Complex solutions for the scalar field model of the Universe

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The Hartle-Hawking proposal is implemented for Hawking's scalar field model of the Universe. For this model the complex saddle-point geometries required by the semiclassical approximation to the path integral cannot simply be deformed into real Euclidean and real Lorentzian sections. Approximate saddle points are constructed which are fully complex and have contours of real Lorentzian evolution. The semiclassical wave function is found to give rise to classical spacetimes at late times and extra terms in the Hamilton-Jacobi equation do not contribute significantly to the potential.

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I. INTRODUCTION

The Euclidean path-integral formulation of quantum gravity has proved useful in attempting to describe the origin and evolution of our Universe. The "no boundary proposal" (NBP) of Hartle and Hawking [1] is a combination of path-integral and canonical methods, where one defines the wave function of the Universe by

$$\Psi_0(h_{ij},\phi,\partial M)=\sum_M \int D(g_{\mu\nu},\Phi)\exp(-I[g_{\mu\nu},\Phi,M]) , \tag{1}$$

where I is the Euclidean action for the metric and matter field configurations which induce the three-metric h_{ij} and matter field ϕ on the surface ∂M. The path integral is taken over all four-metrics and regular matter fields on compact manifolds M whose only boundary is ∂M.

Although the Euclidean formulation has advantages over other formulations [2,3] it has several disadvantages. In particular the choice of integration contour is not uniquely defined by the NBP [4]. As it stands, the path integral does not even converge. This arises because derivatives of the conformal factor exist in the volume term of the action, so the Euclidean action can be made arbitrarily negative. A suggestion [5] to cure this involves rotating the integration contour in the conformal integral, but this process does not extend to compact four-manifolds [6]. However, to use the integral at all one must admit complex-valued configurations to obtain a convergence. Moreover, classical space times are only predicted by the wave function when it becomes oscillatory, as described below.

Despite the difficulties in defining the path integral, predictions can be extracted from it by use of the semiclassical approximation. The wave function can be expressed approximately using saddle points of the path integral:

$$\Psi\sim\sum_n A_n e^{-I_n/\hbar} . \tag{2}$$

The n configurations which minimize the action have actions I_n and the prefactors A_n are determined by an integral over fluctuations around the solution [7]. The saddle points for the NBP are solutions of the equations of motion with the no boundary condition at the initial surface and the real arguments of the wave function at the final surface.

The configuration space of quantum cosmology in its full generality is infinite dimensional; therefore minisuperspace models are often used to restrict the degrees of freedom to a finite number in order to make the problem tractable. As long as the minisuperspace equations of motion coincide with the Einstein equations, the lowest order of the semiclassical wave function will coincide for minisuperspace and the full theory [8,9].

Lorentzian minisuperspace actions for bosonic matter generally have the form [10]

$$S[q^\alpha(t),N(t)]=\int_0^1 dt\, N\left[\frac{1}{2N^2}f_{\alpha\beta}(q)\dot q^\alpha\dot q^\beta-U(q)\right] . \tag{3}$$

The finite number of degrees of freedom are represented by the q^α. The minisupermetric $f_{\alpha\beta}$ has the signature $(-,+,+,\dots)$, the negative part coming from the conformal part of the gravitational field [11]. This is a constrained system as the momentum conjugate to the lapse N vanishes and gives rise to the Hamiltonian constraint

$$\frac{1}{2N^2}f_{\alpha\beta}\dot q^\alpha\dot q^\beta+U(q)=0 . \tag{4}$$

Following the canonical quantization procedure of Dirac [12], one constructs the operator form of the constraint to obtain the Wheeler–DeWitt equation [11] for the minisuperspace wave function:

$$[-\tfrac{1}{2}\nabla^2+U(q)]\Psi(q^\alpha)=0 . \tag{5}$$

Here factor ordering considerations are ignored and the Laplacian is with respect to the minisupermetric. This equation contains all the dynamics of the system [13] in the form of a time-independent Schrödinger equation which is characteristic of time reparametrization-invariant theories such as general relativity. In the full theory there would also be momentum constraints, but

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the restriction to minisuperspace eliminates these.

