Stationary solutions in five-dimensional gravity with a magnetic field

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Using the potential formalism, six new stationary axisymmetric solutions of the five-dimensional Kaluza-Klein field equations are constructed. It is supposed that each potential depends only on one parameter which satisfies the Laplace equation. All the solutions have a scalar potential and some of them possess magnetic fields which represent a magnetic monopole, dipole, and quadrupole.

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I. INTRODUCTION

In the 1920s, a five-dimensional relativity theory was proposed by Kaluza and Klein [1] in an attempt to achieve a unified theory of gravitation and electromagnetism, the only interactions well understood at that time. In this theory the electromagnetic field does not have any fundamental significance of its own, but is considered merely as a component of gravity in a world which is assumed to have an extra spatial dimension which is curled up to form a circle so small as to be unobservable. Recently, the Kaluza-Klein idea has been generalized to higher dimensions and has been merged with supergravity and the string ideas of the 1960s into a yet grander theory: the theory of superstrings which holds promise of the first consistent quantum theory of gravitation. Hence searching for exact solutions in higher-dimensional framework is becoming of great interest nowadays. But in spite of this fact, there are only a few known exact solutions to the Kaluza-Klein theories so far.

In this paper we shall construct a set of six new stationary axisymmetric solutions to the vacuum field equations of the five-dimensional Kaluza-Klein theory. The full metric of this theory can be written in the form [2]

$$\gamma_{MN}=\begin{bmatrix} I^{-1}g_{\mu\nu}+I^2A_\nu A_\mu & I^2A_\mu \\ I^2A_\nu & I^2\end{bmatrix}$$

$$M,N=1,\ldots,5,\quad \nu,\mu=1,\ldots,4 \tag{1}$$

where A_μ is the electromagnetic four-potential, $g_{\nu\mu}$ is the space-time metric tensor, and I^2 is the scalar potential. In the source-free region the field equations are characterized by the vanishing of the five-dimensional Ricci tensor, i.e.,

$$\hat{R}_{MN}=0\ . \tag{2}$$

We put a caret on $\hat{R}_{MN}$ to distinguish them from the corresponding four-dimensional quantities $R_{\mu\nu}$. The potential formalism introduced by Neugebauer [3] in five-dimensional gravity shall be employed in the present paper; therefore, a summary of that formalism is provided in Sec. II as a necessary reference for the remaining sections where several solutions to the equations (2) are constructed.

II. POTENTIAL FORMALISM

When the five-dimensional metric contains two no-null Killing vector fields X^μ and Y^μ with $Y^\mu Y_\mu<0$ (stationary case), five real potentials $\psi^A=(\kappa,f,\psi,\chi,\epsilon)$, $A=1,\ldots,5$ can be defined covariantly in the form [3]

$$\begin{aligned}&\kappa^{4/3}=I^2=X^\nu X_\nu\ ,\quad f=-I^2Y^\mu Y_\mu+I^{-1}(X^\mu Y_\mu)^2\ ,\\ &\psi=-I^{-2}X_\nu Y^\nu\ ,\quad \chi_{,\mu}=\epsilon_{\alpha\beta\gamma\delta\mu}X^\alpha Y^\beta X^{\gamma;\delta}\ ,\\ &\epsilon_{,\mu}=\epsilon_{\alpha\beta\gamma\delta\mu}X^\alpha Y^\beta Y^{\gamma;\delta}\ ,\end{aligned} \tag{3}$$

where $\epsilon_{\alpha\beta\gamma\delta\mu}$ is the five-dimensional Levi-Civita pseudotensor. In the coordinate system with $X^\nu=\delta^\nu{}_5$, $Y^\mu=\delta^\mu{}_4$, one finds that f,ψ,χ,ϵ are, respectively, the gravitational, electrostatic, magnetostatic, and rotational potentials. These potentials are analogous to the Ernst potentials [4] of the Einstein-Maxwell theory and define a Riemannian space $V_p{}^5$ with the metric [5]

$$\begin{aligned}dS_p^2&=G_{AB}d\psi^Ad\psi^B\\ &=\frac{1}{2f^2}[df^2+(d\epsilon+\psi\,d\chi)^2]+\frac{1}{2f}\left[\kappa^2d\psi^2+\frac{1}{\kappa^2}d\chi^2\right]\\ &\quad+\frac{2}{3}\frac{d\kappa^2}{\kappa^2}\ .\end{aligned} \tag{4}$$

