Black holes in higher-derivative gravity theories

Salvatore Mignemi*
Department of Physics, University of Newcastle-Upon-Tyne, Newcastle-Upon-Tyne NE1 7RU, United Kingdom and Istituto Nazionale di Fisica Nucleare, Sezione di Cagliari, Via Ada Negri 18, 09127 Cagliari, Italy

David L. Wiltshire†
Department of Physics and Mathematical Physics, University of Adelaide, GPO Box 498, Adelaide, South Australia 5001, Australia

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We study static spherically symmetric solutions of Einstein gravity plus an action polynomial in the Ricci scalar R of arbitrary degree n in an arbitrary dimension D. The global properties of all such solutions are derived by studying the phase space of field equations in the equivalent theory of gravity coupled to a scalar field, which is obtained by a field redefinition and conformal transformation. The following uniqueness theorem is obtained: Provided that the coefficient a_2 of the R^2 term in the Lagrangian polynomial is positive then the only static spherically symmetric asymptotically flat solution with a regular horizon in these models is the Schwarzschild solution. Other branches of solutions with regular horizons, which are asymptotically anti–de Sitter, or de Sitter, are also found. An exact Schwarzschild–de Sitter–type solution is found to exist in the $R + aR^2$ theory if $D > 4$. If terms of cubic or higher order in R are included in the action, then such solutions also exist in four dimensions. The general Schwarzschild–de Sitter–type solution for arbitrary D and n is given. The fact that the Schwarzschild solution in these models does not coincide with the exterior solution of physical bodies such as stars has important physical implications which we discuss. As a byproduct, we classify all static spherically symmetric solutions of D-dimensional gravity coupled to a scalar field with a potential consisting of a finite sum of exponential terms.

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I. INTRODUCTION

Theories of gravity involving higher powers of the Riemann tensor in the Lagrangian have been proposed in several different contexts since the first days of general relativity. They were first introduced by Weyl in his affine theory, which aimed to unify gravity and electromagnetism [1]. Such models have become attractive again in recent years following the demonstration that the addition to the Einstein-Hilbert Lagrangian of terms quadratic in the Ricci tensor leads to a renormalizable theory [2]. Unfortunately a massive spin-2 "ghost" is present in the linearized spectrum, leading to an instability of the theory and a loss of unitarity. It has been suggested that this problem would disappear in a full nonperturbative treatment of the model, or even that the ghost states in the perturbative expansion could be a gauge artifact [3]. However, a nonperturbative formulation is still a distant goal, while the latter possibility seems to have been ruled out [4].

The effects of higher-derivative gravity have also proven to be useful in cosmology, beginning with the early work of Starobinsky [5] and Kerner [6] who introduced higher-derivative terms with a view to obtaining solutions which avoid the initial singularity. Later on it was realized that such models can lead to inflationary expansion driven only by gravity [7–9]. Higher-derivative models have also been studied in the context of quantum cosmology [10]. In particular, it has been argued that the introduction of quadratic terms may solve some of the problems due to the nonpositiveness of the ordinary Einstein-Hilbert action for Euclidean quantum cosmology [11]. It is also interesting to note that quadratic corrections to the action are obtained from quantum wormhole effects [12].

Finally, we should mention that higher-order Lagrangians arise naturally in higher-dimensional theories, such as Kaluza-Klein and string models. In the first case they are introduced in order to obtain spontaneous compactification from purely gravitational higher-dimensional theories, and in this context the ghost-free Gauss-Bonnet actions have attracted much interest [13, 14]. In the second case, they are obtained as an effective low-energy action [15].

While the cosmology of these models has been largely studied, both in four [5–9,16] and higher dimensions [17, 18], comparatively little is known about black hole solutions. The weak-field limit has been studied to some extent for the $R + aR^2$ theory in four dimensions [19, 20], but the properties of the full solutions in higher-derivative theories remain largely unexplored. On di-

*Present address: Laboratoire de Physique Théorique, Institut Henri Poincaré, 11, rue P. et M. Curie, 75231 Paris Cedex 05, France. Electronic address: mignemi@ccr.jussieu.fr
†Electronic address: dlw@physics.adelaide.edu.au

mensional grounds, one would expect that the higher-derivative terms become dominant in the proximity of the singularity. However, it is still possible that horizons exist and that, contrary to the assumptions of Refs. [19] and [20], analogues of the usual uniqueness theorems for stationary axisymmetric black holes can be derived. Indeed, such conclusions can be immediately drawn in the case of the $R + aR^2$ theory in four dimensions as a result of the "no-hair" theorem proved by Whitt [21], as we show in Appendix A. Thus the question of the nature of the static spherically symmetric solutions of more general higher-derivative models is very interesting from the point of view of general relativity.