A WKB solution of (5) is

$$\Psi(q) \sim A(q) e^{-I(q)/\hbar} , \tag{6}$$

where the prefactor A and the rapidly varying action of the solution I will be complex. Inserting this into (5) and equating powers of the Planck mass one can obtain in the first two orders the equations

$$-\tfrac{1}{2}(\nabla I)^2 + U(q) = 0 , \tag{7}$$

$$2\nabla I \cdot \nabla A + A \nabla^2 I = 0 . \tag{8}$$

The gradient and dot product are again with respect to the minisupermetric. The real and imaginary parts of (7) are then

$$-\tfrac{1}{2}(\nabla I^{\mathrm{Re}})^2 + \tfrac{1}{2}(\nabla I^{\mathrm{Im}})^2 + U(q) = 0 , \tag{9}$$

$$\nabla I^{\mathrm{Re}} \cdot \nabla I^{\mathrm{Im}} = 0 . \tag{10}$$

If the semiclassical wave function is such that $[(\nabla I^{\mathrm{Re}})^2| \ll |(\nabla I^{\mathrm{Im}})^2|$, then by (9) I^{Im} will be an approximate solution to the Lorentzian Hamilton-Jacobi equation and so defines classical trajectories along integral curves of $\partial/\partial t \equiv \nabla I^{\mathrm{Im}} \cdot \nabla$, which by Eq. (10) are curves with $I^{\mathrm{Re}} = \mathrm{const}$. In particular, the usual classical equations of motion are obtained [10]. $\exp(-I^{\mathrm{Re}})$ provides a measure on the set of classical trajectories [8].

II. COMPLEX MINISUPERSPACE MODEL

A minisuperspace model of this procedure can be constructed, attempting to find complex solutions inducing the real end points which are the arguments of the wave function. Previously, some calculations with the NBP have been performed by splitting such complex solutions into real Euclidean and real Lorentzian sections, matching such solutions at the junction. However, as was pointed out by Halliwell and Hartle [6], not all complex solutions can be decomposed in this way and in fact the following model is such a case.

As the Universe appears to be isotropic and homogeneous on large scales, consider the minisuperspace model consisting of a closed Robertson-Walker universe with a scale factor a and massive minimally coupled scalar field Ψ of mass m, both functions of coordinate time only. In previous studies this model has proved physically reasonable, as it predicts classical inflationary space times with the most probable value of the density parameter being unity [14,15].

In order to consider complex solutions to the field equations, the metric is expressed in Euclidean form which will then be analytically continued:

$$ds^2 = \sigma^2 [N(t)^2 dt^2 + a(t)^2 d\Omega_3^2] , \tag{11}$$

where $\sigma^2 = 2/(3\pi m_P^2)$, N is the lapse, and $d\Omega_3^2$ is the usual three-sphere metric. Expressing the scalar field as $\sqrt{2}\pi\sigma\phi$ and assuming the potential $2\pi^2\sigma^2 m^2 \phi^2$, the Euclidean action is

$$I = -\tfrac{1}{2}\int dt \left[\frac{a}{N}\dot{a}^2 + Na - \frac{a^3}{N}\dot{\phi}^2 - m^2 N \phi^2 a^3 \right] . \tag{12}$$

In the ansatz (3), this action corresponds to the minisupermetric with $f_{aa} = -a$, $f_{\phi\phi} = a^3$, and $U = a(1 - a^2 m^2 \phi^2)/2$. Variations with respect to ϕ, a, and N give the Euclidean equations of motion:

$$\frac{d^2\phi}{d\tau^2} + \frac{3}{a}\frac{da}{d\tau}\frac{d\phi}{d\tau} - m^2\phi = 0 , \tag{13}$$

$$\frac{d^2 a}{d\tau^2} + a m^2 \phi^2 + 2a \left[\frac{d\phi}{d\tau} \right]^2 = 0 , \tag{14}$$

$$\left[\frac{da}{d\tau} \right]^2 - a^2 \left[\frac{d\phi}{d\tau} \right]^2 - 1 + a^2 m^2 \phi^2 = 0 , \tag{15}$$

where the variable τ is defined such that $d\tau = N\,dt$. Consider now a complex metric on the real manifold. Then the above equations are to be solved in the complex τ plane with the analytic continuation of the usual NBP initial conditions for this model [14]:

$$a=0, \quad \frac{da}{d\tau}=1, \quad \frac{d\phi}{d\tau}=0, \quad \phi=\phi_0 \ \text{ for } \tau=0 . \tag{16}$$

