extremum is the paths $X_{\rm cl}(t)$, $\xi_{\rm cl}(t)$ satisfying the equations of motion

$$D_{(+)}X \equiv \ddot{X} + 2\gamma\dot{X} + \omega_R^2X = 0 , \quad (4.26)$$

$$D_{(-)}\xi \equiv \ddot{\xi} - 2\gamma\dot{\xi} + \omega_R^2\xi = 0 , \quad (4.27)$$

subject to the boundary conditions

$$X(0) = X_0, \quad X(\tau) = X_f, \quad \xi(0) = \xi_0, \quad \xi(\tau) = \xi_f . \quad (4.28)$$

The solutions are

$$X_{\rm cl}(t) = \frac{e^{-\gamma t}}{\sin\omega\tau}[X_f e^{\gamma\tau}\sin\omega t + X_0\sin\omega(\tau-t)] , \quad (4.29)$$

$$\xi_{\rm cl}(t) = \frac{e^{\gamma t}}{\sin\omega\tau}[\xi_f e^{-\gamma\tau}\sin\omega t + \xi_0\sin\omega(\tau-t)] , \quad (4.30)$$

where $\omega^2 = \omega_R^2 - \gamma^2$. The action $\tilde{S}$ evaluated on these solutions is

$$\tilde{S}_{\rm cl} = \tilde{K}(\tau)X_f\xi_f + \hat{K}(\tau)X_0\xi_0 - L(\tau)X_0\xi_f - N(\tau)X_f\xi_0 , \quad (4.31)$$

where

$$\tilde{K}(\tau) = -\tfrac{1}{2}M\gamma + \tfrac{1}{2}M\omega\cot\omega\tau , \quad (4.32)$$

$$\hat{K}(\tau) = +\tfrac{1}{2}M\gamma + \tfrac{1}{2}M\omega\cot\omega\tau , \quad (4.33)$$

$$L(\tau) = \frac{M\omega e^{-\gamma\tau}}{2\sin\omega\tau} , \quad (4.34)$$

$$N(\tau) = \frac{M\omega e^{\gamma\tau}}{2\sin\omega\tau} . \quad (4.35)$$

Now write

$$X(t) = X_{\rm cl}(t) + \delta X(t), \quad \xi(t) = \xi_{\rm cl}(t) + \delta\xi(t) , \quad (4.36)$$

where

$$\delta X(0) = 0, \quad \delta X(\tau) = 0, \quad \delta\xi(0) = 0, \quad \delta\xi(\tau) = 0 . \quad (4.37)$$

The path integral (4.21) now becomes

$$J(X_f,\xi_f,\tau|X_0,\xi_0,0)=\exp(i\tilde{S}_{\rm cl})\int\mathcal{D}(\delta X)\mathcal{D}(\delta\xi)\exp\left[-i\frac{M}{2}\int dt\,\delta X D_{(-)}\delta\xi-\phi[\xi_{\rm cl}+\delta\xi]\right]. \qquad (4.38)$$

However, since the exponent is just linear in δX, the integral over δX is readily performed to pull down a δ functional $\delta[D_{(-)}\delta\xi]$. Integrating over $\delta\xi$, the only contribution thus comes from $\delta\xi=0$, and a prefactor of $(\det[D_{(-)}])^{-1}$ appears. This prefactor was evaluated by Caldeira and Leggett, and we denote the result $F^2(\tau)$. The final result is therefore of the form

$$J(X_f,\xi_f,\tau|X_0,\xi_0,0)=F^2(\tau)\exp\{i\tilde{S}_{\rm cl}-\phi[\xi_{\rm cl}(t)]\}\ . \qquad (4.39)$$

Here, $\phi[\xi_{\rm cl}(t)]$ has the form

$$\phi[\xi_{\rm cl}(t)]=A(\tau)\xi_f^2+B(\tau)\xi_f\xi_0+C(\tau)\xi_0^2\ . \qquad (4.40)$$

Explicit (but rather lengthy) expressions for the coefficients A, B, C are given in Ref. [16], and we do not give them here. However, they simplify enormously in the Fokker-Planck limit considered here, in which case they are given by

$$A(\tau)=\frac{2M\gamma kTe^{-2\gamma\tau}}{\sin^2\omega\tau}\left[\frac{1}{4\gamma}(e^{2\gamma\tau}-1)-I\right], \qquad (4.41)$$

$$B(\tau)=\frac{2M\gamma kTe^{-\gamma\tau}}{\sin^2\omega\tau}\left[-\frac{\cos\omega\tau}{2\gamma}(e^{2\gamma\tau}-1)+I\cos\omega\tau+J\sin\omega\tau\right], \qquad (4.42)$$

$$C(\tau)=\frac{M\gamma kT}{\sin^2\omega\tau}\left[\frac{1}{2\gamma}(e^{2\gamma\tau}-1)-I\cos2\omega\tau-J\sin2\omega\tau\right], \qquad (4.43)$$

where

$$I=\tfrac{1}{2}\gamma(\gamma^2+\omega^2)^{-1}(e^{2\gamma\tau}\cos2\omega\tau-1)+\tfrac{1}{2}\omega(\gamma^2+\omega^2)^{-1}e^{2\gamma\tau}\sin2\omega\tau\ , \qquad (4.44)$$

$$J=-\tfrac{1}{2}\omega(\gamma^2+\omega^2)^{-1}(e^{2\gamma\tau}\cos2\omega\tau-1)+\tfrac{1}{2}\gamma(\gamma^2+\omega^2)^{-1}e^{2\gamma\tau}\sin2\omega\tau\ . \qquad (4.45)$$

For future reference, we note that, in the short time limit, each of $A(\tau)$, $B(\tau)$, and $C(\tau)$ are approximately equal to $\frac{2}{3}M\gamma kT\tau+O(\tau^2)$.

V. THE DECOHERENCE FUNCTIONAL FOR THE CALDEIRA-LEGGETT MODEL

We are going to calculate the decoherence functional (2.21) for the system described in the previous section, consisting of a distinguished harmonic oscillator coupled in an environment consisting of a thermal bath of harmonic oscillators to provide decoherence. The projection operators will be projections onto the position of the distinguished oscillator. For mathematical simplicity, we will use Gaussian projections, (2.5).

A. The decoherence functional

The decoherence functional is written down most readily using the path-integral form (2.36). In our case it is

$$\begin{aligned}D[\bar{x}_k,\bar{y}_k]=\int & dx_f dy_f d\mathbf{R}_f d\mathbf{Q}_f dx_0 dy_0 d\mathbf{R}_0 d\mathbf{Q}_0 \mathcal{D}x\,\mathcal{D}y\,\mathcal{D}\mathbf{Q}\,\mathcal{D}\mathbf{R}\,\delta(x_f-y_f)\delta(\mathbf{R}_f-\mathbf{Q}_f)\rho_A(x_0,y_0)\rho_B(\mathbf{R}_0,\mathbf{Q}_0)\\ &\times\exp(iS_A[x]-iS_A[y]+iS_B[\mathbf{R}]-iS_B[\mathbf{Q}]+iS_I[x,\mathbf{R}]-iS_I[y,\mathbf{Q}])\\ &\times\exp\left[-\sum_{k=1}^{n}\frac{[x(t_k)-\bar{x}_k]^2}{\sigma_k^2}-\sum_{k=1}^{n}\frac{[y(t_k)-\bar{y}_k]^2}{\sigma_k^2}\right].\end{aligned} \qquad (5.1)$$

For convenience, we will omit preexponential factors in Secs. V A–V C. (These can always be deduced, if desired, by appealing to normalization conditions.) The sum is over histories $(x(t),y(t),\mathbf{R}(t),\mathbf{Q}(t))$, where t runs from $t=t_0$ to $t=t_f=t_{n+1}$, and the histories satisfy the boundary conditions.

$$x(t_0)=x_0,\ \ y(t_0)=y_0,\ \ x(t_f)=x_f,\ \ y(t_f)=y_f\ , \qquad (5.2)$$

$$\mathbf{R}(t_0)=\mathbf{R}_0,\ \ \mathbf{Q}(t_0)=\mathbf{Q}_0,\ \ \mathbf{R}(t_f)=\mathbf{R}_f,\ \ \mathbf{Q}(t_f)=\mathbf{Q}_f\ . \qquad (5.3)$$

On the initial surface, $t=t_0$, the initial density matrix of the system is folded in and is taken to have the form (4.10) and (4.14); on the final surface at $t=t_f$, the δ functions enforce $x_f=y_f$, $\mathbf{R}_f=\mathbf{Q}_f$, and then x_f and $\mathbf{R}_f$ are summed over. The histories are obliged to pass through the Gaussian slits at positions $\bar{x}_k,\bar{y}_k$ at times $t=t_k$, for $k=1,\ldots,n$. It will be convenient to work always in the Fokker-Planck limit.

Because the projections refer only to system A and not the environment B, the environment coordinates may be completely integrated out. One thus obtains

$$D[\bar{x}_k,\bar{y}_k]=\int dx_f dy_f dx_0 dy_0 \mathcal{D}x\,\mathcal{D}y\,\delta(x_f-y_f)\rho_A(x_0,y_0)\exp(iS_A[x]-iS_A[y])\mathcal{F}[x,y;\tau) \times\exp\left[-\sum_{k=1}^{n}\frac{[x(t_k)-\bar{x}_k]^2}{\sigma_k^2}-\sum_{k=1}^{n}\frac{[y(t_k)-\bar{y}_k]^2}{\sigma_k^2}\right], \tag{5.4}$$

where $\mathcal{F}[x,y;\tau)$ is the influence functional introduced in the previous section (in the Fokker-Planck limit). We then have

$$D[\bar{x}_k,\bar{y}_k]=\int dx_f dy_f dx_0 dy_0 \mathcal{D}x\,\mathcal{D}y\,\delta(x_f-y_f)\rho_A(x_0,y_0)\exp(i\tilde{S}[x,y]-\phi[x,y]) \times\exp\left[-\sum_{k=1}^{n}\frac{[x(t_k)-\bar{x}_k]^2}{\sigma_k^2}-\sum_{k=1}^{n}\frac{[y(t_k)-\bar{y}_k]^2}{\sigma_k^2}\right], \tag{5.5}$$

where $\tilde{S}[x,y]$ and $\phi(x,y)$ are given by (4.22) and (4.23), respectively.

