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Lorentzian wormholes in higher-derivative gravity and the weak energy condition

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The possibility of the existence of a traversible wormhole solution which does not break the weak energy condition is examined in a higher-derivative theory of gravity. No such solution is found, suggesting that a Lorentzian wormhole without the violation of the weak energy condition is incompatible with a wide class of gravitational theories. We show this in two simple examples of spherically symmetric wormhole solutions.

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I. INTRODUCTION

Recently special classical solutions, called wormholes, of the Euclidean Einstein equation have attracted a great deal of attention of many authors both in quantum gravity and in quantum cosmology. Several types of such solutions are found [1]. The most important effect of this wormhole is quantum tunneling among disconnected universes if there exist many universes other than our Universe. And this interaction among universes via wormholes brings a significant effect on our world, the determination of the fundamental parameters of the theory (the cosmological constant, etc.) [2].

On the other hand, Lorentzian metric wormhole solutions also attract many authors since they bring the possibility of constructing a time machine [3] by which we can travel to our past. However causality would be violated as a result. The configuration of this wormhole is a compact S^2 tube connecting two asymptotically flat spaces. Such a wormhole is called traversible. However, we cannot find such solutions in the Einstein equation satisfying the weak energy condition (WEC) because the wormhole throat is sustained by exotic matter whose energy density is negative. In order to expand the wormhole radius along the radial direction, the derivative of the S^2 radius with respect to the coordinate l must be positive near the throat. On the other hand, since this derivative is proportional to $-$(the energy-momentum tensor on radially directed null geodesic)$=-(T_{\hat{t}\hat{t}}+T_{\hat{l}\hat{l}})$, the wormhole collapses to a zero radius as long as the energy-momentum tensor of the matter field is positive. In order to find a wormhole solution which does not violate the WEC, we must introduce something new which could provide either an effective, negative-energy density or an equivalent effect in the Einstein equation. Such an example is the Casimir effect, the quantum effect of the nongravitational fields, which has been examined in Ref. [3]. But it requires too small of a distance between the two conducting S^2 plates and it is not realistic as a matter of fact.

There is another possible candidate which could provide a traversible wormhole solution. If the radius of the wormhole throat is very small, the curvature is very large near the throat. Then the terms of higher powers of curvature, if they exist in the field equation, would be essential near the throat. So it is worthwhile studying whether or not a theory containing higher-curvature terms can provide a wormhole satisfying the WEC.

An interesting approach to the construction of a traversible wormhole has been performed in terms of the surgery and junction conditions [4]. In this approach, δ-function singularities appear in the Einstein equation. If we apply this approach to a gravitational theory containing terms such as the nth power R^n of the scalar curvature for some $n \geq 2$, higher powers of the δ function would appear in the field equation. So, it seems that this approach is not applicable to a theory containing R^n terms. However, Hochberg [5] has applied this method to a higher curvature theory by rewriting it to Einstein gravity with a normal scalar field in terms of the conformal transformation $\bar{g}_{\mu\nu}=\Omega^2(R)g_{\mu\nu}$. But this approach is not valid in the case where the conformal factor $\Omega^2(R)$ is negative or its inverse transformation is nonanalytic. As seen later, the case of $\Omega^2(R)$ negative is interesting in higher curvature gravity, because it is known that the tachyonic scalar mode of the metric fluctuation appears in the flat-space limit. This situation is similar to Einstein gravity with exotic matter. Then we can expect an appearance of a term which corresponds to the negative-energy density. It is therefore interesting to study this case if the existence of this mode is justified in the theory. So we study higher curvature theories directly without performing any transformation from the original one to another.

The purpose of this paper is to find a traversible wormhole solution, which is compatible with the WEC, in a higher curvature theory. In Sec. II we briefly review the incompatibility of a Lorentzian wormhole solution with the WEC in the case of the usual Einstein gravity. In Sec. III the possibility of the existence of a wormhole solution which does not break the WEC is examined in a higher curvature theory of gravity. The conclusion and discussions are given in the final section.

II. EINSTEIN GRAVITY AND WEAK ENERGY CONDITION

Here we briefly review that spherically symmetric, Lorentzian wormhole solutions in Einstein gravity are not compatible with the WEC.

The spherically symmetric wormhole $\mathcal{W}$ is a spacetime represented by the coordinate (t,l,θ,ϕ) with $-\infty<t<\infty$, $-\infty<l<\infty$, $0\leq\theta<\pi$, and $0\leq\phi\leq 2\pi$, and whose metric is

$$ds^2=g_{tt}dt^2+2g_{tl}dt\,dl+dl^2+r^2(d\theta^2+\sin^2\theta\,d\phi^2)\ , \tag{2.1}$$

where g_{tt},g_{tl},r are function on t,l with $g_{tt}<0$ and $r>0$. We also assume that $\mathcal{W}$ is (i) geodesically complete and (ii) asymptotically flat; that is, (i) every geodesic can be extended to arbitrary values of its affine parameter and (ii) any component of the Riemannian tensors with respect to an orthonormal basis for the tangent space over $\mathcal{W}$ converges to zero as $l\to\pm\infty$. In particular, the asymptotic flatness implies that, for any t, $r(t,l)\to\infty$ as $l\to\pm\infty$.

Let p_1: $\mathcal{W}\to\mathbf{R}$ and p_2: $\mathcal{W}\to\mathbf{R}$ be the projections to the t factor and the l factor, respectively. For a fixed t_0, the t_0-sliced hypersurface or the t_0 slice in $\mathcal{W}$ is written as $S=p_1^{-1}(t_0)$. For each l, $F_l=S\cap p_2^{-1}(l)$ is a spacelike two-sphere. Since $r(t_0,l)\to\infty$ as $l\to\pm\infty$, the function $r(t_0,l)$ on l has the minimal value. We may assume that $b_0=r(t_0,0)\leq r(t_0,l)$ for all l. Then F_0 is called the *wormlike throat* in S of radius b_0. (See Fig. 1.) We say that the energy-momentum tensor $T_{\mu\nu}$ on $\mathcal{W}$ satisfies the WEC if $T_{\mu\nu}V^\mu V^\nu\geq 0$ for any non-spacelike vector V^μ. Morris, Thorne, and Yurtsever [3] showed that $T_{\mu\nu}$ on any spherically symmetric, asymptotically flat wormhole satisfying the Einstein equation,

$$8\pi T_{\mu\nu}=R_{\mu\nu}-\tfrac{1}{2}Rg_{\mu\nu} \tag{2.2}$$

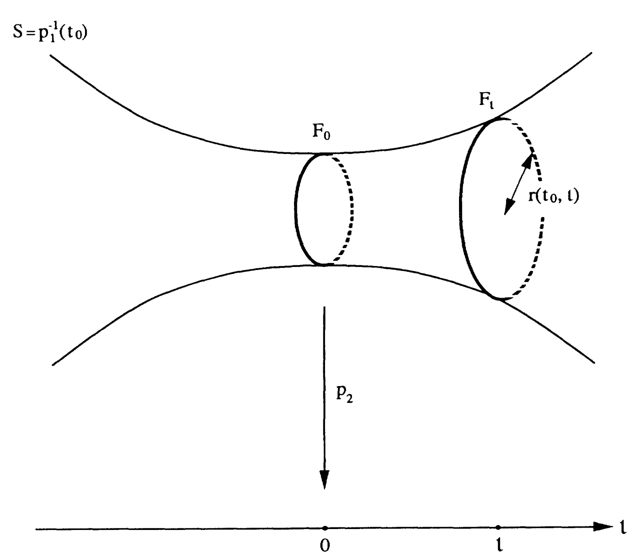

FIG. 1. The projection p_2 from the t_0 slice $S=p_1^{-1}(t_0)$ of $\mathcal{W}$ to the l factor.

violates the WEC. If $T_{\mu\nu}$ satisfies the Einstein equation (2.2), then for any null vector K^μ, $8\pi T_{\mu\nu}K^\mu K^\nu=R_{\mu\nu}K^\mu K^\nu$. Therefore, if the WEC is satisfied, we have the inequality

$$R_{\mu\nu}K^\mu K^\nu\geq 0\ . \tag{2.3}$$

The following proposition (the wormhole version of the Hawking-Penrose theorem [6]) implies a statement given in Ref. [3].

