Coalescing binary systems of compact objects to (post)$^{5/2}$-Newtonian order. II. Higher-order wave forms and radiation recoil

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Using formulas developed by Blanchet, Damour, and Iyer, we obtain a symmetric trace-free multipolar expansion of the gravitational radiation from a coalescing binary system which is sufficiently accurate to allow a post-Newtonian calculation of the linear momentum carried off by the gravitational radiation prior to a binary coalescence. We briefly examine the structure of the post-quadrupole corrections to the wave form for an orbiting binary system near coalescence. The post-Newtonian correction to the momentum ejection allows a more accurate calculation of the system recoil velocity (radiation rocket effect). We find that the higher-order correction actually reduces the net momentum ejection. Furthermore, the post-Newtonian correction to the momentum flux has only a weak dependence on the mass ratio of the objects in the binary, suggesting that previous test mass calculations may be quite accurate. We estimate an upper bound of the center-of-mass velocity of 1 $\mathrm{km\,s^{-1}}$ for neutron star binaries very near coalescence. In an appendix we give a self-contained (albeit less rigorous) derivation of the gravitational wave form using the Epstein-Wagoner formalism.

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I. INTRODUCTION AND SUMMARY

Binary systems containing compact objects, such as neutron stars, may prove to be the most fruitful systems for obtaining information about relativistic gravity in strong-field, dynamical regimes. For example, the binary pulsar PSR 1913+16 has given us our first evidence that gravitational waves exist, and new binary pulsars, such as PSR 1534, may give better accuracy. When laser-interferometric gravitational wave detectors are built, coalescing binary systems will be the leading candidates for detectable sources. For example, Schutz [1] has shown that by examining the gravitational wave "chirp" of a coalescing system it will be possible to determine the distance to the system and even estimate the Hubble constant. Unfortunately, to date, gravitational wave detectors have been unable to definitively observe any events. However, Thorne [2] has suggested that the next generation of laser-interferometric detectors should not only have sufficient sensitivity to detect gravitational waves, but also have sufficient resolution to allow detailed study of the actual shape of the wave form emitted by a given event. This motivates our search for accurate predictions of the gravitational wave form emitted during a binary coalescence.

In addition the direct detection of an emitted gravitational wave, and the decay or coalescence of the system caused by the gravitational energy radiated away, there is another potentially observable effect of gravitational radiation from binary systems. The emitted radiation can also carry off linear momentum, giving rise to a recoil of the center of mass of the system. This effect was first studied by Bekenstein [3] and then treated in more detail by Fitchett [4] and Fitchett and Detweiler [5]. It has even been suggested that this recoil velocity may become sufficiently large in the late states of a black-hole–black-hole coalescence to reach galactic escape velocities. For an overview of this "gravitational-radiation rocket effect" see Redmount and Rees [6].

Our approach to the problem of finding the gravitational radiation from a coalescing binary system, as well as its subsequent recoil motion, is to use approximate equations based on the assumption of weak fields and slow motion. In order to obtain some degree of confidence in the results as we evolve the orbit to the closest possible separation, where the fields may not be so weak nor the motions so slow, we use equations carried to the highest practical order in a post-Newtonian expansion. Roughly speaking, the post-Newtonian expansion is an expansion of corrections to Newtonian gravitational theory in powers of $\epsilon^2 \approx (m/r) \approx v^2$. (We use units in which $G=c=1$.) We emphasize that our calculation is not a "test mass" calculation, in fact, we consider cases where the binary masses are comparable (e.g., neutron stars). This approximation scheme is certainly not as accurate as a fully general relativistic 3+1 numerical solution, but it has the advantage of being available now. Such a full-blown numerical solution is probably still many years in the future.

The details of our analysis of binary systems fall into three separate but related parts: the motion of the system, the radiated wave form, and the ejected momentum.

To evolve the orbital motion of the binary system, we use the Damour-Deruelle equations of motion [7]. These equations of motion include all terms up to, and including, radiation reaction at the (post)$^{5/2}$-Newtonian level. It is the radiation reaction term that gives rise to a secular decay and circularization of the orbit. Lincoln and Will [8] studied the late-time evolution of binary systems using the Damour-Deruelle equations of motion in detail, and discussed the range of validity of the Damour-Deruelle equations for different types of binary systems. We also mention that a completely equivalent set of equations of motion have been derived by Grishchuk and Kopejkin [9].

In order to obtain an expression for the wave form of sufficient accuracy to allow a post-Newtonian calculation to the momentum ejection we explore two techniques. In Sec. II we use the mathematically rigorous technique developed by Blanchet and Damour [10–13] and Damour and Iyer [14] to construct the wave form. [Throughout this paper the formalism developed in Refs. [10–14] will be referred to as the Blanchet-Damour-Iyer (BDI) formalism.] Appendix A also contains a separate, but nonrigorous, derivation of the wave form, which uses the formalism of Epstein and Wagoner [15] (EW).

In Sec. II we assemble and use a number of key results of the Blanchet-Damour-Iyer formalism. In particular Blanchet and Damour [12] have developed integral expressions for the lowest order and first post-Newtonian correction for all the "mass multipole moments" of the gravitational radiation. Similarly, Damour and Iyer [14] have obtained a formal expression for the "current quadrupole moment" of the radiation. We then evaluate these very general formulas in the point-mass limit to obtain explicit formulas for the necessary multipoles. The advantage of constructing the wave form with the BDI formalism is that it relies only on rigorous and well-defined mathematical steps. Specifically, the final results only contain integrals over the compact material source. This is in contrast with the Epstein-Wagoner formalism. The disadvantage of constructing the wave form in this rigorous way is that the development is not particularly transparent. Indeed the formulas discussed above rely heavily on a number of mathematical preliminaries which are presented elsewhere [10,11]. Therefore, in the Appendix we also present a less rigorous, but self-contained derivation of the wave form.

In Appendix A we extend the Epstein-Wagoner (EW) [15] formalism to the $(\text{post})^{3/2}$-Newtonian order beyond the usual quadrupole radiation. However, in using the EW formalism we must resolve several technical issues. Our equations of motion, the Damour-Deruelle equations, are expressed in de Donder or harmonic gauge, whereas the EW formalism is carıied out in a gauge that is only approximately de Donder (they agree only to first order in the metric perturbation). Furthermore, the definition of the fields used by EW is different than those generally used in the expression of the de Donder gauge condition. Problems arising from the differences in the gauge and differences in the field definitions could probably be sidestepped by showing that the two approaches actually agree out to some order in the post-Newtonian expansion. However, we deal with the problem directly by explicitly showing that the EW derivation of the radiation carries through essentially unchanged in de Donder gauge. Working consistently in de Donder gauge also allows us to compare the results of this derivation unambiguously with the derivation in Secs. II and III.

A feature of the EW formalism is the appearance of integrals over the infinite extent of the effective stress energy of the gravitational fields. Some of these integrals are formally divergent, thus calling into question the validity of the entire formalism. (This is precisely why we also give a rigorous derivation of the wave form.) Wagoner and Will [16], performing this calculation to $(\text{post})^1$-Newtonian order, encountered this same obstacle of divergent quantities, but found that in all cases the divergent quantities appeared only in terms with vanishing transverse-traceless part; thus they did not contribute to the physical radiation. Even at the higher $(\text{post})^{3/2}$-Newtonian order we find the same is true.

In developing the wave form (by either BDI or EW) we neglect the hereditary contributions to the radiation. These hereditary contributions are primarily due to the "tail" of the gravitational waves. The term "hereditary" is used to reflect that these contributions depend on the entire past history of the system. (See Eq. (4.4) of Kovacs and Thorne [17], or Eq. (39) of Crowley and Thorne [18] or Eq. (1.4) of Blanchet and Damour [13(b)].) It is true that these hereditary terms do enter the wave forms at the $(\text{post})^{3/2}$-Newtonian order; however, we show that these terms do not enter the momentum ejection calculation at the order we are considering. The effect recently discussed by Christodoulou [13,19–21] is also an heredity effect; however it has been shown to enter the wave form only at the $(\text{post})^{5/2}$-Newtonian level [13(b),21], and therefore is neglected in this wave-form calculation. The contribution of these hereditary effects to the wave form of a coalescing binary system will be explored in a future publication [22].

Schematically our wave form has the form

$$h^{ij}_{\text{TT}}=\frac{2\mu}{R}[Q^{ij}+(\delta m/m)P^{ij}_{1/2}+P^{ij}_{1}+(\delta m/m)P^{ij}_{3/2}+P^{ij}_{\text{hered}}+O(\varepsilon^6)]_{\text{TT}}\ . \tag{1.1}$$

Here, μ is the reduced mass of the binary system, R is the distance between the system and the observer, Q^{ij} represents the usual quadrupole term (two time derivatives of the quadrupole moment tensor), which is of order $O(\varepsilon^2)$, $P^{ij}_{1/2}$ represents the $(\text{post})^{1/2}$-Newtonian correction of $O(v\approx\varepsilon)$ smaller than the quadrupole term, etc. The notation TT denotes the transverse traceless part [cf. Eq. (2.2) below]. The factor $\delta m/m$ (the difference in mass of the two bodies divided by the total mass m) is explicitly displayed in Eq. (1.1) to emphasize that if the binary constituents have equal masses, then these odd-half-order terms are shut off. More importantly, in an observed wave form, information about the difference in the masses of the two objects will reside in these odd-half-order terms. The term P^{ij}_{hered} represents the hereditary contribution to the wave form which we neglect in this calculation. We emphasize that the hereditary contribution does enter the wave form at the $(\text{post})^{3/2}$-Newtonian order, but does not enter the momentum ejection calculation.

Figure 1 shows the total wave form (modulo P^{ij}_{hered}) during a coalescence of a system with $m_1=10m_2$, plotted as a function of time. Figures 2(a)–2(e) show the relative contribution of the first four terms in Eq. (1.1) to the total wave form. (Figures 2 are plotted against orbital phase.)

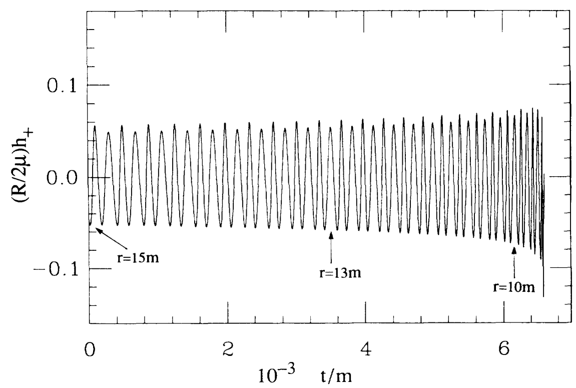

FIG. 1. Contributions to the "+" polarization of the wave form. Plotted is $(R/2\mu)h_+$ against time. The time is arbitrarily set to zero at an orbital separation of 15 m. Corresponding orbital separation is marked. The plot is terminated when the separation is 3 m. In this plot $m_1/m_2=10$, $\Theta=90°$, $\Phi=0°$. Note the piling up of the peaks ("chirp") at late time.

Notice that, although the composite wave form [Fig. 2(a)] basically follows the quadrupole term [Fig. 2(b)] until fairly late in the evolution, where the approximation scheme is becoming questionable, the higher-order terms [Figs. 2(c)–2(e)] do produce a noticeable change [e.g., the "sawtooth" structure in Fig. 2(a)]. To what extent the contributions of these higher-order corrections of the wave form will be detectable by laser interferometric gravity-wave detectors such as the Laser Interferometric Gravitational Observatory (LIGO) will be the topic of future research. On the one hand, the higher-order terms have different phase and frequency signatures (especially the odd-half-order terms), which enhances their detectability. On the other hand, they are all of considerably smaller amplitude than the quadrupole term until the very large stages of the coalescence.

As the binary constituents spiral toward each other the emitted gravitational radiation carries away linear momentum if the two masses in the binary system are unequal. This causes a recoil of the system's center of mass.

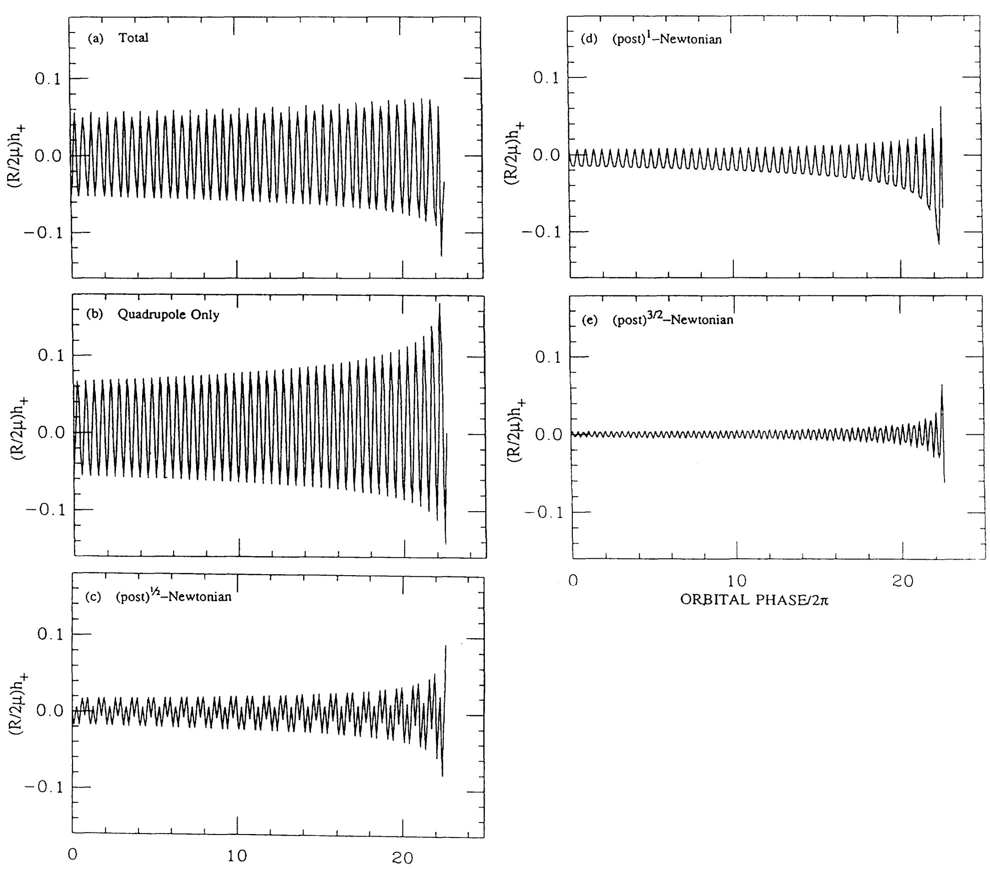

FIG. 2. Gravitational wave form plotted against orbital phase. Plotted is $(R/2\mu)h_+$ with $(m_1/m_2)=10$, $\Theta=90°$, $\Phi=0°$. The growing amplitude in each plot is caused by the shrinking orbital radius. The orbital phase is arbitrarily set to zero at an orbital separation of 15 m. After 10 (20) revolutions the separation has shrunk to 13 m (9 m). The plots are terminated when the separation is 4 m. (a) Composite wave form plotted against phase angle. (b) Lowest-order contribution (pure quadrupole). Note the difference between this contribution and the composite wave form. (c)–(e) $\text{Post}^{1/2}$-Newtonian, $(\text{post})^1$-Newtonian, and $(\text{post})^{3/2}$-Newtonian contribution, respectively.

The physical intuition behind this effect is quite simple. If the two objects are in a circular orbit the lighter mass will be moving faster than the heavier mass, and will therefore be more effective in "forward beaming" its gravitational radiation. This gives a net momentum ejection in the direction of motion of the lighter mass, thus causing a recoil of the system in the opposite direction (see Fig. 3).