The Jordan theory and the Brans-Dicke theory are contained in (4) after an appropriate conformal transformation. We assume that the five potentials ψ^A depend only on two variables X^1 and X^2. Let these coordinates be the Weyl canonical ones ρ and ζ, and z the complex variable $z=\rho+i\zeta$. The fields equations (2) can be derived in function of the potentials ψ^A from the Lagrangian [3,5]

$$\mathcal{L}=\frac{\rho}{2f^2}[f_{,a}f^{,a}+(\epsilon_{,a}+\psi\chi_{,a})(\epsilon^{,a}+\psi\chi^{,a})]+\frac{\rho}{2f}(\kappa^2\psi_{,a}\psi^{,a}+\kappa^{-2}\chi_{,a}\chi^{,a})+\frac{2}{3}\frac{\rho}{\kappa^2}\kappa_{,a}\kappa^{,a}\,, \quad a=x^1,x^2\,. \tag{5}$$

The variation $\delta\mathcal{L}/\delta\psi^A=0$ leads to

$$(\rho\psi^A{}_{,z})_{,\bar z}+(\rho\psi^A{}_{,\bar z})_{,z}+2\rho\left\{\begin{matrix}A\\ B\ \ C\end{matrix}\right\}\psi^B{}_{,z}\psi^C{}_{,\bar z}=0\,. \tag{6}$$

An overbar denotes complex conjugation, and $\{{}^{\,B}_{A\ C}\}$ are the Christoffel symbols of the metric (4). If it is supposed that the five real potentials ψ^A depend only on one parameter $\lambda=\lambda(z,\bar z)$, the potential space $V_p{}^5$ is reduced to a one-dimensional subspace, and the field equations (6) become

$$2\rho\left[\psi^A{}_{,\lambda\lambda}+\left\{\begin{matrix}A\\ B\ \ C\end{matrix}\right\}\psi^B{}_{,\lambda}\psi^C{}_{,\lambda}\right]\lambda_{,z}\lambda_{,\bar z}+\psi^A{}_{,\lambda}[(\rho\lambda_{,z})_{,\bar z}+(\rho\lambda_{,\bar z})_{,z}]=0\,. \tag{7}$$

When the parameter λ is taken as a solution of the generalized Laplace equation

$$(\rho\lambda_{,z})_{,\bar z}+(\rho\lambda_{,\bar z})_{,z}=0\,, \tag{8}$$

the field equations (7) reduce to

$$\psi^A{}_{,\lambda\lambda}+\left\{\begin{matrix}A\\ B\ \ C\end{matrix}\right\}\psi^B{}_{,\lambda}\psi^C{}_{,\lambda}=0\,. \tag{9}$$

For the axisymmetric case these last equations can be rewritten as [6]

$$(\rho g_{,z}g^{-1})_{,\bar z}+(\rho g_{,\bar z}g^{-1})_{,z}=0\,, \tag{10}$$

which is called the chiral-field equation. The matrix g is an element of the group SL(3,R); i.e., g is a real matrix with a determinant equal to one, and g and its transpose g^T are the same. An apposite parametrization of the matrix g in terms of ψ^A is given by

$$g=-\frac{2}{f\kappa^{2/3}}\begin{vmatrix} f^2+\epsilon^2-f\kappa^2\psi^2 & -\epsilon & -\frac{1}{2\sqrt{2}}(\epsilon\chi+f\kappa^2\psi)\\ -\epsilon & 1 & \frac{1}{2\sqrt{2}}\chi\\ -\frac{1}{2\sqrt{2}}(\epsilon\chi+f\kappa^2\psi) & \frac{1}{2\sqrt{2}}\chi & \frac{1}{8}(\chi^2-\kappa^2 f)\end{vmatrix}\,. \tag{11}$$