Because the projections reside only the discrete set of slices $t=t_k$, for $k=1,2,\ldots,n$, it is convenient to rewrite (5.5) in terms of integrals on these slices and propagation between them. It is also useful to go to the variable X, ξ, defined by $X=x+y$, $\xi=x-y$. We then have

$$D[\bar{X}_k,\bar{\xi}_k]=\int dX_{n+1}d\xi_{n+1}dX_n d\xi_n\cdots dX_0 d\xi_0\delta(\xi_{n+1})\rho_A(X_0,\xi_0,t_0) \times\prod_{k=0}^{n}J(X_{k+1},\xi_{k+1},t_{k+1}|X_k,\xi_k,t_k)\exp\left[-\sum_{k=1}^{n}\frac{(X_k-\bar{X}_k)^2}{\sigma_k^2}-\sum_{k=1}^{n}\frac{(\xi_k-\bar{\xi}_k)^2}{\sigma_k^2}\right]. \tag{5.6}$$

Here,

$$J(X_{k+1},\xi_{k+1},t_{k+1}|X_k,\xi_k,t_k) =\int\mathcal{D}X\,\mathcal{D}\xi\exp\{i\tilde{S}[X,\xi;t_{k+1},t_k)-\phi[X,\xi;t_{k+1},t_k)\} \tag{5.7}$$

where $\tilde{S}[X,\xi;t_{k+1},t_k]$ and $\phi[X,\xi;t_{k+1},t_k]$ denote the quantities (4.22) and (4.23), respectively, but with the integration domain $[0,\tau]$ replaced by $[t_k,t_{k+1}]$. Also, for convenience we have made the redefinition $2\sigma_k^2\to\sigma_k^2$. As in the previous section, Eq. (5.7) may be evaluated exactly with the result

$$J(X_{k+1},\xi_{k+1},t_{k+1}|X_k,\xi_k,t_k) =F_{k+1,k}^2\exp(i\tilde{S}_{k+1,k}-\phi_{k+1,k}), \tag{5.8}$$

where $F_{k+1,k}=F(t_{k+1}-t_k)$ with $F(t)$ as in Eq. (4.39). Also,

$$\tilde{S}_{k+1,k}\equiv\tilde{S}_{\rm cl}(X_{k+1},\xi_{k+1},t_{k+1}|X_k,\xi_k,t_k) =\tilde{K}_{k+1,k}X_{k+1}\xi_{k+1}+\hat{K}_{k+1,k}X_k\xi_k -L_{k+1,k}X_k\xi_{k+1}-N_{k+1,k}X_{k+1}\xi_k, \tag{5.9}$$

where $\tilde{K}_{k+1,k}=\tilde{K}(t_{k+1}-t_k)$ with $K(t)$ given by (4.32), and similarly for $\hat{K}_{k+1,k},L_{k+1,k}$ and $N_{k+1,k}$. Likewise,

$$\phi_{k+1,k}=A_{k+1,k}\xi_{k+1}^2+B_{k+1,k}\xi_{k+1}\xi_k+C_{k+1,k}\xi_k^2 \tag{5.10}$$

with $A_{k+1,k}=A(t_{k+1}-t_k)$, etc.

As an aside, we note the following point. The propagator J from $t=0$ to $t=\tau$ is given by Eq. (4.12), which involves the influence functional (4.13). This, in turn, involves the density matrix of the reservoir B at time $t=t_0$, given by Eq. (4.14). The propagator between slices $t=t_k$ and $t=t_{k+1}$, Eq. (5.7), comes from expressions of identical form, but with the change of domains of integration noted above. It is perhaps surprising, however, that the propagator from t_k to t_{k+1} should involve the density matrix ρ_B at $t=t_0$. The reason for this is that the environment in the Caldeira-Leggett model is taken to be essentially infinite. This means that, although the system A is itself affected substantially by its interaction with the environment B, A has negligible effect on the dynamics of the environment. To a good approximation therefore, the environment is always in thermal equilibrium, described by the density matrix (4.14) for all time.

Our task now is to evaluate the decoherence functional (5.6) for various choices of initial density matrix ρ_A. All the integrations are Gaussian and may therefore be carried out in closed form. A direct assault on the integrations is possible, but for the purposes of exhibiting the qualitative features of the decoherence functional (our aim in this section), we have found the following method to be convenient. Recall that, in Sec. IV, the evaluation of the propagator J was considerably eased by the simple observation that X occurs linearly in the exponent. Because of the presence of the projections, X does not occur linearly in the exponent of the decoherence functional. However, the following trick turns out to be extremely useful. Write

$$\exp\left[-\frac{(X_k-\bar{X}_k)^2}{\sigma_k^2}\right] =\frac{1}{\sqrt{\pi}}\int dP_k\exp\left[-P_k^2+\frac{2iP_k}{\sigma_k}(X_k-\bar{X}_k)\right]. \tag{5.11}$$

Now inserting (5.8) and (5.11), the decoherence functional becomes

$$D[\bar{X}_k,\bar{\xi}_k]=\int dX_{n+1}d\xi_{n+1}dX_0d\xi_0d^nXd^n\xi d^nP\delta(\xi_{n+1})\rho_A(X_0,\xi_0,t_0)\exp\left[\sum_{k=0}^{n}[i\tilde{S}_{k+1,k}-\phi_{k+1,k}]\right]$$
$$\times\exp\left[-\sum_{k=1}^{n}\left[P_k^2+\frac{2iP_k}{\sigma_k}(X_k-\bar{X}_k)+\frac{(\xi_k-\bar{\xi}_k)^2}{\sigma_k^2}\right]\right]. \tag{5.12}$$

The exponent of the decoherence functional is now entirely linear in the variables X_k, and we may proceed with the evaluation, beginning with the integral over X_k.

A change of variables is useful. Consider the classical solution for $X(t)$ connecting the initial and final points, (4.29). Write it as

$$X_{\rm cl}(t)=X_{n+1}\alpha(t)+X_0\beta(t). \tag{5.13}$$

Here, $\alpha(t)$ and $\beta(t)$ are solutions to the field equations for X whose exact form may be found by comparison with (4.29). They satisfy the boundary conditions

$$\alpha(t_0)=0,\quad \alpha(t_f)=1, \tag{5.14}$$

$$\beta(t_0)=1,\quad \beta(t_f)=0. \tag{5.15}$$

Now perform the change of variables

$$\begin{aligned}X_k&=X_k^{\rm cl}+\delta X_k\\&=X_{n+1}\alpha_k+X_0\beta_k+\delta X_k,\end{aligned} \tag{5.16}$$

where $\alpha_k=\alpha(t_k)$, $\beta_k=\beta(t_k)$. It follows from the above that δX_k obey the boundary conditions

$$\delta X_0=0=\delta X_{n+1}. \tag{5.17}$$

Under this shift of integration variables, one finds that

$$\sum_{k=1}^{n}\tilde{S}_{k+1,k}=\tilde{S}_{\rm cl}(X_f,\xi_f,t_f|X_0,\xi_0,t_0)+\tilde{S}^{(1)}, \tag{5.18}$$

where $\tilde{S}_{\rm cl}$ is given [from (4.13)] by

$$\tilde{S}_{\rm cl}=\hat{K}(\tau)\xi_0X_0-N(\tau)\xi_0X_f \tag{5.19}$$

and

$$\tilde{S}^{(1)}=\sum_{k=1}^{n}[-L_{k+1,k}\xi_{k+1}+(\hat{K}_{k+1,k}+\tilde{K}_{k,k-1})\xi_k-N_{k,k-1}\xi_{k-1}]\delta X_k. \tag{5.20}$$

Using the above results, the decoherence functional may now be written

$$D[\bar{X}_k,\bar{\xi}_k]=\int dX_{n+1}dX_0d\xi_0d^n(\delta X)d^n\xi d^nP\rho_A(X_0,\xi_0,t_0)$$
$$\times\exp\left[i\hat{K}(\tau)X_0\xi_0-iN(\tau)X_{n+1}\xi_0+i\tilde{S}^{(1)}-\sum_{k=0}^{n}\phi_{k+1,k}(\xi_k,\xi_{k+1})\right.$$
$$\left.-\sum_{k=1}^{n}\left[P_k^2+\frac{2iP_k}{\sigma_k}(X_{n+1}\alpha_k+X_0\beta_k+\delta X_k-\bar{X}_k)+\frac{(\xi_k-\bar{\xi}_k)^2}{\sigma_k^2}\right]\right] \tag{5.21}$$

The integrals over X_{n+1} and δX_k pull down the δ functions

$$\delta^{(n)}\left[-L_{k+1,k}\xi_{k+1}+(\hat{K}_{k+1,k}+\tilde{K}_{k,k-1})\xi_k-N_{k,k-1}\xi_{k-1}-2\frac{P_k}{\sigma_k}\right]\delta\left[N(\tau)\xi_0+2\sum_{k=1}^{n}\frac{\alpha_kP_k}{\sigma_k}\right]. \tag{5.22}$$

The integrations over ξ_0 and $\delta\xi_k$ may then be performed. The only contributions come from

$$\xi_0=-\frac{2}{N(\tau)}\sum_{k=1}^{n}\frac{\alpha_kP_k}{\sigma_k} \tag{5.23}$$

and from the value of ξ_k satisfying the difference equation

$$-L_{k+1,k}\xi_{k+1}+(\hat{K}_{k+1,k}+\tilde{K}_{k,k-1})\xi_k-N_{k,k-1}\xi_{k-1}=2\frac{P_k}{\sigma_k} \tag{5.24}$$

for $k=1,2,\ldots,n$, with the boundary conditions that $\xi_{n+1}=0$. Equation (5.24) may be solved explicitly, but the exact form of the solution will not be needed. It will be linearly dependent on ξ_0 and P_k. We will hereafter assume the integrations over ξ_0 and ξ_k have been done, and use ξ_0 to denote the right-hand side of (4.23) and $\delta\xi_k$ to denote the solution to (5.24). The feature of ξ_0 and ξ_k to keep in mind is that they are both linear in P_k. We now have

$$D[\bar{X}_k,\bar{\xi}_k]=\int dX_0 d^nP\,\rho_A(X_0,\xi_0,t_0)\exp\left[i\hat{K}(\tau)X_0\xi_0-\sum_{k=0}^{n}\phi_{k+1,k}(\xi_k,\xi_{k+1})-\sum_{k=1}^{n}\left[P_k^2+\frac{2iP_k}{\sigma_k}(X_0\beta_k-\bar{X}_k)+\frac{(\xi_k-\bar{\xi}_k)^2}{\sigma_k^2}\right]\right] . \tag{5.25}$$

We will now consider the evaluation of this expression for various different forms for the initial density matrix.

B. Wave-packet initial states

We first consider an initial density matrix corresponding to a pure state consisting of a wave packet of approximate momentum p centered about the point $\bar{x}_0$. One thus has

$$\rho_A(X_0,\xi_0,t_0)=\exp\left[ip\xi_0-\frac{(X_0-\bar{X}_0)^2}{\sigma^2}-\frac{\xi_0^2}{\sigma^2}\right] , \tag{5.26}$$

where $\bar{X}_0=2\bar{x}_0$. Inserting this into the decoherence functional, the integration over X_0 may be performed, and one obtains

$$D[\bar{X}_k,\bar{\xi}_k]=\int d^nP\exp\left[ip\xi_0-\frac{\xi_0^2}{\sigma^2}-\sum_{k=0}^{n}\phi_{k+1,k}(\xi_k,\xi_{k+1})+i\bar{X}_0\left[\hat{K}(\tau)\xi_0,2\sum_{k=1}^{n}\frac{\beta_kP_k}{\sigma_k}\right]-\frac{\sigma^2}{4}\left[\hat{K}(\tau)\xi_0-2\sum_{k=1}^{n}\frac{\beta_kP_k}{\sigma_k}\right]^2-\sum_{k=1}^{n}\left[P_k^2-\frac{2iP_k}{\sigma_k}\bar{X}_k+\frac{(\xi_k-\bar{\xi}_k)^2}{\sigma_k^2}\right]\right] . \tag{5.27}$$