The rate of momentum ejection is obtained by integrating the momentum flux over a sphere far from the system:

$$\frac{dP^k}{dt}=\int \frac{dE}{d\Omega\, dt} n^k d\Omega \ , \tag{1.2}$$

where

$$\frac{dE}{dt\, d\Omega}=\frac{R^2}{32\pi}(\dot{h}^{ij}_{\mathrm{TT}}\dot{h}^{ij}_{\mathrm{TT}}) \ , \tag{1.3}$$

where n^k is the component of the unit normal vector in the kth direction, an overdot denotes d/dt, and h^{ij} is schematically represented in Eq. (1.1). To lowest order the momentum ejection comes from the cross term of $\dot{Q}$ with $\dot{P}_{1/2}$; for this term the intuitive argument given above holds. The next-order correction comes from cross terms of $\dot{Q}$ with $\dot{P}_{3/2}$, and $\dot{P}_{1/2}$ and $\dot{P}_1$. (Also notice these cross terms will be multiplied by the factor $\delta m/m$.) The cross term of Q^{ij} with P^{ij}_{hered} does not survive the angular integration in Eq. (1.2). Surprisingly, we find that the surviving correction terms actually reduce the net momentum ejection. In fact, when the objects spiral in to a separation of approximately 7 m, the correction term begins to dominate, suggesting that the approximation has broken down. Note that in de Donder coordinates the horizon of an isolated black hole is at a radius $r=m$, not the usual $r=2m$, as in Schwarzschild coordinates. Based on test mass calculations, Fitchett [4] and Fitchett and Detweiler [5] have argued that the lowest-order momentum calculation should underestimate the total momentum ejection when the object separation is quite small. In Sec. IV we discuss the relevance of our finding to this claim.

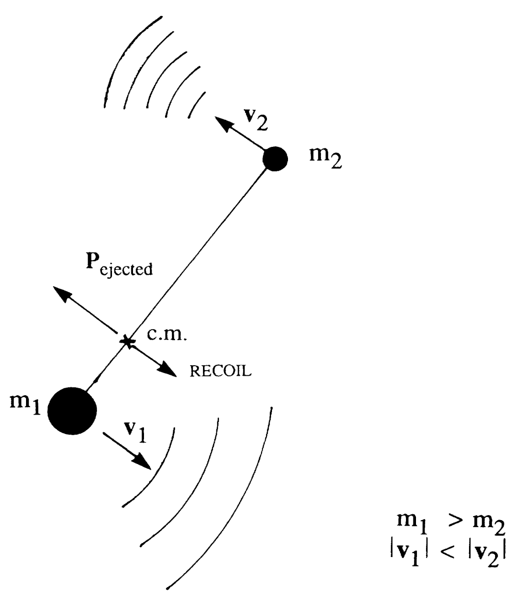

FIG. 3. Momentum ejection and subsequent recoil of a binary system.

Using the momentum ejection formula obtained from Eq. (1.2), we are able to estimate the center-of-mass recoil motion of a coalescing binary system. In the case of a coalescence of two neutron stars the center-of-mass velocity is kept very small by the factor $\delta m/m$, which is typically ≤ 0.05. In Sec. IV we show $|\mathbf{V}_{\mathrm{c.m.}}|\approx 1 \ \mathrm{km\,s^{-1}}$ at neutron star separations of approximately 9 m. In the case of a coalescence of two black holes or a neutron star and a black hole the momentum ejection and center-of-mass recoil velocity are maximized for a mass ratio $m_1/m_2=2.6$. At separations of approximately 9 m we estimate the recoil velocity to be of order 3 $\mathrm{km\,s^{-1}}$. This is certainly well below galactic escape velocities, which are of order 1000 $\mathrm{km\,s^{-1}}$. We also point out that our velocity estimates are independent of the total mass of the system. Unfortunately, we are unable to make reliable estimates of the recoil velocity just prior to the coalescence of two black holes.

In the remainder of this paper we show the details of the calculations. In Sec. II, we develop the general expression for wave forms for sources consisting of N "point" masses. In Sec. III we examine the two-body equation of motion and then specialize the general waveform results of Sec. II to the special case of binary systems ($N=2$). Section IV treats the linear momentum carried off from a binary system by the gravitational radiation. Appendix A contains a separate nonrigorous derivation of the wave form. Appendix B contains some of the mathematical details required for evaluating the multipole integrals which arise in Appendix A.

II. MULTIPOLAR EXPRESSION FOR THE GRAVITATIONAL WAVE FORM: THE POINT PARTICLE LIMIT

The gravitational radiation in the far zone is given by the "transverse-traceless" projection of the spatial portion of the metric perturbation:

$$h_{\alpha\beta}\equiv g_{\alpha\beta}-\eta_{\alpha\beta} \ . \tag{2.1}$$

(See Thorne [23] (henceforth referred to as Th80) and Refs. [24] and [25] for our notation and conventions.) We define the "far zone" as $R \gg \lambda >$ characteristic dimension of the source. Here $R\equiv|\mathbf{x}|$ is the distance from the source to the field point, $\mathbf{x}$ is the spatial position vector of the distant field point relative to the center of mass of the system, and λ is the characteristic wavelength of the radiation. The "transverse-traceless" (TT) projection is computed in the following way:

$$h^{ij}_{\mathrm{TT}}=P^{ik}P^{jl}h^{kl}-\tfrac{1}{2}P^{ij}P^{kl}h^{kl} \ , \tag{2.2}$$

where

$$P^{ij}=\delta^{ij}-n^i n^j \ , \tag{2.3}$$

$\mathbf{n}\equiv\mathbf{x}/R$ is the outward directed unit vector, and summation over repeated spatial indices is assumed. In this paper the most frequently used properties of the transverse-traceless operator are

$$(\delta^{ij})_{TT}=(f^i n^j)_{TT}=(f^j n^i)_{TT}=0 \ . \tag{2.4}$$

These follow easily from Eq. (2.2).

In the far zone the metric perturbation (i.e., the radiation) can be expanded in terms of radiative "mass" multipoles denoted by $I^{ij\cdots}$, and the "current" multipoles denoted by $J^{ij\cdots}$. These multipoles are functions of the retarded time only. From Eq. (4.8) of Th80 we write the radiation

$$h^{ij}=\frac{2}{R}\left\{ {}^{(2)}I^{ij}+\frac{1}{3}{}^{(3)}I^{ijk}n^k+\frac{1}{12}{}^{(4)}I^{ijkl}n^k n^l+\frac{1}{60}{}^{(5)}I^{ijklm}n^k n^l n^m+\cdots \right. \\ \left. +\varepsilon^{kl(i}\left[\frac{4}{3}{}^{(2)}J^{j)k}n^l+\frac{1}{2}{}^{(3)}J^{j)km}n^l n^m+\frac{2}{15}{}^{(4)}J^{j)kmn}n^l n^m n^n+\cdots\right]\right\} . \tag{2.5}$$

The presuperscript number on each multipole denotes the order of differentiation with respect to retarded time. The ellipses in Eq. (2.5) represents higher order multipoles which do not enter the wave form at the $(\text{post})^{3/2}$-Newtonian order.

In order to have a wave form of sufficient accuracy for our momentum calculation we must compute the multipoles I^{ij}, I^{ijk}, and J^{ij} to lowest order plus their first post-Newtonian correction. The remaining multipoles, I^{ijkl}, I^{ijklm}, J^{ijk}, and J^{ijkl}, we need only to their lowest Newtonian order. As we have mentioned several times thus far, we are neglecting "hereditary" contributions to the wave form. We can now make this very precise: we are neglecting $(\text{post})^{3/2}$-Newtonian corrections to the radiative mass quadrupole moment (I^{ij}). To have wave forms which are formally accurate to the $(\text{post})^{3/2}$-Newtonian order these would need to be included. However, in the general expression for the rate of momentum ejection given in Sec. IV it is clear that this omission will not affect the momentum calculation at the order we are considering.

In order to have a complete description of the gravitational radiation from the system we now need to relate the radiative multipoles in Eq. (2.5) to the dynamics of the source. To ensure our wave-form formula is on firm mathematical footing we use the technique developed by Blanchet and Damour [10–13] and Damour and Iyer [14] (the BDI formalism). Although we present essentially none of the details of their derivation, we give the following summary of their technique. (1) Outside the material source, expand the general solution of the source-free Einstein field equations (i.e., the metric) in terms of arbitrary symmetric trace-free (STF) radiative multipoles. (See [Th80] or the Appendix of Ref. [10] for a discussion of STF multipoles.) (2) Obtain a post-Newtonian expansion of the near-zone metric. This expansion can be expressed solely in terms of retarded-time integrals over the finite extent of the material source (i.e., the "source moments"). (3) Match the two solutions in the weak-field-near-zone overlap region. Thus the radiative moments in step (1) can be related to the well defined source moments in step (2).

We now assemble a collection of results from several sources which give the necessary radiative multipoles in terms of integrals over the finite extent of the material source. The radiative mass multipoles given by Eq. (3.34) of Ref. [12] ar accurate to $(\text{post})^1$-Newtonian order beyond their lowest-order contribution

$$I^L(u)=\int (x^L)^{\text{STF}}\sigma(\mathbf{x},u)d^3x+\frac{1}{2(2l+3)}\frac{d^2}{du^2}\int |x|^2(x^L)^{\text{STF}}\sigma(\mathbf{x},u)d^3x-\frac{4(2l+1)}{(l+1)(2l+3)}\frac{d}{du}\int (x^{iL})^{\text{STF}}\sigma^i(\mathbf{x},u)d^3x \ . \tag{2.6}$$

Here the superscript L denotes a multi-index. (See Ref. [24].) The superscript STF denotes that only the symmetric trace-free part is to be taken. A similar expression for the radiative current quadrupole moment is given by Eq. (5.36) of Ref. [14]:

$$J^{ij}=\left[\varepsilon^{jab}\int d^3x\left\{\sigma^b x^{ai}+\frac{1}{2}\sigma^b x^{ai}P_{(-1)}-\frac{1}{2}\sigma^s x^{ai}P^{bs}_{(-1)}+\frac{7}{4}\sigma^b x^a P^i_{(0)}+\frac{1}{4}\sigma^s x^a P^{bsi}_{(0)}+\frac{7}{2}\sigma^b x^i P^a_{(0)}+\frac{11}{4}\sigma^a P^{bi}_{(1)}-\frac{7}{4}\sigma^i x^a P^b_{(0)}\right.\right. \\ +\frac{d}{dt}\left[\frac{5}{56}\sigma x^{ais}P^{sb}_{(-1)}-\frac{1}{56}\sigma x^{ass}P^{ib}_{(-1)}-\frac{3}{112}\sigma x^{as}P^{sbi}_{(0)}+\frac{9}{112}\sigma x^{ai}P^b_{(0)}\right. \\ \left.\left.\left.-\frac{9}{112}\sigma x^a P^{bi}_{(1)}-\frac{1}{14}\sigma x^{aiss}P^b_{(-2)}+\frac{3}{28}x^{ass}T^{bi}-\frac{1}{28}x^{asi}T^{bs}\right]\right\}\right]^{\text{STF}} . \tag{2.7}$$

The multiple index on x represents a product (e.g., $x^{ai} \equiv x^a x^i$). The sources in Eq. (2.6) and Eq. (2.7) are given by

$$\sigma(\mathbf{x},t) = T^{00} + T^{kk} \ , \tag{2.8a}$$

$$\sigma^i(\mathbf{x},t) \equiv T^{0i} \ . \tag{2.8b}$$

The $P^L_{(\alpha)}(\mathbf{x})$ in Eq. (2.7) is the generalized tensorial potential

$$P^L_{(\alpha)}(\mathbf{x}) \equiv \int \sigma(\mathbf{y}) r^\alpha_{xy} n^{i_l}_{xy} \cdots n^{i_l}_{xy} d^3y \ , \tag{2.8c}$$

where $r_{xy} \equiv |\mathbf{x}-\mathbf{y}|$ and $n^i_{xy} \equiv (x^i - y^i)/r_{xy}$. [Notice, in Eq. (2.8c) the α on the right side is actually an exponent.] In Eqs. (2.8) $T^{\mu\nu}$ is the stress energy of the matter in the source. The second term in Eq. (2.8a), T^{kk}, represents the spatial trace of $T^{\mu\nu}$. The remaining spin multipoles can be computed (to sufficient accuracy) from the Newtonian formula

$$J^{ijk} = \{ \varepsilon^{kab} \int \sigma^b x^{aij} d^3x \}^{\mathrm{STF}} \ , \tag{2.9a}$$

$$J^{ijkl} = \{ \varepsilon^{lab} \int \sigma^b x^{aijk} d^3x \}^{\mathrm{STF}} \ . \tag{2.9b}$$

If we now restrict our attention to the case of N well-separated bodies, and neglect the internal structure of the bodies, we can collect the relevant formulas from Appendix C of Ref. [14]. In this case the stress energy tensor of the matter for N "point masses" can be formally represented by

$$T^{\mu\nu}(\mathbf{x},t) = \sum_A m_A \frac{dx^\mu_A}{dt} \frac{dx^\nu_A}{dt} \frac{1}{\sqrt{-g}} \frac{dt}{d\tau} \delta(\mathbf{x} - \mathbf{x}_A(t)) \ . \tag{2.10}$$

The sum over A represent the sum over the N body labels. The mass and position of the Ath body are represented by m_A and $\mathbf{x}_A(t)$. If we neglect the nonradiative infinite self-interaction terms the sources in Eqs. (2.8) become

$$\sigma = \sum_A m_A (1 - U_A + \tfrac{3}{2} v_A^2) \delta(\mathbf{x} - \mathbf{x}_A) \ , \tag{2.11a}$$

$$\sigma^i = \sum_A m_A v^i_A (1 - U_A + \tfrac{1}{2} v_A^2) \delta(\mathbf{x} - \mathbf{x}_A) \ , \tag{2.11b}$$

$$T^{ij} = \sum_A m_A v^i_A v^j_A (1 - U_A + \tfrac{1}{2} v_A^2) \delta(\mathbf{x} - \mathbf{x}_A) \ . \tag{2.11c}$$

The quantity v^A represents the bulk velocity of body A, $v_A = |d\mathbf{x}_A/dt|$. The potential U_A is the Newtonian gravitational potential at the position of body A produced by all the other bodies:

$$U_A = \sum_{B \neq A} \frac{m_B}{|\mathbf{x}_A - \mathbf{x}_B|} \ . \tag{2.12}$$

Substituting Eqs. (2.11) into Eqs. (2.6), (2.7) and (2.9), we obtain to the necessary order

$$I^{ij} = \sum_A^N m_A \left\{ \left[\left[1 + \frac{3}{2} v_A^2 - \sum_{B \neq A} \frac{m_B}{r_{AB}} \right] x^i_A x^j_A + \frac{1}{14} \frac{d^2}{dt^2} (|\mathbf{x}_A|^2 x^i_A x^j_A) \right]^{\mathrm{STF}} - \frac{20}{21} \frac{d}{dt} [(x^i_A x^j_A x^k_A)^{\mathrm{STF}} v^k] \right\} , \tag{2.13a}$$

$$I^{ijk} = \sum_A m_A \left\{ \left[\left[1 + \frac{3}{2} v_A^2 - \sum_{B \neq A} \frac{m_B}{r_{AB}} \right] x^i_A x^j_A x^k_A + \frac{1}{18} \frac{d^2}{dt^2} (|\mathbf{x}_A|^2 x^i_A x^j_A x^k_A) \right]^{\mathrm{STF}} - \frac{7}{9} \frac{d}{dt} [(x^i_A x^j_A x^k_A x^l_A)^{\mathrm{STF}} v^l] \right\} , \tag{2.13b}$$

$$I^{ijkl} = \sum_A m_A (x^i_A x^j_A x^k_A x^l_A)^{\mathrm{STF}} \ , \tag{2.13c}$$

$$I^{ijklm} = \sum_A m_A (x^i_A x^j_A x^k_A x^l_A x^m_A)^{\mathrm{STF}} \ . \tag{2.13d}$$

Similarly for the current moments of the radiation Damour and Iyer [14] have obtained

$$\begin{aligned}
J^{ij} = \Bigg\{ \varepsilon_{jab} \sum_A m_A \Bigg[& x^{ai}_A v^b_A \left[1 + \frac{1}{2} v_A^2 \right] \\
& + \sum_{B \neq A} m_B \Bigg\{ \Bigg[-\frac{1}{2} \frac{v^b_A x^{ai}_A}{|\mathbf{x}_A - \mathbf{x}_B|} - \frac{1}{2} \frac{v^s_A x^{ai}_A n^{bs}_{AB}}{|\mathbf{x}_A - \mathbf{x}_B|} + \frac{7}{4} v^b_A x^a_A n^i_{AB} + \frac{1}{4} v^s_A x^a_A n^{bsi}_{AB} \\
& \qquad + \frac{7}{2} v^b_A x^i_A n^a_{AB} + \frac{11}{4} v^a_A n^{bi}_{AB} |\mathbf{x}_A - \mathbf{x}_B| - \frac{7}{4} v^i_A x^a_A n^b_{AB} \Bigg] \\
& \qquad + \frac{d}{dt} \Bigg[\frac{5}{56} \frac{x^{ais}_A n^{sb}_{AB}}{|\mathbf{x}_A - \mathbf{x}_B|} - \frac{1}{56} \frac{x^{ass}_A n^{ib}_{AB}}{|\mathbf{x}_A - \mathbf{x}_B|} - \frac{3}{112} x^{as}_A n^{sbi}_{AB} \\
& \qquad + \frac{9}{112} x^{ai}_A n^b_{AB} - \frac{9}{112} x^a_A n^{bi}_{AB} |\mathbf{x}_A - \mathbf{x}_B| - \frac{1}{14} \frac{x^{aiss}_A n^b_{AB}}{|\mathbf{x}_A - \mathbf{x}_B|^2} \Bigg] \Bigg\} \\
& + \frac{d}{dt} \left[\frac{3}{28} x^{ass}_A v^{bi}_A - \frac{1}{28} x^{asi}_A v^{bs}_A \right] \Bigg] \Bigg\}^{\mathrm{STF}} ,
\end{aligned} \tag{2.14a}$$

$$J^{ijk}=\sum_A m_A(\varepsilon^{iab}x_A^a v_A^b x_A^j x_A^k)^{\rm STF}\ , \tag{2.14b}$$

$$J^{ijkl}=\sum_a m_A(\varepsilon^{iab}x_A^a v_A^b x_A^j x_A^k x_A^l)^{\rm STF}\ . \tag{2.14c}$$

The symbol $n_{AB}^i \equiv (x_A^i - x_B^i)/|\mathbf{x}_A - \mathbf{x}_B|$, and $n_{AB}^{ij} \equiv n_{AB}^i n_{AB}^j$. These multipoles can be substituted into Eq. (2.5) to obtain the wave form; however, we will delay this until we specialize these results to the two-body case.