The ansatz $\psi^A=\psi^A(\lambda(z,\bar z))$ transforms the chiral-field equation (10) into

$$(g_{,\lambda}g^{-1})_{,\lambda}=0\,.$$

Let us define now a constant 3×3 matrix A by

$$g_{,\lambda}g^{-1}=A\,. \tag{12}$$

Since g belongs to the group SL(3,R), the following properties for the matrix A,

$$\text{(a) } Ag=gA^T\,,\quad \text{(b) } A=\bar A\,,\quad \text{(c) } \mathrm{Tr}A=0\,, \tag{13}$$

hold. The field equations (10) are unchanged under the transformation $g'=CgC^T$ provided that the constant matrix C also belongs to the group SL(3,R). Under this transformation the matrix A changes to $A'=CAC^{-1}$. There are two classes of equivalence induced by the transformation of the matrix A. Hence each matrix with the properties (13b) and (13c) is equivalent to one of the following two normal forms of the matrix A which are the representatives of the classes:

$$\begin{bmatrix}0&1&0\\0&0&1\\d&e&0\end{bmatrix},\quad \begin{bmatrix}q&0&0\\0&0&1\\0&2q^2&-1\end{bmatrix}, \tag{14}$$

where d is the determinant of A and $e=\frac{1}{2}\mathrm{Tr}A^2$. Then it is enough to work with these two cases to solve the differential equation (12). Working with the first matrix in (14) one has different cases according to the roots of its characteristic polynomial. With the second matrix one has two cases $q=0$ and $a\neq0$. All the cases have been solved recently [7]. We are concerned with three of them.

Case 1. All the roots r_1,r_2,r_3 are real ($r_1+r_2+r_3=0$); therefore, we can use a diagonal form for the matrix A. The solution of the equation (12) is given by

$$\kappa^2=(4b)^{1/3}\exp(3r_3\lambda/2)\,,\quad \psi=\epsilon=\chi=0\,,\quad f=-2ab\exp[(r_3/2+r_1)\lambda]\,. \tag{15}$$

Case 2. The three roots are again real but $r_1=r_2$, $r_3=-2r_1$. The invariant factors of dimension one and two are equal to unity. Thus one can use a Jordan normal form for the matrix A. The solution of (12) is then

$$\kappa^2=\left[\frac{4\exp(r_1\lambda)}{a^{1/3}(b\lambda+c)}\right]^{3/2}, \quad f=\frac{\sqrt{a}\,\exp(-3r_1\lambda/2)}{P^{1/2}}$$
$$\chi=\frac{2\sqrt{2}b}{P}, \quad \psi=\epsilon=0, \quad \text{and } P=b\lambda+c \; . \tag{16}$$

Case $q=0$. Now employing the second matrix in (14) when $q=0$, the real potential ψ^A which satisfies the equation (12) is given by

$$\kappa^2=\frac{8b}{\sqrt{-a}\,(-P)^{3/2}}, \quad f=[|b|(-P)^{1/2}]^{-1},$$
$$\chi=\frac{2\sqrt{2}b}{|P|}, \quad \psi=\epsilon=0, \quad a=-\frac{1}{b^2}, \tag{17}$$

where a,b,c are constants and λ satisfies $(\rho\lambda_{,z})_{,\bar z}+(\rho\lambda_{,\bar z})_{,z}=0$.

We have started assuming that the five-dimensional metric depends on two variables x^1 and x^2. Thus the five-dimensional line element reads

$$dS^2=H(x^1,x^2)[(dx^1)^2+(dx^2)^2]+\gamma_{ab}dx^a dx^b, \tag{18}$$

where a,b run over the three values 3,4,5. From the definitions (3) together with the specific expressions of the potentials ψ^A in the three cases mentioned above, one is able to get the components of the matrix

$$\gamma=\begin{vmatrix} I^{-1}g_{33}+I^2A_3^2 & I^{-1}g_{34}+I^2A_3A_4 & I^2A_3 \\ I^{-1}g_{34}+I^2A_3A_4 & I^{-1}g_{44}+I^2A_4^2 & I^2A_4 \\ I^2A_3 & I^2A_4 & I^2 \end{vmatrix}. \tag{19}$$

In order to have the full metric one would need to find the function H, which can be obtained by integration of [8]

$$(\ln H)_{,z}=\frac{1}{(\ln\rho)_{,z}}[(\ln\rho)_{,zz}+\tfrac{1}{4}\mathrm{Tr}D^2], \tag{20}$$

where the matrix $D=\gamma_{,z}\gamma^{-1}$. In the following sections we shall find the components of the matrix γ and integrate the function H for the three cases we have mentioned for different parameters λ satisfying the Laplace equation (8).