The important step now is to organize the exponent into terms quadratic and linear in P_k. We therefore write the decoherence functional in the form

$$D[\bar{X}_k,\bar{\xi}_k]=\int d^nP\exp\left[-\sum_{k=1}^{n}\sum_{j=1}^{n}P_kM_{kj}P_j+\sum_{k=1}^{n}(U_k+iV_k)P_k-\sum_{k=1}^{n}\frac{\bar{\xi}_k^2}{\sigma_k^2}\right] , \tag{5.28}$$

where

$$\sum_{k=1}^{n}\sum_{j=1}^{n}P_kM_{kj}P_j=\frac{\xi_0^2}{\sigma^2}+\sum_{k=0}^{n}\phi_{k+1,k}(\xi_k,\xi_{k+1})+\frac{\sigma^2}{4}\left[\hat{K}(\tau)\xi_0-2\sum_{k=1}^{n}\frac{\beta_kP_k}{\sigma_k}\right]^2+\sum_{k=1}^{n}\left[P_k^2+\frac{\xi_k^2}{\sigma_k^2}\right] , \tag{5.29}$$

$$\sum_{k=1}^{n}U_kP_k=2\sum_{k=1}^{n}\frac{\xi_k}{\sigma_k^2}\bar{\xi}_k , \tag{5.30}$$

and

$$\sum_{k=1}^{n}V_kP_k=p\xi_0+\bar{X}_0\left[\hat{K}(\tau)\xi_0-2\sum_{k=1}^{n}\frac{\beta_kP_k}{\sigma_k}\right]+2\sum_{k=1}^{n}\frac{P_k}{\sigma_k}\bar{X}_k . \tag{5.31}$$

In particular,

$$V_k=\frac{2}{\sigma_k}\left[\bar{X}_k-\frac{p}{N(\tau)}\alpha_k-\bar{X}_0\left[\frac{\hat{K}(\tau)}{N(\tau)}\alpha_k+\beta_k\right]\right] . \tag{5.32}$$

Now let

$$Y_k=\frac{p}{N(\tau)}\alpha_k+\left[\frac{\hat{K}(\tau)}{N(\tau)}\alpha_k+\beta_k\right]\bar{X}_0 . \tag{5.33}$$

The significance of this is as follows. Consider the classical solution for $X(t)$ given by (5.13). This is the solution for fixed initial and final X. However, Hamilton-Jacobi theory together with Eq. (4.31) give

$$P_\xi(t_0)=-\frac{\partial\tilde{S}}{\partial\xi_0}=-\hat{K}(\tau)X_0+N(\tau)X_f , \tag{5.34}$$

where P_ξ is the momentum conjugate to ξ, and we can use this relation to obtain the classical solution for fixed initial X and P_ξ:

$$X_{\rm cl}(t)=\frac{P_\xi(t_0)}{N(\tau)}\alpha(t)+\left[\frac{\hat{K}(\tau)}{N(\tau)}\alpha(t)+\beta(t)\right]X_0 . \tag{5.35}$$

When the decoherence functional is diagonal, $x=y=\frac{1}{2}X$, and since $P_\xi=\frac{1}{2}M\dot{X}$, we can identify $p=M\dot{x}$, the momentum conjugate to x, with P_ξ. We therefore have the result that $Y_k=X_{\rm cl}(t_k)=2x_{\rm cl}(t_k)$, where $x_{\rm cl}(t)$ is the classical solution with initial position $\bar{x}_0$ and initial momentum p.

The integral over P_k may be carried out, with the formal result

$$D[\bar{X}_k,\bar{\xi}_k]=\exp\left[+\frac{1}{4}U^TM^{-1}U-\frac{i}{2}U^TM^{-1}V-\frac{1}{4}V^TM^{-1}V-\sum_{k=1}^{n}\frac{\bar{\xi}_k^2}{\sigma_k^2}\right] \quad (5.36)$$

in an obvious matrix notation. Equation (5.36) may be rearranged into the form

$$D[\bar{X}_k,\bar{\xi}_k]=\exp\left[-\frac{1}{4}\sum_{kj}\bar{\xi}_k\tilde{M}_{kj}\bar{\xi}_j-\frac{i}{2}U^TM^{-1}V-\sum_{kj}\frac{\bar{X}_k-Y_k}{\sigma_k}M_{kj}^{-1}\frac{\bar{X}_j-Y_j}{\sigma_j}\right], \quad (5.37)$$

where $\tilde{M}_{kj}$ may be found from the above. It will be positive definite because the decoherence functional is by construction normalizable. Similarly, it follows from (5.29) that M_{kj} is positive definite.

We may now see that the decoherence functional has the expected qualitative features. The first term in the exponent of (5.37) shows that the decoherence functional is small for large values of $\bar{\xi}_k$, i.e., that distinct histories decohere. The second term, which is linear in $\bar{\xi}_k$, is purely imaginary. It does not affect decoherence and, in fact, vanishes when $\bar{\xi}_k$ is set to zero. The third term clearly shows that the diagonal part of the decoherence functional is peaked when the slit positions $\bar{X}_k$ lie along the classical trajectory, $\bar{X}_k=Y_k$.

Note, however, that this is the decoherence functional specifically for the wave-packet initial state and it is yet to be seen whether these features continue to hold for more general initial states. Furthermore, it should be noted that the peaking of the (modulus of the) decoherence functional about $\bar{\xi}_k=0$ is, at best, a crude qualitative indication of a tendency towards decoherence. A much better quantitative indication is the condition (3.4), and this is what we shall use in what follows.

At this stage, the full advantage of writing the slit projections in X in terms of their Fourier transform is clear. The qualitative features of the decoherence functional—decoherence of distinct histories, and peaking about classical trajectories—are clearly exhibited. The detailed expressions for the widths of the peaks are rather complicated. But use of the identity (5.11) leads to a clean separation of the terms giving the configuration about which the decoherence functional is peaked from the terms giving the width of the peaks: the former are linear in P_k in (5.28) and the latter are quadratic in P_k. It seems likely that this simple trick will be similarly useful in calculations of more complicated decoherence functionals.

C. General initial states: The Wigner function

For more general initial states, we have found that some of the qualitative features of the decoherence functional may be exhibited using the Wigner transform of the initial density matrix. We therefore write the initial density matrix,

$$\rho_A(X_0,\xi_0,t_0)=\int dp_0 e^{ip_0\xi_0}W(p_0,X_0)\ , \quad (5.38)$$

where $W(p_0,X_0)$ is the Wigner function and is obtained in terms of ρ_A using the inverse of (5.38). The Wigner function has many properties shared by classical phase-space distributions and has often been proposed as an interpretational tool [18,19]. Inserting (5.38) into (5.25), one obtains

$$D[\bar{X}_k,\bar{\xi}_k]=\int dp_0dX_0d^nP\ W(p_0,X_0)\exp\left[i[p_0+\hat{K}(\tau)X_0]\xi_0-\sum_{k=0}^{n}\phi_{k+1,k}(\xi_k,\xi_{k+1})-\sum_{k=1}^{n}\left[P_k^2+\frac{2iP_k}{\sigma_k}(X_0\beta_k-\bar{X}_k)+\frac{(\xi_k-\bar{\xi}_k)^2}{\sigma_k^2}\right]\right], \quad (5.39)$$

where, recall, ξ_0 and ξ_k are given by (5.23) and (5.24). Inserting the expression for ξ_0, some elementary arrangement of the terms yields an expression very similar to (5.28):

$$D[\bar{X}_k,\bar{\xi}_k]=\int dp_0dX_0d^nP\ W(p_0,X_0)\times\exp\left[-\sum_{k=1}^{n}\sum_{j=1}^{n}P_kM_{kj}P_j+\sum_{k=1}^{n}(U_k+iV_k)P_k-\sum_{k=1}^{n}\frac{\bar{\xi}_k^2}{\sigma_k^2}\right]. \quad (5.40)$$

Here, U_k is given as before by (5.30), but M_{kj} is given by

$$\sum_{k=1}^{n}\sum_{j=1}^{n}P_kM_{kj}P_j=\sum_{k=0}^{n}\phi_{k+1,k}(\xi_k,\xi_{k+1})+\sum_{k=1}^{n}\left[P_k^2+\frac{\xi_k^2}{\sigma_k^2}\right]. \quad (5.41)$$

Also, $V_k=2(\bar{X}_k-Y_k)/\sigma_k$, where Y_k is given by

$$Y_k=\frac{p_0}{N(\tau)}\alpha_k+\left[\frac{\hat{K}(\tau)}{N(\tau)}\alpha_k+\beta_k\right]X_0\ . \quad (5.42)$$

This differs from (5.33) only in as much as p and $\bar{X}_0$ have been replaced by p_0 and X_0.

Again, one can formally carry out the integration over

P_k, with the result

$$D[\bar{X}_k,\bar{\xi}_k]=\int dp_0 dX_0 W(p_0,X_0)$$
$$\times\exp\left[+\frac{1}{4}U^T M^{-1}U-\frac{i}{2}U^T M^{-1}V -\frac{1}{4}V^T M^{-1}V-\sum_{k=1}^{n}\frac{\bar{\xi}_k^2}{\sigma_k^2}\right]. \tag{5.43}$$

In particular, setting $\xi_k=0$, we see that the diagonal part of the decoherence functional is given by

$$p[\bar{X}_k]=\int dp_0 dX_0 W(p_0,X_0)$$
$$\times\exp\left[-\sum_{kj}\frac{\bar{X}_k-Y_k}{\sigma_k}M_{kj}^{-1}\frac{\bar{X}_j-Y_j}{\sigma_j}\right]. \tag{5.44}$$

This, then, is the formal result for an arbitrary initial density matrix.

The form of (5.44) is suggestive of an ensemble of classical paths, with the Wigner function of the initial density matrix giving the probability distribution of their initial values of coordinates and momenta. This cannot be quite correct, however. First, the Wigner function is not always positive, whereas (5.44) is by construction. Second, (5.44) is a probability distribution on a sequence of position samples and makes no reference to momenta. The connection with phase-space distributions is obtained by considering histories consisting of position samplings at two moments of time. By taking the time very close together, one thus obtains an approximate position sampling together with a time-of-flight momentum sampling over a short time interval. The resulting probability distribution turns out to be the Wigner function smeared over an $\hbar$-sized region of phase space—just sufficient to make it positive. These results are described in more detail in a separate paper [20].

D. Decoherence

The complexity of expressions such as (5.37) makes it difficult to obtain more than qualitative information about decoherence and classical peaking. More precise quantitative calculations for simpler cases will be the subject of the following sections. Here we note one particular simple case showing some important quantitative features of decoherence. First of all, take the projections onto the distinguished system to be at every moment of time, from t_0 to t_f. Secondly, take their widths to zero, so that the histories for the distinguished system are completely fine grained. From (5.5), one thus obtains

$$D[\bar{x}(t),\bar{y}(t)]=\delta(\bar{x}_f-\bar{y}_f)\exp(i\tilde{S}[\bar{x},\bar{y}]-\phi[\bar{x},\bar{y}])$$
$$\times\rho_A(\bar{x}_0,\bar{y}_0). \tag{5.45}$$

Using the density matrix inequality (3.15) for ρ_A, and using the explicit form for ϕ, (4.23), one finds that the decoherence functional satisfies the approximate decoherence condition

$$|D[\bar{x},\bar{y}]|\le\exp\left[-2M\gamma kT\int dt[\bar{x}-\bar{y}]^2\right]$$
$$\times(D[\bar{x},\bar{x}]D[\bar{y},\bar{y}])^{1/2}. \tag{5.46}$$

This indicates that paths separated by distances of order l decohere on a time scale of order

$$t_D\sim(2M\gamma kTl^2)^{-1}. \tag{5.47}$$

As noted by Zurek, this time can be very short indeed [21].