III. TWO-BODY MOTION AND WAVE FORM

A. Two-body equation of motion

We now restrict our treatment of the radiation to the astrophysically relevant case of a binary system ($N=2$). In addition to being the relevant case, this restriction has two computational advantages: (1) We will be able to use the well established Damour-Deruelle two-body equations of motion [7] to evolve the orbital system, and to simplify higher-order time derivatives created by the derivatives in Eq. (2.5). (2) If we establish a coordinate system with its origin at the "center of mass" of the binary system then Eqs. (2.13) and Eqs. (2.14) can be considerably simplified.

The Damour-Deruelle (DD) equations [7] describe the motion of a binary system of compact objects. These equations of motion include terms through the dissipative radiation-reaction term of (post)$^{5/2}$-Newtonian order. They can be schematically represented

$$\begin{aligned} a_1^i &= A_{\rm N}^i(m_2,\mathbf{x}_1-\mathbf{x}_2)+A_{\rm PN}^i(m_1,m_2,\mathbf{x}_1-\mathbf{x}_2,\mathbf{v}_1,\mathbf{v}_2)\\ &\quad + A_{\rm PPN}^i(m_1,m_2,\mathbf{x}_1-\mathbf{x}_2,\mathbf{v}_1,\mathbf{v}_2)\\ &\quad + A_{{\rm P}^{5/2}{\rm N}}^i(m_1,m_2\mathbf{x}_1-\mathbf{x}_2,\mathbf{v}_1,\mathbf{v}_2)\ . \end{aligned} \tag{3.1}$$

Here a_1^i is the ith component of the acceleration of body No. 1, $A_{\rm N}^i$ represents the Newtonian contribution to the acceleration, $A_{\rm PN}^i$ represents the post-Newtonian contribution, etc. A similar formula for the acceleration of the second body can be obtained by interchanging the labels 1 and 2. The DD equations also contain corrections for spinning bodies. However, to be consistent with our assumption of static structure, we will neglect spin effects in this discussion. We also point out that the DD equations exceed the accuracy of the wave forms that we have computed. Our wave forms contain terms through $O(\varepsilon^3)$; the DD equations contain terms of $O(\varepsilon^5)$. We do, however, expect that our wave forms coupled with the DD equations of motion will give a reasonably good quantitative approximation of the gravitational radiation emitted, including the secular changes as the orbit decays.

From the Damour-Deruelle equations of motion one can derive a constant of the motion [to (post)^2-Newtonian order], which can be interpreted as the "center of mass" of a binary system. Choosing this as the origin of the coordinate system we introduce the relative position vector $\mathbf{x}\equiv\mathbf{x}_1-\mathbf{x}_2$, where $\mathbf{x}_1$ ($\mathbf{x}_2$) represents the position of body 1 (2) relative to the center of mass. Explicitly,

$$\mathbf{x}_1=\left[\frac{m_2}{m}+\left[\frac{\eta}{2}\frac{\delta m}{m}\right]\left[v^2-\frac{m}{r}\right]+O(\varepsilon^4)\right]\mathbf{x}\ , \tag{3.2a}$$

$$\mathbf{x}_2=\left[-\frac{m_1}{m}+\left[\frac{\eta}{2}\frac{\delta m}{m}\right]\left[v^2-\frac{m}{r}\right]+O(\varepsilon^4)\right]\mathbf{x}\ . \tag{3.2b}$$

Here $m\equiv m_1+m_2$, $\mu\equiv m_1 m_2/m$, $\eta\equiv\mu/m$, $\delta m\equiv m_1-m_2$, $r\equiv|\mathbf{x}|$, and $\mathbf{v}\equiv\dot{\mathbf{x}}$. The velocities of the individual bodies can be obtained by differentiating Eqs. (3.2) and using the Newtonian equation of motion to eliminate the acceleration that arises in the post-Newtonian correction

$$\begin{aligned}\mathbf{v}_1&=\left[\frac{m_2}{m}+\left[\frac{\eta}{2}\frac{\delta m}{m}\right]\left[v^2-\frac{m}{r}\right]\right]\mathbf{v}\\ &\quad-\frac{\eta}{2}\frac{\delta m}{m}\frac{m}{r^2}\dot{r}\mathbf{x}+O(\varepsilon^5)\ ,\end{aligned} \tag{3.3a}$$

$$\begin{aligned}\mathbf{v}_2&=\left[-\frac{m_1}{m}+\left[\frac{\eta}{2}\frac{\delta m}{m}\right]\left[v^2-\frac{m}{r}\right]\right]\mathbf{v}\\ &\quad-\frac{\eta}{2}\frac{\delta m}{m}\frac{m}{r^2}\dot{r}\mathbf{x}+O(\varepsilon^5)\ .\end{aligned} \tag{3.3b}$$

Using the transformation above, Lincoln and Will [8] have reduced the DD equations to the form

$$\mathbf{a}=\mathbf{a}_1-\mathbf{a}_2=(m/r^2)[(-1+A)\hat{\mathbf{r}}+B\mathbf{v}]\ , \tag{3.4}$$

where $\hat{\mathbf{r}}\equiv\mathbf{x}/r$. Here A and B can be written as the sum of (post)^1-, (post)^2-, and (post)$^{5/2}$-Newtonian correction terms. That is, $A=A_1+A_2+A_{5/2}$ and $B=B_1+B_2+B_{5/2}$, where

$$A_1=2(2+\eta)\frac{m}{r}-(1+3\eta)v^2+\frac{3}{2}\eta\dot{r}^2\ , \tag{3.5a}$$

$$\begin{aligned}A_2&=-\frac{3}{4}(12+29\eta)\left[\frac{m}{r}\right]^2-\eta(3-4\eta)v^4\\ &\quad-\frac{15}{8}\eta(1-3\eta)\dot{r}^4+\frac{3}{2}\eta(3-4\eta)v^2\dot{r}^2\\ &\quad+\frac{1}{2}\eta(13-4\eta)\frac{m}{r}v^2+(2+25\eta+2\eta^2)\frac{m}{r}\dot{r}^2\ ,\end{aligned} \tag{3.5b}$$

$$A_{5/2}=\frac{8}{5}\eta\frac{m}{r}\dot{r}\left[3v^2+\frac{17}{3}\frac{m}{r}\right]\ , \tag{3.5c}$$

$$B_1=2(2-\eta)\dot{r}\ , \tag{3.5d}$$

$$\begin{aligned}B_2&=\frac{1}{2}\dot{r}\left[\eta(15+4\eta)v^2-(4+41\eta+8\eta^2)\frac{m}{r}\right.\\ &\quad\left.-3\eta(3+2\eta)\dot{r}^2\right]\ ,\end{aligned} \tag{3.5e}$$

$$B_{5/2}=-\frac{8}{5}\eta\frac{m}{r}\left[v^2+3\frac{m}{r}\right]. \tag{3.5f}$$

B. Two-body wave-form equation

We now use Eqs. (3.2) and (3.3) to simplify the multipole expressions; substituting into Eqs. (2.13) and summing over the two bodies, we obtain

$$I^{ij}=\mu\left[\left[1+\frac{29}{42}(1-3\eta)v^2-\frac{1}{7}(5-8\eta)\frac{m}{r}\right]x^ix^j -\frac{4}{7}(1-3\eta)r\dot{r}x^iv^j+\frac{11}{21}(1-3\eta)r^2v^iv^j\right]^{\rm STF}, \tag{3.6a}$$

$$I^{ijk}=-\mu\frac{\delta m}{m}\left[\left[1+\frac{5-19\eta}{6}v^2-\frac{5-13\eta}{6}\frac{m}{r}\right]x^ix^jx^k +(1-2\eta)(r^2v^iv^jx^k-r\dot{r}v^ix^jx^k)\right]^{\rm STF}, \tag{3.6b}$$

$$I^{ijkl}=\mu(1-3\eta)(x^ix^jx^kx^l)^{\rm STF}, \tag{3.6c}$$

$$I^{ijklm}=-\mu\frac{\delta m}{m}(1-2\eta)(x^ix^jx^kx^lx^m)^{\rm STF}. \tag{3.6d}$$

Similarly for the current multipoles we substitute into Eqs. (2.14) and find

$$J^{ij}=-\mu\frac{\delta m}{m}\left\{\varepsilon^{jab}\left[\left[1+\frac{1-5\eta}{2}v^2+2(1+\eta)\frac{m}{r}\right]x^ix^av^b +\frac{1}{28}\frac{d}{dt}[(1-2\eta)\times(3r^2v^i-r\dot{r}x^i)x^av^b]\right]\right\}^{\rm STF}, \tag{3.7a}$$

$$J^{ijk}=\mu(1-3\eta)(\varepsilon^{iab}x^av^bx^jx^k)^{\rm STF}, \tag{3.7b}$$

$$J^{ijkl}=-\mu\frac{\delta m}{m}(1-2\eta)(\varepsilon^{iab}x^av^bx^jx^kx^l)^{\rm STF}. \tag{3.7c}$$

The mass quadrupole term Eq. (3.6a) agrees exactly with Eq. (3.37) of Blanchet and Schäfer [26]. The lowest-order term in the mass octupole Eq. (3.6b) also agrees with Blanchet and Schafer. However, the explicit two-body post-Newtonian correction in Eq. (3.6b) has not been previously published. The mass multipoles, I^{ijkl} and I^{ijklm} are just Newtonian mass multipoles of the source; both can be easily computed from Th80 Eq. (5.28a). Damour and Iyer [14] have computed a post$^{3/2}$-Newtonian expression for the current quadrupole J^{ij}. After their result is carefully reduced to the two-body case, it agrees with Eq. (3.7a). The current multipoles J^{ijk} and J^{ijkl} are just the Newtonian current multipoles; both can be computed from Th80, Eq. (5.28b). In Eqs. (3.6) and (3.7) we have made repeated use of simple identities, such as $m_1^2+m_2^2=m^2(1-2\eta)$, $m_1^3+m_2^3=m^3(1-3\eta)$.

These expressions for the multipoles can now be substituted into Eq. (2.5). When the time derivatives produce an acceleration, we substitute the DD equations of motion to the necessary order. The result is the wave form

$$h^{ij}_{\rm TT}=\frac{2\mu}{R}[Q^{ij}+(\delta m/m)P^{ij}_{1/2}+P^{ij}_1+(\delta m/m)P^{ij}_{3/2}]_{\rm TT}, \tag{3.8a}$$

where

$$Q^{ij}=2\left[v^iv^j-\frac{m}{r}\frac{x^ix^j}{r^2}\right], \tag{3.8b}$$

$$P^{ij}_{1/2}=3\mathbf{n}\cdot\hat{\mathbf{r}}\frac{m}{r}\left[\frac{x^iv^j+v^ix^j}{r}-\dot{r}\frac{x^ix^j}{r^2}\right]+\mathbf{n}\cdot\mathbf{v}\left[\frac{m}{r}\frac{x^ix^j}{r^2}-2v^iv^j\right], \tag{3.8c}$$

$$\begin{aligned}P^{ij}_1=&\frac{1}{3}(1-3\eta)\left\{(\mathbf{n}\cdot\hat{\mathbf{r}})^2\frac{m}{r}\left[\left[3v^2-15\dot{r}^2+7\frac{m}{r}\right]\frac{x^ix^j}{r^2}+15\dot{r}\frac{x^iv^j+v^ix^j}{r}-14v^iv^j\right]\right.\\&\left.+\mathbf{n}\cdot\hat{\mathbf{r}}\,\mathbf{n}\cdot\mathbf{v}\frac{m}{r}\left[12\dot{r}\frac{x^ix^j}{r^2}-16\frac{x^iv^j+v^ix^j}{r}\right]+(\mathbf{n}\cdot\mathbf{v})^2\left[6v^iv^j-2\frac{m}{r}\frac{x^ix^j}{r^2}\right]\right\}\\&+\frac{1}{3}\left[3(1-3\eta)v^2-2(2-3\eta)\frac{m}{r}\right]v^iv^j+\frac{2}{3}\frac{m}{r}\dot{r}(5+3\eta)\frac{x^iv^j+v^ix^j}{r}\\&+\frac{1}{3}\frac{m}{r}\left[3(1-3\eta)\dot{r}^2-(10+3\eta)v^2+29\frac{m}{r}\right]\frac{x^ix^j}{r^2},\end{aligned} \tag{3.8d}$$

$$P_{3/2}^{ij}=(1-2\eta)\left\{(\mathbf{n}\cdot\hat{\mathbf{r}})^3\frac{m}{r}\left[\frac{5}{4}\left[3v^2-7\dot{r}^2+6\frac{m}{r}\right]\dot{r}\frac{x^ix^j}{r^2}-\frac{1}{12}\left[21v^2-105\dot{r}^2+44\frac{m}{r}\right]\frac{x^iv^j+v^ix^j}{r}-\frac{17}{2}\dot{r}v^iv^j\right]\right.$$
$$+\frac{1}{4}(\mathbf{n}\cdot\hat{\mathbf{r}})^2(\mathbf{n}\cdot\mathbf{v})\frac{m}{r}\left[\left[-9v^2+45\dot{r}^2-28\frac{m}{r}\right]\frac{x^ix^j}{r^2}-54\dot{r}\frac{x^iv^j+v^ix^j}{r}+58v^iv^j\right]$$
$$\left.+\frac{3}{2}(\mathbf{n}\cdot\hat{\mathbf{r}})(\mathbf{n}\cdot\mathbf{v})^2\frac{m}{r}\left[5\frac{x^iv^j+v^ix^j}{r}-3\dot{r}\frac{x^ix^j}{r^2}\right]+\frac{1}{2}(\mathbf{n}\cdot\mathbf{v})^3\left[\frac{m}{r}\frac{x^ix^j}{r^2}-4v^iv^j\right]\right\}$$
$$+\frac{1}{12}\mathbf{n}\cdot\hat{\mathbf{r}}\frac{m}{r}\left[\frac{x^iv^j+v^ix^j}{r}\left[\dot{r}^2(63+54\eta)-\frac{m}{r}(128-36\eta)+v^2(33-18\eta)\right]\right.$$
$$\left.+\frac{x^ix^j}{r^2}\dot{r}\left[-v^2(63-54\eta)+\frac{m}{r}(242-24\eta)+\dot{r}^2(15-90\eta)\right]-(186+24\eta)\dot{r}v^iv^j\right]$$
$$+\mathbf{n}\cdot\mathbf{v}\left[\frac{1}{2}v^iv^j\left[-2v^2(1-5\eta)+\frac{m}{r}(3-8\eta)\right]-\frac{x^ix^j}{r^2}\frac{m}{r}\left[\frac{3}{4}(1-2\eta)\dot{r}^2+\frac{1}{3}(26-3\eta)\frac{m}{r}-\frac{1}{4}(7-2\eta)v^2\right]\right.$$
$$\left.-\frac{1}{2}\frac{x^iv^j+v^ix^j}{r}\frac{m}{r}\dot{r}(7+4\eta)\right]. \qquad (3.8e)$$

This now makes the schematic representation of the wave form [Eq. (1.1) or (3.8a)] actually an explicit representation. The first term Q^{ij}, now given by Eq. (3.8b), is just the standard quadrupole term. The second term $P_{1/2}^{ij}$, now given by Eq. (3.8c), is the (post)$^{1/2}$-Newtonain correction term. The next term (P_1^{ij}) is the (post)^1-Newtonain correction to the wave form. Through (post)^1-Newtonian order the wave form is identical to the Wagoner-Will wave form [16]. The final term ($P_{3/2}^{ij}$) is the (post)$^{3/2}$-Newtonian correction to the wave form. Figure 2 shows the relative contribution of these terms to the total wave form for the "+" polarization (see Eq. (4.3a) for a definition of the "+" polarization). Once again we emphasize that Eq. (3.8) is still not a full description of the wave form through (post)$^{3/2}$-Newtonian order; hereditary contributions to the gravitational waves have been omitted.