Here a is set to zero at $\tau=0$ in order to close off the geometry and regularity determines the other initial conditions. In the absence of poles of analytic $a(\tau)$ and $\phi(\tau)$, Cauchy's theorem allows the deformation of a general contour in the complex τ plane into one with straight sections with $N = \mathrm{const}$ along them (e.g., along the Euclidean time axis with $N=1$ then parallel to the Lorentzian time axis with $N=i$). This will not change the action and so represents a complex diffeomorphism; diffeomorphically related geometries should be represented only once in the contribution to the semiclassical wave function to avoid an infinite gauge volume. Note that the contours referred to in this context are not the same as the contours over field configurations used in the definition of the path integral; instead we are merely considering the complex saddle points used to approximate the path integral.

For cases such as the de Sitter space model considered by Halliwell and Hartle [6], using the NBP, the complex solutions can be deformed into a real Euclidean section along the real τ axis (corresponding to $N=1$), followed by a real Lorentzian section (corresponding to $N=i$) along an imaginary τ direction. Such real solutions must be matched at the junction where N changes its value. The junction condition $K_{ij}=0$ is equivalent to $dq^\alpha/dt=0$, where the q^α represent minisuperspace coordinates and t is the coordinate time along the Lorentzian section.

It should be noted that these conditions in turn follow simply from the consideration that a and ϕ should be analytic functions of τ. Analyticity of a implies

$$\frac{da}{d\tau} = \frac{1}{N_1}\frac{da}{dt_1} = \frac{1}{N_2}\frac{da}{dt_2} , \tag{17}$$

where the Euclidean section has $N = N_1 = 1$ and coordinate time t_1 and the Lorentzian section has $N = N_2 = i$ and coordinate time t_2. Requiring that a be real on both sections when N has different complex values means that both derivatives must vanish in the above equation. The same argument applies to the other q^α in general, which recovers the result of Halliwell and Hartle.

However, this separation of contour cannot be performed in general. For the NBP model considered here one can see this by attempting to find $\dot{a}=\dot{\phi}=0$ for a and ϕ real along a Euclidean contour from the origin. Equation (13) implies that ϕ is a monotonic function on this contour so $\dot{\phi}=0$ only at the origin, but $\dot{a}\neq 0$ there.

In general, the q^α must be fully complex and one looks for solutions inducing real end points which are nearly Lorentzian in the vicinity of the end points. This latter condition is necessary to get oscillatory behavior from the semiclassical wave function, which will correspond to classical space time via the WKB method.

III. APPROXIMATE COMPLEX SOLUTIONS

One has only the freedom to choose the initial complex value of ϕ_0 in this model. However, this freedom is sufficient to obtain a one-parameter family of approximate solutions with real end points. By judicious choice of the imaginary part of ϕ_0, one can obtain such solutions for most values of the real part of ϕ_0.

In analogy with the inflationary solution of the real version of this model, the approximate solution is

$$\phi \simeq \phi_0 + i\frac{m\tau}{3}, \quad a \simeq a_0 \exp(-mi\phi_0\tau + \tfrac{1}{6}m^2\tau^2) \tag{18}$$

and for large $|\phi_0^{\rm Re}| \gg |\phi_0^{\rm Im}|$ this solution is analytic and attracting in the region $1/m\phi_0^{\rm Re} \ll \tau^{\rm Im} \ll (3\phi_0^{\rm Re}-1)/m$. (The superscripts denote real and imaginary parts.) One can see this by considering a small departure χ from the approximate solution for ϕ; linear perturbation analysis gives

$$\ddot{\chi} - 3i\left[\phi_0 + i\frac{m\tau}{3}\right]\dot{\chi} \simeq 0\ . \tag{19}$$

Solutions to this either represent a change in the initial value of ϕ or behave like

$$\dot{\chi} \simeq \exp[-3m\phi_0^{\rm Re}\tau^{\rm Im} - 3m\phi_0^{\rm Im}\tau^{\rm Re} - \tfrac{1}{2}m^2(\tau^{\rm Re})^2 + \tfrac{1}{2}m^2(\tau^{\rm Im})^2] \times (\text{oscillating part})\ . \tag{20}$$

Perturbations to the approximate solution decay exponentially like $1/a^3$ in a Lorentzian direction ($\tau^{\rm Re}=\text{const}$). In a Euclidean direction, the exponent cannot grow above a factor of order unity. Once $\phi^{\rm Re}$ has fallen to a value comparable with $\dot{\phi}$, the approximation will no longer hold and so the region of validity ends when $\tau^{\rm Im} \sim 3\phi_0^{\rm Re}/m$.