This simple case therefore explicitly indicates the general tendency of the environment to induce decoherence. But it also illustrates a subtlety. To obtain decoherence of the set of histories $\{\bar{x}(t)\}$ to some degree $\epsilon<1$, it is necessary that

$$\exp\left[-2M\gamma kT\int dt[\bar{x}-\bar{y}]^2\right]\le\epsilon. \tag{5.48}$$

However, the set of histories $\{\bar{x}(t)\}$ are completely fine grained. It follows that it will always be possible to find pairs of histories $\bar{x}(t),\bar{y}(t)$ which are distinct, yet for which $\int dt[\bar{x}-\bar{y}]^2$ is so close to zero that (5.48) cannot be satisfied. Clearly what is needed is further coarse graining of the histories $\{\bar{x}(t)\}$, so that $\bar{x}$ has significance only up to some length scale l, say [22]. The moral of this, therefore, is that, to satisfy an approximate decoherence condition of the form (3.4), in this model *both* types of coarse graining are necessary—tracing out the environment *and* smearing over position.

VI. EXPLICIT EVALUATION OF SOME SPECIAL CASES

In Sec. V, we evaluated the decoherence functional for the case of an arbitrary number of projections in the Caldeira-Leggett model. Or rather, we evaluated it to the point where its qualitative features could be seen: decoherence and peaking about classical trajectories. However, we were not able to evaluate it to the point where we could obtain a *quantitative* idea of the degree of decoherence. In this and the next section, therefore, we will evaluate the decoherence functional completely for the simplest nontrivial case, namely, the case of histories characterized by projections at just two moments of time. This involves evaluating (5.6) for the case $n=2$.

As we have seen already, the decoherence functional has the property that it is diagonal in the final projection, although in the present case this is only true approximately because the Gaussian slit projectors obey the mutually exclusive property only approximately. Nevertheless, to the extent that it is true, Eq. (5.6) for the case $n=2$ reduces to

$$D(\bar{X}_1,\bar{X}_2,\bar{\xi}_1)=\int dX_2 d\xi_2 dX_1 d\xi_1 dX_0 d\xi_0 \delta(\xi_2)\rho_A(X_0,\xi_0,t_0)F_{2,1}^2 F_{1,0}^2 \exp(i\tilde{S}_{2,1}-\phi_{2,1}+i\tilde{S}_{1,0}-\phi_{1,0}) \times \exp\left[-\frac{(X_2-\bar{X}_2)^2}{\sigma_2^2}-\frac{(X_1-\bar{X}_1)^2}{\sigma_1^2}-\frac{(\xi_1-\bar{\xi}_1)^2}{\sigma_1^2}\right], \quad (6.1)$$

where F, $\tilde{S}$, and ϕ are defined by Eqs. (5.8)–(5.10), and for convenience we have performed the redefinitions $2\sigma_k^2 \to \sigma_k^2$ for $k=1,2$. The difference between exact diagonality in the final projection [assumed in (6.1)] and approximate diagonality [exhibited by (5.6)] amounts to an overall factor of $\exp(-\bar{\xi}_2^2/\sigma_2^2)$. Our approximation therefore involves taking it to be equal to one.

For the initial density matrix, we will take a general Gaussian

$$\rho_A(X_0,\xi_0,t_0)=\exp(-\alpha_0 X_0^2-\beta_0\xi_0^2-\gamma_0 X_0\xi_0 +\mu_0 X_0+\nu_0\xi_0+\epsilon_0) . \quad (6.2)$$

Here, α_0 and β_0 are real and γ_0 is imaginary. Clearly $\alpha_0>0$ for normalizability, $\mathrm{Tr}\rho=1$. The density operator $\hat{\rho}_A$ must be a positive operator, i.e., $\langle\psi|\hat{\rho}_A|\psi\rangle\geq 0$ for all normalizable states $|\psi\rangle$. This may be shown to imply that $\beta_0>\alpha_0$ [20].

To evaluate the decoherence functional (6.1), we could, of course, just use the method used for the general case in Sec. V and quite simply evaluate the final expression for this particular case. However, this turns out to be rather cumbersome and we have found it easier to employ a different method. In particular, we shall proceed as follows.

Step (i). Perform the integrations over X_0 and ξ_0, thus obtaining the evolution of the reduced density matrix from t_0 to t_1:

$$\tilde{\rho}(X_1,\xi_1,t_1)=\int dX_0 d\xi_0 F_{1,0}^2 \exp(i\tilde{S}_{1,0}-\phi_{1,0}) \times\rho_A(X_0,\xi_0,t_0) . \quad (6.3)$$

Step (ii). Multiply by the projectors at time t_1 and then evolve to time t_2; i.e., calculate the quantity

$$\tilde{D}(\bar{X}_1,X_2,\bar{\xi}_1)=\int dX_1 d\xi_1 F_{2,1}^2 \exp(i\tilde{S}_{2,1}-\phi_{2,1}) \times\left[-\frac{(X_1-\bar{X}_1)^2}{\sigma_1^2}-\frac{(\xi_1-\bar{\xi}_1)^2}{\sigma_1^2}\right] \times\tilde{\rho}(X_1,\xi_1,t_0) , \quad (6.4)$$

where we may use the fact that $\xi_2=0$ in $\tilde{S}_{2,1}$ and $\phi_{2,1}$.

Step (iii). Finally, multiply by the single projector at time t_2 and integrate over X_2:

$$D(\bar{X}_1,\bar{X}_2,\bar{\xi}_1) =\int dX_2 \exp\left[-\frac{(X_2-\bar{X}_2)^2}{\sigma_2^2}\right]\tilde{D}(\bar{X}_1,X_2,\bar{\xi}_1) . \quad (6.5)$$

Beginning with step (i), a tedious but straightforward calculation yields the result

$$\tilde{\rho}(X_1,\xi_1,t_1)=\pi F_{1,0}^2\Delta_{1,0}^{-1/2}\exp(-\alpha_1 X_1^2-\beta_1\xi_1^2 -\gamma_1 X_1\xi_1+\mu_1 X_1 +\nu_1\xi_1+\epsilon_1) , \quad (6.6)$$

where

$$\Delta_{1,0}=\alpha_0(\beta_0+C_{1,0})-\tfrac{1}{4}(\gamma_0-i\hat{K}_{1,0})^2 \quad (6.7)$$

and

$$\alpha_1=\frac{N_{1,0}^2\alpha_0}{4\Delta_{1,0}} , \quad (6.8)$$

$$\beta_1=\frac{1}{4\Delta_{1,0}}[-B_{1,0}^2\alpha_0+L_{1,0}^2(\beta_0+C_{1,0}) +iL_{1,0}B_{1,0}(\gamma_0-i\hat{K}_{1,0})]+A_{1,0} , \quad (6.9)$$

$$\gamma_1=\frac{1}{4\Delta_{1,0}}[-2iN_{1,0}B_{1,0}\alpha_0 -L_{1,0}N_{1,0}(\gamma_0-i\hat{K}_{1,0})]-i\tilde{K}_{1,0} , \quad (6.10)$$

$$\mu_1=\frac{1}{2\Delta_{1,0}}[-iN_{1,0}\alpha_0\nu_0+\frac{i}{2}N_{1,0}\mu_0(\gamma_0-i\hat{K}_{1,0})] , \quad (6.11)$$

$$\nu_1=\frac{1}{2\Delta_{1,0}}[-iL_{1,0}(\beta_0+C_{1,0})\mu_0-B_{1,0}\alpha_0\nu_0 +\tfrac{1}{2}(\gamma_0-i\hat{K}_{1,0})(iL_{1,0}\nu_0+B_{1,0}\mu_0)] , \quad (6.12)$$

$$\epsilon_1=\frac{1}{4\Delta_{1,0}}[(\beta_0+C_{1,0})\mu_0^2+\alpha_0\nu_0^2 -(\gamma_0-i\hat{K}_{1,0})\mu_0\nu_0]+\epsilon_0 . \quad (6.13)$$

This completes step (i).

As an aside, and by way of a check, we compare these results with the calculations of Caldeira and Leggett for the evolution of the reduced density matrix [16]. They took as their initial state a wave packet of approximate momentum p, centered around $x=0$ and with width σ. The corresponding initial density matrix is

$$\rho_A(X_0,\xi_0,t_0) =(2\pi\sigma^2)^{-1/2}\exp\left[ip\xi_0-\frac{X_0^2+\xi_0^2}{8\sigma^2}\right] . \quad (6.14)$$

From the above, we find the reduced density matrix at time t_1 to be

$$\tilde{\rho}(X_1,\xi_1,t_1)=\pi F_{1,0}^2\Delta_{1,0}^{-1/2}\exp\left[-\frac{N_{1,0}^2}{32\sigma^2\Delta_{1,0}^2}\left[X_1-\frac{p}{N_{1,0}}\right]^2-\beta_1\xi_1^2\right]$$
$$\times\exp\left[i\hat{K}_{1,0}X_1\xi_1-i\frac{N_{1,0}}{16\sigma^2\Delta_{1,0}}(4\sigma^2\hat{K}_{1,0}L_{1,0}-B_{1,0})\left[X_1-\frac{p}{N_{1,0}}\right]\xi_1\right], \tag{6.15}$$

where

$$\beta_1=A_{1,0}+2\sigma^2L_{1,0}^2-\frac{1}{32\sigma^2\Delta_{1,0}}(B_{1,0}-4\sigma^2L_{1,0}\hat{K}_{1,0})^2 \tag{6.16}$$

and

$$32\sigma^2\Delta_{1,0}=8\sigma^2\hat{K}_{1,0}^2+4\left[C_{1,0}+\frac{1}{8\sigma^2}\right]. \tag{6.17}$$

This agrees with the results of Caldeira and Leggett (up to a number of numerical factors which we take to be typographical errors in their paper).