Proposition 1. With the notation as above, the inequality (2.3) does not hold somewhere in $\mathcal{W}$. In particular, the energy-momentum tensor $T_{\mu\nu}$ violates the WEC.

Here we shall give a sketch of the proof, which will be helpful for the reader to understand why the WEC is not compatible with the Einstein equation on $\mathcal{W}$.

Sketch of Proof. Contrarily, suppose that (2.3) is satisfied everywhere in $\mathcal{W}$. For a fixed t_0, let S be the t_0-sliced hypersurface. We set $r(t_0,l)=r_0(l)$. Since $r_0(l)\to\infty$ as $l\to-\infty$, for some $s<0$,

$$\left.\frac{dr_0}{dl}\right|_{l=s}<0\ . \tag{2.4}$$

At any point p of $F_s=S\cap p_2^{-1}(s)$, let $\gamma(v)$ be the future-directed null geodesic in $\mathcal{W}$ with an affine parameter such that (i) $\gamma(0)=p$, (ii) $\mathbf{N}\cdot\partial/\partial\theta=\mathbf{N}\cdot\partial/\partial\phi=0$, and (iii) $\mathbf{N}\cdot\partial/\partial l>0$, where $\mathbf{N}=\mathbf{N}(v)$ are the vectors tangent to $\gamma(v)$ for all $v\geq 0$. We may assume that $g_{tl}=0$ along $\gamma(v)$, that is, the t coordinate of $\mathcal{W}$ is chosen so that $\partial/\partial t$ is orthogonal to t slices at any point of $\gamma(v)$. Then $\mathbf{N}$ is represented as

$$\mathbf{N}=\beta\left[\frac{1}{\alpha}\frac{\partial}{\partial t}+\frac{\partial}{\partial l}\right]\ , \tag{2.5}$$

where $\alpha=[-g_{tt}(\gamma(v))]^{1/2}$ and $\beta=\beta(v)$ is a positive function. The volume expansion $\hat{\theta}=\chi_{\mu\nu}g^{\mu\nu}$ on $\gamma(v)$ is defined by

$$\hat{\theta}\equiv\frac{1}{r^2}\nabla_\theta N_\theta+\frac{1}{r^2\sin^2\theta}\nabla_\phi N_\phi\ ,$$

where $\chi_{\mu\nu}$ is the null second fundamental tensor with respect to $\mathbf{N}$, see Ref. [7], Sec. 4.4. By the definition of (2.5)

$$\hat{\theta}=\frac{2\beta}{r}\left[\frac{\dot{r}}{\alpha}+r'\right]\ , \tag{2.6}$$

where $\dot{r}=dr/dt$ and $r'=dr/dl$. We may assume that $\dot{r}\leq 0$ at $\mathcal{F}_s$. Then, by (2.4), $\hat{\theta}<0$ at F_s, that is, F_s is an outer trapped surface. (In the case where $\dot{r}>0$ at F_s, we can complete the proof by the parallel argument except reversing the time direction.) By the Raychaudhuri equation (Ref. [7], proposition 4.4.4) and (2.3), the function $\hat{\theta}(v)=\hat{\theta}(\gamma(v))$ satisfies

$$\frac{d}{dv}\hat{\theta}=-R_{\mu\nu}N^\mu N^\nu-2\hat{\sigma}^2-\frac{1}{2}\hat{\theta}^2\leq-\frac{1}{2}\hat{\theta}^2\ , \tag{2.7}$$

where $\hat{\sigma}$ is the function defined in Ref. [7], Sec. 4.2. Since $c=\hat{\theta}(0)<0$, by (2.7), the value of $\hat{\theta}(v)$ takes $-\infty$ at some $v=v_0$ in $0<v_0\leq-2/c$. By (2.6) $r(\gamma(v_0))$ is zero. This

contradicts that $r(t,l)>0$ everywhere in $\mathcal{W}$. Therefore the inequality (2.3) does not hold somewhere in $\mathcal{W}$.

In this proof, the inequality (2.3) is used essentially. But, in a higher derivative gravity, the inequality derived from the WEC is not as crucial as (2.3). For example, if the gravitational Lagrangian is given by $L_G=\sqrt{-g}\,(R-\omega R^2)$, then by the field equation (3.11) in Sec. III, we have, for any null vector K^μ,

$$8\pi T_{\mu\nu}K^\mu K^\nu=[(1-2\omega R)R_{\mu\nu}+2\omega\nabla_\mu\nabla_\nu R]K^\mu K^\nu .$$

Therefore the WEC implies only

$$R_{\mu\nu}K^\mu K^\nu \geq 2\omega(RR_{\mu\nu}-\nabla_\mu\nabla_\nu R)K^\mu K^\nu . \tag{2.8}$$

If the inequality (2.8) holds and if the right-hand side (RHS) of (2.8) is non-negative, then the inequality (2.3) holds. This contradicts proposition 1. Therefore, if the RHS of (2.8) is positive, the inequality (2.8) does not hold. Then the wormhole solution $\mathcal{W}$ violates WEC. But, in general, the RHS of (2.8) is negative somewhere. Such an example will be given in Sec. III; see Fig. 4. Thus, in the higher derivative case, we need other arguments to study the compatibility of the field equation with the WEC.

It should be noticed here that the violation of the WEC on $\mathcal{W}$ can be proved by (2.7) even in the case when the RHS of (2.8) is negative if its absolute value is small enough. If we assume $-$[the RHS of (2.8)] $\leq \frac{1}{2}\epsilon^2$ of some positive ϵ with $\epsilon \leq |\hat{\theta}(0)|$, then we can show that the value of $\hat{\theta}(v)$ takes $-\infty$ at some finite v. This means that if $|\omega|$ is sufficiently small, we cannot have any wormhole solutions, and the higher curvature terms do not play any important role in the field equation.

III. HIGHER DERIVATIVE GRAVITY AND WEC

In order to study the case including higher derivative terms, we consider a concrete form of wormholes. Let $\mathcal{W}$ be a static, spherically symmetric wormhole whose throat radius is $b_0(>0)$ and having the metric (Ref. [8], Sec. III.A)

$$ds^2=-e^{2\Phi}dt^2+\frac{dr^2}{1-b/r}+r^2(d\theta^2+\sin^2\theta\, d\phi^2) , \tag{3.1}$$

where $\Phi=\Phi(r)$ and $b=b(r)$ are smooth functions defined in $b_0\leq r<\infty$ such that $0<b/r\leq 1$ and the equality $b/r=1$ holds if and only if $r=b_0$. We require that the wormhole is asymptotically flat, that is, both Φ and b/r converge to 0 as $r\to\infty$ (Ref. [8], Sec. III.B4). Note that the metric (3.1) represents half $\mathcal{W}_+$ of $\mathcal{W}$, and $\mathcal{W}$ is the union of $\mathcal{W}_+$ and its mirror image $\mathcal{W}_-$ along the throat at $r=b_0$. According to Ref. [8] in Sec. III.C3, if we use the l coordinate defined by

$$l(r)=\pm\int_{b_0}^{r}\frac{dr}{\sqrt{1-b/r}}$$

instead of the r coordinate, then the metric on $\mathcal{W}$ is represented by the form as in (2.1). It is easily checked that for all positive integer n, there exist both the limits

$$\lim_{l\to+0}\frac{d^n r}{dl^n}$$

and

$$\lim_{l\to-0}\frac{d^n r}{dl^n} ,$$

and they are equal to each other. Therefore $r=r(l)$ is a smooth function in $-\infty<l<\infty$, and hence, in particular, $\mathcal{W}$ is a smooth wormhole. Any singularity such as a δ function does not appear in the energy-momentum tensor near the wormhole throat.

Since the wormhole $\mathcal{W}$ is static, it might be possible to make a time machine. Since the wormhole solution breaks the WEC as shown above, such a time machine cannot be stabilized without exotic matter in the Einstein equation (2.2). So we extend here the theory by adding higher curvature terms to the Einstein action, and study in the following whether the WEC can be satisfied.