One of the reasons for the lack of attention to the black hole problem is undoubtedly the difficulty of solving the higher-order differential equations arising in higher-derivative models. Some progress can be made, however, by using the fact that higher-derivative theories are equivalent, by redefinition of the metric, to ordinary Einstein gravity coupled to a scalar plus a massive spin-2 field. This result was first proven by Higgs [22], and later rediscovered by Whitt [21], in the case of the $R+aR^2$ theory in four dimensions, for which only the scalar field is present in the effective theory. More recently the equivalence has been extended to the case of actions containing powers of the Ricci and Riemann tensor [23].[1] The importance of this equivalence is that one can now use the formalism of ordinary general relativity to study the more general higher-derivative theories.

Our analysis in this paper will make use of the equivalence of the general D-dimensional action

$$S = \int d^D x \frac{\sqrt{-g} f(R)}{4\kappa^2} \tag{1.1}$$

to Einstein gravity coupled to a scalar field [23]. Here f is an arbitrary function of the Ricci scalar R, and κ^2 denotes the gravitational constant in D dimensions. If we define σ by

$$\frac{2\kappa\sigma}{\sqrt{D-1}} = \ln[\varepsilon f'(R)], \tag{1.2a}$$

where

$$\varepsilon = \begin{cases} 1 & \text{if } f' > 0, \\ -1 & \text{if } f' < 0, \end{cases} \tag{1.2b}$$

and make a conformal transformation

$$\hat{g}_{ab} = [\varepsilon f'(R)]^{2/(D-2)} g_{ab}, \tag{1.3}$$

then the field equations derived from (1.1) are equivalent to those derived from the action

$$\hat{S} = \int d^D \hat{x} \sqrt{-\hat{g}} \left(\frac{\hat{R}}{4\kappa^2} - \frac{1}{D-2} \hat{g}^{ab} \partial_a \sigma \partial_b \sigma - \mathcal{V}(\sigma) \right), \tag{1.4a}$$

where

$$\mathcal{V} = \frac{\varepsilon}{4\kappa^2} [\varepsilon f'(R)]^{-D/(D-2)} (Rf' - f), \tag{1.4b}$$

with R, f, and f' defined implicitly in terms of σ via (1.2). For the quadratic theory, for example, with $f = R + aR^2$ we find [18]

$$\mathcal{V} = \frac{\varepsilon}{16\kappa^2 a} \exp\left(\frac{2(D-4)\kappa\sigma}{(D-2)\sqrt{D-1}}\right) \times \left[1 - \varepsilon \exp\left(\frac{-2\kappa\sigma}{\sqrt{D-1}}\right)\right]^2. \tag{1.5}$$

Similarly, for the cubic theory with $f = R + aR^2 + bR^3$ we find [16]

$$\mathcal{V} = \frac{\varepsilon}{54\kappa^2 b^2} \exp\left(\frac{-2D\kappa\sigma}{(D-2)\sqrt{D-1}}\right) \left(\pm \left\{ a^2 - 3b \left[1 - \varepsilon \exp\left(\frac{2\kappa\sigma}{\sqrt{D-1}}\right)\right]\right\}^{3/2} - a^3 + \frac{9}{2} ab \left[1 - \varepsilon \exp\left(\frac{2\kappa\sigma}{\sqrt{D-1}}\right)\right] \right). \tag{1.6}$$

For other actions polynomial in R,

$$f(R) = R + \sum_{p=2}^{n} a_p R^p, \tag{1.7}$$

and f' can be inverted to give a precise analytic expression for R in terms of σ, and hence for $\mathcal{V}(\sigma)$, for a general polynomial only if $n \le 5$.

The general higher-order field equations obtained from the action (1.1), with f of the form (1.7), are given by

$$R_{ab} - \tfrac{1}{2} g_{ab} R + \sum_{p=2}^{n} a_p \Big(pR^{p-1} R_{ab} - p(p-1) R^{p-3} \Big[RR_{;ab} + (p-2) R_{;a} R_{;b} \Big] + g_{ab} \{ p(p-1) R^{p-3} [R \Box R + (p-2) R^{;c} R_{;c}] - \tfrac{1}{2} R^p \} \Big) = 0. \tag{1.8}$$

[1]For a different approach to the problem see [24].