III. GETTING THE MATRIX γ

It is easy to verify that $\det\gamma=g_{33}g_{44}-g_{34}=-\rho^2$, and from the defintions (3), one finds for our three cases that

$$I^2=\kappa^{4/3}, \quad g_{44}=-f, \quad g_{33}=-\rho^2/g_{44}, \quad A_4=-\psi, \quad g_{34}=0, \tag{21a}$$

$$A_{3,z}=\frac{\rho}{fI^3}\chi_{,\bar z}. \tag{21b}$$

In the Boyer-Lindquist coordinate system (r,θ,ϕ,t) [4],

$$x^1=\rho=\sqrt{r^2-2mr}\,\sin\theta,$$
$$x^2=\zeta=(r-m)\cos\theta.$$

Using the explicit expressions for χ given by (16) and (17), (21b) takes the form

$$A_{3,r}=\frac{\sqrt{2}}{4}\sin\theta\lambda_{,\theta},$$
$$A_{3,\theta}=-\frac{\sqrt{2}}{4}(r^2-2mr)\sin\theta\lambda_{,r}. \tag{22}$$

It is important to note that A_1, A_2, A_4 always vanish. The integrability of (22) is guaranteed by (8). So giving a λ solution of (8) one can get the matrix γ from (21).

IV. INTEGRATION OF H

In the Boyer-Lindquist coordinate system the five-dimensional line element reads

$$dS^2=H(r,\theta)\Delta\left[\frac{dr^2}{r^2-2mr}+d\theta^2\right]+\gamma_{ab}dx^a dx^b, \tag{23}$$

$$a,b=3,4,5,$$
$$\Delta=(r-m)^2-m^2\cos^2\theta.$$

In order to determine the function H, it is necessary to integrate the differential equation (20); this equation and its complex conjugate yield a differential equation system which shall be written explicitly for our three cases. A straightforward calculation leads to

Case 1:

$$(\ln\{H\exp[(r_1+r_3)\lambda]\})_{,\theta}=\frac{\rho^2}{\Delta}\Lambda_0\{(r-m)\lambda_{,r}\lambda_{,\theta}+\tfrac{1}{2}\cot\theta[(\lambda_{,\theta})^2-Z(\lambda_{,r})^2]\},$$
$$(\ln\{H\exp[(r_1+r_3)\lambda]\})_{,r}=\frac{\rho^2}{\Delta}\Lambda_0\left[\cot\theta\lambda_{,r}\lambda_{,\theta}-\frac{r-m}{2Z}[(\lambda_{,\theta})^2-Z(\lambda_{,r})^2]\right], \quad \Lambda_0=r_1^2+r_2^2+r_1r_3; \tag{24}$$

Case 2:

$$\{\ln[HP^{-1}\exp(-r_1\lambda)]\}_{,\theta}=\frac{\rho^2}{\Delta}3r_1^2\{(r-m)\lambda_{,r}\lambda_{,\theta}+\tfrac{1}{2}\cot\theta[(\lambda_{,\theta})^2-Z(\lambda_{,r})^2]\},$$
$$\{\ln[HP^{-1}\exp(-r_1\lambda)]\}_{,r}=\frac{\rho^2}{\Delta}3r_1^2\left[\cot\theta\lambda_{,r}\lambda_{,\theta}-\frac{r-m}{2Z}[(\lambda_{,\theta})^2-Z(\lambda_{,r})^2]\right], \tag{25}$$

where $Z=r^2-2mr$.

Case $q=0$:

It was proven [9] that in this case, for any parameter λ, the function H always has the form

$$H(r,\theta)=I_0 I^{-2} , \tag{26}$$

where $I_0=-4b^2P_0$, P_0 being an arbitrary constant and

$$I^2=-\frac{4b^2}{P} .$$

V. EXACT SOLUTIONS WITH SCALAR POTENTIALS

For case 1 we have only the scalar potential given by (15). We integrate Eq. (24) for different parameters λ satisfying the Laplace equation (8) in this section. In this way, three new stationary axisymmetric exact solutions to the five-dimensional Kaluza-Klein field equations (2) are obtained. These solutions read

$$dS^2=H\left[1-\frac{2m}{r}+\frac{m^2\sin^2\theta}{r^2}\right]\left[\frac{dr^2}{1-\frac{2m}{r}}+r^2d\theta^2\right]+I^{-1}(g_{33}d\phi^2+g_{44}dt^2)+I^2dy^2 . \tag{27}$$