Now consider step (ii). With the results of step (i), Eq. (6.4) may be written

$$\tilde{D}(\bar{X}_1,X_2,\bar{\xi}_1)=\pi F_{2,1}^2F_{1,0}^2\Delta_{1,0}^{-1/2}\times\int dX_1d\xi_1\exp(i\tilde{S}_{2,1}-\phi_{2,1})\times\exp(-\tilde{\alpha}_1X_1^2-\tilde{\beta}_1\xi_1^2-\gamma_1X_1\xi_1+\tilde{\mu}_1X_1+\tilde{\nu}_1\xi_1+\tilde{\epsilon}_1), \tag{6.18}$$

where

$$\tilde{\alpha}_1=\alpha_1+\frac{1}{\sigma_1^2},\quad \tilde{\beta}_1+\frac{1}{\sigma_1^2}, \tag{6.19}$$

$$\tilde{\mu}_1=\mu_1+\frac{2\bar{X}_1}{\sigma_1^2},\quad \tilde{\nu}_1=\nu_1+\frac{2\bar{\xi}_1}{\sigma_1^2},\quad \tilde{\epsilon}_1=\epsilon_1-\frac{\bar{X}_1^2+\bar{\xi}_1^2}{\sigma_1^2}. \tag{6.20}$$

But the integral (6.18) is now of the same form as (6.3), and we may again use the results of step (i), recalling that we may set $\xi_2=0$ in the expressions for $\tilde{S}_{2,1}$ and $\phi_{2,1}$. We thus obtain

$$\tilde{D}(\bar{X}_1,X_2,\bar{\xi}_1)=\pi^2F_{2,1}^2F_{1,0}^2\Delta_{2,1}^{-1/2}\Delta_{1,0}^{-1/2}\times\exp(-\alpha_2X_2^2+\mu_2X_2+\epsilon_2), \tag{6.21}$$

where

$$\Delta_{2,1}=\tilde{\alpha}_1(\tilde{\beta}_1+C_{2,1})-\tfrac{1}{4}(\gamma_1-i\hat{K}_{2,1})^2 \tag{6.22}$$

and

$$\alpha_2=\frac{N_{2,1}^2\tilde{\alpha}_1}{4\Delta_{2,1}}, \tag{6.23}$$

$$\mu_2=\frac{1}{2\Delta_{2,1}}\left[-iN_{2,1}\tilde{\alpha}_1\tilde{\nu}_1+\frac{i}{2}N_{2,1}\tilde{\mu}_1(\gamma_1-i\hat{K}_{2,1})\right], \tag{6.24}$$

$$\epsilon_2=\frac{1}{4\Delta_{2,1}}[(\tilde{\beta}_1+C_{2,1})\tilde{\mu}_1^2+\tilde{\alpha}_1\tilde{\nu}_1^2-(\gamma_1-i\hat{K}_{2,1})\tilde{\mu}_1\tilde{\nu}_1]+\tilde{\epsilon}_1. \tag{6.25}$$

We could at this stage proceed to step (iii), but it turns out to be easier to first simplify expression (6.21). Some lengthy algebra leads to the result

$$\tilde{D}(\bar{X}_1,X_2,\bar{\xi}_1)=\pi^2F_{2,1}^2F_{1,0}^2\Delta_{2,1}^{-1/2}\Delta_{1,0}^{-1/2}\exp\left[-\left[\frac{1}{\sigma_1^2}-\frac{\tilde{\alpha}_1}{\sigma_1^4\Delta_{2,1}}\right]\bar{\xi}_1^2\right]$$
$$\times\exp\left\{-i\frac{N_{2,1}\tilde{\alpha}_1}{\sigma_1^2\Delta_{2,1}}X_2\bar{\xi}_1+\frac{1}{4\sigma_1^2\Delta_{2,1}}\left[4\tilde{\alpha}_1\nu_1-2(\gamma_1-i\hat{K}_{2,1})\left[\mu_1+2\frac{\bar{X}_1}{\sigma_1^2}\right]\right]\bar{\xi}_1\right\}$$
$$\times\exp\left[-\frac{N_{2,1}^2}{4\sigma_1^4\Delta_{2,1}}\left[X_2+i\frac{\nu_1}{N_{2,1}}-i(\gamma_1-i\hat{K}_{2,1})\frac{\bar{X}_1}{N_{2,1}}\right]^2\right.$$
$$-\frac{\alpha_1N_{2,1}^2}{4\Delta_{2,1}}\left[X_2+i\frac{\nu_i}{N_{2,1}}-i(\gamma_1-i\hat{K}_{2,1})\frac{\mu_1}{2\alpha_1N_{2,1}}\right]^2$$
$$\left.-\frac{\alpha_1}{\sigma_1^2\Delta_{2,1}}(C_{2,1}+\tilde{B}_1)\left[\bar{X}_1-\frac{\mu_1}{2\alpha_1}\right]^2+\epsilon_1+\frac{\mu_1^2}{4\alpha_1}\right]. \tag{6.26}$$

The final step, step (iii), is now readily performed using the identity

$$\int dX_2 \exp\left[-\frac{(X_2-\bar{X}_2)^2}{\sigma_2^2}-a(X_2-\alpha)^2-b(X_2-\beta)^2+icX_2\bar{\xi}_1\right]$$
$$=\pi^{1/2}(\sigma_2^{-2}+a+b)^{-1/2}\exp\left\{\frac{1}{\sigma_2^{-2}+a+b}\left[-\frac{c^2}{4}\bar{\xi}_1^{-2}+ic\left[\frac{\bar{X}_2}{\sigma_2^2}+\alpha a+\beta b\right]\bar{\xi}_1 -\frac{a}{\sigma_2^2}(\bar{X}_2-\alpha)^2-\frac{b}{\sigma_2^2}(\bar{X}_2-\beta)^2-ab(\alpha-\beta)^2\right]\right\}. \quad (6.27)$$

Using the above identity, we obtain the final result, which is conveniently written in the form

$$D(\bar{X}_1,\bar{X}_2,\bar{\xi}_1)=\pi^{5/2}F_{2,1}^2F_{1,0}^2\Delta_{2,1}^{-1/2}\Delta_{1,0}^{-1/2}(\sigma_2^{-2}+\alpha_2)^{-1/2}$$
$$\times\exp\left[i\frac{\bar{\xi}_1N_{2,1}}{\sigma_1^2(\alpha_2\sigma_2^2+1)\Delta_{2,1}}\left[-\tilde{\alpha}_1(\bar{X}_2-Y_2)+i\frac{\gamma_1-i\hat{K}_{2,1}}{\sigma_1^2N_{2,1}}(\bar{X}_1-Y_1)\right]\right]$$
$$\times\exp\left[-l^{-2}\bar{\xi}_1^{-2}-(\bar{X}-Y)^TM(\bar{X}-Y)+\epsilon_1+\frac{\mu_1^2}{4\alpha_1}\right]. \quad (6.28)$$

Here,

$$l^{-2}=\frac{1}{\sigma_1^2}-\frac{\tilde{\alpha}_1}{\sigma_1^4\Delta_{2,1}}+\frac{N_{2,1}^2\tilde{\alpha}_1^2}{4(\alpha_2+\sigma_2^{-2})\sigma_1^4\Delta_{2,1}^2}$$
$$=\frac{1}{\sigma_1^2}-\frac{\tilde{\alpha}_1}{\sigma_1^4(\alpha_2\sigma_2^2+1)\Delta_{2,1}}. \quad (6.29)$$

As in Sec. IV, we have introduced the notation

$$\bar{X}=\begin{bmatrix}\bar{X}_1\\ \bar{X}_2\end{bmatrix},\quad Y=\begin{bmatrix}Y_1\\ Y_2\end{bmatrix}, \quad (6.30)$$

where

$$Y_1=\frac{\mu_1}{2\alpha_1}, \quad (6.31)$$

$$Y_2=-i\frac{\nu_1}{N_{2,1}}+i(\gamma_1-i\hat{K}_{2,1})\frac{\mu_1}{2\alpha_1N_{2,1}}. \quad (6.32)$$

Also

$$M=\begin{bmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix}, \quad (6.33)$$

where

$$M_{11}=\frac{1}{\sigma_1^2}-\frac{C_{2,1}+\tilde{\beta}_1}{\sigma_1^4\Delta_{2,1}}+\frac{(\gamma_1-i\hat{K}_{2,1})^2N_{2,1}^2}{16\sigma_1^4(\alpha_2+\sigma_2^{-2})\Delta_{2,1}^2}$$
$$=\frac{1}{\sigma_1^2}-\frac{C_{2,1}+\tilde{\beta}_1+(1/4)\sigma_2^2N_{2,1}^2}{\sigma_1^4(\alpha_2\sigma_2^2+1)\Delta_{2,1}}, \quad (6.34)$$

$$M_{12}=M_{21}=-i\frac{(\gamma_1-i\hat{K}_{2,1})N_{2,1}}{4(\alpha_2\sigma_2^2+1)\sigma_1^2\Delta_{2,1}}, \quad (6.35)$$

$$M_{22}=\frac{N_{2,1}^2\tilde{\alpha}_1}{4(\alpha_2\sigma_2^2+1)\Delta_{2,1}}. \quad (6.36)$$

Using this notation, and also using (6.8)–(6.13), Eq. (6.28) may be rewritten

$$D(\bar{X}_1,\bar{X}_2,\bar{\xi}_1)=\pi^{5/2}F_{2,1}^2F_{1,0}^2\Delta_{2,1}^{-1/2}\Delta_{1,0}^{-1/2}(\sigma_2^{-2}+\alpha_2)^{-1/2}$$
$$\times\exp\left[-4i\frac{\bar{\xi}_1}{\sigma_1^2N_{2,1}}[M_{22}(\bar{X}_2-Y_2)+M_{12}(\bar{X}_1-Y_1)]\right]$$
$$\times\exp\left[-l^{-2}\bar{\xi}_1^{-2}-(\bar{X}-Y)^TM(\bar{X}-Y)+\epsilon_0+\frac{\mu_0^2}{4\alpha_0}\right]. \quad (6.37)$$

Note that the matrix M must be positive definite by construction since the decoherence functional must satisfy the normalization conditions (2.24) and (2.25b). Equation (6.37) is the decoherence functional for the class of initial density matrices (6.2) for histories characterized by position projections at two moments of time. We note that the decoherence functional for histories characterized by projections at three or more moments of time could be calculated by recursive use of the relations (6.8)–(6.13), but we will not pursue that here. In the next section we will evaluate the expression (6.37) for particular initial states.

VII. DECOHERENCE AND CLASSICAL CORRELATIONS

We will now evaluate the decoherence functional (6.37) for specific choices of the initial state contained in the Gaussian ansatz, Eq. (6.2). We will look for decoherence and for the degree of peaking about the classical paths.

A. Single wave packet

Let the initial state be a wave packet centered around point a, with width σ, and momentum centered around p:

$$\Psi_0(x)=\exp\left[ipx-\frac{(x-a)^2}{\sigma^2}\right] . \tag{7.1}$$

The associated density matrix is

$$\rho_0=\exp\left[-\frac{X^2+\xi^2}{2\sigma^2}+\frac{2a}{\sigma^2}X+ip\xi-\frac{2a^2}{\sigma^2}\right] . \tag{7.2}$$

That is, it is of the form Eq. (6.2), with

$$\alpha_0=\beta_0=\frac{1}{2\sigma^2},\quad \gamma_0=0 , \tag{7.3}$$

$$\mu_0=\frac{2a}{\sigma^2},\quad \nu_0=ip,\quad \epsilon_0=-\frac{2a^2}{\sigma^2} . \tag{7.4}$$

One can now calculate all the terms entering Eq. (6.37). One finds that α_1, β_1, and μ_1 are real, γ_1 and ν_1 are imaginary, and ϵ_1 is complex. The quantities Y_1 and Y_2 in Eqs. (6.31) and (6.32) are real. Denote by $x_{\rm cl}(t)$ the classical solution at time t with initial position a and initial momentum p. Then, $Y(t_1)=2x_{\rm cl}(t_1)$ and $Y(t_2)=2x_{\rm cl}(t_2)$. The coefficient of ξ_1 in the decoherence functional (6.37) is purely imaginary.

Consider now the condition the decoherence functional must satisfy for the probability sum rules to be satisfied to order ϵ. It is given by (3.4), which in the present case reads

$$|{\rm Re}D(\bar{x}_1,\bar{x}_2|\bar{y}_1,\bar{x}_2)| < \epsilon[D(\bar{x}_1,\bar{x}_2|\bar{x}_1,\bar{x}_2)D(\bar{y}_1,\bar{x}_2|\bar{y}_1,\bar{x}_2)]^{1/2} \tag{7.5}$$

for $\bar{x}_1\neq\bar{y}_1$. Inserting the expression (6.37) for the decoherence functional, it is not difficult to show that this condition will be satisfied if

$$\exp[-(\bar{x}_1-\bar{y}_1)^2(l^{-2}-M_{11})]<\epsilon \tag{7.6}$$

(apart from prefactors, which are of order 1).