Since the arbitrary-dimensional Schwarzschild solution

$$ds^2 = -\left(1 - \frac{2GM}{r^{D-3}}\right) dt^2 + \left(1 - \frac{2GM}{r^{D-3}}\right)^{-1} dr^2 + d\Omega^2_{D-2} \qquad (1.9)$$

has $R = 0$ throughout the domain of outer communications, it is clear that the Schwarzschild solution solves the equations (1.8) for any choice of the constants a_p. Thus the problem before us is to determine whether the Schwarzschild solution is the only static spherically symmetric asymptotically flat solution with a regular horizon. On account of the equivalence of the higher-order theory to the theory decribed by the action (1.4) this uniqueness problem can be regarded as the problem of establishing a "no-hair theorem" for the latter model.

Whitt established such a theorem in the case of the $R + aR^2$ theory in four dimensions[2] [21], by demonstrating that all asymptotically flat stationary axisymmetric solutions to the higher-order vacuum equations must have $R = 0$ in the domain of outer communications. If one adds extra matter fields to the action one still finds that $R = 0$ for stationary, axisymmetric, asymptotically flat solutions provided that the energy-momentum tensor is traceless and satisfies the matter circularity condition. For such solutions, therefore, (1.8) becomes equivalent to the usual Einstein equations, and the usual uniqueness theorems and no-hair theorems will carry over to the fourth-order theory. If one considers an arbitrary polynomial in R of the form (1.7), however, then Whitt's argument breaks down, as we demonstrate in Appendix A. Consequently a different approach is called for.

Our approach here will differ not only from that of Whitt, but also from other standard approaches to no-hair theorems [26, 27], in that we will solve the problem by studying the phase space of the field equations obtained from (1.4). We will take advantage of the fact that by making a judicious choice of coordinates these equations may be written in the form of a five-dimensional autonomous system of ordinary first-order differential equations, so that all the global properties of the solutions can be derived. Our approach not only has the advantage that it can be used to establish a black hole uniqueness theorem for general actions polynomial in R, but it will also enable us to determine the nature of other solutions in these models which have regular horizons but which are not asymptotically flat.

The analysis we will use here is very similar to that developed in Refs. [28] and [29], where we derived the global properties of static spherically symmetric solutions in models of gravity which arise from the dimensional reduction of certain higher-dimensional gravity theories. The scalar field in (1.4) then corresponds to the radius of the extra dimensions (the "compacton"), and the potential $\mathcal{V}(\sigma)$ contains one or two exponential terms.

The analysis of [28] and [29] is based on the fact that the appropriate field equations can be reduced to a five-dimensional autonomous system of first-order ordinary differential equations. This is possible essentially due to the fact that the field equations form a system very similar to those of a Toda lattice when written in terms of appropriate coordinates [30] (the Toda lattice being an integrable system). Since the metric and scalar fields are related to the functions X, Y, V, Z, and W of the five-dimensional phase space $\mathcal{M}$, they are necessarily regular at all points of the integrals curves apart from critical points. Consequently, in order to determine the global properties of all solutions, namely, the structure of their singularities, horizons, and asymptotic regions, it suffices to study the properties of the solutions at critical points of $\mathcal{M}$. The determination of which critical points are connected to which other ones by integral curves requires a careful analysis of the structure of the space $\mathcal{M}$: in particular of surfaces corresponding to particular subspaces, which separate integral curves corresponding to spacetimes of different causal structures. This method represents a powerful analytical tool for these (and possibly other) models for which it is not possible to write down the general static spherically symmetric solutions in a closed analytic form.

One should mention that on account of the way in which the phase-space functions are constructed, the metric functions (with signature $- + + \cdots +$) are necessarily positive. Thus for the Schwarzschild solution the integral curves obtained correspond to the domain of outer communications only. A similar analysis of the Reissner-Nordström solution yields a phase space with distinct regions which correspond to (i) integral curves in the domain of outer communications, and (ii) integral curves in the region between the singularity and the inner horizon. (These two regions of the phase space are separated by a surface which corresponds to the Robinson-Bertotti solutions.) Thus the method described does not pick out regions of the spacetimes in which the Killing vector $\partial/\partial t$ is spacelike. Such regions can only be obtained by continuation of such solutions as are described here.