Throughout this section, we shall use the same notation as in Ref. [8], and the Lagrangian is given as

$$L=\sqrt{-g}\,f(R)+L_M ,$$

where $f(R)$ is a function of the scalar curvature R and L_M represents the matter Lagrangian. In order to avoid the problem of unitarity, we do not consider the higher derivative term $R_{\mu\nu}R^{\mu\nu}$, which induces the massive ghost. From the above Lagrangian, the following field equation is obtained:

$$8\pi T_{\mu\nu}=f'(R)R_{\mu\nu}-\tfrac{1}{2}f(r)g_{\mu\nu}-f''(R)(\nabla_\mu\nabla_\nu R-\Box R g_{\mu\nu}) -f'''(R)(\nabla_\mu R\nabla_\nu R-\nabla^\alpha R\nabla_\alpha R g_{\mu\nu}) , \tag{3.2}$$

where $\Box\equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ and $f'(R)=df/dR$, etc.

First we show that, if $f'(R)>0$ and $f'(0)\neq 0$, the wormhole solution is not compatible with the WEC. Then, in fact, the theory given by the above Lagrangian is equivalent to Einstein gravity with an additional scalar field.

Proposition 2. With the notation as above, if $f'(R)>0$ everywhere in $\mathcal{W}$ and $f'(0)\neq 0$, then the energy-momentum tensor $T_{\mu\nu}$ satisfying (3.2) violates the WEC.

Proof. Since $f'(R)>0$, we can rewrite Eq. (3.2) in terms of the conformal transformation [5],

$$\bar{g}_{\mu\nu}=\Omega^2(R)g_{\mu\nu} , \tag{3.3}$$

$$\Omega^2(R)=f'(R)>0 , \tag{3.4}$$

as

$$\bar{G}_{\mu\nu}=\tfrac{1}{2}[\nabla_\mu\phi\nabla_\nu\phi-\tfrac{1}{2}\bar{g}_{\mu\nu}\nabla_\lambda\phi\nabla^\lambda\phi-\bar{g}_{\mu\nu}V(\phi)]+8\pi\bar{T}_{\mu\nu} , \tag{3.5}$$

$$\bar{T}_{\mu\nu}=\Omega^{-2}(R)T_{\mu\nu} , \tag{3.6}$$

$$\phi=\sqrt{3}\ln f'(R) , \tag{3.7}$$

where $\bar{G}_{\mu\nu}=\bar{R}_{\mu\nu}-\frac{1}{2}\bar{g}_{\mu\nu}\bar{R}$. Since $\mathcal{W}$ is asymptotically flat, $\bar{g}_{\mu\nu}$ is well approximated by $f'(0)g_{\mu\nu}$ in a region far from the throat. So the wormhole $\bar{\mathcal{W}}$ with underlying space $\mathcal{W}$ and with metric $\bar{g}_{\mu\nu}$ is also asymptotically flat. We set $\tilde{t}_{\mu\nu}\equiv\bar{G}_{\mu\nu}-8\pi\bar{T}_{\mu\nu}$. Then, by (3.5), for any null vector K^μ,

$$\tilde{t}_{\mu\nu}K^\mu K^\nu=\tfrac{1}{2}(\nabla_\mu\phi K^\mu)^2\geq 0 .$$

If the WEC on $\mathcal{W}$ were satisfied, then, by (3.6)

$$\overline{T}_{\mu\nu}K^{\mu}K^{\nu}=\Omega^{-2}(R)T_{\mu\nu}K^{\mu}K^{\nu}\geq 0 \ .$$

In particular, by (3.5), we would have

$$\overline{R}_{\mu\nu}K^{\mu}K^{\nu}=\overline{G}_{\mu\nu}K^{\mu}K^{\nu} =(\tilde{t}_{\mu\nu}+8\pi\overline{T}_{\mu\nu})K^{\mu}K^{\nu}\geq 0 \ . \quad (3.8)$$

Proposition 1 in Sec. II implies that the wormhole solution $\overline{\mathcal{W}}$ does not satisfy the inequality (3.8) corresponding to (2.3) for $\mathcal{W}$. This gives a contradiction. Thus $T_{\mu\nu}$ on $\mathcal{W}$ must violate the WEC.

Here we suppose that

$$f(R)=R-\omega R^2 \ , \quad (3.9)$$

where the gravitational constant, the coefficient of R in Eq. (3.9), is taken to be the unity; hence, $f'(0)=1$, and ω is an arbitrary constant. The dimensional quantity would be measured by b_0 hereafter. Then the theory is given by the Lagrangian

$$L=\sqrt{-g}\,(R-\omega R^2)+L_M \ . \quad (3.10)$$

Then the field equation (3.2) is represented as

$$8\pi T_{\mu\nu}=R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R+2\omega[\nabla_\mu\nabla_\nu R-RR_{\mu\nu} +(R^2/4-\Box R)g_{\mu\nu}] \ . \quad (3.11)$$

Since $\mathcal{W}$ is asymptotically flat, $|R|$ is very small in a region far from the throat. Therefore $f'(R)=1-2\omega R$ is positive in this region. On the other hand, R is positive and very large near the throat since the throat radius is expected to be very small. Therefore, if $\omega\leq 0$, $1-2\omega R$ is positive also near the throat. So, by proposition 2, in the case of $\omega\leq 0$, the energy-momentum tensor (3.2) violates the WEC. This means that, if the wormhole solution (3.11) is compatible with the WEC, then ω must be positive.

Throughout the remainder of this section, we assume that

$$\omega>0 \ .$$

In this case, $f'(R)=1-2\omega R$ could be negative near the throat, so we cannot apply the above argument. Furthermore, it is well known that the theory (3.10) with $\omega>0$ contains the tachyonic scalar mode in the fluctuations of the metric around the flat space, so the asymptotically flat background metric on $\mathcal{W}$ is unstable against this fluctuation. To avoid that this metric is destabilized by such a fluctuation, we must modify the theory so that the field equation is effectively equivalent to that induced from $\sqrt{-g}\,(R-\omega' R^2)$ for some $\omega'>0$. Such a modified theory can be obtained by adding the loop corrections ($L_{\rm loop}$) of conformally invariant matter fields to the original Lagrangian (3.10) as follows [9]:

$$L_{\rm eff}=\sqrt{-g}\,(R-\omega R^2)+L_M+L_{\rm loop} \ .$$

From this effective Lagrangian $L_{\rm eff}$ we obtain the field equation which has the same form with (3.11) except that ω is replaced by

$$\omega'=\omega+\sum_i N_i\gamma^i/96\pi^2 \ ,$$

where N_i (γ^i) represents the number (the calculable constant) of the ith loop-corrected matter field. Since $L_{\rm loop}$ comes from the matter which is coupled in the conformally invariant way to the gravity, it does not produce any undesirable scalar-fluctuation mode of the metric field. So we consider hereafter the theory given by $L_{\rm eff}$ and drop the prime of the modified ω' for simplicity. The value of the modified ω depends on the spin γ^i and the number N_i of the matter fields in $L_{\rm loop}$. Except for the case of conformally flat metrics, the loop corrections bring in general new terms to the equation of motion. We add these terms to the original energy-momentum tensor. From now on, we regard this added one as the energy-momentum tensor and consider the case of modified $\omega>0$.

Since the conformally transformed equation cannot be used for positive ω, we must use Eq. (3.11) directly to study the compatibility of the wormhole solution (3.1) with the WEC.

The dependence on b_0, which is the typical dimensionful parameter in our formulation in Eq. (3.11), is seen as follows by replacing r by rb_0. After the replacement, r has no dimension and represents the radius with the scale of the unit b_0. Since the Einstein term is proportional to b_0^{-2} and the R^2 term is proportional to b_0^{-4}, the R^2 term could exceed the Einstein term if $\omega/b_0^2>1$. This can be realized for sufficiently small b_0 even if ω is very small. For simplicity, we take $b_0=1$ hereafter. Then the magnitude of b_0 can be estimated by the value of ω. So a large value of ω means a small value of b_0 and vice versa.