IV. MOMENTUM FLUX AND RADIATION RECOIL

With the wave form in hand, we now proceed to the calculation of the momentum carried off by the gravitational radiation. There are two equivalent approaches to this problem: (1) using Eqs. (3.8) for the wave form, directly compute the momentum ejected by integrating the momentum flux [i.e., directly compute the integral in Eq. (1.2)]; (2) with the symmetric trace-free multipole decomposition of the wave form use the formula in Th80 to evaluate the momentum. We choose the first; the second technique is used at the end of the section for the simplified case of circular orbits as a check.

The net k component of the momentum flux carried across a distant sphere S, which is centered on the source is given by

$$\dot{P}^k=\frac{R^2}{32\pi}\int_S(\dot{h}^{ij}_{\rm TT}\dot{h}^{ij}_{\rm TT})n^k d\Omega\ . \qquad (4.1)$$

If we choose a coordinate system with the orbit in the xy plane, and use the TT operator Eq. (2.2) and Eq. (2.3), this formula can be written in terms of the two polarizations "+" and "×" (e.g., see Ref. [27]):

$$\dot{P}^k=\frac{R^2}{16\pi}\int_S[(\dot{h}_+)^2+(\dot{h}_\times)^2]n^k d\Omega\ , \qquad (4.2)$$

where

$$h_+=h^{\Theta\Theta}=-h^{\Phi\Phi}=Ah^{xx}+Bh^{yy}+Ch^{xy}\ , \qquad (4.3a)$$

$$h_\times=h^{\Theta\Phi}=h^{\Phi\Theta}=-D(h^{xx}-h^{yy})+Eh^{xy}\ , \qquad (4.3b)$$

and

$$A=\tfrac{1}{4}[\cos2\Phi(1+\cos^2\Theta)-\sin^2\Theta]\ , \qquad (4.4a)$$

$$B=-\tfrac{1}{4}[\cos2\Phi(1+\cos^2\Theta)+\sin^2\Theta]\ , \qquad (4.4b)$$

$$C=\tfrac{1}{2}\sin2\Phi(1+\cos^2\Theta)\ , \qquad (4.4c)$$

$$D=\tfrac{1}{2}\sin2\Phi\cos\Theta\ , \qquad (4.4d)$$

$$E=\cos2\Phi\cos\Theta\ . \qquad (4.4e)$$

Here Θ and Φ are the azimuth and colatitude of the field point, and h^{xx}, h^{yy}, and h^{xy} are the wave forms computed using Eqs. (3.8). Notice that by choosing the x-y plane to contain the orbit, $z=0$ and $v^z=0$; hence, h^{zz}, h^{xz}, and h^{yz} vanish. Also notice that by using Eqs. (4.3) we do not need to apply the transverse-traceless operator in Eq. (4.2). The TT operator is in effect built into these equations. Substituting Eqs. (4.3) into Eq. (4.2), and using formulas such as

$$\int_S A^2 n^x n^x d\Omega = \frac{3\pi}{70}, \quad \int_S A^2 n^x n^x n^y d\Omega = 0 , \quad \int_S (A^2+D^2) n^x n^x n^y n^y d\Omega = \frac{8\pi}{315} \tag{4.5}$$

we find, after considerable manipulation,

$$\begin{aligned}
\dot{P}^x = \frac{8}{105}\frac{\delta m}{m}\eta^2(m/r)^4 [& v_x^3(-45\hat{x}^4+93\hat{x}^2-50)+v_x^2 v_y(-135\hat{x}^2+131)\hat{x}\hat{y} \\
& +v_x v_y^2(135\hat{x}^4-118\hat{x}^2-12)+v_y^3(45\hat{x}^2+10)\hat{x}\hat{y}+(m/r)[(12\hat{x}^2-8)v_x+12v_y\hat{x}\hat{y}] \\
& +\tfrac{1}{36}((m/r)^2\{v_x[8(3\hat{x}^2+34)\eta-12(295\hat{x}^2-252)]+12v_y\hat{x}\hat{y}(2\eta-295)\} \\
& \quad +(m/r)\{v_y^3[8(292\hat{x}^2+425)\hat{x}\hat{y}\eta-2(12\,301\hat{x}^2+854)\hat{x}\hat{y}] \\
& \qquad +v_x v_y^2[12(584\hat{x}^4-101\hat{x}^2-248)\eta-6(12\,301\hat{x}^4-10\,615\hat{x}^2-22)] \\
& \qquad +v_x^2 v_y[6(12\,301\hat{x}^2-9783)\hat{x}\hat{y}-48(146\hat{x}^2-117)\hat{x}\hat{y}\eta] \\
& \qquad +v_x^3[2(12\,301\hat{x}^4-21\,252\hat{x}^2+8163)-4(584\hat{x}^4-1419\hat{x}^2+729)\eta]\} \\
& \quad +v_y^5[12(1843\hat{x}^4-852\hat{x}^2-140)\hat{x}\hat{y}-12(1036\hat{x}^4-195\hat{x}^2-62)\hat{x}\hat{y}\eta] \\
& \quad +v_x v_y^4[6(18\,430\hat{x}^6-22\,519\hat{x}^4+4534\hat{x}^2+332)-12(5180\hat{x}^6-5426\hat{x}^4+437\hat{x}^2+70)\eta] \\
& \quad +v_x^2 v_y^3[12(10\,360\hat{x}^4-9394\hat{x}^2+537)\hat{x}\hat{y}\eta-24(9215\hat{x}^4-9044\hat{x}^2+1269)\hat{x}\hat{y}] \\
& \quad +v_x^3 v_y^2[12(10\,360\hat{x}^6-18\,296\hat{x}^4+8255\hat{x}^2-249)\eta \\
& \qquad -12(18\,430\hat{x}^6-32\,087\hat{x}^4+14\,789\hat{x}^2-1035)] \\
& \quad +v_x^4 v_y[12(9215\hat{x}^4-13\,828\hat{x}^2+4771)\hat{x}\hat{y}-12(5180\hat{x}^4-8419\hat{x}^2+3349)\hat{x}\hat{y}\eta] \\
& \quad +v_x^5[6(3686\hat{x}^6-8331\hat{x}^4+5622\hat{x}^2-925)-24(2\hat{x}^2-1)(259\hat{x}^4-514\hat{x}^2+259)\eta])] ,
\end{aligned} \tag{4.6}$$

$$\dot{P}^y = \dot{P}^x(\text{exchange } x \text{ and } y) , \tag{4.7}$$

$$\dot{P}^z = 0 . \tag{4.8}$$

Here $\hat{x}$ and $\hat{y}$ represent x/r and y/r. The first two lines of Eq. (4.6) represent the lowest-order term in the momentum ejected. If a Keplerian orbit is inserted into this portion we obtain the results of Fitchett [4]. [Note the last term in the first line of Fitchett's Eq. (2.23) should be $\frac{1}{2}e^3(20\cos\theta+90\cos^3\theta)$]. The remainder of Eq. (4.6) (i.e., $\frac{1}{36}(\ )$) is the next post-Newtonian-order correction to the momentum flux.

Figure 4 shows the relative contribution of both terms to the momentum ejected during the late stages of a coalescence. It clearly shows that the correction term is $\approx 180°$ out of phase until the final plunge has begun. In essence the correction term reduces the momentum flux. This effect is also evident in Fig. 5 where we plot $|d\mathbf{P}/dt|$ with and without the post-Newtonian correction term included.

The effect can also be seen analytically if we consider the case of circular motion. In order to maintain a circular orbit the relative coordinate velocity must be (to the necessary post-Newtonian order)

$$\mathbf{v} = \left[\frac{m}{r}\right]^{1/2}\left[1-\frac{3-\eta}{2}\frac{m}{r}\right](-\sin\phi, \cos\phi, 0) . \tag{4.9}$$

Here ϕ is the orbital phase angle. Substituting (4.9) into (4.6) we find a drastic simplification

$$\frac{d\mathbf{P}}{dt} = \frac{8}{105}\frac{\delta m}{m}\eta^2(m/r)^{11/2}[58-\tfrac{1}{9}(5583+182\eta)(m/r)](\sin\phi, -\cos\phi, 0) . \tag{4.10}$$

This same equation is obtained at the end of the section by using the decomposed STF multipoles [Eqs. (3.6) and (3.7)] and the momentum formula from Th80. Notice that the correction term in Eq. (4.10) overtakes the lowest-order term when the orbit decays through $r \approx 10$ m. In Fig. 4 the correction overtakes the lowest-order term at $r \approx 7$ m. The discrepancy is due to the inclusion of still higher-order corrections to the motion in the Damour-Deruelle equations. In particular, the motion that produced the graph was a decaying, quasicircular orbit.

The fact that the rate of momentum ejection is reduced by the post-Newtonian correction is a curious effect, but not completely unexpected. A similar effect is seen in

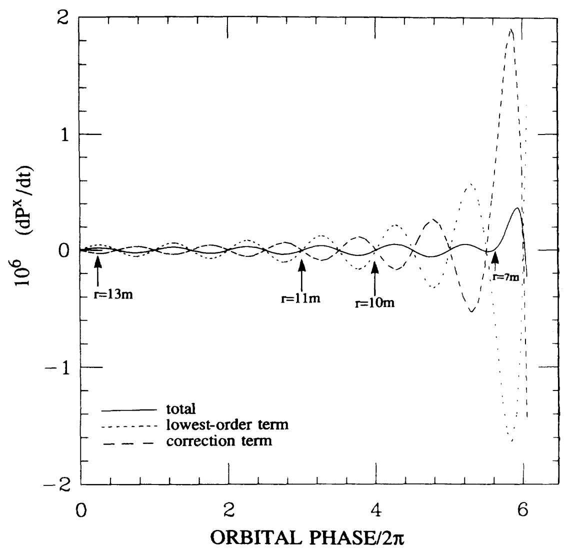

FIG. 4. Relative contributions to the x component of the ejected momentum flux plotted against orbital phase angle. Corresponding orbital separation is marked. Total momentum flux (solid curve) is the sum of the lowest-order term (dotted curve) and the next-order correction (dashed curve). Notice that the total momentum ejection basically follows the lowest-order contribution until late in the evolution ($r \approx 7$ m), at which point it begins to follow the correction term, signifying that the approximation has broken down.

post-Newtonian calculations of luminosity (i.e., the rate of energy ejection). The post-Newtonian corrections to the luminosity formula are negative and therefore they reduce the rate of energy ejection. In fact for binary systems at separations less than approximately 9 m the post-Newtonian terms begin to dominate and eventually the computed luminosity turns negative, signifying that the approximate calculation is no longer valid [see the discussion in Lincoln and Will [8] following their Eq. (4.2)]. Also the "exact" test-mass calculations given by Fitchett and Detweiler [5] show that the rate of momentum ejection for binaries in circular orbits is actually somewhat less than the rate predicted by the Newtonian contribution for $r > 9$ m. In essence their "higher-order" contributions to the momentum ejection also reduce the momentum ejection while the binary separation is still large.

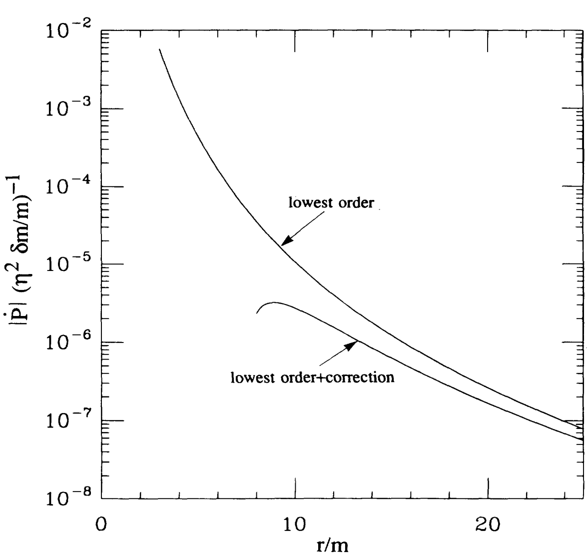

FIG. 5. Magnitude of ejected momentum flux plotted against orbital separation. The upper curve is the lowest-order contribution. The lower curve is the sum of the lowest-order term and the correction term. Compare Fitchett and Detweiler [5] (Fig. 2).

On the other hand, Fitchett and Detweiler [5] also showed that for $r < 9$ m the opposite occurs: the Newtonian contribution to the momentum ejection underestimates the "exact" test-mass momentum ejection. They use this fact to estimate very large recoil velocities for black holes near coalescence. Unfortunately, at these very late states of a black-hole coalescence (i.e., separations ≤ 5 m) our approximation has broken down, and it is not possible to make reliable predictions of the recoil velocities. However, in support of conclusions drawn from test mass calculations, we note the very weak dependence on the mass ratio in the post-Newtonian correction term in Eq. (4.10). The mass ratio enters through the variable η which has a maximum value of 0.25 when $m_1 = m_2$. This results in less than a 1% change in the post-Newtonian correction in Eq. (4.10).

To obtain the resulting trajectory of the center of mass (c.m.) of the system we integrate Newton's second law:

$$-\frac{d\mathbf{P}}{dt} = F_{\text{c.m.}} = m\ddot{\mathbf{X}}_{\text{c.m.}} . \tag{4.11}$$

In order to perform the integration we numerically evolve the orbit of the binary system with the Damour-Deruelle equations Eqs. (3.4) and (3.5). We supply Eqs. (4.6) and (4.7) with the relative position $\mathbf{x}$ and $\mathbf{v}$ from the solutions of the Damour-Deruelle equations. We then numerically integrate Eq. (4.11) to find the position and velocity of the center of mass. As the binary orbit decays to separations much below 9 m we have very little confidence in our post-Newtonian expression for the momentum ejection; however, at separations greater than 9 m the lowest-order contribution should give a realistic upper bound on the rate of momentum ejection.

Figure 6 shows the speed of the recoiling center of mass as a function of orbital separation for two coalescing neutron stars. Recall that for neutron stars the momentum ejection, and hence the recoil velocity, is suppressed by the factor $\delta m / m \leq 0.05$. Notice that at a separation just prior to the point where the neutron stars "touch" and hydrodynamic processes control the dynamics, (i.e., $r \approx 9$ m), the recoil velocity is only ≈ 1 km s^{-1}. Numerical hydrodynamic evolutions of neutron-star coalescences show wave forms and luminosity that decrease very rapidly as hydrodynamic processes proceed [28]. The momentum ejection must also rapidly attenuate as the stars coalesce. Thus, the estimation of a 1-km s^{-1} recoil velocity is probably a realistic upper bound for a neutron-star–neutron-star coalescence. Using test mass calculations, Fitchett and Detweiler [5] have estimated recoil velocities on the order of 100 km s^{-1} for

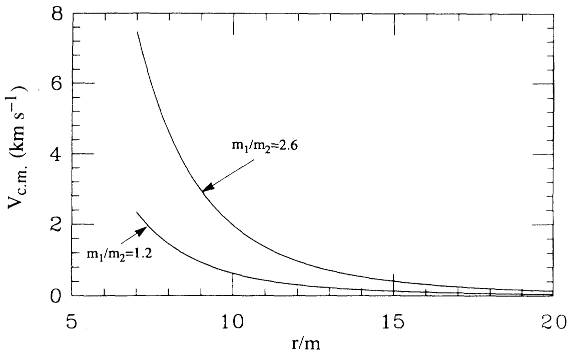

FIG. 6. Center of mass recoil velocity plotted against orbital separation.