The first step of our analysis here will be to generalize the work of [28] and [29] to establish the properties of solutions to the equations derived from (1.4) with a potential consisting of a sum of s exponential terms

$$\mathcal{V}(\sigma) = \frac{-1}{4\kappa^2} \sum_{i=1}^{s} \lambda_i \exp\left(\frac{-4g_i\kappa\sigma}{D-2}\right), \qquad (1.10)$$

the λ_i and g_i, $i = 1, \ldots, s$ being constants. Of the higher-derivative theories, only the $R + aR^2$ theory has an action which is precisely of this form. However, for each n there will be a special choice of the constants a_p for which the potential (1.4b) reduces to this form. This will be so if the $(n-1)$th root of f' is linear in R. In particular, if we choose

[2] In fact, a black hole uniqueness theorem for the more restricted case of pure R^2 theory, without an Einstein-Hilbert term, was obtained much earlier on by Buchdahl [25].

$$a_p = \frac{(2a_2)^{p-1}(n-2)!}{(n-1)^{p-2}p!(n-p)!}, \qquad 3 \le p \le n, \tag{1.11}$$

then

$$f' = \left(1 + \frac{2a_2 R}{n-1}\right)^{n-1}, \tag{1.12}$$

and we find that the potential takes the form

$$\mathcal{V} = \frac{(n-1)^2}{8n\kappa^2 a_2} \exp\left(\frac{-2D\kappa\sigma}{(D-2)\sqrt{D-1}}\right)\left[\tilde{\varepsilon}\exp\left(\frac{2n\kappa\sigma}{(n-1)\sqrt{D-1}}\right) - \frac{n}{n-1}\exp\left(\frac{2\kappa\sigma}{\sqrt{D-1}}\right) + \frac{\tilde{\varepsilon}^{n-1}}{n-1}\right], \tag{1.13a}$$

where[3]

$$\tilde{\varepsilon} = \begin{cases} 1 & \text{if } 1 + 2a_2R/(n-1) > 0, \\ -1 & \text{if } 1 + 2a_2R/(n-1) < 0. \end{cases} \tag{1.13b}$$

We shall call this class of models "special polynomial R theories."

Before proceeding further, we note that the definition of asymptotic flatness in our analysis requires some clarification. On account of the conformal transformation (1.3) the radial coordinate of the higher-derivative theory, r, is related to the radial coordinate of the effective theory, $\hat{r}$, by

$$r = [\varepsilon f'(R)]^{-1/(D-2)}\,\hat{r} = \exp\left(\frac{-2\kappa\sigma}{(D-2)\sqrt{D-1}}\right)\hat{r}. \tag{1.14}$$

Thus solutions which are asymptotically flat in the higher-derivative theory need not be asymptotically flat in the effective theory, since it is the dependence on r as measured by the metric g_{ab} which is of physical importance rather than the dependence on $\hat{r}$ as measured by the metric $\hat{g}_{ab}$. Moreover, the definition of the asymptotic region can vary between the physical and the effective theories. The definitions of spatial infinity and of asymptotic flatness will only coincide when $\sigma \to$ const as $\hat{r} \to \infty$. However, since σ is not a physical field here (unlike the Kaluza-Klein case [28, 29]), we need place no requirements on its asymptotic form at spatial infinity in the effective theory. Thus a discussion of the existence of static spherically symmetric black holes in the higher-derivative models requires an examination of *all* static spherically symmetric solutions in the effective theory, and not merely the ones which are asymptotically flat according to the effective theory.

We will begin by deriving the global properties for the theory with the potential (1.10)—almost all cases are dealt with in Sec. II. For special values of the parameters, which include the cases of the $R + aR^2$ theory and all potentials (1.13), the structure of the phase space is modified. Such solutions will be discussed in Sec. III. We will then generalize the analysis to include the potential (1.6) of the $R + aR^2 + bR^3$ theory in Sec. IV. This generalization will be found to produce only minor changes to the results of Secs. II and III. We will further demonstrate that our results can be extended to the case of the polynomial of arbitrary degree. Some physical implications of our work are discussed in Sec. V.

[3]Note that $\varepsilon = \tilde{\varepsilon}^{n-1}$ with these definitions: for n even ε and $\tilde{\varepsilon}$ have the same sign. However, if n is odd then $\varepsilon = +1$, and the sign of $\tilde{\varepsilon}$ corresponds instead, for example, to the plus or minus sign multiplying the first term in (1.6).