Let $\mathbf{e}_{\hat{t}},\mathbf{e}_{\hat{r}},\mathbf{e}_{\hat{\theta}},\mathbf{e}_{\hat{\phi}}$ be the orthonormal basis for the tangent space over $\mathcal{W}$ defined as follows:

$$\mathbf{e}_{\hat{t}}=e^{-\phi}\frac{\partial}{\partial t} \ , \quad \mathbf{e}_{\hat{r}}=\sqrt{1-a}\,\frac{\partial}{\partial r} \ ,$$

$$\mathbf{e}_{\hat{\theta}}=r^{-1}\frac{\partial}{\partial\theta} \ , \quad \mathbf{e}_{\hat{\phi}}=(r\sin\theta)^{-1}\frac{\partial}{\partial\phi} \ ,$$

where we set $a\equiv b/r$.

It can be shown that the energy-momentum tensor $T_{\mu\nu}$ is diagonal with respect to this basis. So it is a tensor of type I (see Ref. [7], Sec. 4.3). Therefore the energy-momentum tensor satisfies the WEC if and only if $T_{\hat{t}\hat{t}}\geq 0, T_{\hat{t}\hat{t}}+T_{\hat{r}\hat{r}}\geq 0$, and $T_{\hat{t}\hat{t}}+T_{\hat{\theta}\hat{\theta}}=T_{\hat{t}\hat{t}}+T_{\hat{\phi}\hat{\phi}}\geq 0$. We denote by $E_{\hat{t}}$, $E_{\hat{r}}$, $E_{\hat{\theta}}$ the Einstein terms in $8\pi T_{\hat{t}\hat{t}}$, $8\pi(T_{\hat{t}\hat{t}}+T_{\hat{r}\hat{r}})$, $8\pi(T_{\hat{t}\hat{t}}+T_{\hat{\theta}\hat{\theta}})$, respectively.

First we consider the case where Φ is a constant function. Then the nonzero components of the Ricci tensor with respect to the basis $\mathbf{e}_{\hat{t}},\mathbf{e}_{\hat{r}},\mathbf{e}_{\hat{\theta}},\mathbf{e}_{\hat{\phi}}$ are [8]

$$R_{\hat{r}\hat{r}}=\frac{a'}{r} \ , \quad R_{\hat{\theta}\hat{\theta}}=R_{\hat{\phi}\hat{\phi}}=\frac{a'}{2r}+\frac{a}{r^2} \ ,$$

where $a'=da/dr$. The scalar curvature and its first and second derivatives on r are

$$R=\frac{2a'}{r}+\frac{2a}{r^2} \ , \quad R'=\frac{2a''}{r}-\frac{4a}{r^3} \ ,$$

$$R''=\frac{2a'''}{r}-\frac{2a''}{r^2}-\frac{4a'}{r^3}+\frac{12a}{r^4} \ .$$

Other quantities appearing in Eq. (3.11) can be written in the normalized form as

$$\nabla_{\hat r}\nabla_{\hat r}R=\frac{2(1-a)}{r}\eta(r)-\frac{a'}{r}\left[a''-\frac{2a}{r^2}\right],$$

$$\nabla_{\hat\theta}\nabla_{\hat\theta}R=\nabla_{\hat\phi}\nabla_{\hat\phi}R=\frac{(1-a)}{r^2}\left[2a''-\frac{4a}{r^2}\right],$$

where

$$\eta(r)=a'''-\frac{a''}{r}-\frac{2a'}{r^2}+\frac{6a}{r^3}\ . \tag{3.12}$$

From (3.11) we obtain

$$8\pi T_{\hat t\hat t}=\tfrac{1}{2}f(R)-f''(R)(\Box R)$$
$$=E_{\hat t}+2\omega v_{\hat t}\ , \tag{3.13}$$

$$8\pi(T_{\hat t\hat t}+T_{\hat r\hat r})=f'(R)(R_{\hat t\hat t}+R_{\hat r\hat r})-f''(R)\nabla_{\hat r}\nabla_{\hat r}R$$
$$=E_{\hat r}+2\omega v_{\hat r}\ , \tag{3.14}$$

$$8\pi(T_{\hat t\hat t}+T_{\hat\theta\hat\theta})=8\pi(T_{\hat t\hat t}+T_{\hat\phi\hat\phi})$$
$$=f'(r)R_{\hat\theta\hat\theta}-f''(R)\nabla_{\hat\theta}\nabla_{\hat\theta}R$$
$$=E_{\hat\theta}+2\omega v_{\hat\theta}\ , \tag{3.15}$$

where

$$E_{\hat t}=\frac{1}{r}\left[a'+\frac{a}{r}\right], \tag{3.16}$$

$$E_{\hat r}=\frac{a'}{r}\ , \tag{3.17}$$

$$E_{\hat\theta}=\frac{a'}{2r}+\frac{a}{r^2}\ , \tag{3.18}$$

$$v_{\hat t}=-\left[\frac{a'}{r}+\frac{a}{r^2}\right]^2+\frac{2(1-a)}{r}\eta(r)-\frac{a'}{r}\left[a''-\frac{2a}{r^2}\right], \tag{3.19}$$

$$v_{\hat r}=\frac{2(1-a)}{r}\eta(r)-\frac{a'}{r}\left[a''+\frac{2a'}{r}\right], \tag{3.20}$$

$$v_{\hat\theta}=-\left[\frac{2a'}{r}+\frac{2a}{r^2}\right]\left[\frac{a'}{2r}+\frac{a}{r^2}\right]+\frac{(1-a)}{r}\left[\frac{a''}{r}-\frac{4a}{r^3}\right]. \tag{3.21}$$

Now we return to the general case where Φ is not always assumed to be constant. Then the Einstein terms are given as [8]

$$E_{\hat t}=\frac{1}{r}\left[a'+\frac{a}{r}\right], \tag{3.22}$$

$$E_{\hat r}=\frac{a'}{r}+S_{\hat r}\ , \tag{3.23}$$

$$E_{\hat\theta}=\frac{1}{r}\left[\frac{a'}{2}+\frac{a}{r}\right]+S_{\hat\theta}\ . \tag{3.24}$$

where $S_{\hat r}$ and $S_{\hat\theta}$ are the Φ-dependent terms given as

$$S_{\hat r}(r)=2(1-a)\frac{\Phi'}{r}\ , \tag{3.25}$$

$$S_{\hat\theta}(r)=(1-a)\left[\Phi''-\frac{\Phi'}{2(1-a)}\left[a'+\frac{2a}{r}-\frac{2}{r}\right]+\Phi'^2\right]. \tag{3.26}$$

The situation of $\mathcal{W}$ near the throat seems to be quite different from that far from the throat. So we divide $\mathcal{W}$ into the two parts: the small-r region with r coordinate $\le r_1$ and the large-r region with r coordinate $\ge r_1$ for some $r_1>1$.

First we consider the large-r region. If Φ were constant, $S_{\hat r}$ would be zero. Then, since a' is negative somewhere in this region, $E_{\hat r}$ would also be negative there. As seen below, $E_{\hat r}$ is the main part of $8\pi(T_{\hat t\hat t}+T_{\hat r\hat r})$, so that $E_{\hat r}<0$ would imply $8\pi(T_{\hat t\hat t}+T_{\hat r\hat r})<0$. Therefore we need to suppose that $\Phi(r)$ is not constant in $r\ge r_1$. We choose $a(r)$ and $\Phi(r)$ such that, for any r with $r\ge r_1$, (i) $a\gg\max\{r|a'|,r|\Phi'|,r^2|\Phi''|\}$ and (ii) $\Phi'>|a'|/2(1-a)$. Then by (3.22) and (i) we have $E_{\hat t}>0$, and by (3.23) and (ii) $E_{\hat r}>0$. By (3.26) and (i) we have $\max\{|a'|/2r,|S_{\hat\theta}|\}\ll a/r^2$. So Eq. (3.24) implies $E_{\hat\theta}>0$.