To see what this implies, we need to be precise about what is meant by "$\bar{x}_1\neq\bar{y}_1$" in condition (7.5). Recall that we are not using true projections but the Gaussian slit projections, (2.3). At a fixed moment of time, these projections partition the configuration space into regions with size of order a few times the width σ_1. The variables $\bar{x}_1$ and $\bar{y}_1$ label the regions, and thus have significance only up to a few times the width. It follows that "$\bar{x}_1\neq\bar{y}_1$" means that $|\bar{\xi}_1|$ should be greater than a few times the width. How many times the width? The Gaussian slit projections are exclusive only to the extent that $\exp(-\bar{\xi}_1^2/2\sigma_1^2)$ is approximately zero, and we should not expect to obtain decoherence to a degree better than this. If we seek to obtain decoherence to order ϵ, therefore, we should choose $|\bar{\xi}_1|$ to be sufficiently large that

$$\exp\left[-\frac{\bar{\xi}_1^2}{2\sigma_1^2}\right]\ll\epsilon \tag{7.7}$$

to ensure that the nonexclusivity of the projections cleanly separates from the issue of obtaining decoherence. The rapid decay of the exponential will ensure that this condition is readily satisfied.

Inserting the explicit forms for l^{-2} and M_{11}, one finds

$$\sigma_1^2(l^{-2}-M_{11})=\frac{C_{2,1}+\beta_1-\alpha_1+(1/4)\sigma_2^2N_{2,1}^2}{\sigma_1^2(\alpha_2\sigma_2^2+1)\Delta_{2,1}} \tag{7.8}$$

and some straightforward manipulation shows that

$$0\leq\sigma_1^2(l^{-1}-M_{11})\leq 1 . \tag{7.9}$$

It follows that the decoherence is most effective when $\sigma_1^2(l^{-2}-M_{11})$ is very close to 1.

After decoherence to the requisite degree is achieved, we are interested in determining the degree to which the diagonal part of the decoherence functional is peaked about the classical paths. The diagonal part is given by

$$p(\bar{X}_1,\bar{X}_2)=\frac{(\det M)^{1/2}}{\pi}\exp[-(\bar{X}-Y)^TM(\bar{X}-Y)] . \tag{7.10}$$

The degree of peaking is determined by the size of the eigenvalues of the matrix M, in comparison to quantities of the form $(\bar{X}_1-Y_2)^2$, $(\bar{X}_2-Y_2)^2$. The latter quantities, when nonzero, are greater than a few times σ_1^2,σ_2^2, because $\bar{X}_1,\bar{X}_2$ are defined only up to these widths. A convenient measure of the degree of peaking, therefore, is the quantity

$$\sigma_1^2\sigma_2^2\det M=\left[1+\frac{1}{\sigma_1^2\alpha_1}+\frac{4\Delta_{2,1}}{\sigma_2^2\alpha_1N_{2,1}^2}\right]^{-1} . \tag{7.11}$$

One has

$$\sigma_1^2\sigma_2^2\det M\leq 1 \tag{7.12}$$

and thus the probability measure (7.10) is most strongly peaked when $\sigma_1^2\sigma_2^2\det M$ is very close to 1.

We now evaluate expressions (7.9) and (7.11) in a variety of interesting cases and see whether the requirements of decoherence and classical peaking are met. We will consider the cases of the free particle and the harmonic oscillator, with and without environment, in the limits of the time intervals (t_2-t_1) and (t_1-t_0) both large and small.

(1) No environment. In the case of no environment, we may set A, B, and C to zero, and also the dissipation γ to zero. In the short time limit, with both (t_2-t_1) and (t_1-t_0) small, the free-particle and harmonic-oscillator cases coincide, and one has

$$\tilde{K}(\tau)=\hat{K}(\tau)=L(\tau)=N(\tau)=\frac{M}{2\tau} . \tag{7.13}$$

Also, $\Delta_{2,1}\approx\hat{K}_{2,1}^2/4$, $\Delta_{1,0}\approx\hat{K}_{1,0}^2/4$, and $\alpha_1\approx\alpha_0$. It follows that

$$\sigma_1^2(l^{-2}-M_{11})\approx\left[1+\sigma_1^2\left[\frac{1}{2\sigma^2}+\frac{1}{\sigma_2^2}\right]\right]^{-1} \tag{7.14}$$

and

$$\sigma_1^2\sigma_2^2\det M\approx\left[1+2\sigma^2\left[\frac{1}{\sigma_2^2}+\frac{1}{\sigma_1^2}\right]\right]^{-1} . \tag{7.15}$$

Each of the quantities has to be close to 1. Equation (7.14) indicates one should take $\sigma \gg \sigma_1$, while (7.15) indicates one should take $\sigma_1 \gg \sigma$ and $\sigma_2 \gg \sigma$. There is, therefore, a certain amount of conflict between the demands of decoherence and classical peaking, but a compromise is possible. For example, if one takes $2\sigma \sim \sigma_1 \sim \sigma_2$, then

$$\sigma_1^2(l^{-2}-M_{11})\approx\sigma_1^2\sigma_2^2\det M\approx\tfrac{1}{2} , \tag{7.16}$$

which can be sufficient for satisfactory decoherence and classical peaking.

In the long time limit for the free particle, $\tilde{K}$, $\hat{K}$, L, and n all go to zero, as do α_1, β_1, γ_1, and $\Delta_{2,1}$. One thus has

$$\sigma_1^2(l^{-2}-M_{11})\to 0 \tag{7.17}$$

and therefore there is no decoherence. Similarly,

$$\sigma_1^2\sigma_2^2\det M\to 0 . \tag{7.18}$$

Both of these features might have been anticipated given the spreading of the wave packet for the free particle. However, by choosing the mass of the particle to be sufficiently large, one could ensure that it remains decohered and peaked about the classical path for a long period of time.

For the harmonic oscillator, the quantities $\sigma_1^2(l^{-2}-M_{11})$ and $\sigma_1^2\sigma_2^2\det M$ oscillate without tending to fixed values in the long time limit, but return to their short time limit values when both $\omega(t_2-t_1)$ and $\omega(t_1-t_0)$ are simultaneously equal to integer multiples of 2π.

(2) With environment. In the short time limit, the quantities A, B, and C are all linear in time (see Sec. IV), and it is not difficult to see that all dependence on the environment drops out, reducing to the case of no environment. In the long time limit, β_1 and γ_1 tend to oscillatory functions, $C(\tau)$ and $N(\tau)$ grow like $e^{2\gamma\tau}$, and

$$\alpha_1\approx\frac{N_{1,0}^2}{4C_{1,0}}\approx\frac{M(\gamma^2+\omega^2)}{8KT} . \tag{7.19}$$

The quantity $\sigma_1^2(l^{-2}-M_{11})$ is dominated by $C_{2,1}$ and $N_{2,1}$, and one has

$$\sigma_1^2(l^{-2}-M_{11})\sim\frac{1}{\alpha_1\sigma_1^2+1} . \tag{7.20}$$

Similarly, it is readily shown that

$$\sigma_1^2\sigma_2^2\det M\sim\frac{\alpha_1^2\sigma_1^2\sigma_2^2}{(\alpha_1\sigma_1^2+1)(\alpha_1\sigma_2^2+1)} . \tag{7.21}$$

Decoherence and classical peaking are therefore controlled by the quantity $\alpha_1\sigma_1^2$ (or $\alpha_1\sigma_2^2$). Loosely speaking, this is the ratio of the energy of the particle to the thermal energy of the environment. Classical peaking is obtained when this quantity is large. Physically, this is not surprising since it is the condition that the particle has sufficient inertia to resist the thermal fluctuations of the environment. However, decoherence demands that $\alpha_1\sigma_1^2$ be small. This is again to be expected physically because, on general grounds, decoherence demands a certain amount of interaction from the environment. Again, therefore, there is a certain amount of competition between decoherence and classical peaking, but again a compromise can be reached if the parameters of the models are chosen such that $\alpha_1\sigma_1^2\sim 1$.

An important feature to note is that the quantity (7.20) controlling decoherence is independent of the initial density matrix. We have therefore exhibited the degree of decoherence as a function of the coarse graining for the class of initial states contained in the Gaussian ansatz (6.20), not just for wave-packet initial states.

It should also be noted that the fact that we obtained decoherence without an environment in the short time limit is a feature peculiar to the initial state consisting of a single wave packet. The density matrix for this initial state is peaked along the history traced out by the wave-packet evolution and is essentially zero elsewhere. The off-diagonal terms of the decoherence functional essentially sample the density matrix along two different histories. But if the density matrix is nonzero along one and only one history, the off-diagonal term of the decoherence functional will clearly be small.

B. States corresponding to a set of classical solutions

Because of the special nature of wave-packet initial states, it is important to consider other initial states more representative of the general case. A more general initial state leading to classical behavior will generally predict not just one classical solution, but a set of classical solutions, with a probability measure on that set. A simple example of a wave function of this more general variety is one of the form

$$\Psi_0(x)=\exp(-Fx^2) . \tag{7.22}$$

For the special value $F=\frac{1}{2}M\omega$, this is, of course, the ground state of the harmonic oscillator and remains in this state under unitary evolution. However, if $F=F_R+iF_I$ is allowed to be an arbitrary complex number, with F_R small and F_I large, F will evolve from its initial value. Wave functions of this type arise as wave functions for scalar field fluctuations in inflationary universe models. An earlier heuristic analysis suggests a prediction of a set of classical histories, satisfying $M\dot{x}=p=-2F_Ix$, and with probability proportional to $\exp(-2F_Rx^2)$ for a given initial value of x [18]. We will show how these features emerge from the present approach.

An initial wave function of the form (7.22) gives an initial density matrix of the Gaussian form (6.2), with

$$\alpha_0=\beta_0=\tfrac{1}{2}F_R,\quad \gamma_0=iF_I \tag{7.23}$$

and $\mu_0=\nu_0=\epsilon_0=0$. It follows that $Y_1=Y_2=0$. Again, the coefficient of $\bar{\xi}_1$ in (6.37) is purely imaginary. The decoherence condition is again (7.6).

In the short time limit,

$$\sigma_1^2(l^{-2}-M_{11}) \approx \left[1+\sigma_1^2\left[\alpha_0+\frac{1}{\sigma_2^2}\right]\right]^{-1} \tag{7.24}$$

so decoherence can be achieved if $\alpha_0\sigma_1^2 \ll 1$ and $\sigma_1 \ll \sigma_2$. In the long time limit, with an environment, the degree of decoherence becomes independent of the initial conditions, and the discussion reduces to that of the single wave-packet case discussed above.

The diagonal part of the decoherence functional is given by

$$p(\bar{X}_1,\bar{X}_2)=\frac{(\det M)^{1/2}}{\pi}\exp(-\bar{X}^T M\bar{X}) . \tag{7.25}$$

This probability is not peaked about a particular classical path, but as mentioned above, we anticipate that it predicts a set of classical paths with a certain initial distribution of x. To illustrate this, we proceed as follows.