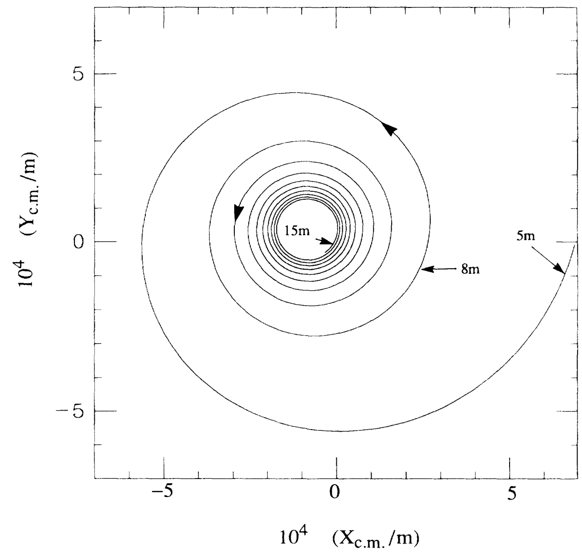

FIG. 7. Center of mass trajectory during a coalescence showing the center of mass spiraling outward as the orbit spirals inward. The three marks denote the orbital separation when the center of mass is at the marked position. Notice the center of mass only moves a small fraction of the orbital separation even during the late stages of the coalescence. The small offset of the center of the spiral is due to the initial conditions.

coalescing black holes at separations of $r \approx 5$ m. Unfortunately, we have little confidence in our approximation at such small separations, and therefore we are unable to make a reliable estimate for the recoil velocity for black holes near coalescence.

Figure 7 shows the trajectory of the binary system's center of mass as the orbital separation shrinks. Qualitatively, it has the form expected; as the binary constituents spiral inward due to the radiation reaction force, the center of mass spirals outward, reflecting the increasing rate of momentum ejection and the shrinking orbital period.

As a final check of our momentum calculation, we now demonstrate the use of the general momentum ejection formula of Th80. With the STF multipoles of the radiation we can now use the formula in Th80 to write down expressions for the linear momentum carried off by the radiation [Th80, Eq. (4.20′)]:

$$\frac{dP^i}{dt} = \frac{10}{315}\,{}^{(4)}I^{ijk}\,{}^{(3)}I^{jk} + \frac{112}{315}\varepsilon^{ijk}\,{}^{(3)}I^{jl}\,{}^{(3)}J^{kl} + \frac{1}{18}\frac{5}{315}\,{}^{(5)}I^{ijkl}\,{}^{(4)}I^{jkl} + \frac{1}{2}\frac{5}{315}\varepsilon^{ijk}\,{}^{(4)}I^{jlm}\,{}^{(4)}J^{klm} + \frac{20}{315}\,{}^{(4)}J^{ijk}\,{}^{(3)}J^{jk} + \cdots \; . \tag{4.12}$$

The presuperscript on each multipole denotes the order of time differentiation. The ellipsis denotes that we have truncated the series of multipole products. We have retained enough terms to give the lowest-order term for the momentum ejection [the first two terms in Eq. (4.12)], and the first correction term [the last three terms in Eq. (4.12) and the post-Newtonian corrections to the first two terms]. The first correction term is of $O(\varepsilon^2)$ relative to the lowest-order term. Notice that a correction of $O(\varepsilon^3)$ in I^{ij} or I^{ijk} (i.e., the tail corrections discussed above) would produce a correction of $O(\varepsilon^3)$ beyond the lowest-order term in the momentum formula. In this momentum calculation we have only been keeping terms of $O(\varepsilon^2)$ beyond the lowest order, and therefore we may neglect the contribution to the momentum ejection from the heredity terms.

Substituting the STF multipoles into Eq. (4.12), and using the post-Newtonian equation of motion to eliminate the higher-order derivatives, and restricting the motion to a circular orbit in the x-y plane, we find, for the x component of the first two terms in Eq. (4.12),

$$\frac{10}{315}\,{}^{(4)}I^{xjk}\,{}^{(3)}I^{jk} + \frac{112}{315}\varepsilon^{xjk}\,{}^{(3)}I^{jl}\,{}^{(3)}J^{kl} = \frac{8}{105}\frac{\delta m}{m}\eta^2 (m/r)^{11/2}\left[58 - \frac{46031 - 19\,576\eta}{63}\frac{m}{r}\right]\sin\phi \; . \tag{4.13}$$

Similarly for the last three terms in Eq. (4.12),

$$\frac{1}{18}\frac{5}{315}\,{}^{(5)}I^{xjkl}\,{}^{(4)}I^{jkl} + \frac{1}{2}\frac{5}{315}\varepsilon^{xjk}\,{}^{(4)}I^{jlm}\,{}^{(4)}J^{klm} + \frac{20}{315}\,{}^{(4)}J^{ijk}\,{}^{(3)}J^{jk} = \frac{8}{105}\frac{\delta m}{m}\eta^2 (m/r)^{11/2}\left[\frac{6950}{63}(1-3\eta)\frac{m}{r}\right]\sin\phi \; . \tag{4.14}$$

The two pieces combined, together with similar results for the y component, give a total momentum ejection of

$$\frac{d\mathbf{P}}{dt} = \frac{8}{105}\frac{\delta m}{m}\eta^2\left[\frac{m}{r}\right]^{11/2}\left[58 - \frac{5583 + 182\eta}{9}\frac{m}{r}\right](\sin\phi, -\cos\phi, 0) \; . \tag{4.15}$$

This agrees exactly with Eq. (4.10) above. Notice that the correction due to the higher-order multipoles Eq. (4.14) [i.e., the last three terms in Eq. (4.12)] actually adds to the momentum ejection, but the post-Newtonian corrections to lower multipoles [i.e., the post-Newtonian corrections in Eq. (4.13)] actually subtract from the momentum ejection. Thus it is not the inclusion of higher-Newtonian multipoles in the momentum formula that reduces the momentum ejection. It is the higher-order corrections to the lower-order multipoles which causes the reduction.

V. CONCLUDING REMARKS

Using the formalism of Blanchet, Damour and Iyer [10–14], we have derived gravitational wave forms accurate to $(\text{post})^{3/2}$-Newtonian order beyond the usual quadrupole wave forms (modulo the hereditary corrections). With these higher-order wave forms we have obtained a post-Newtonian correction to the rate of momentum ejection from a coalescing binary system. The higher-order momentum calculation has allowed us to estimate an upper bound of $|\mathbf{V}_{\text{c.m.}}|=1\ \text{km}\,\text{s}^{-1}$ for the recoil velocity of a coalescing neutron-star system. We have noted that the higher-order correction term in the momentum ejection formula has only a weak dependence on the mass ratio of the objects in the binary [see Eq. (4.10)]. This suggests that previous test mass calculations may in fact be quite accurate. We have also noted that the higher order correction to the momentum ejection actually reduces the net rate of momentum ejection, in effect stabilizing the center-of-mass motion of the system.

To improve these results would require extending the wave-form calculation to the $(\text{post})^{5/2}$-Newtonian order and including the tail terms that we have omitted. This calculation would be quite tedious, and the still higher-order corrections to the wave form would probably not be discernible for even the most sensitive gravitational-wave detectors, nevertheless, it would allow the calculation of $(\text{post})^{2}$-Newtonian corrections to the luminosity formula and the rate of momentum ejection.

At the higher order we expect not only an improvement in the formal accuracy of the expressions for luminosity and momentum ejection, but also a qualitative improvement in their behavior as the binary separation shrinks, i.e., we would expect them to behave in the test mass limit more like the exact test-mass calculations given by Fitchett and Detweiler [5]. At the end of Sec. IV we showed that in the case of circular motion it is not the inclusion of higher multipoles that causes the higher-order correction to subtract from the momentum ejection, but rather the post-Newtonian corrections to the lower multipoles. The fact that these terms are so negative is in large part due to the use of the $(\text{post})^{1}$-Newtonian circular velocity formula Eq. (4.9). In the regime of interest $r=5-10$ m this formula drastically underestimates the speed required to hold a circular orbit compared to speed predicted either by the more accurate $(\text{post})^{2}$-Newtonian equation of motion or, in the test mass limit, the exact Schwarzschild equation of motion. The post-Newtonian formula for the luminosity shows similar behavior to our post-Newtonian momentum formula; that is, it turns over and goes negative as the binary separation decays through $r\approx 9$ m. This again is caused by the omission of higher terms in the calculation. This problem might clear up at the next order where we can consistently use a more accurate (higher-order) equation of motion.

We also point out that we have neglected effects due to the spin of the bodies. Inclusion of spin will affect the momentum ejection calculation through both the motion of the system and the wave form. For a counter-rotating test mass the innermost stable circular orbit can be as large as $r=9$ m (in Boyer-Lindquist coordinates), inside which the orbit decays rapidly (see Ref. [25], p. 911). This rapid, nondissipative decay of the orbit will enter the momentum ejection calculation through the equation of motion. Also, the wave form for spinning bodies will contain h^{zz}, h^{xz}, h^{yz} components resulting from components of spin not orthogonal to the orbital x-y plane [see the discussion following Eq. (4.4)]. This could result in secular ejection of momentum out of the orbital plane.

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APPENDIX A: POST-NEWTONIAN GRAVITATIONAL RADIATION: THE EPSTEIN-WAGONER FORMALISM

1. Foundations

In this appendix we give an alternative derivation of the wave form Eq. (3.8). Although this approach lacks the "rigor" of the derivation outlined in Secs. II and III, it is self-contained and substantially simpler.

In this first subsection we present general formulas for the far-zone gravitational radiation of a fluid source. Most of the results presented here can be found in Th80, but the development is closely patterned after Sec. II of Epstein and Wagoner [15] (EW). However, unlike EW, we work strictly in the de Donder gauge. The only assumption we make here is that the field is weak enough that space-time can be covered by a coordinate system that satisfied the de Donder gauge condition,

$$\bar{h}^{\alpha\beta}{}_{,\beta}=0\ , \tag{A1}$$

where $\bar{h}^{\alpha\beta}$ is the potential defined by

$$\bar{h}^{\alpha\beta}\equiv-(-g)^{1/2}g^{\alpha\beta}+\eta^{\alpha\beta}\ . \tag{A2}$$

In this gauge the exact Einstein field equations take the form [Th80]

$$\Box \bar{h}^{\alpha\beta} = -16\pi\tau^{\alpha\beta} , \tag{A3}$$

where $\Box \equiv (-\partial^2/\partial t^2 + \nabla^2)$ is the flat-space wave operator. The "effective" stress-energy pseudotensor $\tau^{\alpha\beta}$ is given by

$$\tau^{\alpha\beta} = (-g)(T^{\alpha\beta} + t^{\alpha\beta}_{\rm LL}) + (1/16\pi)[\bar{h}^{\alpha\mu}{}_{,\nu}\bar{h}^{\beta\nu}{}_{,\mu} - \bar{h}^{\alpha\beta}{}_{,\mu\nu}\bar{h}^{\mu\nu}] , \tag{A4}$$

and $t^{\alpha\beta}_{\rm LL}$ is the Landau-Lifshitz pseudotensor (Ref. [25], Eq. (20.22)), and $T^{\alpha\beta}$ is the stress-energy of the matter. Although Eq. (A3) above has exactly the same structure as EW Eq. (9), we do not assert that the effective stress-energy pseudotensor $\tau^{\alpha\beta}$ that appears in Eq. (A3) is the same as the one that appears in the EW calculation. The reason is that EW define their potential by $\theta_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$, where $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$, and then impose the gauge condition $\theta^{\mu\nu}{}_{,\nu} = 0$. To first order in $\theta^{\mu\nu}$ this agrees with Eqs. (A1) and (A2), but not to higher order. In Appendix A 3 we construct a $(\text{post})^{3/2}$-Newtonian expression for $\tau^{\alpha\beta}$ in de Donder gauge, and show how it is related to the $(\text{post})^{3/2}$-Newtonian expression for $\tau^{\alpha\beta}$ that appears in EW.

Equation (A3) is just the standard wave equation, and we can immediately write down a formal solution for an outgoing wave, which is valid at any field $\mathbf{x}$:

$$\bar{h}^{\alpha\beta}(\mathbf{x},t) = 4\int \frac{\tau^{\alpha\beta}(\mathbf{x}',t')\delta(t'-t+|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|} d^4x' = 4\int \frac{\tau^{\alpha\beta}(\mathbf{x}',t_{\rm ret})}{|\mathbf{x}-\mathbf{x}'|} d^3x' , \tag{A5}$$

where $t_{\rm ret} = t - |\mathbf{x}-\mathbf{x}'|$. Unfortunately, this is an integral equation for $\bar{h}^{\alpha\beta}$ since $\bar{h}^{\alpha\beta}$ is itself embedded in $\tau^{\alpha\beta}$ [see Eq. (A4)]. In the next section we use an iterative process to solve for $\bar{h}^{\alpha\beta}$.

In the Introduction we mentioned that we neglect contributions to the radiation from the tail of the gravitational waves. Generally these tail contributions are thought of as arising from the radiation scattering off the background curvature of space-time. Therefore, one might conclude that the use of the flat-space Green's function (the Dirac delta function) in Eq. (A5) to write the formal solution to the field equations signifies the omission of the tail contributions. However, this is not the case. The flat-space Green's function in Eq. (A5) arises solely because we have chosen to write the field equations with a flat-space wave operator [see Eq. (A3)]. Crowley and Thorne [18] have shown that when the field equations are formulated in terms of a flat-space wave operator (as we have done here), the tail terms arise from the last term in the effective stress-energy pseudotensor [i.e., the term $\bar{h}^{\alpha\beta}{}_{,\mu\nu}\bar{h}^{\mu\nu}$ in Eq. (A4)]. Therefore, our omission of the tail contributions can be summarized as the omission of contributions to the effective stress-energy pseudotensor from the term $\bar{h}^{\alpha\beta}{}_{,\mu\nu}\bar{h}^{\mu\nu}$. For a full discussion of tail contributions in the flat-space formulation of the field equations see Crowley and Thorne [18] [particularly the discussion surrounding their Eq. (39)]. In Appendix B we show that the tail of the radiation does not enter the wave form until the $(\text{post})^{3/2}$-Newtonian order, and we show it does not contribute to the momentum ejection at the order we are considering.

2. Slow-motion sources and a multipole expansion

Just as in our previous development, the gravitational radiation far from the material source is given by the "transverse-traceless" projection of the spatial portion of the metric perturbation

$$h_{\alpha\beta} \equiv g_{\alpha\beta} - \eta_{\alpha\beta} . \tag{A6}$$

In the far zone, where $|\bar{h}^{\mu\nu}| \ll 1$, $\bar{h}^{\mu\nu}$ reduces to the "trace-reversed" metric perturbation, i.e.,

$$\bar{h}^{\alpha\beta} = h^{\alpha\beta} - \tfrac{1}{2}h\eta^{\alpha\beta} \text{ (far zone)} . \tag{A7}$$

We also follow the convention of EW and let the subscript TT denote that the quantity is not only contracted with the projection operator Eq. (2.4), but also is to be evaluated in the far zone (large R).