For example,

$$a(r)=qr^{-1/n}\ \text{and}\ \Phi(r)=-Nr^{-1/n}\quad(r\ge r_1) \tag{3.27}$$

satisfy the above conditions (i) and (ii), where q and N are positive constants with $N\gg q$ and n is a natural number sufficiently larger than q,N. These forms (3.27) of $a(r)$ and $\Phi(r)$ determine the higher-order terms as follows by neglecting the terms of order $(1/n,r^{-4-1/n})$:

$$v_{\hat t}=\frac{1}{r^4}a(4-5a)\ , \tag{3.28}$$

$$v_{\hat r}=\frac{12}{r^4}a(1-a)\ , \tag{3.29}$$

$$v_{\hat\theta}=-\frac{2}{r^4}[1+2a(1-a)]\ . \tag{3.30}$$

Since $v_{\hat r}>0$, $E_{\hat r}+2\omega v_{\hat r}>0$. Although $v_{\hat t}$ is negative somewhere and $v_{\hat\theta}$ is negative everywhere, they are proportional to r^{-4}. On the other hand, both $E_{\hat t}$ and $E_{\hat\theta}$ are proportional to r^{-2}. Therefore we have $E_{\hat t}+2\omega v_{\hat t}>0$ and $E_{\hat\theta}+2\omega v_{\hat\theta}>0$ in $r\ge r_1$.

Next we consider the small-r region. Recall that in the pure Einstein theory, this region necessarily contains exotic matter. But, if the throat radius is very small, then the curvature is very large in this region. So it can be expected that the higher curvature term could contribute largely to the field equation. In contrast with the large-r case, we may assume that Φ is constant for simplicity, and concentrate our attention on the contribution of higher curvature terms in $1\le r\le r_1$. It is easily seen that the form (3.27) of $a(r)$ cannot satisfy the WEC near $r=1$. In fact, we get

$$T_{\hat{r}} \equiv 8\pi(T_{\hat{t}\hat{t}}+T_{\hat{r}\hat{r}}) = -\frac{q}{n}\left[1+2\omega\frac{q}{n}\right]+O(n^{-3})$$

at the wormhole throat $r=1$, and it shows the breaking of the WEC. So we need new forms of $a(r)$ other than (3.27) in $1 \le r \le r_1$, which could satisfy the WEC at least at $r=1$. In order to evade unnecessary complexities, we choose forms of $a(r)$ as simple as possible. Throughout two simple examples of $a(r)$ in the following, we show the incompatibility of the WEC with wormhole solutions satisfying (3.11).

Example 1. For $r_1=2$, we define the form of $a(r)$ as

$$a(r)=\begin{cases}(2-r)^m p+(2-r)tq+s & \text{if } 1\le r\le 2 ,\\ qr^{-1/n} & \text{if } r\ge 2 ,\end{cases} \qquad (3.31)$$

where m is a positive integer, and p,t,q,s are positive constants satisfying the following:

$$p+\frac{2^{-1-1/n}}{n}q+s=1 , \quad q=2^{1/n}s , \quad \text{and } t=\frac{2^{-1-1/n}}{n} .$$

These relations are obtained from $a(1)=1$ and the continuity conditions of $a(r)$ and $a'(r)$ at $r=2$, respectively. We comment on the discontinuity of $a''(r)$ at $r=2$ in the remark in the Appendix, where it is pointed out that this discontinuity produces no difficulty. We note that

$$0<p<1-s , \quad \lim_{n\to\infty} p=1-s , \quad \lim_{n\to\infty} q=s , \quad \text{and } \lim_{n\to\infty} t=0 .$$

We take n sufficiently large, say $n=5000$, so that p,q,t can be identified, respectively, with their limits, that is,

$$p=1-s , \quad q=s , \quad \text{and } t=0 . \qquad (3.32)$$

Then $a(r)$ is a function with the mutually independent parameters m and p.

It is possible to choose parameters (m,p) so that $T_{\hat{r}}$ is positive everywhere in the wormhole. Such an example will be given in the Appendix, where $m=4$ and $p=1/72$.

Although this example realizes the positivity of $T_{\hat{r}}$, it does not satisfy the WEC. This is seen as follows by imposing the WEC at $r=1$. By (3.13), $T_{\hat{t}\hat{t}} \ge 0$ at $r=1$ implies

$$(1-mp)[1-2\omega(1-mp)] \ge 0 .$$

If $1-mp \le 0$ held, we would have $1-2\omega(1-mp) \le 0$ and hence $1 \le 2\omega(1-mp) \le 0$, a contradiction. So we have

$$1-mp \ge 0 , \qquad (3.33)$$

$$1-2\omega(1-mp) \ge 0 . \qquad (3.34)$$

By (3.14) and (3.15), $T_{\hat{t}\hat{t}}+T_{\hat{r}\hat{r}} \ge 0$ and $T_{\hat{t}\hat{t}}+T_{\hat{\theta}\hat{\theta}} \ge 0$ at $r=1$ imply

$$2\omega mp(m-3) \ge 1 , \qquad (3.35)$$

$$1-2\omega(2-mp) \ge 0 . \qquad (3.36)$$

By (3.35), we have $m>3$, that is, $m=4,5,6,\ldots$. From the inequalities (3.33)–(3.36), the following bounds for p are obtained:

$$\frac{2}{m(m-2)} \le p \le \frac{1}{4} .$$

In particular, if $m=4$, the WEC at $r=1$ requires $p=1/4$. In the Appendix we have taken $p=1/72<1/4$, so the WEC at $r=1$ is not satisfied. Even in the case of $m=4$ and $p=1/4$, $v_{\hat{r}}$ takes a negative value somewhere in $1<r<2$, as shown in Fig. 3, and then $T_{\hat{r}}$ also does since $T_{\hat{r}}<v_{\hat{r}}$.

Thus the WEC at $r=1$, (3.33)–(3.36), give severe restrictions on the parameters. In Fig. 2 the region satisfying (3.33)–(3.36) for $p=1/12$ is shown by the shaded area in the m-ω plane, where A is the intersection point of the two curves a and c. The shaded area in Fig. 2 represents a typical shape of the allowed region of (m,ω) for $p<1/4$, and it shrinks to the point A when $p=1/4$. The m coordinate $m_A(p)$ (for short m_A) of A is $1+\sqrt{1+2/p}$. If (m,ω) is in the shaded area, then

$$m_A \le m \le 1/p , \qquad (3.37)$$

as shown in Fig. 2. Hereafter we try to find a pair (m,p) satisfying (3.37) such that the associated wormhole solu-

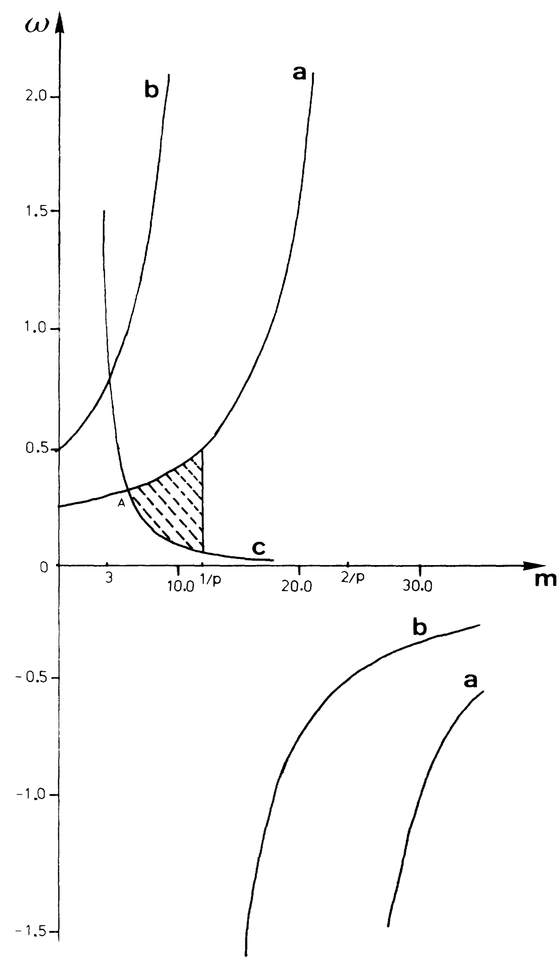

FIG. 2. The allowed region by the WEC at $r=1$ in the m-ω plane is shown by the shaded area for $p=1/12$ in the case of example 1. The curves a, b, and c represent $1/[2(2-mp)]$, $1/[2(1-mp)]$, and $1/[2mp(m-3)]$, respectively.