After taking $2ab=-1$ and $(4b)^{2/9}=I_0^2$, one arrives at solution A:

$$\begin{aligned}
\lambda&=\frac{\lambda_0}{2}\ln\left[\frac{1-\cos\theta}{1+\cos\theta}\right] ,\\
H&=\left[\frac{1-\cos\theta}{1+\cos\theta}\right]^{-(r_1+r_3)\lambda_0/2}(\sin\theta)^{\lambda_0^2\Lambda_0/2}\\
&\quad\times[(r-m)^2-m^2\cos^2\theta]^{-\lambda_0^2\Lambda_0/2} ,\\
I^2&=I_0^2\left[\frac{1-\cos\theta}{1+\cos\theta}\right]^{r_3\lambda_0/2} ,\\
g_{44}&=-\left[\frac{1-\cos\theta}{1+\cos\theta}\right]^{(r_3/2+r_1)\lambda_0/2} .
\end{aligned} \tag{28}$$

As above $g_{33}=-\rho^2/g_{44}$. Throughout the paper, this relation holds.

Solution B:

$$\begin{aligned}
\lambda&=\lambda_0(r-m)\cos\theta ,\\
H&=\exp[-\lambda_0^2\Lambda_0\rho^2/4-(r_1+r_3)\lambda_0\zeta] ,\\
I^2&=I_0^2\exp(r_3\lambda_0\zeta) ,\\
g_{44}&=-\exp[(r_1+r_3/2)\lambda_0\zeta] .
\end{aligned} \tag{29}$$

Solution C:

$$\begin{aligned}
\lambda&=\lambda_0\ln\rho^2 ,\\
H&=\rho^{2[\lambda_0^2\Lambda_0-\lambda_0(r_1+r_3)]} ,\\
g_{44}&=-\rho^{2\lambda_0(r_1+r_3/2)} .
\end{aligned} \tag{30}$$

VI. EXACT SOLUTIONS WITH MAGNETIC FIELD

The exact stationary axisymmetric solutions to the five-dimensional Kaluza-Klein theory constructed from the one-dimensional subspaces (16) and (17) (cases 2 and $q=0$) have a scalar potential and magnetic field. In both cases the magnetic fields are the same and can be obtained by integration of (22) but the scalar potential and the function H for each case are different.

For case 2 the metric reads

$$\begin{aligned}
dS^2&=H\left[1-\frac{2m}{r}+\frac{m^2\sin^2\theta}{r^2}\right]\left[\frac{dr^2}{1-\frac{2m}{r}}+r^2d\theta^2\right]\\
&\quad+I^{-1}(g_{33}d\phi^2+g_{44}dt^2)+I^2(A_3d\phi^2+dy^2) ,
\end{aligned} \tag{31}$$

where $g_{33}=-\rho^2/g_{44}$ and y is the extra spatial dimension which is compactified to a circle.

For the case $q=0$ the metric reads

$$dS^2=I^{-2}d\Lambda-dt^2+I^2(A_3d\phi^2+dy^2) , \tag{32}$$

where

$$\begin{aligned}
d\Lambda&=\left[1-\frac{2m}{r}+\frac{m^2\sin^2\theta}{r^2}\right]\left[\frac{dr^2}{1-\frac{2m}{r}}+r^2d\theta^2\right]\\
&\quad+\left[1-\frac{2m}{r}\right]r^2\sin^2\theta\,d\phi^2 ,
\end{aligned} \tag{33}$$

which for $r\gg m$ becomes

$$d\Lambda\approx dr^2+r^2(d\theta^2+\sin^2\theta\,d\theta^2)=d\Omega .$$

A. Magnetic fields for different λ's

We show now the corresponding gauge fields for different parameters λ which satisfy the generalized Laplace equation (8) in Table I.

We want to point out that since Eq. (8) is a linear differential one, a linear combination of different parameters λ_j also satisfies Eq. (8) and A_3 would be a linear combination of the corresponding $A_{3(j)}$. The first gauge field in Table I is clearly of a magnetic monopole which

TABLE I. The corresponding gauge fields for different parameters λ which satisfy the generalized Laplace equation (8). λ_0 is a constant, $\rho^2=(r^2-2mr)\sin^2\theta$, $\Delta=(r-m)^2-m^2\cos^2\theta$, and P_n is a Legendre polynomial with $x=(r-m)/m$, $y=\cos\theta$.