For $|\phi_0^{\rm Re}| \gg 1 \gg |\phi_0^{\rm Im}$, examination of a Taylor-series expansion about the origin shows that ϕ is approximately constant with a fractional error $\sim 1/(\phi_0^{\rm Re})^2$. One can therefore use the approximate solution

$$\phi \simeq \phi_0^{\rm Re}, \quad a \simeq \frac{\sin m\phi_0^{\rm Re}\tau}{m\phi_0^{\rm Re}} \tag{21}$$

in this region. Analytically continuing to the imaginary axis and matching to the approximate solution (18) shows that the constant a_0 has an imaginary part of order $i/m\phi_0^{\rm Re}$ and a much smaller real part.

One can use this approximate solution to find solutions which have a and ϕ simultaneously real. These would be the saddle points which contribute to the semiclassical wave function. The ϕ equation implies that in a Lorentzian direction only $\phi^{\rm Re}$ varies, whereas in a Euclidean direction only $\phi^{\rm Im}$ varies. In particular, there is a vertical line $\tau=-3\phi_0^{\rm Im}/m + i\tau^{\rm Im}$ for which $\phi^{\rm Im}=0$. The a equation suggests that a is oscillatory in a Euclidean direction when ϕ_0 is approximately large and real. Thus, one expects to find a roughly Lorentzian line for which $a^{\rm Im}=0$. Together, these observations suggest that it may be possible to obtain nearly coincident real a and real ϕ curves in the complex plane.

By choosing $\phi_0^{\rm Im}$ so that

$$\phi_0^{\rm Im} = -\frac{\pi}{6\phi_0^{\rm Re}}(1+2n) \tag{22}$$

where n is an integer, Lorentzian lines given by

$$\tau = \frac{\pi(1+2n)}{2m\phi_0^{\rm Re}} + i\tau^{\rm Im} \tag{23}$$

are obtained for which both a and ϕ are approximately real. One can see this by eliminating τ from (18) to obtain

$$a \simeq \frac{i}{m\phi_0^{\rm Re}} e^{3(\phi_0^2-\phi^2)/2}\ . \tag{24}$$

If a and ϕ are real then $3\phi_0^{\rm Re}\phi_0^{\rm Im} = -\pi/2 - n\pi$, which gives condition (22), and the solution for ϕ immediately gives the line in the complex plane where a and ϕ are real. By continuity of the approximate solution it is evident that successively finer tuning of $\phi_0^{\rm Im}$ will obtain exactly real end points a and ϕ. Furthermore, in this approximation the condition which fixes a and ϕ to be almost real will not depend on the time parameter along the Lorentzian line, so that a and ϕ will be approximately real along the whole line. The extent to which this fails is determined by the time taken for the solution to reach the attractor; from the decay of the ϕ perturbations this will be a small time of order $1/m_0^{\rm Re}$ along the Lorentzian line.

This method corresponds to the approach of Laflamme and Shellard [16], where they obtain Lorentzian solutions by analytically continuing from the maximum in a at $\tau^{\rm Re} \simeq \pi/2m\phi_0^{\rm Re}$ along the Euclidean axis, assuming real a and ϕ with ϕ approximately constant. However, it can now be seen that a complex initial scalar value is required and also that there are many complex solutions which interpolate between the NBP initial conditions and the given a and ϕ. In fact, for all values of n in (22) which have $\phi_0^{\rm Im} \ll 1$ the approximation above is good and the results hold. It may be that one can continue to find solutions with real end points for larger values of $\phi_0^{\rm Im}$, but this approximation cannot determine them.

The behavior of a and ϕ along the lines (23) is just like that of the real inflationary solution for the scalar field model, so all the familiar results of inflation can be obtained from this complex model. Positive values of a are obtained for even values of n. Note also that along these lines

$$a \simeq |a_0| e^{3[(\phi_0^{\rm Re})^2 - \phi^2]/2} \tag{25}$$

so that

$$\phi_0^{\rm Re} \simeq (\tfrac{2}{3}\ln a + \phi^2)^{1/2} \tag{26}$$

determines which initial conditions to fix at $\tau=0$ for given end points a and ϕ.