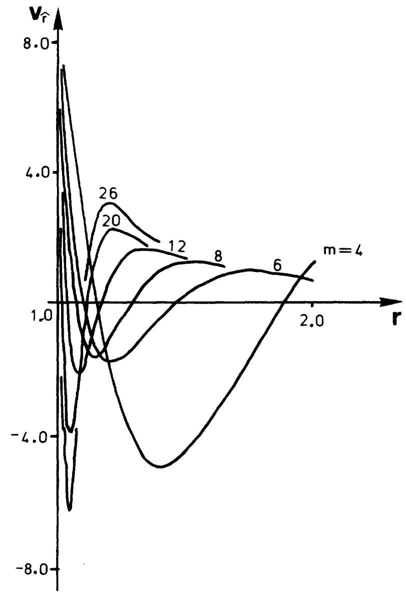

FIG. 3. The curves represent $v_{\hat r}$ for the cases of $m=4$, 6, 8, 12, 20, and 26, where $n=5000$ and p is determined by $m=1+\sqrt{1+2/p}$. The values of $v_{\hat r}$ are normalized as 8.0 at $r=1.01$.

tion admits the WEC for all $r\geq 1$. Since $a(r)$ depends on m and p by (3.20) $v_{\hat r}$ does also. We set $v_{\hat r}(r)=v_{\hat R}[m,p](r)$ to express the explicit (m,p) dependence of $v_{\hat r}$.

Let $\mu(m,p)$ be the minimal value of $v_{\hat r}[m,p](r)$ in $1\leq r\leq 2$ for any pair (m,p) satisfying (3.37). For any fixed p, $\mu(m,p)$ attains the maximum value at $m=m_A(p)$, but this value is still negative. This is seen in Fig. 3, where the graphs of $v_{\hat r}=v_{\hat r}[m,p]$ are illustrated for various values of (m,p) with $m=m_A(p)$. All curves in Fig. 3 are normalized to be 8.0 at $r=1$. Then we conclude that it is impossible to find any pair (m,p) such that the associated wormhole solution satisfies WEC.

Finally we comment on the inequality (2.8) in this example. If the RHS of (2.8) is positive, we can prove the

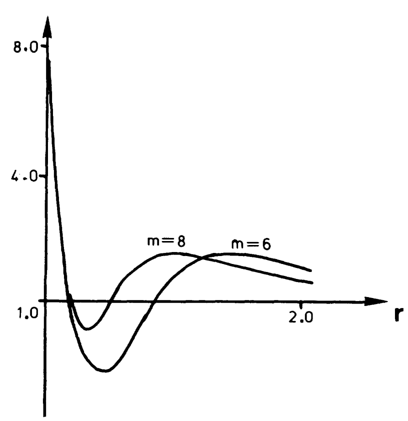

FIG. 4. $RR_{\hat r\hat r}-\nabla_{\hat r}\nabla_{\hat r}R$ for $m=6,8$, and other situations are the same with the case in Fig. 3.

incompatibility of the WEC with the wormhole solution. We show the numerical calculations of the quantity $RR_{\hat r\hat r}-\nabla_{\hat r}\nabla_{\hat r}R$ proportional to the RHS of (2.8); see Fig. 4. From these it can be seen that, in general, the inequality (2.8) cannot induce the inequality (2.7), which was crucial in the proof of proposition 1.

Example 2. For a sufficiently large $r_1>1$, we define $a(r)$

$$a(r)=\begin{cases} d\exp[u(1-r)]+(r_1-r)^2p+(r_1-r)t+s \\ \qquad\qquad \text{if } 1\leq r\leq r_1 , \\ qr^{-1/n} \ \text{if } r\geq r_1 . \end{cases} \tag{3.38}$$

Since the leading term of $a(r)$ is an exponential function, $a(r)$ decreases as $r\to r_1$ more rapidly than the function used in example 1. The parameters d, t, p, and $s(<1)$ are positive constants related to each other as follows:

$$d=1-(r_1-1)^2p-(r_1-1)t-s ,$$

$$t=s/nr_1 ,$$

$$p=s/2nr_1^2 ,$$

and

$$s=qr_1^{1/n} .$$

These relations are obtained from the continuity conditions for $a(r)$, $a'(r)$, and $a''(r)$ at $r=r_1$ and from the condition $a(1)=1$. Here we neglected the higher-order terms with respect to n^{-1} and r_1^{-1} by taking $n=5000$ and $r_1=10$. Since $a''(r)$ is continuous, we do not need to consider here the point discussed in remark of the Appendix. Since we have fixed the values of r_1 and n, the parameters s, t, p, and q are determined by d. The WEC at $r=1$ constrains the values of the mutually independent parameters d and u.

By (3.13)–(3.15), the WEC at $r=1$ is given in terms of d, u, and ω as

$$\omega\leq\frac{1}{4(1-du)} , \tag{3.39}$$

$$\omega\leq d_0\frac{d-1/u}{d^2-d_+^2} , \tag{3.40}$$

and

$$\omega\geq\frac{1}{2du(u-2)} , \tag{3.41}$$

where

$$d_0=\frac{u}{2u^2(u-1)}$$

and

$$d_+=\frac{1}{u^2(u-1)} .$$

Note that the inequality (3.41) implies $u>2$. The region in the d-ω plane restricted by (3.39)–(3.41) is shown in Fig. 5 for $u=3.0$, where B is the intersection point of the two curves a and c. The shaded area represents a typical

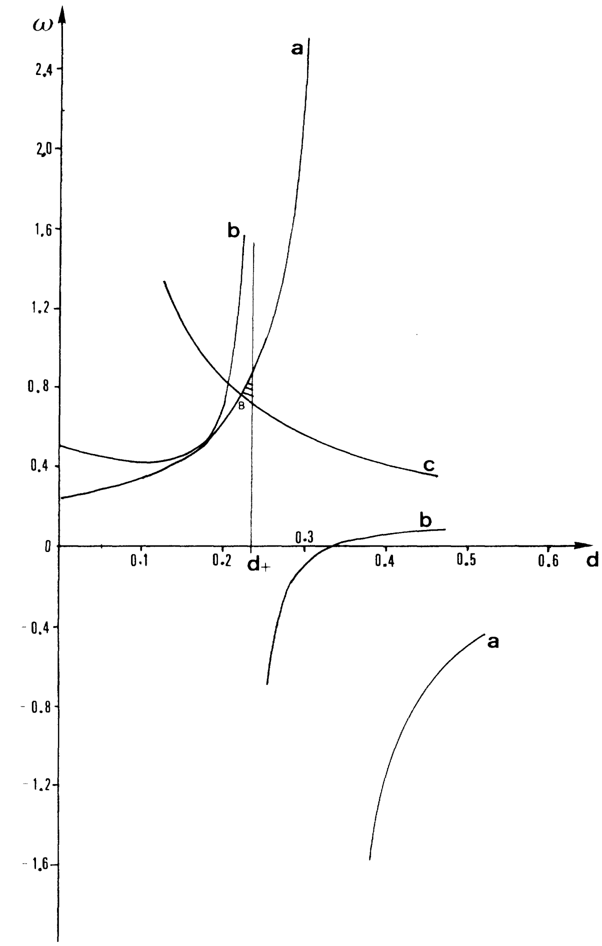

FIG. 5. The allowed region by the WEC at $r=1$ in the d-ω plane is shown by the shaded area for $u=3.0$ in the case of example 2. The curves a, b, and c represent $1/[4(1-du)]$, $d_0(d-1/u)/(d^2-d_+^2)$, and $1/[2du(u-2)]$, respectively.

shape of the allowed region. From Fig. 5 we can see that the value of ω is restricted to a very narrow and small region. As in example 1, we set $v_{\hat{r}}(r)=v_{\hat{r}}[d,u](r)$ if necessary, and let $\mu(d,u)$ be the minimum value of $v_{\hat{r}}[d,u](r)$ in $1\le r\le 10$. The d coordinate d_B of B satisfies $d_B=2/u^2$. For any fixed $u,\mu(d,u)$ $(d_B\le d\le d_+)$ attains the maximum value at $d=d_B$. This is seen from Fig. 6.