Using Eq. (A7) and expanding Eq. (A5), we obtain, to lowest order in $|\mathbf{x}'|/R$,

$$h^{ij}_{\rm TT} = \bar{h}^{ij}_{\rm TT} = \frac{4}{R}\int \tau^{ij}(\mathbf{x}', t-R+\mathbf{n}\cdot\mathbf{x}')d^3x'_{\rm TT} . \tag{A8}$$

In order to find a multipole expansion of the radiation we now restrict the discussion to slow motion sources. This allows us to expand Eq. (A8) in the series

$$h^{ij}_{\rm TT} = \bar{h}^{ij}_{\rm TT} = \frac{4}{R}\sum_{m=0}^{\infty}\frac{1}{m!}\frac{\partial^m}{\partial t^m}\int \tau^{ij}(\mathbf{x}', t-R)(\mathbf{n}\cdot\mathbf{x}')^m d^3x'_{\rm TT} . \tag{A9}$$

As a consequence of the conservation law [Th80]

$$\tau^{\alpha\beta}{}_{,\beta} = 0 , \tag{A10}$$

τ^{ij} satisfies the identifies

$$\tau^{ij} = \tfrac{1}{2}(\tau^{00}x^ix^j)_{,00} + (\tau^{ki}x^j + \tau^{kj}x^i)_{,k} - \tfrac{1}{2}(\tau^{kl}x^ix^j)_{,kl} , \tag{A11a}$$

$$\tau^{ij}x^k = \tfrac{1}{2}(\tau^{0i}x^jx^k + \tau^{0j}x^ix^k - \tau^{0k}x^ix^j)_{,0} + \tfrac{1}{2}(\tau^{li}x^jx^k + \tau^{lj}x^ix^k - \tau^{lk}x^ix^j)_{,l} . \tag{A11b}$$

Using these identifies in the series equation Eq. (A9) generates the multipole expansion

$$h^{ij}_{\rm TT} = \bar{h}^{ij}_{\rm TT} = \frac{2}{R}\frac{d^2}{dt^2}\sum_{m=0}^{\infty} n_{k_1}\cdots n_{k_m} I_{\rm EW}^{ijk_1\cdots k_m}(t-R)_{\rm TT} , \tag{A12}$$

where

$$I^{ij}_{\rm EW} = \int \tau^{00}x^ix^jd^3x , \tag{A13a}$$

$$I^{ijk}_{\rm EW} = \int (\tau^{0i}x^jx^k + \tau^{0j}x^ix^k - \tau^{0k}x^ix^j)d^3x , \tag{A13b}$$

$$I^{ijkl}_{\rm EW} = \int \tau^{ij}x^kx^ld^3x , \tag{A13c}$$

$$I^{ijklm}_{\rm EW}=\frac{1}{3}\frac{d}{dt}\int \tau^{ij}x^k x^l x^m d^3x \ . \tag{A13d}$$

The last two formulas in Eqs. (A13) are obtained from the general formula

$$I^{ijk_1\cdots k_m}_{\rm EW}=\frac{2}{m!}\frac{d^{m-2}}{dt^{m-2}}\int \tau^{ij}x^{k_1}\cdots x^{k_m}d^3x \quad (m\geq 2)\ . \tag{A13e}$$

In Eqs. (A13a) and (A13b) we have assumed that surface terms produced by the integration by parts of the divergences in Eq. (A11a) and (A11b) vanish. This assumption was verified (albeit not rigorously) by EW; since the verification they gave is not gauge dependent, it suffices for our discussion. Throughout the remainder of this paper we neglect surface terms after integrating by parts. The subscript "EW" denotes that the "multipoles" in Eqs. (A13) are defined in a way unique to EW and this paper; they are different from the symmetric-trace-free (STF) multipole expansion we used in Secs. II and III. In Appendix A 5 we carefully show how these quantities are related to the standard mass and current multipoles presented in Sec. III.

3. The effective stress-energy tensor

The previous subsection shows that the radiation problem is essentially reduced to finding expressions for the effective stress-energy tensor $\tau^{\alpha\beta}$ and its time evolution. In this section we employ a weak-field approximation and a perfect fluid model for the matter to obtain approximate expressions for the $\tau^{\alpha\beta}$. The approximate expressions are sufficiently accurate to give radiation formulas which are accurate to the (post)$^{3/2}$-Newtonian order beyond the usual quadrupole term.

Turning first to $T^{\alpha\beta}$ the stress-energy of the matter, we assume that the matter is an isentropic perfect fluid described by

$$T^{\alpha\beta}=[\rho_0(1+\Pi)+p]u^\alpha u^\beta+pg^{\alpha\beta}\ , \tag{A14}$$

where ρ_0 is the baryon rest-mass density, p is the pressure, Π is the specific internal energy, and u^μ is the fluid four-velocity. Through the necessary order, $T^{\alpha\beta}$ can be written [29]

$$T^{00}=\rho_0(1+\Pi+v^2+2U)+O(\rho_0\varepsilon^4)\ , \tag{A15a}$$

$$T^{0i}=\rho_0(1+\Pi+v^2+2U)v^i+pv^i+O(\rho_0\varepsilon^5)\ , \tag{A15b}$$

$$T^{ij}=\rho_0(1+\Pi+v^2+2U)v^iv^j +p[v^iv^j+(1-2U)\delta^{ij}]+O(\rho_0\varepsilon^6)\ , \tag{A15c}$$

where

$$U\equiv\int\frac{\rho_0(\mathbf{x}',t)d^3x'}{|\mathbf{x}-\mathbf{x}'|}\ ,\quad U_{,kk}=-4\pi\rho_0\ , \tag{A16}$$

and

$$v^i=\frac{dx^i}{dt}=\frac{u^i}{u^0}\ . \tag{A17}$$

We now turn our attention to $t^{\alpha\beta}_{\rm LL}$ and $(-g)$, which are both constituents of $\tau^{\alpha\beta}$. Thorne and Kovacs [30] have carried out expansions of $t^{\alpha\beta}_{\rm LL}$ and $(-g)$ in terms of the potential $\bar h^{\alpha\beta}$: i.e.,

$$(-g)=1-\bar h+\tfrac{1}{2}[\bar h^2-\bar h^{\alpha\beta}\bar h_{\alpha\beta}]+O(\bar h^3)\ , \tag{A18}$$

$$t^{\alpha\beta}_{\rm LL}=(16\pi)^{-1}\{\tfrac{1}{2}\eta^{\alpha\beta}\eta_{\lambda\mu}\bar h^{\lambda\mu}{}_{,\rho}\bar h^{\rho\mu}{}_{,\nu}+\eta_{\lambda\mu}\eta^{\nu\rho}\bar h^{\alpha\lambda}{}_{,\nu}\bar h^{\beta\mu}{}_{,\rho}-(\eta^{\alpha\lambda}\eta_{\mu\nu}\bar h^{\beta\nu}{}_{,\rho}\bar h^{\mu\rho}{}_{,\lambda}+\eta^{\beta\lambda}\eta_{\mu\nu}\bar h^{\alpha\nu}{}_{,\rho}\bar h^{\mu\rho}{}_{,\lambda}) +\tfrac{1}{8}(2\eta^{\alpha\lambda}\eta^{\beta\mu}-\eta^{\alpha\beta}\eta^{\lambda\mu})(2\eta_{\nu\rho}\eta_{\sigma\tau}-\eta_{\rho\sigma}\eta_{\nu\tau})\bar h^{\nu\tau}{}_{,\lambda}\bar h^{\rho\sigma}{}_{,\mu}\}+O(\bar h^3)\ . \tag{A19}$$

Using these results we can see from Eq. (A4), that, to lowest order,

$$\tau^{00}=\rho_0+O(\rho_0\varepsilon^2)\ , \tag{A20a}$$

$$\tau^{0i}=\rho_0v^i+O(\rho_0\varepsilon^3)\ , \tag{A20b}$$

$$\tau^{ij}=O(\rho_0\varepsilon^2)\ . \tag{A20c}$$

Substituting these into Eq. (A5) we obtain the potentials

$$\bar h^{00}=4U+O(\varepsilon^4)\ , \tag{A21a}$$

$$\bar h^{0i}=4V^i+O(\varepsilon^5)\ , \tag{A21b}$$

$$\bar h^{ij}=O(\varepsilon^4)\ . \tag{A21c}$$

Here V^i is the standard post-Newtonian potential [31] given by

$$V^i=\int\frac{\rho_0(\mathbf{x}',t)v^{i\prime}d^3x'}{|\mathbf{x}-\mathbf{x}'|},\quad V^i{}_{kk}=-4\pi\rho_0v^i\ . \tag{A22}$$

By virtue of the gauge condition $\bar h^{\alpha\beta}{}_{,\beta}=0$ we see that

$$U_{,0}+V^n{}_{,n}=0\ . \tag{A23}$$

Using these expressions for the potentials in Eq. (A19) we obtain

$$t^{00}_{\rm LL}=(-7/8\pi)U_{,k}U_{,k}+O(\rho_0\varepsilon^4)\ , \tag{A24a}$$

$$t^{0i}_{\rm LL}=(-1/\pi)U_{,k}V^i{}_{,k}+(1/\pi)U_{,k}V^k{}_{,i} +(3/4\pi)U_{,0}U_{,i}+O(\rho_0\varepsilon^5)\ , \tag{A24b}$$

$$t^{ij}_{\rm LL}=(1/4\pi)(U_{,i}U_{,j}-\tfrac{1}{2}\delta^{ij}U_{,k}U_{,k})+O(\rho_0\varepsilon^4)\ . \tag{A24c}$$

Substituting Eqs. (A24), (A21), (A15), and (A18) into Eq. (A4) we finally obtain the effective stress-energy pseudotensor

$$\tau^{00}=\rho_0(1+\Pi+v^2+6U)+(-7/8\pi)U_{,k}U_{,k}+O(\rho_0\varepsilon^4)\ , \tag{A25a}$$

$$\tau^{0i}=\rho_0(1+\Pi+v^2+6U)v^i+pv^i-(1/\pi)U_{,k}V^i{}_{,k} +(1/\pi)U_{,k}V^k{}_{,i}+(3/4\pi)U_{,0}U_{,i}+O(\rho_0\varepsilon^5)\ , \quad \text{(A25b)}$$

$$\tau^{ij}=\rho_0 v^i v^j+(1/4\pi)U_{,i}U_{,j}+\delta^{ij}[p-(1/8\pi)U_{,k}U_{,k}] +O(\rho_0\varepsilon^4)\ . \quad \text{(A25c)}$$

We have freely raised and lowered the spatial indices with the δ^{ij}. We have also neglected contributions from the term $\bar{h}^{\alpha\beta}{}_{,\mu\nu}$ in Eq. (A4). We argued earlier that these contributions are the source of the hereditary part of the radiation, which we are neglecting in this discussion.

Before proceeding to the multipole calculation, we make one final comment about Eq. (A25a)–(A25c). These expressions for $\tau^{\alpha\beta}$ are not the same as the expressions obtained by EW [see the remarks following Eq. (5.33) in Th80]. However, after some manipulation it can be seen that

$$\tau^{00}=\tau^{00}_{\rm EW}+(-1/4\pi)(U^2)_{,kk}\ , \quad \text{(A26a)}$$

$$\tau^{0i}=\tau^{0i}_{\rm EW}+(1/2\pi)[(U_{,k}V^k)_{,i}-(UV^i)_{,kk}]\ , \quad \text{(A26b)}$$

$$\tau^{ij}=\tau^{ij}_{\rm EW}+(1/4\pi)[U^2{}_{,ij}-\delta^{ik}(U^2)_{,kk}]\ . \quad \text{(A26c)}$$

The quantities $\tau^{\alpha\beta}_{\rm EW}$ in Eqs. (A26) denote the expressions given by EW [EW, Eqs. (37)] for the effective stress-energy pseudotensor. When the radiation multipoles are computed using Eq. (A13), the additional terms in Eqs. (A26) all produce terms with vanishing transverse-traceless part; therefore, they will not contribute to the radiation to the post-Newtonian order considered. It is possible that this is representative of a more general property of the effective stress-energy pseudotensor when written with different definitions of the potentials (e.g., our $\bar{h}^{\mu\nu}$ as opposed to EW's $\theta^{\mu\nu}$). See the discussion in Popova and Petrov [32]. For the remainder of this paper we use Eqs. (A25) for our effective stress-energy pseudotensor, although these remarks show that had we used EW's effective stress-energy pseudotensor there would be no observable change in the resultant radiation multipoles.

Lincoln and Will [8] have used the Wagoner-Will [16] wave forms, which were derived using the gauge of Epstein-Wagoner, along with the Damour-Deruelle equations of motion, which are derived using de Donder gauge, to study the nature of the radiation from a coalescing binary system. The discussion above shows that there is no formal inconsistency due to the different gauges.

4. Compact object approximation

In Secs. A 1–A 3 we found expressions for the radiation multipoles in terms of integrals of the effective stress-energy tensor of a perfect fluid source. Our only assumption about the configuration of the material source was that it had finite extent. Now, in order to evaluate the multipole integrals explicitly [Eq. (A25) with Eq. (A13)], we further assume that the matter consists of N static, spherically symmetric, perfect fluid "point" masses. We will later set $N=2$ to work out the radiation of a binary system.

Wagoner and Will [16] have carried out a similar point-mass calculation, but they terminated the approximation of $\tau^{\alpha\beta}$ at $O(\rho_0\varepsilon^2)$. As we move to a higher-order approximation, more care must be taken to avoid divergent integrals and other meaningless quantities in the calculation. In particular, this calculation includes terms of $O(\rho_0\varepsilon^3)$, and consequently the internal pressure will explicitly appear in the integrand of I^{ijk} [see Eqs. (A25b) and (A13b)]. From this it appears that the radiation could depend on the internal equation of state of each body, and not just on the "mass" of each body. Furthermore, examining Eq. (A13) and Eq. (A25) shows that post-Newtonian corrections in $\tau^{\alpha\beta}$ will produce terms of the form $\int_A \rho_0 U\, d^3x$ in the multipole integrals (A represents integration over the Ath body). These integrals can lead to an infinite self-interaction in the point-mass limit, and thus require careful mass renormalization. In fact, it is not immediately clear that these integrals are well defined at all. Although our treatment is not as mathematically rigorous as Damour's derivation of the equation of motion for point masses [7], we do carefully specify our point-mass approximation, and use virial relations to eliminate the pressure from the integrand. In the end, the wave form depends only on the "mass" of objects.

In broad strokes, our "point-mass" approximation is as follows: We first assume that the separation between bodies is large compared to the dimensions of the individual bodies, but we do not assume that the bodies are mathematical "points"; i.e., we never define the particles as Dirac delta functions. This means that we ignore tidal interactions between the bodies; as a consequence we can treat the body as static and spherical, as seen in a suitable comoving reference frame. This allows us to integrate over the material source in the usual way. This, in turn, allows us to use well-defined integral virial relations to remove the pressure, and the post-Newtonian definition of the mass absorbs the self-interaction terms. Only after we have integrated over the material source do we formally shrink the body dimensions to negligible size to perform the integrations over the external fields. In the end, the final expressions for the radiation multipoles depend only on the "mass" of the objects; all the dependence on the internal structure having been either eliminated or absorbed in the definition of the mass.

We begin our formal treatment of the point-mass limit by defining the conserved effective mass, center-of-mass world line, and the velocity of the Ath body. Following Will (Ref. [31], p. 146), and assuming static structure, we can write

$$m_A\equiv\int_A \rho^*(1+\Pi-\tfrac{1}{2}U_A)d^3x\ , \quad \text{(A27)}$$

$$\mathbf{x}_A\equiv\frac{1}{m_A}\int_A \rho^*(1+\Pi-\tfrac{1}{2}U_A)\mathbf{x}\,d^3x\ , \quad \text{(A28)}$$

$$\mathbf{v}_A=\frac{d\mathbf{x}_A}{dt}=\mathbf{v}\ , \quad \text{(A29)}$$

where

$$U_A = \int_A \frac{\rho_0(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|} d^3x' , \tag{A30}$$

and ρ^* is the "conserved" energy density, given by

$$\rho^* = \rho_0(1+\tfrac{1}{2}v^2+3U)+O(\rho_0\epsilon^4) . \tag{A31}$$

Equation (A29) shows that by assuming static structure the fluid velocity is constant over the body. This allows us to pull the velocity out of the integrand.