Halliwell and Hartle have argued [6] that in order to recover quantum field theory in curved space time from quantum cosmology one should restrict $N^{\rm Re} \geq 0$. The $\sqrt{g}$ term in the complex action renders N double valued and hence pairs of saddle points must be considered; but the matter action for the sign-reversed action will be negative definite and the quantum field theory is not normalizable. This implies for the present model that $n \geq 0$; i.e., solutions must have end points lying in the $\tau^{\rm Re} > 0$ plane.

IV. CLASSICAL SPACETIMES IN THE COMPLEX MODEL

The complex action of a solution to the equations of motion obtained from (12) is

$$I = \int d\tau\, a[a^2 m^2 \phi^2 - 1] \ . \tag{27}$$

To evaluate this for one saddle point using the approximate solution, consider the τ contour which goes along the real τ axis to $\tau = \pi/2m\phi_0^{\rm Re}$, then proceeds in an imaginary direction, going up a Lorentzian direction along the contour where a and ϕ are real until the end points are reached (see Fig. 1). This contour is only useful for $n=0$ since the approximation $\phi \simeq \phi_0^{\rm Re}$ along the Euclidean axis fails beyond the maximum of a (ϕ blows up as a recollapses).

For large $\phi_0^{\rm Re}$ one can use the approximate solution (21) to evaluate the integral along the Euclidean section:

$$I_1 = \int_0^{\pi/2m\phi_0^{\rm Re}} d\tau^{\rm Re} a(a^2m^2\phi^2 - 1) \simeq -\frac{1}{3m^2(\phi_0^{\rm Re})^2} \ . \tag{28}$$

Continuing along the second section (now Lorentzian), one can use the same approximation, but the integral contributes only a negligible imaginary piece (integration proceeds only to $\tau^{\rm Im} = 1/m\phi_0^{\rm Re}$, before entering the region of validity of the solution). Lastly, the approximate solution is used to obtain the integral along the contour of real a and ϕ to the end points. As the scale factor exponentiates in this section, one can ignore the second term in the action; an integration by parts and ignoring factors of order unity yields

$$I_3 \simeq \frac{im\phi a^3}{3} + \frac{im^2|a_0^3|}{9} \times \int_{1/m\phi_0^{\rm Re}}^{3(\phi_0^{\rm Re}-\phi)/m} d\tau^{\rm Im} \exp[3m\phi_0^{\rm Re}\tau^{\rm Im} - \tfrac{1}{2}m^2(\tau^{\rm Im})^2] \ . \tag{29}$$

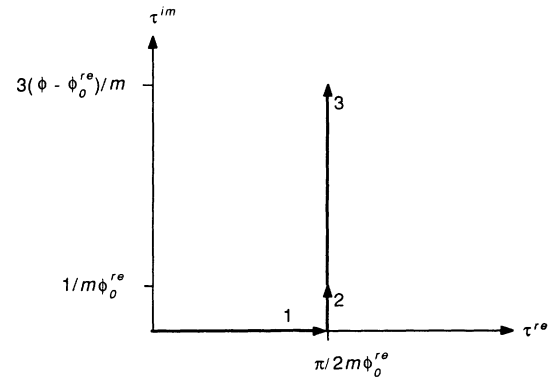

FIG. 1. Integration contour for the action in the complex time plane.

The upper limit of the integral gives the Lorentzian time at which the desired end point is reached. Completing the square in the exponent and using the result [17]

$$\frac{2}{\sqrt{\pi}}\int_0^x dt\, e^{-t^2} \simeq 1 - \frac{e^{-x^2}}{\sqrt{\pi}x} \quad \text{for } x \gg 1 \tag{30}$$

one finally obtains

$$I_3 \simeq \frac{mi\phi a^3}{3} + \left[\text{a term} \leq \frac{mia^3}{27\phi}\right] \quad \text{for } \phi_0^{\rm Re} \gg 1 \text{ and } \phi \gg 1 \ . \tag{31}$$

As an approximation is being used to derive the action, the errors in the evaluation of $I^{\rm Re}$ can be larger than the value derived here. As discussed above ϕ_0 can be finely tuned so that the integrand in (29) is entirely imaginary. Only for a time of order $1/m\phi_0^{\rm Re}$ will this not be the case and may there be a real error term of order $1/m^2(\phi_0^{\rm Re})^2$. This is in fact of the same order as the real action already calculated. Note that the exponential decay of perturbations from the attractor is sufficient to make these contributions to the integrand decay exponentially, despite the exponential growth of a^3.