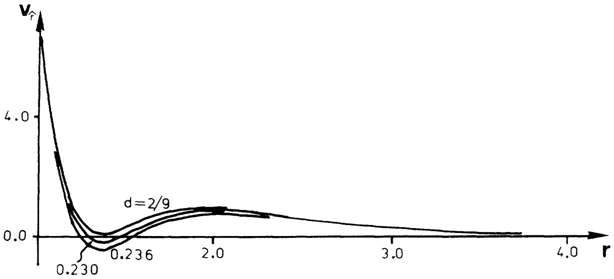

FIG. 6. The curves represent $v_{\hat{r}}$ for the cases of $d=2/9$, 0.230, and 0.236, where $n=5000$ and $u=3.0$. The values of $v_{\hat{r}}$ are normalized as 8.0 at $r=1.0$.

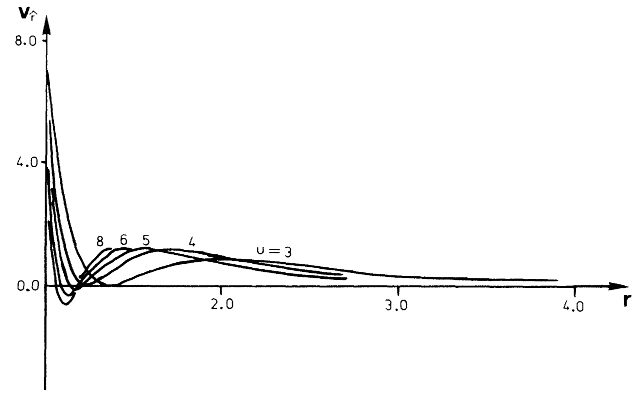

FIG. 7. The curves represent $v_{\hat{r}}$ for the cases of $u=3$, 4, 5, 6, and 8, where $n=5000$ and $d=2/u^2$. The values of $v_{\hat{r}}$ are normalized as 8.0 at $r=1.0$.

In Fig. 7 the graphs of $v_{\hat{r}}=v_{\hat{r}}[d,u]$ are illustrated for various values of (d,u) with $d=2/u^2$.

In contrast with the higher-order term of $T_{\hat{r}}$ in example 1, the minima of $v_{\hat{r}}[2/u^2,u]$ for the case of $u=3$ and 4 are positive, but they are very small. Moreover, since the ω coordinate of any point in the allowed region is not so large, we have $2\omega v_{\hat{r}}<-E_{\hat{r}}$ or equivalently $T_{\hat{r}}<0$. In Fig. 8 we show the numerical results of $T_{\hat{r}}$ for these cases, where the values of ω are the ω coordinates, $1/4(1-2/u)$, of B. Then we can conclude that the wormhole solutions in this example also violate the WEC.

We have seen throughout the above two examples that it was impossible to find an appropriate wormhole solution $a(r)$ which satisfies the WEC even if ω is positive. The reason why it was impossible could be summed up as follows. First impose the WEC at the wormhole throat, $r=1$, which is given in general as

$$(a'+1)-2\omega[(a'+1)^2+a'(a''-2)]\ge 0\ , \tag{3.42}$$

$$a'[1-2\omega(2a'+a'')]\ge 0\ , \tag{3.43}$$

$$(1+a'/2)[1-4\omega(a'+1)]\ge 0\ , \tag{3.44}$$

where $a'=a'(1)$ and $a''=a''(1)$. These inequalities give severe constraints for a', a'', and ω. The inequality (3.43) implies $a''>-a'$, which could explain why $v_{\hat{r}}$ decreases rapidly from the value at $r=1$ as r increases, and $v_{\hat{r}}$ soon

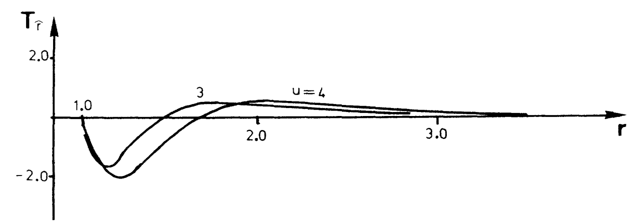

FIG. 8. The curves represent $T_{\hat{r}}$ for the case of $u=3$ and 4, where $n=5000$, $d=2/u^2$, and $\omega=1/[4(1-2/u)]$. Here $T_{\hat{r}}$ is rescaled appropriately for either u.

reaches at its small minimum. On the other hand, the value of ω is constrained to be so small by the WEC at $r=1$, (3.42)–(3.44), that the value of $T_{\hat{r}}$ could not be positive near the point where $v_{\hat{r}}$ attains the minimum value. Therefore we could conclude that the higher derivative terms cannot prevent the wormhole throat from collapsing.

Although the above numerical analysis was done by imposing the WEC at $r=1$, it would be possible to get similar results by imposing the WEC at any other point near the wormhole throat. However, we did not do such analyses.

IV. CONCLUSION AND DISCUSSIONS

Since the wormhole solution is not compatible with the WEC in the case of Einstein gravity, exotic matter or some quantum effect, which could provide a negative-energy density, is necessary near the wormhole throat for the existence of stable traversible wormholes. If the size of the wormhole throat is small, the curvature is very large near the throat. Then we could expect that the R^2 term plays an important role in the Einstein equation near the throat. We studied the gravitational theory with the Lagrangian $L_G=\sqrt{g}(R-\omega R^2)$ in order to see whether or not the higher curvatures could overcome the above incompatibility. If $\omega<0$, the classical field equation of our theory is equivalent to that of Einstein gravity with a normal scalar field constructed by the metric. Then the incompatibility of wormhole solutions with the WEC can be shown in terms of the conformally transformed equation. So we concentrated our attention to the case of $\omega>0$, where the conformally transformed equation is not well defined.

Whatever the sign of ω is, we can prove the braking of the WEC on the wormhole if the right-hand side of the inequality (2.8) is bounded from below by some sufficiently small negative number. Since such a bound does not exist except for the case of very small $|\omega|$, we studied directly the equation of motion admitting a spherically symmetric wormhole solution. Two examples of such solutions with a few parameters are considered. We restricted the range of the parameters by imposing the WEC at the wormhole throat, and we tried to find a wormhole solution which is compatible with the WEC everywhere in the wormhole. However, we could not find such a solution, and we concluded that the wormhole is not compatible with the WEC in the theory with the R^2 term.

The part proportional to ω in the RHS of (3.11) can be regarded as an effective energy-momentum tensor $(T_{\text{eff}})_{\mu\nu}$ induced by the R^2 term. We have seen in the previous section that $(T_{\text{eff}})_{\mu\nu}K^\mu K^\nu$ could be negative somewhere in the wormhole (see Fig. 4). So one might expect that the higher derivative terms could provide the same result with that provided by Einstein gravity including exotic matter. However, we could not obtain any evidence supporting this expectation. This means that, since $(T_{\text{eff}})_{\mu\nu}$ is written by the metric itself, in contrast with the case of an exotic matter field, it is difficult to make $(T_{\text{eff}})_{\mu\nu}K^\mu K^{\mu\nu}$ negative near the wormhole throat so as to prevent the wormhole from collapsing. In other words, exotic matter could support the wormhole throat, but $(T_{\text{eff}})_{\mu\nu}$ could not.

In a more general case where the gravitational Lagrangian includes any power series of curvature, the effective energy-momentum tensor would be more complicated and we cannot say easily anything about the existence of a wormhole solution satisfying the WEC. It would be necessary to improve the Raychaudhuri equation so that it is applicable to such higher derivative theories.