Parameter λ	A_3
$\lambda_0\ln(1-2m/r)$	$\frac{\sqrt{2}}{2}\lambda_0 m(\cos\theta-1)$
$\lambda_0 m(r-m)\Delta^{-1}$	$\frac{\sqrt{2}m^3}{4}\lambda_0\frac{\sin^2\theta\cos\theta}{\Delta}$
	$+\frac{\sqrt{2}}{2}m\lambda_0(\cos\theta-1)$
$\lambda_0 m^2\cos\theta\Delta^{-1}$	$\frac{\sqrt{2}}{4}m^2\lambda_0(r-m)\sin^2\theta\Delta^{-1}$
$\lambda_0\ln\rho^2$	$\frac{\sqrt{2}}{2}\lambda_0(r-m)\cos\theta$
$\lambda_0\ln[\tan(\theta/2)]$	$\frac{\sqrt{2}}{4}\lambda_0(r-m)$
$P_n(x)P_n(y)$	$A_3=\sum_n A_3^{(n)}$ where
	$A_3^{(n)}=-\frac{\sqrt{2}mn}{4(2n+1)}[P_{n-1}(y)-yP_n(y)][P_{n+1}(x)-P_{n-1}(x)]$

has appeared in the 1980s in Kaluza-Klein theories. In particular it has been shown that five-dimensional Kaluza-Klein theory embodies magnetic monopole solutions with remarkable properties [9–11]. The magnetic charge is $\sqrt{2m}\,\lambda_0/2$. The first term of the second magnetic field in Table I depends on r as a magnetic quadrupole term and the second term is again the potential of a magnetic monopole. The third magnetic field is of a magnetic dipole of strength $\sqrt{2m^2}\lambda_0/4$. The fourth and fifth magnetic fields in the table do not vanish when $m=0$; they exist independently of the parameter m. Finally the sixth magnetic field is written in terms of the Legendre polynomials; in this case we would need to impose opposite boundary conditions to have a finite magnetic potential; for instance, the solution can be inside a sphere on which A_3 vanishes.

For some parameters λ's in case 2, the corresponding function H, the component g_{44}, and the scalar potential are given. The metrics have the form of (31).

B. Solution A

For the parameter $\lambda=(\lambda_0/2)\ln(1-\cos\theta)/(1+\cos\theta)$

$$H=P\left[\frac{1-\cos\theta}{1+\cos\theta}\right]^{r_1\lambda_0/2}\frac{(\sin\theta)^{\lambda_0^2 3r_1^2/2}}{[(r-m)^2-m^2\cos^2\theta]^{3r_1^2\lambda_0^2/4}},$$
$$g_{44}=-\frac{\sqrt{a}}{\sqrt{|P|}}\left[\frac{1-\cos\theta}{1+\cos\theta}\right]^{-3r_1\lambda_0/2}, \tag{34}$$
$$I^2=\frac{I_0^2}{a^{1/3}P}\left[\frac{1-\cos\theta}{1+\cos\theta}\right]^{\lambda_0 r_1/2}.$$

Its corresponding magnetic field is 5 in Table I. The function $P=b\lambda+c$, with b and c arbitrary constants.

C. Solution B

For the parameter $\lambda=\lambda_0\zeta=\lambda_0(r-m)\cos\theta$,

$$H=P\exp(-r_1\lambda_0\zeta^{-(3r_1^2\lambda_0^2\rho^2)/4}),$$
$$g_{44}=-\frac{\sqrt{a}}{\sqrt{|P|}}\exp(-3r_1\lambda_0\zeta/2), \tag{35}$$
$$I^2=\frac{I_0^2}{a^{1/3}P}\exp(r_1\lambda_0\zeta).$$

D. Solution C

For the parameter $\lambda=\lambda_0\ln\rho^2$,

$$H=P\rho^{2(r_1\lambda_0+3r_1^2\lambda_0^2)},$$
$$g_{44}=-\frac{\sqrt{a}}{\sqrt{|P|}}\rho^{-2(3r_1\lambda_0/2)}, \tag{36}$$
$$I^2=\frac{I_0^2}{a^{1/3}P}\rho^{2r_1\lambda_0}.$$

$$P=b\lambda+c,\quad I_0^2=4b^{4/3},\quad a=-1/b^2.$$

VII. FINAL COMMENT

Further solutions to the five-dimensional Einstein equations can be obtained providing other solutions of the Laplace equation or using the SL(3,R) symmetry transformations $g=Cg_0C^T$.

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