Summarizing, the action for this saddle point is given by

$$I^{\rm Im} \simeq m\phi a^3/3 \text{ and } I^{\rm Re} \sim 1/m^2(\phi_0^{\rm Re})^2 \quad \text{for } \phi_0^{\rm Re} \gg 1 \text{ and } \phi \gg 1 \ . \tag{32}$$

This method is not available for the other saddle points since the scalar field does not remain approximately constant along the whole Euclidean time axis, as mentioned at the beginning of this section. This result invalidates the claim by Halliwell [8] that for the scalar field model there are many saddle points corresponding to chains of expanding and recollapsing Euclidean universes. However, the analytic continuation here reveals that there is indeed a large (if not infinite) number of saddle points for large a and ϕ in the inflationary region.

Using (26), it can be seen that for this saddle point $I^{\rm Re}$ tends to zero as one goes to larger geometries for fixed ϕ, implying that the measure for such trajectories is unity. This is encouraging; one does not want large geometries to be suppressed. The value of $(\nabla I^{\text{Re}})^2$ can now be compared with $(\nabla I^{\text{Im}})^2$. In the inflationary region,

$$(\nabla I^{\text{Re}})^2 \equiv -\frac{1}{a}\left[\frac{\partial I^{\text{Re}}}{\partial a}\right]^2 + \frac{1}{a^3}\left[\frac{\partial I^{\text{Re}}}{\partial \phi}\right]^2 \sim \phi^2/m^4 a^3 (\phi_0^{\text{Re}})^8 \tag{33}$$

and

$$(\nabla I^{\text{Im}})^2 \simeq -m^2\phi^2 a^3 \tag{34}$$

so that Eq. (9) becomes a Lorentzian Hamilton-Jacobi equation to a very good approximation, since $|(\nabla I^{\text{Re}})^2| \ll |(\nabla I^{\text{Im}})^2|$ holds for $\phi_0^{\text{Re}} \gg 1$. Thus the wave function evaluated on this saddle point predicts the emergence of a classical universe, with inflationary evolution.

V. LATE CLASSICAL TIMES

In conclusion, the complex model provides many saddle points for the path integral and the action can be calculated for at least one of these. The form of the action predicts the emergence of a one-parameter family of classical inflationary universes. Eventually the classical real scalar field eventually evolves to a region where it oscillates and behaves as dust, as discussed by Hawking and Page [18]. These authors give approximate solutions in this region. Here, the complex solution will no longer be valid, but the evolution in the inflationary region has already been adjusted so that a and ϕ are real as they evolve. Thus in the dust regime one expects the complex model to reproduce the real scalar field evolution for some time. In addition, the real part of the action I will not change if a and ϕ remain real along the Lorentzian line in the complex τ plane and the value of I^{Re} remains just that corresponding to the inflationary solution from which it evolves (since the integral in the dust phase remains purely imaginary).

For times when the spatial curvature term is negligible,

$$a(\phi_0^{\text{Re}}, t) \simeq m^{2/3} a_0 e^{3(\phi_0^{\text{Re}})^2/2} t^{2/3}, \qquad \phi \simeq \frac{2}{3}\frac{\cos(mt - 3\phi_0^{\text{Re}})}{mt}. \tag{35}$$

Now that a and ϕ are evolving in a different manner, the functional form of I^{Re} will change, in that ϕ_0^{Re} has a different dependence on a and ϕ. Now $(\nabla I^{\text{Re}})^2$ is a term of order a^{-3} at its largest, so that interpreting $(\nabla I^{\text{Re}})^2$ as an extra term in the potential would correspond to matter with an energy density like a^{-6}, which is negligible. Therefore the complex model presented here has the favorable features of an inflationary period followed by a dust phase, without the presence of some extra matter.

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