The probability of finding a given value of $\bar{X}_1$ at time t_1 is

$$\begin{aligned} p(\bar{X}_1) &= \int d\bar{X}_2 p(\bar{X}_1,\bar{X}_1) \\ &= \left[\frac{\det M}{\pi M_{22}}\right]^{1/2}\exp\left[-\frac{\det M}{M_{22}}\bar{X}_1^2\right] . \end{aligned} \tag{7.26}$$

It may be shown that

$$\frac{\det M}{M_{22}}=\frac{\alpha_1}{\sigma_1^2\alpha_1+1} . \tag{7.27}$$

Letting $t_1 \to t_0$, $\alpha_1 \to \alpha_0 = \frac{1}{2}F_R$. Then, it is easily seen that, if $\frac{1}{2}\sigma_1^2 F_R \ll 1$, one has

$$p(\bar{X}_1) \approx \left[\frac{F_R}{2\pi}\right]^{1/2}\exp(-\tfrac{1}{2}F_R\bar{X}_1^2) . \tag{7.28}$$

The initial distribution of x is therefore proportional to $\exp(-2F_R x^2)$ (recalling that $X=2x$).

To see that (7.25) is peaked about the classical path connecting a given value of $\bar{X}_1$ to $\bar{X}_2$, consider the conditional probability of $\bar{X}_2$, given $\bar{X}_1$. This is given by

$$\begin{aligned} p(\bar{X}_2|\bar{X}_1) &= \frac{p(\bar{X}_2,\bar{X}_1)}{p(\bar{X}_1)} \\ &= \left[\frac{M_{22}}{\pi}\right]^{1/2}\exp\left[-M_{22}\left[\bar{X}_2+\frac{M_{12}}{M_{22}}\bar{X}_1\right]^2\right] . \end{aligned} \tag{7.29}$$

The conditional probability (7.29) is peaked about

$$\bar{X}_2=-\frac{M_{12}}{M_{22}}\bar{X}_1=i\frac{\gamma_1-i\hat{K}_{2,1}}{N_{2,1}(\sigma_1^2\alpha_1+1)}\bar{X}_1 . \tag{7.30}$$

This may be shown to be a classical solution in (t_2-t_1). To discover the initial data this solution satisfies, let (t_2-t_1) become small, and let $t_1 \to t_0$. One obtains

$$\bar{x}_2=\frac{1}{1+(1/2)\sigma_1^2 F_R}\left[\bar{x}_1-\frac{2F_I\bar{x}_1}{M}(t_2-t_1)\right] . \tag{7.31}$$

If $\frac{1}{2}\sigma_1^2 F_R \ll 1$, (7.31) is the classical path from $\bar{x}_1$ to $\bar{x}_2$, with initial momentum $p=-2F_I x$. Equations (7.28) and (7.31) therefore indicate that the heuristic interpretation described above may be maintained if $\frac{1}{2}\sigma_1^2 F_R \ll 1$.

Another way to arrive at the same result is to write $\bar{x}_2=\bar{x}_1+\bar{k}(t_2-t_1)/M$, and use (7.25) to derive a probability distribution $p(\bar{x}_1,\bar{k})$, for $\bar{k}$ and $\bar{x}_1$, in the limit of small (t_2-t_1). This yields a joint probability distribution for a position sampling and time-of-flight momentum sampling at approximately the same time. This was briefly mentioned in Sec. V C and carried out in detail in Ref. [20]. These results show that the initial distribution of positions and momenta are given by a smeared version of the Wigner function of the initial state. In particular, the Wigner function (and its smeared version) are peaked about $p=-2F_I x$, if F_I is large and F_R small, consistent with the above analysis.

The degree of peaking about the classical path in (7.29) is determined by the quantity $\sigma_2^2 M_{22}$. In the long time limit, this is given by

$$\sigma_2^2 M_{22}=\frac{\alpha_1\sigma_2^2}{\alpha_1\sigma_2^2+1} . \tag{7.32}$$

Comparing with (7.21), we therefore see that the discussion of classical peaking (and the tension between classical peaking and decoherence) is essentially the same as that of the single wave-packet case.

VIII. SUPERPOSITIONS

We now study an important but simple illustrative case, namely, that in which the initial state is taken to be a superposition of two wave packets. This example shows most clearly how interference is an obstruction to assigning probabilities to histories, and how interference is destroyed by coupling to an environment. This example is essentially the double-slit experiment but paired down to its most basic form.

A. Without environment

Consider a particle moving in one dimension, in a pure state whose wave function at $t=t_0$ is a superposition of wave packets far apart, but moving towards each other. So

$$|\Psi(t_0)\rangle = |\Psi_+(t_0)\rangle + |\Psi_-(t_0)\rangle , \tag{8.1}$$

where $|\Psi_+(t_0)\rangle$ is a wave packet at $x=L>0$, with width $\sigma \ll L$, and with momentum $-p$. Similarly, $|\Psi_-(t_0)\rangle$ is located at $x=-L$, has the same width, but momentum p. Explicitly,

$$\langle x|\Psi_+(t_0)\rangle = \exp\left[-ipx-\frac{(x-L)^2}{\sigma^2}\right] , \tag{8.2}$$

$$\langle x|\Psi_-(t_0)\rangle = \exp\left[ipx-\frac{(x+L)^2}{\sigma^2}\right] . \tag{8.3}$$

The wave packets are therefore approximately orthogonal at $t=t_0$, up to terms of order $\exp(-L^2/\sigma^2)$. Let them meet at the origin at time t_1. We will assume that the parameters such as the mass of the particle are chosen so that the wave packets do not spread appreciably. In fact, we could consider a harmonic oscillator in which they do not spread at all.

The form of the wave function might tempt one to ascribe definite properties to the history of the particle. In particular, one might wish to say that the particle is in the neighborhood of either $x=L$ or $-L$ at time t_0, and then in the neighborhood of the origin at time t_1, with some probability for each of these two histories. We shall show explicitly, however, that this view is not tenable because this pair of histories does not form a decoherent set.

At time t_0, it is sufficient to ask whether the particle lies on the positive or negative x axis. This is effected through the projections

$$P_+=\int_0^\infty dx\,|x\rangle\langle x|,\quad P_-=\int_{-\infty}^0 dx\,|x\rangle\langle x| . \tag{8.4}$$

It is easily seen that

$$P_\pm|\Psi_\pm(t_0)\rangle\approx|\Psi_\pm(t_0)\rangle,\quad P_\pm|\Psi_\mp(t_0)\rangle\approx 0 \tag{8.5}$$

up to terms of order $\exp(-L^2/\sigma^2)$. At time t_1, when the wave packets meet, we will ask whether the particle lies in a region of size Δ around the origin, where Δ is much less than the wave-packet width σ. This proposition is effected by the projection

$$P_\Delta=\int_{-\Delta/2}^{\Delta/2} dx\,|x\rangle\langle x| . \tag{8.6}$$

One has

$$P_\Delta|\Psi_\pm(t_1)\rangle\approx|x=0\rangle\langle x=0|\Psi_\pm(t_1)\rangle . \tag{8.7}$$

An exhaustive set of alternatives at time t_1 consists of P_Δ together with its complement $1-P_\Delta$.

The candidate probabilities for the histories in which the particle was either in $x<0$ or $x>0$ at t_0, and then near the origin at t_1 are given by the diagonal elements of the decoherence functional:

$$D(\pm,\pm)=\mathrm{Tr}[P_\Delta e^{-iH(t_1-t_0)}P_\pm|\Psi(t_0)\rangle\langle\Psi(t_0)|P_\pm e^{iH(t_1-t_0)}] . \tag{8.8}$$

using (8.5), and then evolving to time t_1, this becomes

$$\begin{aligned}D(\pm,\pm)&\approx\mathrm{Tr}[P_\Delta|\Psi_\pm(t_1)\rangle\langle\Psi_\pm(t_1)|]\\&\approx|\langle\Psi_\pm(t_1)|x=0\rangle|^2 .\end{aligned} \tag{8.9}$$

But if these probabilities are to satisfy the probability sum rules, it is necessary that the off-diagonal terms of the decoherence functional are zero, or at least small. The off-diagonal terms are given by

$$\begin{aligned}D(\pm,\mp)&=\mathrm{Tr}[P_\Delta e^{-iH(t_1-t_0)}P_\pm|\Psi(t_0)\rangle\langle\Psi(t_0)|P_\mp e^{iH(t_1-t_0)}]\\&\approx\mathrm{Tr}[P_\Delta e^{-iH(t_1-t_0)}|\Psi_\pm(t_0)\rangle\langle\Psi_\mp(t_0)|e^{iH(t_1-t_0)}]\\&\approx\langle\Psi_\mp(t_1)|x=0\rangle\langle x=0|\Psi_\pm(t_1)\rangle ,\end{aligned} \tag{8.10}$$

again using (8.5). But $\langle x=0|\Psi_+(t_1)\rangle$ and $\langle x=0|\Psi_-(t_1)\rangle$ are essentially equal. One therefore has

$$|D(\pm,\mp)|\approx D(+,+)\approx D(-,-) \tag{8.11}$$

and it is *not* possible to satisfy the condition of approximate decoherence:

$$|D(\pm,\mp)|\ll[D(+,+)D(-,-)]^{1/2} . \tag{8.12}$$

The set of histories are therefore not decoherent and the assertion that "the particle was either in $x<0$ or $x>0$ at t_0, and then near the origin at t_1" is meaningless.

B. With environment

Suppose we now couple this system to an environment using the Caldeira-Leggett model described in earlier sections. The main difference is that the evolution of the initial density matrix is no longer unitary, but is instead described by the Caldeira-Leggett propagator, (4.39). A second difference is that the environment introduces dissipation into the classical equations of motion (according to which the wave packets move), and the time t_1 at which the wave packets meet is modified. The initial density matrix has the form

$$\rho=\rho_{(++)}+\rho_{(--)}+\rho_{(+-)}+\rho_{(-+)} , \tag{8.13}$$

where, from (8.2) and (8.3),

$$\rho_{(++)}(x_0,y_0)=\exp\left[ip(x_0-y_0)-\frac{(x_0-L)^2}{\sigma^2}-\frac{(y_0-L)^2}{\sigma^2}\right] , \tag{8.14}$$

$$\rho_{(+-)}(x_0,y_0)=\exp\left[ip(x_0+y_0)-\frac{(x_0-L)^2}{\sigma^2}-\frac{(y_0+L)^2}{\sigma^2}\right] . \tag{8.15}$$

$\rho_{(--)}$ and $\rho_{(-+)}$ are obtained by letting $p\to-p$ and $L\to-L$ in (8.14) and (8.15), respectively.

The decoherence functional is given by

$$D(\alpha,\alpha')=\int_{-\Delta/2}^{\Delta/2}dx_1\int_{\alpha}dx_0\int_{\alpha'}dy_0\rho(x_0,y_0,t_0)J(x_1,x_1,t_1|x_0,y_0,t_0) \ . \tag{8.16}$$

Here α and α' denote the integration ranges at time t_0, which may be the positive or negative axes. The four possible terms of the decoherence functional $D(\pm,\pm)$ and $D(\pm,\mp)$, are thus obtained by integrating over each of the four quadrants in the x_0y_0 plane. J is the Caldeira-Leggett propagator, (4.39), which we write

$$J(x_1,x_1,t_1|x_0,y_0,t_0)=\exp[i\tilde{S}-C(x_0-y_0)^2]$$

where $C=C_{1,0}$ and is given by (4.43) with $\tau=t_1-t_0$. The explicit form for $\tilde{S}$ will not be needed.