Assuming the bodies are small compared to the separation distance also simplifies our expression for the potential

$$U = \int \frac{\rho_0(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|} d^3x' . \tag{A32}$$

Since ρ_0 vanishes outside the bodies we can write the potential

$$U = U_A + \sum_{B\neq A}^{N} \int_B \frac{\rho_0(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|} d^3x' = U_A + \sum_{B\neq A}^{N} U_B , \tag{A33}$$

where we have singled out a generic body A. Assuming the separation is large compared to the dimensions of body A, the second term in Eq. (A33) can be approximated for field points within A by its value at the center of A, i.e.,

$$U(\mathbf{x}) = U_A(\mathbf{x}) + \sum_{B\neq A}^{N} \frac{m_B}{r_{AB}} \quad (\mathbf{x}\in A) , \tag{A34}$$

where

$$r_{AB} \equiv |\mathbf{r}_{AB}| \quad \text{and} \quad \mathbf{r}_{AB} = \mathbf{x}_A - \mathbf{x}_B . \tag{A35}$$

By looking at the definition of the mass Eq. (A27), it is clear that irrelevant terms of $O(\epsilon^4)$ have been omitted from Eq. (A34).

Using Eq. (A29) and the same arguments as above we can also find similar expressions for the post-Newtonian potential V^i:

$$V^i = v_A^i U_A + \sum_{B\neq A}^{N} v_B^i U_B . \tag{A36}$$

If the field point is inside the body A, Eq. (A36) can be written

$$V^i(\mathbf{x}) = v_A^i U_A(\mathbf{x}) + \sum_{B\neq A}^{N} v_B^i \frac{m_B}{r_{AB}} \quad (\mathbf{x}\in A) . \tag{A37}$$

Using the formulas above, we now explicitly carry out the multipole integrations. Substituting Eq. (A25a) into Eq. (A13a), we now write

$$(I_{\rm EW}^{ij})_{\rm TT} = \int [\rho_0(1+\Pi+v^2+6U) - (7/8\pi)U_{,n}U_{,n} + O(\rho_0\epsilon^4)]x^i x^j d^3x_{\rm TT} . \tag{A38}$$

Integrating by parts and then using Eq. (A31) and Eq. (A16) and neglecting terms with vanishing transverse-traceless part we obtain

$$(I_{\rm EW}^{ij})_{\rm TT} = \int \rho^*[1+\Pi+\tfrac{1}{2}v^2-\tfrac{1}{2}U+O(\rho_0\epsilon^4)]x^i x^j d^3x_{\rm TT} . \tag{A39}$$

The region of integration extends over all space, but the integrand vanishes except on the interior of the bodies. This allows us to rewrite Eq. (A38) as

$$(I_{EW}^{ij})_{\rm TT} = \sum_A \int_A [\rho^*(1+\Pi-\tfrac{1}{2}U_A)+\tfrac{1}{2}\rho^* v^2 - \tfrac{1}{2}\rho^*(U-U_A)+O(\rho_0\epsilon^4)]x^i x^j d^3x_{\rm TT} . \tag{A40}$$

The term $(U-U_A)$ is just the potential at body A produced by the other bodies. Assuming that body A is small compared to the separation we approximate this by Eq. (A34). Using spherical symmetry and Eq. (B1), the product $x^i x^j$ slips out of the integrand as $x_A^i x_A^j$. Finally, using Eq. (A27) and Eq. (A29) we have

$$(I_{\rm EW}^{ij})_{\rm TT} = \sum_A m_A \left[1+\frac{1}{2}v_A^2 - \frac{1}{2}\sum_{B\neq A}^{N}(m_B/r_{AB})+O(\epsilon^4)\right] \times (x_A^i x_A^j)_{\rm TT} . \tag{A41}$$

This result has been obtained by Wagoner and Will [16]. Later in this section we specialize this result to the two-body case.

We now turn to the more formidable three-index multipole $I_{\rm EW}^{ijk}$ Eq. (A13b):

$$(I_{\rm EW}^{ijk})_{\rm TT} \equiv \int (\tau^{0i}x^j x^k + \tau^{0j}x^i x^k - \tau^{0k}x^i x^j) d^3x_{\rm TT} . \tag{A42}$$

Wagoner and Will [16] evaluated this term only to lowest post-Newtonian order, obtaining the $P_{1/2}$ contribution the wave form in Eq. (1.1); here we carry the computation to the next order, to obtain part of the $P_{3/2}$ contribution to the wave form. In Eq. (A42) the transverse-traceless operator (TT) contracts only with the indices i and j; i.e., it will always be understood that there is a contraction of the index k with the unit-vector components n^k. To eliminate unnecessary manipulation of unwieldy expressions, we only work with one of the three permutations of ijk in Eq. (A42), i.e.,

$$(\tilde{I}_{\rm EW}^{ijk})_{\rm TT} \equiv \int \tau^{0i}x^j x^k d^3x_{\rm TT} . \tag{A43}$$

After we find $(\tilde{I}_{\rm EW}^{ijk})_{\rm TT}$ we will permute the indices and reassemble Eq. (A42). The three relationships in Eq. (2.4) are sufficient to show that if a term has vanishing transverse-traceless part and is dropped in this permutation, then it will also have vanishing transverse-traceless part in the other two permutations of ijk that make up Eq. (A42).

Substituting Eq. (A25b) for τ^{0i}, integrating by parts, using Eq. (A31) for ρ^*, and dropping terms with vanishing transverse-traceless part and terms of $O(\rho_0\epsilon^5)$ we have

$$(\tilde{I}^{ijk}_{\rm EW})_{\rm TT}=\int\{[\rho^*(1+\Pi+\tfrac{1}{2}v^2+U)v^i+pv^i -2\rho^*V^i-(1/4\pi)U_{,ni}V^n]x^jx^k +(3/4\pi)U_{,i}(V^jx^k+V^kx^j)\}d^3x_{\rm TT}\ . \tag{A44}$$

In accordance with our point-mass approximation we separate the region of integration into two regions: the material source (MS), and the free space (FS) outside the material source. The terms proportional to ρ^* will not contribute to the free space integral. We write Eq. (A44) as

$$(\tilde{I}^{ijk}_{\rm EW})_{\rm TT}=(\tilde{I}^{ijk}_{\rm MS})_{\rm TT}+(\tilde{I}^{ijk}_{\rm FS})_{\rm TT}\ , \tag{A45}$$

where

$$(\tilde{I}^{ijk}_{\rm MS})_{\rm TT}=\sum_A^N\int_A\{[\rho^*(1+\Pi+\tfrac{1}{2}v^2+U)+p]v^ix^jx^k -[2\rho^*V^ix^jx^k]-[(1/4\pi)U_{,ni}V^nx^jx^k] +[(3/4\pi)U_{,i}(V^jx^k+V^kx^j)]\}d^3x_{\rm TT}\ , \tag{A46}$$

and

$$(\tilde{I}^{ijk}_{\rm FS})_{\rm TT}=\int_{\rm FS}\{[(-1/4\pi)U_{,ni}V^nx^jx^k] +[(3/4\pi)U_{,i}(V^jx^k+V^kx^j)]\}d^3x_{\rm TT}\ . \tag{A47}$$

The first two set of square brackets in Eq. (A46) can be treated using the same techniques that were used in the I^{ij} calculation. The third [ ] is more subtle. Using Eqs. (A34) and (A37) we expand

$$\sum_A^N\int_A U_{,ni}V^nx^jx^kd^3x_{\rm TT}=\sum_A^N\left[v_A^n\int_A U_{A,ni}U_Ax^jx^kd^3x+\sum_{B\neq A}^N v_B^n(m_B/r_{AB})\int_A U_{A,ni}x^jx^kd^3x +\sum_{C\neq A}^N v_A^n(m_C/r_{AC})_{,ni}\int_A U_Ax^jx^kd^3x +\sum_{B\neq A}^N\sum_{C\neq A}^N v_B^n(m_B/r_{AB})(m_C/r_{AC})_{,ni}\int_A x^jx^kd^3x\right]_{\rm TT}\ . \tag{A48}$$

The third and the fourth integral in Eq. (A48) vanish as the volume of body A shrinks (point mass limit). By spherical symmetry of body A we can write

$$U_{A,ni}=-4\pi\rho_0n_A^nn_A^i+r_A^{-1}U_{A,r_A}(\delta^{ni}-3n_A^nn_A^i)\ , \tag{A49}$$

where n_A^i and r_A are a unit vector and the distance from the center of body A to a point in body A. Only the first term in Eq. (A49) survives in the first and second integrals in Eq. (A48) and the result is similar to ones treated in the $I^{ij}_{\rm EW}$ case. The fourth [ ] in Eq. (A46) is easily shown to vanish in the point-mass limit, by splitting up the potential, as in Eq. (A48). We now collect the result

$$(\tilde{I}^{ijk}_{\rm MS})_{\rm TT}=\sum_A^N\left[m_Av_A^ix^{A^j}x_A^k\left[1+\frac{1}{2}v_A^2+\sum_{B\neq A}^N(m^b/r_{AB})\right] -\frac{5}{3}\sum_{B\neq A}^N m_Av_B^ix_A^jx_A^k(m_B/r_{AB})+v_A^ix_A^jx_A^k\left[-\frac{1}{6}\int_A\rho^*U_Ad^3x+\int_A pd^3x\right]\right]_{\rm TT}\ . \tag{A50}$$

Note that the integrals in the last term exactly cancel by the virial theorem (see Ref. [31], pp. 148 and 245).

Using Eqs. (A34) and (A37) we split up the free-space portion of the integral

$$(\tilde{I}^{ijk}_{\rm FS})_{\rm TT}=\sum_A^N\left\{\frac{1}{4\pi}\left[v_A^n\int_{\rm FS}U_{A,ni}U_Ax^jx^kd^3x+\sum_{B\neq A}^N v_B^n\int_{\rm FS}U_{A,ni}U_Bx^jx^kd^3x\right] +\frac{3}{4\pi}\left[v_A^j\int_{\rm FS}U_{A,i}U_Ax^kd^3x+\sum_{B\neq A}^N v_B^i\int_{\rm FS}U_{A,i}U_Bx^kd^3x\right] +\frac{3}{4\pi}\left[v_A^k\int_{\rm FS}U_{A,i}U_Ax^jd^3x+\sum_{B\neq A}^N v_B^k\int_{\rm FS}U_{A,i}U_Bx^jd^3s\right]\right\}_{\rm TT}\ . \tag{A51}$$

All of the A-combined-with-A terms vanish by spherical symmetry and transverse-traceless arguments. The remaining integrals are treated in Appendix B. Assembling $\tilde{I}^{ijk}$ from the results above we have

$$(\tilde{I}^{ijk}_{\rm EW})_{\rm TT}=(\tilde{I}^{ijk}_{\rm MS})_{\rm TT}+(\tilde{I}^{ijk}_{\rm FS})_{\rm TT}$$

$$=\sum_A^N \left[m_A v_A^i x_A^j x_A^k \left[1+\frac{1}{2}v_A^2+\sum_{B\neq A}^N (m^B/r_{AB})\right]-\frac{5}{3}\sum_{B\neq A}^N m_A v_B^i x_A^j x_A^k (m_B/r_{AB}) \right.$$

$$\left. -\frac{1}{4\pi}\sum_{B\neq A}^N v_B^n Q_{AB}^{nijk}+\frac{3}{4\pi}\sum_{B\neq A}^N [v_B^j Q_{AB}^{ik}+v_B^k Q_{AB}^{ij}] \right]_{\rm TT}, \qquad \text{(A52)}$$

where the Q's are defined in Appendix B. Shortly we will give a more explicit formula, where we have specialized to the binary case.

The evaluation of the other multipoles $I^{ijkl}_{\rm EW}$ and $I^{ijklm}_{\rm EW}$ [Eqs. (A13c) and (A13d)] is tedious, but routine. Both are at least of order ε^2, and therefore the post-Newtonian self-interaction that plagued $I^{ij}_{\rm EW}$ and $I^{ijk}_{\rm EW}$ will not affect these terms. The messy details are carried out in Appendix B. After contraction with the normal vectors the results are

$$(I^{ij}_{\rm EW})_{\rm TT}=\sum_A^N m_A x_A^i x_A^j \left[1+\frac{1}{2}v_A^2-\frac{1}{2}\sum_{B\neq A}^N m_B/r_{AB}\right]_{\rm TT}, \qquad \text{(A53a)}$$

$$(n^k I^{ijk}_{\rm EW})_{\rm TT}=(n^k\tilde{I}^{ijk}_{\rm EW}+n^k I^{jik}_{\rm EW}-n^k\tilde{I}^{kij}_{\rm EW})_{\rm TT}, \qquad \text{(A53b)}$$

$$(n^k n^l I^{ijkl}_{\rm EW})_{\rm TT}=\sum_A^N m_A v_A^i v_A^j (\mathbf{n}\cdot\mathbf{x}_A)^2-\frac{1}{12}\sum_A^N\sum_{B\neq A}^N m_A m_B \frac{r_{AB}^i r_{AB}^j}{r_{AB}}\left[1-\frac{(\mathbf{r}_{AB}\cdot\mathbf{n})^2}{r_{AB}^2}+\frac{6(\mathbf{x}_A\cdot\mathbf{n})^2}{r_{AB}^2}\right]_{\rm TT}, \qquad \text{(A53c)}$$

$$(n^k n^l n^m I^{ijklm}_{\rm EW})_{\rm TT}=\frac{1}{3}\frac{d}{dt}\left\{\sum_A^N m_A v_A^i v_A^j (\mathbf{n}\cdot\mathbf{x}_A)^3\right.$$

$$\left. -\frac{1}{4}\sum_A^N\sum_{B\neq A}^N m_A m_B \frac{r_{AB}^i r_{AB}^j}{r_{AB}}\left[\frac{2(\mathbf{x}_A\cdot\mathbf{n})^3-3(\mathbf{x}_A\cdot\mathbf{n})^2(\mathbf{r}_{AB}\cdot\mathbf{n})+2(\mathbf{x}_A\cdot\mathbf{n})(\mathbf{r}_{AB}\cdot\mathbf{n})^2}{r_{AB}^2}+(\mathbf{x}_A\cdot\mathbf{n})\right]\right\}_{\rm TT}. \qquad \text{(A53d)}$$

5. Two-body EW multipoles

These multipole expressions can be reduced to the two-body case in exactly the same way the STF multipoles in Sec. II were handled:

$$(I^{ij}_{\rm EW})_{\rm TT}=\mu\left\{1+\frac{1-3\eta}{2}v^2-\frac{1-2\eta}{2}\frac{m}{r}\right\}(x^i x^j)_{\rm TT}, \qquad \text{(A54a)}$$

$$(n^k I^{ijk}_{\rm EW})_{\rm TT}=\mu\frac{\delta m}{m}\left\{-[(\mathbf{n}\cdot\mathbf{r})(v^i x^j+v^j x^i)-(\mathbf{n}\cdot\mathbf{v})x^i x^j]+\left[-\frac{1-5\eta}{2}v^2-\frac{12\eta+7}{6}\frac{m}{r}\right](\mathbf{n}\cdot\mathbf{r})(v^i x^j+v^j x^i)\right.$$

$$\left. +\left[\left[\frac{1-5\eta}{2}v+\frac{12\eta+17}{6}\frac{m}{r}\right](\mathbf{n}\cdot\mathbf{v})+\frac{1-6\eta}{6}\frac{m}{r}\frac{(\mathbf{n}\cdot\mathbf{r})}{r}\dot{r}\right]x^i x^j\right\}_{\rm TT}, \qquad \text{(A54b)}$$

$$(n^k n^l I^{ijkl}_{\rm EW})_{\rm TT}=\mu\left\{(1-3\eta)(\mathbf{n}\cdot\mathbf{r})^2 v^i v^j-\frac{1}{6}\frac{m}{r}\left[1+2(1-3\eta)\frac{(\mathbf{n}\cdot\mathbf{r})^2}{r^2}\right]x^i x^j\right\}_{\rm TT}, \qquad \text{(A54c)}$$

$$(n^k n^l n^m I^{ijklm}_{\rm EW})_{\rm TT}=\mu\frac{\delta m}{m}\frac{d}{dt}\left\{-\frac{1}{3}[(1-2\eta)(\mathbf{n}\cdot\mathbf{r})^3 v^i v^j]+\frac{1}{12}\left[\left[1+(1-2\eta)\frac{(\mathbf{n}\cdot\mathbf{r})^2}{r^2}\right]\frac{m}{r}\mathbf{n}\cdot\mathbf{r}\,x^i x^j\right]\right\}_{\rm TT}$$

$$=\mu\frac{\delta m}{m}\left\{-(1-2\eta)(\mathbf{n}\cdot\mathbf{r})^2(\mathbf{n}\cdot\mathbf{v})v^i v^j+\frac{1}{12}\left[1+5(1-2\eta)\frac{(\mathbf{n}\cdot\mathbf{r})^2}{r^2}\right]\frac{m}{r}(\mathbf{n}\cdot\mathbf{r})(v^i x^j+v^j x^i)\right.$$

$$\left. +\frac{1}{12}\left[1+3(1-2\eta)\frac{(\mathbf{n}\cdot\mathbf{r})^2}{r^2}\right]\left[(\mathbf{n}\cdot\mathbf{v})-\frac{\mathbf{n}\cdot\mathbf{r}}{r}\dot{r}\right]\frac{m}{r}x^i x^j\right\}_{\rm TT}. \qquad \text{(A54d)}$$

In Eq. (A54d) we have explicitly carried out the time differentiation, and used the Newtonian equation of motion to eliminate the accelerations.