ACKNOWLEDGMENTS

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APPENDIX

Here we show that $T_{\hat{r}}>0$ for the parametrization of Eq. (3.31) with $s=71/72$ and $m=4$.

First, we define $\Phi(r)$ so as to make $T_{\hat{r}}$ positive in the large-r region. For any $\epsilon>0$, there exists a positive integer n_0 such that for any integer n with $n\geq n_0$, a smooth function $\Phi(r)=\Phi_n(r)$ can be defined as

$$\Phi(r)=\begin{cases}-36a(r) & \text{if } r\geq 2\ ,\\ -\frac{71}{2} & \text{if } 1\leq r\leq 2-\epsilon\ ,\end{cases} \tag{A1}$$

and $|\Phi^{(k)}(r)|<\epsilon$ for $k=1,2,3,4$ and for all r with $2-\epsilon\leq r\leq 2$.

Here we suppose that $r\geq 2$. If we neglect the terms of order $n^{-1}r^{-2-1/n}$, then the Einstein term $E_{\hat{r}}$ of $T_{\hat{r}}$ is

$$R_{\hat{t}\hat{t}}+R_{\hat{r}\hat{r}}=\frac{71}{n}qr^{-1-1/n}\left[1-\frac{72}{71}a\right] .$$

Therefore $E_{\hat{r}}>0$ if $a<71/72$, or equivalently if $r>2$. The higher curvature terms are shown to be positive in this region as in example 1. Thus we have $T_{\hat{r}}>0$ in $r\geq 2$.

Next we suppose that $1\leq r<2$. Then $a(r)\geq 71/72$. Since $-a'/r\geq 0$, by (3.12) and (3.31) for $m=4$,

$$\eta(r)\geq a'''-\frac{a''}{r}+\frac{6}{8}\frac{71}{72}\geq -36p+\frac{71}{96} .$$

Since $p<1/72$,

$$\eta(r)>\frac{23}{96} . \tag{A2}$$

By the definition of $\Phi(r)$, for $k=1,2,3,4$, $|\Phi^{(k)}(r)|<\epsilon$ in $2-\epsilon\leq r<2$, and $|\Phi^{(k)}(r)|=0$ in $1\leq r\leq 2-\epsilon$. For all r sufficiently near 2, $2(1-a)\eta(r)/r$ is close to $1\times 6\times\frac{71}{72}\times 8\times 72=\frac{71}{6912}$. When ϵ is sufficiently small, or equivalently n sufficiently large, $\Phi(r)$ does not contribute to $\text{sgn}(v_{\hat{r}}(r))$ in $1\leq r<2$. If we neglect any effects of Φ, then $2\omega[v_{\hat{r}}-2(1-a)\eta(r)/r]$ can be identified with the second term $\xi(r)$ of (3.20). Then we have

$$\xi(r) = -\frac{a'}{r}\left[a'' + \frac{2a'}{r}\right]$$
$$\geq \frac{P(r)}{r}\left[12(2-r)^2 p - \frac{8(2-r)^3}{r} - \frac{2^{1/n} q}{nr}\right]$$
$$\geq \frac{P(r)}{r}\left[4(2-r)^2 p - 2^{1/n}\frac{q}{n}\right],$$

where $P(r) = 4(2-r)^3 p + 2^{-1-1/n} q/n$. For any $\epsilon > 0$, there exists a positive integer n_0 and a positive number δ with $\delta < \epsilon$ such that, for any $n \geq n_0$, $\xi(r) = \xi_n(r) > 0$ in $r \leq 2-\delta$ and $\xi(r) > -\epsilon$ in $2-\delta \leq r < 2$. By this fact and (A2), we have $v_{\hat{r}} > 0$. Thus, if ω is sufficiently large, then $T_{\hat{r}}(r) > 0$ for all r with $1 \leq r < 2$.

Remark. In the above example, a C^1 function $a(r)$ is used to make our argument simple, so $a''(r)$ is discontinuous at $r = 2$. By (3.31) for $m = 4$,

$$\delta\langle a''' \rangle = 2^{-2-1/n}\left[1 + \frac{1}{n}\right]\frac{q}{n}\delta(r-2),$$

where $\delta\langle h(r) \rangle$ denotes the δ-function term of a function $h(r)$. The Φ-dependent term $S_R(r)$ of R is $-2(S_{\hat{r}} + S_{\hat{\theta}})$; see Ref. [8]. By (3.25) and (3.26), $S_R(r)$ contains the $a'(r)$ term $-\Phi' a'$, so that $\delta\langle S_R''(r) \rangle = -\Phi'\delta\langle a''' \rangle$. Therefore,

$$\delta\langle T_{\hat{r}} \rangle = 2\omega\left[\frac{2(1-a)}{rr} + \Phi'\right]\delta\langle a''' \rangle$$
$$= 2^{-1-1/n}\omega\left[\frac{1}{72} + \frac{36}{n}2^{-1-1/n}\right]$$
$$\times\left[1 + \frac{1}{n}\right]\frac{q}{n}\delta(r-2).$$

Strictly, we need to consider the "junction condition" as in Sec. 21.13 of Ref. [10] and Refs. [4,5]. In fact, the surface energy-momentum tensor for $\Sigma = \{(t,r,\theta,\phi) : r = 2\}$ in $\mathcal{W}$ is

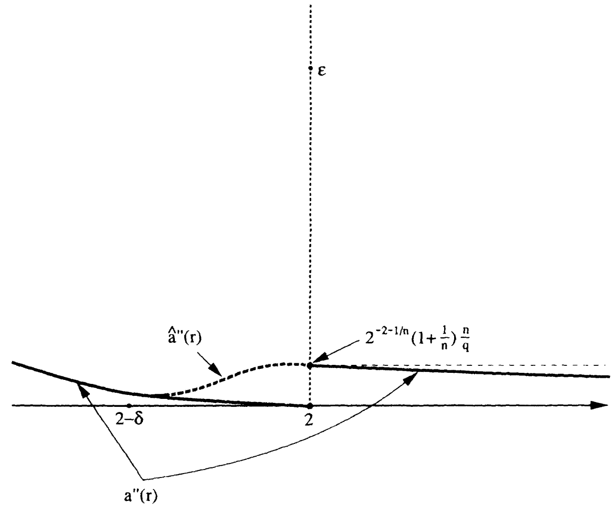

FIG. 9. The smooth function $\hat{a}''(r)$ is obtained by taking the broken curve instead of the underlying part in the graph of $a''(r)$.

$$\lim_{\epsilon \to 0}\int_{2-\epsilon}^{2+\epsilon} T_{\hat{r}}(r)dr = 2^{-1-1/n}\omega\left[\frac{1}{72} + \frac{36}{n}2^{-1-1/n}\right]$$
$$\times\left[1 + \frac{1}{n}\right]\frac{q}{n} > 0.$$

One can obtain a C^∞ function $\hat{a}''(r)$ by smoothing $a''(r)$ near $r = 2$ so as to eliminate the singularities of $a''(r)$ and $a'''(r)$. Furthermore, for any $\epsilon > 0$, there exists δ with $0 < \delta < \epsilon$ and so does n_0 such that for all $n \geq n_0$, $|\hat{a}''(r)|, |\hat{a}'''(r)| < \epsilon$ in $2-\delta \leq r \leq 2$, and $\hat{a}''(r) = a''(r)$ in both $1 \leq r \leq 2-\delta$ and $r \geq 2$ (Fig. 9). We refer to Hirsch (Ref. [11], Chap. 2) for the standard technique of smoothing functions by using "bump functions." The function $\hat{a}'(r)$ is defined by integrating $\hat{a}''(r)$ backward from $+\infty$ to r. The function $\hat{a}(r)$ is defined similarly. So $\hat{a}(r) = a(r)$ in $r \geq 2$, but in general, $\hat{a}(1)$ is slightly different from $a(1) = 1$. Then the energy-momentum tensor for the C^∞ wormhole $\hat{\mathcal{W}}$ defined by using $\hat{a}(r)/\hat{a}(1)$ instead of $a(r)$ satisfies $T_{\hat{r}}(r) > 0$ everywhere in $\mathcal{W}$.

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