The desired result will be obtained by focusing on the size and location of the maxima of the integrand of (8.16) in the x_0y_0 plane. Let

$$f_{(++)}(x_0,y_0) = \exp[-C(x_0-y_0)^2]|\rho_{(++)}(x_0,y_0)| \ , \tag{8.18}$$

$$f_{(+-)}(x_0,y_0) = \exp[-C(x_0-y_0)^2]|\rho_{(+-)}(x_0,y_0| \ , \tag{8.19}$$

and similarly for $f_{(--)}$ and $f_{(-+)}$. Then, in the coordinates $X=x+y$, $\xi=x-y$, it is readily shown that one has

$$f_{(++)}(x_0,y_0)=\exp\left[-\tilde{C}\xi_0^2-\frac{(X_0-2L)^2}{2\sigma^2}\right] , \tag{8.20}$$

$$f_{(+-)}(x_0,y_0)=\exp\left[-\tilde{C}\left[\xi_0-\frac{L}{\sigma^2\tilde{C}}\right]^2-\frac{X_0^2}{2\sigma^2}\right] \times\exp\left[-2\frac{L^2C}{\sigma^2\tilde{C}}\right] , \tag{8.21}$$

where $\tilde{C}=C+1/2\sigma^2$. Similarly, $f_{(--)}$ and $f_{(-+)}$ are obtained from (8.20) and (8.21) by letting $L\to -L$.

The integrand of (8.16) is the sum of the four f's, apart from phases. From (8.20) and (8.21), it therefore has four peaks: at $x=y=\pm L$ and at $x=-y=\pm L/(2\sigma^2\tilde{C})$. When C is small, the latter pair are close to $x=-y=\pm L$, but for large C they approach the origin. The widths of all four peaks are the same. But, most importantly, the size of the peaks of $f_{(\pm\mp)}$ are suppressed in comparison to the peaks of $f_{(\pm\pm)}$ by the exponential factor.

$$\frac{f^{\max}_{(\pm\mp)}}{f^{\max}_{(\pm\pm)}}=\exp\left[-2\frac{L^2C}{\sigma^2\tilde{C}}\right] , \tag{8.22}$$

where $f^{\max}$ denotes the maximum value of f.

Consider now the evaluation of $D(+,+)$. It is obtained by integrating x_0,y_0 over the first quadrant $x_0>0$, $y_0>0$ in (8.16). Recall that we are assuming that $L\gg\sigma$. The peak of $f_{(--)}$ is far from the integration domain so its contribution will be very small, of order $\exp(-2L^2/\sigma^2)$. The peaks of $f_{(\pm,\mp)}$, on the other hand, can be close to the integration domain (depending on the value of C) but their magnitude is suppressed by the factor (8.22). These terms therefore contribute at worst the same as $f_{(++)}$, but multiplied by (8.22). By far the dominant contribution to the integral, therefore, will come from $f_{(++)}$, whose peak lies well inside the integration domain. Similarly, $D(-,-)$ will be dominated by $f_{(--)}$, and will be the same order of magnitude.

Now consider the off-diagonal term $D(+,-)$. It is obtained by integrating the same integrand over the fourth quadrant $x_0>0$, $y_0<0$. The peak of $f_{(+-)}$ lies inside the integration domain and one would expect this to provide the dominant contribution. The peaks of $f_{(\pm\pm)}$ are far from the integration domain, but they are not suppressed by (8.22). Their contribution would therefore be comparable to that of $f_{(+-)}$. Similarly, the peak of $f_{(-+)}$ may also be comparable, since it can be close to the integration domain. The important point, however, is that it is clear that all four terms are suppressed by the factor (8.22) compared to the contribution $f_{(++)}$ makes when the same integrand is used to calculate $D(+,+)$. A similar argument goes through for $D(-,+)$, and we may therefore write

$$|D(\pm,\mp)|\approx\exp\left[-2\frac{L^2C}{\sigma^2\tilde{C}}\right][D(+,+)D(-,-)]^{1/2} \ . \tag{8.23}$$

In the short time limit, $C\approx\frac{2}{3}M\gamma kT(t_1-t_0)$, so

$$\exp\left[-2\frac{L^2C}{\sigma^2\tilde{C}}\right]\approx\exp[-\tfrac{8}{3}M\gamma kTL^2(t_1-t_0)] \ . \tag{8.24}$$

In the long time limit, C goes to infinity like $e^{2\gamma(t_1-t_0)}$, so

$$\exp\left[-2\frac{L^2C}{\sigma^2\tilde{C}}\right]\approx\exp\left[-\frac{2L^2}{\sigma^2}\right] . \tag{8.25}$$

We therefore have very effective decoherence. Probabilities can be assigned to the histories, and it becomes meaningful to say that "the particle was either in $x<0$ or $x>0$ at t_0, and then near the origin at t_1."

C. The double-slit experiment

Finally, it is perhaps enlightening to comment on how these considerations might affect the fully fledged double-slit experiment. Consider the standard double-slit arrangement, in which one has a source (of electrons, say) incident on a pair of slits with a screen behind, with the whole setup in an evacuated box. The probability distribution of the electron's position at the screen will be the well-known interference pattern. Now ask whether it is possible to think of the electrons producing the interference pattern as having gone through one slit or the other. Differently put, ask whether the probability distribution for the interference pattern might be regarded as a sum of two probabilities—the probabilities for the histories in which the electron went through one or the other slit. This question is affected using projection operators of the form (8.4) at the time at which the electrons were in the neighborhood of the slits (where the x direction is parallel to the screen and the slits). However, from an analysis very similar to that given above, it is readily shown that, due to the presence of interference terms in the decoherence functional, one *cannot* write the interference pattern probability distribution as the sum of these probabilities. It is therefore not possible to think of the electron as having gone through one slit or the other.

Now suppose we gradually introduce an environment into the box, say a gas of photons. As in Sec. VIII B, the environment will induce decoherence of the histories, and then it *will* be possible to assign probabilities to the two possible histories of the electron. The interference pattern may then be written as the sum of the two probabilities. But, the interference pattern will be changed. It will, however, be changed gradually as the environment is introduced into the box. In particular, one will find that there is complementarity between the sharpness of the interference pattern and the degree of decoherence.

This sort of analysis of the double-slit experiment is well known (see, for example, Ref. [23] and references therein). Typical analyses involve the notion of actually *measuring,* to some precision, the position of the electron in the neighborhood of the slits. They thus yield a complementarity relation between the sharpness of the interference pattern and the precision of the measurement.

In the decoherent histories approach, measurements do not play a central role. Precision of the measurement is replaced, in the complementarity relation, by the more fundamental notion of the degree of decoherence—the degree to which probabilities may be assigned. Of course, the distinction is perhaps not so great in that an actual physical measurement might involve observing the photons scattered off the electrons, from which the location of the electron could be deduced. It is, however, perhaps satisfying to see how, in the decoherent histories approach, the notion of complementarity appears, but without reference to any notions of measurement.

Finally, we note that detailed calculations of the full double-slit experiment, exhibiting nondecohering histories, have been given by Omnès [24], but the coupling to an environment in order to induce decoherence was not considered.

IX. SUMMARY AND CONCLUSIONS

The purpose of this paper has been to explore some of the features of a formulation of quantum mechanics for closed systems which deals directly with quantum-mechanical histories. After reviewing the formalism, we addressed the issue of approximate decoherence. A condition for approximate decoherence was proposed. The form of this condition is partially motivated by a simple inequality satisfied by the decoherence functional, which we derived. We argued that our condition ensures that most probability sum rules are satisfied to approximately the same degree. Our argument, however, relied on some assumptions about the statistical distribution of the off-diagonal terms of the decoherence functional. It would be interesting to understand the significance, if any, of the situations in which these assumptions do not hold.

We calculated the decoherence functional for the Caldeira-Leggett model, and derived the general form of the decoherence functional for linear systems, for histories consisting of approximate samplings of position at an arbitrary number of moments of time. It was seen to display the desired formal properties, namely, decoherence, and peaking about classical paths along the diagonal. Both types of the coarse grainings employed (tracing over the environment and smearing over position) were found to be necessary to achieve decoherence. We also found that the probabilities for the histories involved a smeared version of the Wigner function in an essential way.

A more precise evaluation of the decoherence functional was achieved by specializing to the case of histories characterized by approximate position samplings at two moments of time. We studied initial states consisting of a single wave packet, and a wave function corresponding to a set of classical paths. In each case we obtained a *quantitative* measure of the degree of decoherence and classical peaking as a function of the coarse-graining parameters—the temperature of the bath and the width of the position projections. We found that there is an element of conflict between the requirements of classical peaking and decoherence; but, in our cases at least, there seemed to be a compromise regime in which each requirement could be adequately satisfied.

An important case we considered is that of an initial state consisting of a superposition of wave packets. Perhaps more clearly than any other, this example illustrate some of the key features of the decoherent histories approach. First, it provides a very concrete example of a set of histories which do not decohere, and therefore, to which probabilities cannot be assigned. Secondly, it clearly shows how decoherence can be very effectively achieved by coupling the system to a larger environment and then tracing it out.

Some of the work, and in particular that of Secs. V–VIII has much in common with that of Gell-Mann and Hartle [3,8,9]. We have not attempted to be as general as they were, and indeed, some of our results, such as the observation of the tension between decoherence and classical peaking, and the appearance of the Wigner function, are special cases of their results. We have, however, been more explicit and precise in our calculations, and have exhibited in detail the features of the formalism for specific choices of initial state.

Interference in quantum mechanics is best thought of as the failure of the probability sum rules for *histories.* Decoherence as destruction of interference is likewise best understood as the recovery of these rules. In the decoherent histories approach, (exact) decoherence is therefore defined in a precise and unambiguous fashion, namely through the condition (2.32). Decoherence and classical correlations have also been studied using density matrices at a fixed moment of time [15]. In these approaches, the destruction of interference is associated with the tendency of the density matrix towards diagonality, with the establishment of correlations of the system with the environment, and with the stability of certain system states under evolution in the presence of an environment. Although these approaches have a strong intuitive appeal, we feel that none of them supply a set of criteria for decoherence as precise as that supplied by the decoherent histories approach (see however Ref. [25]). In particular, the fact that the probability sum rules are automatically satisfied for histories consisting of events at a single moment of time [see Eq. (2.33)] highlights a possible tension between the decoherent histories approach and the density matrix approach. Reconciliation of these differing approaches will be the topic of future publications.

Note added in proof. Some additional bibliographical remarks are in order. Approaches to quantum mechanics focusing on histories are certainly not new. Wigner has stressed that all of conventional quantum measurement theory is essentially contained in the formula (2.39) (Ref. [26]). It contains both the unitary evolution of the state, together with the collapse of the wave function as a result of a measurement. Approaches based on continuous measurement have also been considered [27,28]. That quantum-mechanical probabilities for histories do not satisfy the probability sum rules was noted by de Broglie [29], and also by Feynman in his seminal paper on the path integral approach to quantum mechanics [30]. See also the later work of Srinivas [31]. All of these approaches are concerned with measurement by an external agency. As stressed in Sec. II F, their interpretational aspects are therefore quite distinct from the decoherent histories approach considered here, although the mathematical machinery is very similar.

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