These expressions for the multipoles can now be substituted into Eq. (A12). When the time derivatives produce an acceleration, we substitute the DD equations of motion to the necessary order. The result checks with the wave form given by Eq. (3.8).

As another check of the considerable manipulation that went into constructing the wave form (in either formalism), we can use the projection integrals given in Th80 [Th80, Eqs. (4.11a) and (4.11b)]. If the wave form is known [i.e., we substitute the EW multipoles Eqs. (A54) into Eq. (A12)], time derivatives of the radiation mass multipoles can be projected out by integrating over the sphere:

$$\frac{d^m}{dt^m} I^{a_1 \cdots a_m} = \left[\frac{m(m-1)(2m+1)!!}{2(m+1)(m+2)} \frac{R}{4\pi} \int h_{\rm TT}^{a_1 a_2} n^{a_3} \cdots n^{a_m} d\Omega \right]^{\rm STF} . \tag{A55a}$$

Similarly for the radiation current multipoles,

$$\frac{d^m}{dt^m} J^{a_l \cdots a_m} = \left[\frac{(m-1)(2m+1)!!}{4(m+2)} \frac{R}{4\pi} \int \varepsilon^{a_1 jk} n^j h_{\rm TT}^{k a_2} n^{a_3} \cdots n^{a_m} d\Omega \right]^{\rm STF} . \tag{A55b}$$

Performing these projections gives Eqs. (3.6) and Eqs. (3.7). Notice that the time derivatives on the left-hand side of the projection equations, Eqs. (A55), are not present in the results Eqs. (3.6) and Eqs. (3.7). Some of the time derivatives "canceled" with time derivatives inherent in the expression of the wave form [see Eqs. (A12) and (A13)]. The remaining time derivatives can be eliminated by simple time integrations. Although Blanchet and Damour [12] showed that the Epstein-Wagoner formalism is equivalent (in a nonrigorous sense), the fact that the wave form can be derived by the two different techniques gives us theoretical and clerical confidence in Eqs. (3.8).

APPENDIX B: EVALUATION OF FIELD INTEGRALS

In this appendix we present some of the mathematical details involved in doing the field integrals in Appendix A.

Variations of the following simple trick are useful when we encounter integrals of the form

$$(A^{ij})_{\rm TT} = \int_A f(r_A) x^i x^j d^3 x^{\rm TT} .$$

Here, the region of the integration is the spherically symmetric body A, and r_A is the radial distance from the center of body A at $\mathbf{x}_A$ to the field point within body A. Using $x^i = x_A^i + r_A^i$, we have

$$(A^{ij})_{\rm TT} = \left\{ x_A^i x_A^j \int_A f(r_A) d^3 r_A + x_A^i \int_A f(r_A) r_A^j d^3 r_A + x_A^j \int_A f(r_A) r_A^i d^3 r_A + \int_A f(r_A) r_A^i r_A^j d^3 r_A \right\}_{\rm TT} .$$

The second and third integrals vanish by spherical symmetry. Also by spherical symmetry, the fourth integral is proportional to δ_{ij}, and thus has vanishing transverse-traceless part [see Eq. (2.4)]. We are left with

$$(A^{ij})_{\rm TT} = (x_A^i x_A^j)_{\rm TT} \int_A f(r_A) d^3 r_A . \tag{B1}$$

In essence the spherical symmetry of A and the transverse-traceless operator allow us to slip $x^i x^j$ out of the integral. This trick is easily generalized to more indices. It also works when the region of integration is outside the spherically symmetric body A.

Also in Appendix A we encountered integrals of the form

$$Q_{AB}^{abcd} = \int U_{A,ab} U_B x^c x^d d^3 x , \tag{B2}$$

where

$$U_A = \frac{m_A}{r_A} , \quad U_{A,ab} = \frac{m_A}{r_A^3} (3 n_A^a n_A^b - \delta^{ab}) , \tag{B3}$$

and

$$U_B = \frac{m_B}{r_B} = \frac{m_B}{|\mathbf{r}_A - \mathbf{r}_{BA}|} . \tag{B4}$$

The region of integration is the entire exterior of the material source. Recall that $\mathbf{x}$ is the position of the field point relative to the origin, $\mathbf{x}_A$ is the position of body A relative to the origin, $\mathbf{r}_A$ is the position of the field point relative body A, and $\mathbf{r}_{BA}$ is the vector from body A to body B; $\mathbf{n}_A$ is formed in the usual way, $\mathbf{n}_A \equiv \mathbf{r}_A / |\mathbf{r}_A|$, likewise for $\mathbf{n}_{BA}$ and $\mathbf{n}$. Equation (B2) can be expanded using $\mathbf{x} = \mathbf{x}_A + \mathbf{r}_A$. That is

$$Q_{AB}^{abcd} = \int U_{A,ab} U_B r_A^c r_A^d d^3 r_A + x_A^d \int U_{A,ab} U_B r_A^c d^3 r_A + x_A^c \int U_{A,ab} U_B r_A^d d^3 r_A + x_A^c x_A^d \int U_{A,ab} U_B d^3 r_A . \tag{B5}$$

Because the individual pieces of Eq. (B5) are useful by themselves, we give each its own identification by

$$Q_{AB}^{abcd} = F_{AB}^{abcd} + x_A^d F_{AB}^{abc} + x_A^c F_{AB}^{abd} + x_A^c x_A^d F_{AB}^{ab} . \tag{B6}$$

The identification of each F in Eq. (B6) with the corresponding integral in Eq. (B5) is clear.

There are several techniques for performing the integrations in Eq. (B5). Using a spherical-harmonic expansion of the potentials is one way. A simpler technique is to write down the general form of the result and solve for the coefficients. For example, the integral F_{AB}^{ab} has only two indices and is symmetric; therefore, it must take the form

$$\alpha n^a_{AB} n^b_{AB} + \beta\delta^{ab} . \quad \text{(B7)}$$

Choosing a particular coordinate system (allowing $\mathbf{n}_{BA}$ to determine the z axis is the simplest choice), and explicitly carrying out the integral for two distinct choices of the indices (a,b) gives the coefficients α and β. For the other F's, which have more indices, the ansatz corresponding to Eq. (B7) is a bit more complicated, but the technique is the same. After considerable manipulation we find

$$F^{abcd}_{AB} = \frac{\pi m_A m_B}{3} r \left\{ 2\frac{r^a_{AB} r^b_{AB} r^c_{AB} r^d_{AB}}{r^4} - 2\delta^{ab}\frac{r^c_{AB} r^d_{AB}}{r^2} + \delta^{cd}\frac{r^a_{AB} r^b_{AB}}{r^2} + \delta^{ac}\frac{r^b_{AB} r^d_{AB}}{r^2} + \delta^{ad}\frac{r^b_{AB} r^c_{AB}}{r^2} + \delta^{bc}\frac{r^a_{AB} r^d_{AB}}{r^2} + \delta^{bd}\frac{r^a_{AB} r^c_{AB}}{r^2} \right\} + \frac{\pi m_A m_B}{3}\tfrac{1}{5}(1-8R^{(\mathrm{I})}_0)\delta^{ab}\delta^{cd} + \frac{\pi m_A m_B}{3}\tfrac{4}{5}(1-3R^{(\mathrm{I})}_0)(\delta^{ac}\delta^{bd}+\delta^{ad}\delta^{bc}) , \quad \text{(B8a)}$$

$$F^{abc}_{AB} = \frac{\pi m_A m_B}{3} \left\{ -3\frac{r^a_{AB} r^b_{AB} r^c_{AB}}{r^3} + 3\delta^{ab}\frac{r^c_{AB}}{r} - 3\delta^{ac}\frac{r^b_{AB}}{r} - 3\delta^{bc}\frac{r^a_{AB}}{r} \right\} , \quad \text{(B8b)}$$

$$F^{ab}_{AB} = \frac{\pi m_A m_B}{3}\frac{1}{r} \left\{ 6\frac{r^a_{AB} r^b_{AB}}{r^2} - 2\delta^{ab} \right\} . \quad \text{(B8c)}$$

In Eqs. (B8) r denotes $r_{AB} = |\mathbf{r}_{AB}|$. Fortunately, when using these results most of the terms involving Kronecker deltas have vanishing transverse-traceless part [see Eq. (2.4)]. The quantity $R^{(\mathrm{I})}_0$ that appears in Eq. (B8a) represents a divergent integral

$$\frac{1}{r}\int_0^\infty \frac{r_A}{r_>} dr_A , \quad r_> = \max\{r_A, r_{AB}\} . \quad \text{(B9)}$$

This quantity has no physical relevance because it always appears in terms that have vanishing transverse-traceless part. The results Eqs. (B8a)–(B8c) can easily be reassembled using Eq. (B5) to find Q^{abcd}_{AB}.

In computing the moment I^{ijklm} a five-index integral of the form treated above appears:

$$F^{abcde}_{AB} = \int U_{A,ab} U_B r^c_a r^d_A r^e_A d^3 r_A . \quad \text{(B10)}$$

The terms in the ansatz solution will have only odd numbers of unit vector components in them, i.e.,

$$F^{abcde}_{AB} = \alpha n^a_{AB} n^b_{AB} n^c_{AB} n^d_{AB} n^e_{AB} + \beta n^a_{AB} n^b_{AB} n^c_{AB} \delta^{de} + \cdots . \quad \text{(B11)}$$

The fact that $\mathbf{n}_{AB} = -\mathbf{n}_{BA}$ immediately gives the only property of F^{abcde}_{AB} that is needed in the calculation. That is

$$\sum_A \sum_{B \neq A} F^{abcde}_{AB} = 0 \quad \text{for all } abcde . \quad \text{(B12)}$$

The final integral we examine appears in the debris of the integration by parts in Appendix A in Eq. (A51). The integration can be done by the technique outlined above. The result is

$$Q^{ab}_{AB} \equiv \int U_{A,a} U_B r^b_A d^3 r_A = \frac{\pi m_A m_B}{r_{AB}}(2r^a_{AB} x^b_A - r^a_{AB} x^b_A) + \frac{\pi m_A m_B}{3}(1-4R^{(\mathrm{I})}_0)\delta^{ab} . \quad \text{(B13)}$$

Here again the region of integration is the exterior of the material source. The quantity $R^{(\mathrm{I})}_0$ represents the same infinite quantity as above, and once again it only appears in quantities that have vanishing transverse-traceless part.

There is one other property of Q^{ab}_{AB} which is useful. Using the fact that $\mathbf{r}_{AB} = \mathbf{x}_A - \mathbf{x}_B$ it is easy to show that the first term in Eq. (B13) sums to zero when summed over the body labels. That is

$$\sum_A \sum_{B \neq A} \frac{\pi m_A m_B}{r_{AB}}(2r^a_{AB} x^b_A - r^a_{AB} x^b_A) = 0 \quad \text{for all } ab . \quad \text{(B14)}$$

Using the field integral above we now briefly outline the evaluation of I^{ijkl}_{EW} [Eqs. (A13c) and (A25c)], and I^{ijklm}_{EW} [Eqs. (A13d) and (A25d)]:

$$I^{ijkl}_{\mathrm{EW}} \equiv \int \tau^{ij} x^k x^l d^2 x = \int \rho_0 v^i v^j x^k x^l d^3 x + \frac{1}{4\pi}\int U_{,i} U_{,j} x^k x^l d^3 x + [\mathrm{TT}=0] . \quad \text{(B15)}$$

The notation $[\mathrm{TT}=0]$ represents terms with vanishing transverse-traceless part that have been omitted. This term is already $O(\varepsilon^2)$; and therefore, post-Newtonian subtleties in the definition of the mass, which plagued the calculation of I^{ij}_{EW} and I^{ijk}_{EW}, are irrelevant. This makes the first integral in Eq. (B15) trivial. After integrating the second term by parts, the self-interaction terms are eliminated by arguments similar to those used in Appendix A. The debris from the integration by parts is easily shown to have vanishing transverse-traceless part. We are left with

$$I_{\rm EW}^{ijkl}=\sum_A m_A v_A^i v_A^j x_A^k x_A^l-\frac{1}{4\pi}\sum_A\sum_{A\neq B}\int U_{,ij}Ux^kx^ld^3x+[{\rm TT}=0]$$
$$=\sum_A m_A v_A^i v_A^j x_A^k x_A^l-\frac{1}{4\pi}\sum_A\sum_{A\neq B}\{F_{AB}^{ijkl}+x_A^lF_{AB}^{ijk}+x_A^kF_{AB}^{ijl}+x_A^kx_A^lF_{AB}^{ij}\}+[{\rm TT}=0]\ . \qquad ({\rm B16})$$

In the second step we have used Eqs. (B2), (B5), and (B6). Explicit expressions for the F's are given in Eqs. (B8c).

Finally we examine $I_{\rm EW}^{ijklm}$ [Eqs. (A13d) and (A25d)],

$$I_{\rm EW}^{ijklm}\equiv\frac{1}{3}\frac{d}{dt}\left\{\int\tau^{ij}x^kx^lx^md^3x\right\}$$
$$=\frac{1}{3}\frac{d}{dt}\left\{\int\rho_0v^iv^jx^kx^lx^md^3x-\frac{1}{4\pi}\int U_{,i}U_{,j}x^kx^lx^md^3x+[{\rm TT}=0]\right\}. \qquad ({\rm B17})$$

Using the same arguments as in the $I_{\rm EW}^{ijkl}$ case we have

$$I_{\rm EW}^{ijklm}=\frac{1}{3}\frac{d}{dt}\left\{\sum_A m_A v_A^i v_A^j x_A^k x_A^l x_A^m-\frac{1}{4\pi}\sum_a\sum_{B\neq A}\int U_{,ij}Ux^kx^lx^md^3x+[{\rm TT}=0]\right\}$$
$$=\frac{1}{3}\frac{d}{dt}\left\{\sum_A m_A v_A^i v_A^j x_A^k x_a^l x_A^m-\frac{1}{4\pi}\sum_A\sum_{B\neq A}\{F_{AB}^{ijklm}+x_A^mF_{AB}^{ijkl}+x_A^lF_{AB}^{ijkm}+x_A^kF_{AB}^{ijlm}+x_A^mx_A^lF_{AB}^{ijk}\right.$$
$$\left.+x_A^mx_A^kF_{AB}^{ijl}+x_A^lx_A^kF_{AB}^{ijm}+x_A^mx_A^lx_A^kF_{AB}^{ij}\}+[{\rm TT}=0]\right\}. \qquad ({\rm B18})$$

The term F^{ijklm} sums to zero by Eq. (B12). In the second step of Eq. (B18) we have used Eqs. (B2), (B5), and (B6). Explicit expressions for the F's are given in Eqs. (B8).

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[24] Our conventions are those of Refs. [23] and [25]. Our notation: greek indices $=0,1,2,3$; latin indices $=1,2,3$; repeated indices are understood to be summed over. $\eta^{\mu\nu}=\eta_{\mu\nu}={\rm diag}(-1,1,1,1)$; $g\equiv\det(g_{\mu\nu})$; $a^{(ij)}\equiv(a^{ij}+a^{ji})/2$; ε^{ijk} is the totally antisymmetric Levi-Civita symbol ($\varepsilon^{123}=+1$); capital letter superscripts denote multi-indices (i.e., $x^L\equiv x^{i_1}x^{i_2}\cdots x^{i_l}$); spatial indices are freely raised and lowered with δ^{ij} and δ_{ij}; the subscript "TT" denotes that the transverse-traceless projection is to be taken; the superscript "STF" denotes that the symmetric trace-free part is to be taken.

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