II. THE GENERAL EXPONENTIAL SUM POTENTIAL

A. The dynamical system

As in [28] and [29] we will begin by choosing coordinates

$$\hat{g}_{ab}d\hat{x}^a d\hat{x}^b = e^{2\hat{u}}\left(-dt^2 + \hat{r}^{2m}d\xi^2\right) + \hat{r}^2\bar{g}_{\alpha\beta}d\bar{x}^\alpha d\bar{x}^\beta, \tag{2.1a}$$

where $m = D-2$, $\hat{u} = \hat{u}(\xi)$, $\hat{r} = \hat{r}(\xi)$, and $\bar{g}_{ab}$ is the metric on an arbitrary m-dimensional Einstein space:

$$\bar{R}_{\alpha\beta} = (m-1)\bar{\lambda}\bar{g}_{\alpha\beta}. \tag{2.1b}$$

Of course we are most interested in the case in which $\bar{g}_{\alpha\beta}$ is the metric on a two-sphere (and $\bar{\lambda} = 1$). However, the structure of the phase space is better revealed by taking the more general ansatz (2.1b). In order to write the equations as a first-order system which is everywhere well defined, we will choose an ordering of the g_i such that

$$g_1 > g_3 > g_4 > \cdots > g_s > g_2. \tag{2.2}$$

If we now define the functions[4] ζ, η, and χ by

$$\zeta = \hat{u} + (m-1)\ln\hat{r}, \tag{2.3}$$

$$\eta = \hat{u} + m\ln\hat{r} - \frac{2g_1\kappa\sigma}{m}, \tag{2.4}$$

$$\chi = \hat{u} + m\ln\hat{r} - \frac{2g_2\kappa\sigma}{m}, \tag{2.5}$$

then the field equations become

[4]These functions correspond, in fact, to the differences of the Toda lattice coordinates [30].

$$\zeta'' = (m-1)^2\bar{\lambda}e^{2\zeta} + \lambda_1 e^{2\eta} + \lambda_2 e^{2\chi} + \sum_{i=3}^{s} \lambda_i e^{2(\alpha_i\chi+\beta_i\eta)}, \tag{2.6a}$$

$$\eta'' = m(m-1)\bar{\lambda}e^{2\zeta} + \frac{1}{m}\left(m+1-g_1{}^2\right)\lambda_1 e^{2\eta} + \frac{1}{m}\left(m+1-g_1g_2\right)\lambda_2 e^{2\chi} + \frac{1}{m}\sum_{i=3}^{s}\left(m+1-g_1g_i\right)\lambda_i e^{2(\alpha_i\chi+\beta_i\eta)}, \tag{2.6b}$$

$$\chi'' = m(m-1)\bar{\lambda}e^{2\zeta} + \frac{1}{m}\left(m+1-g_1g_2\right)\lambda_1 e^{2\eta} + \frac{1}{m}\left(m+1-g_2{}^2\right)\lambda_2 e^{2\chi} + \frac{1}{m}\sum_{i=3}^{s}\left(m+1-g_2g_i\right)\lambda_i e^{2(\alpha_i\chi+\beta_i\eta)}, \tag{2.6c}$$

with the constraint

$$(m+1)\zeta'^2 + \frac{2m\zeta'(g_2\eta'-g_1\chi')}{g_1-g_2} + \frac{1+(m-1)g_2{}^2}{(g_1-g_2)^2}\eta'^2 - 2\frac{1+(m-1)g_1g_2}{(g_1-g_2)^2}\eta'\chi' + \frac{1+(m-1)g_1{}^2}{(g_1-g_2)^2}\chi'^2$$
$$+(m-1)\bar{\lambda}e^{2\zeta} + \frac{\lambda_1}{m}e^{2\eta} + \frac{\lambda_2}{m}e^{2\chi} + \frac{1}{m}\sum_{i=3}^{s}\lambda_i e^{2(\alpha_i\chi+\beta_i\eta)} = 0, \tag{2.6d}$$

where, for $s \geq 3$,

$$\alpha_i = \frac{g_1-g_i}{g_1-g_2}, \qquad i=3,\ldots,s, \tag{2.6e}$$

and

$$\beta_i = 1-\alpha_i = \frac{g_i-g_2}{g_1-g_2}, \qquad i=3,\ldots,s. \tag{2.6f}$$

If $s < 3$ then the last summation term in (2.6a)–(2.6d) vanishes. Note that on account of (2.2), $0 < \alpha_i < 1$, $0 < \beta_i < 1$, and $\alpha_i = \beta_i$ only in the special case in which $s = 3$ and $g_3 = (g_1 + g_2)/2$.

These equations can be recast in the form of a five-dimensional autonomous system of first-order differential equations. If we define variables V, W, X, Y, and Z by

$$V=\chi', \quad W=e^{\chi}, \quad X=\zeta', \quad Y=\eta', \quad Z=e^{\eta}, \tag{2.7}$$

then the constraint (2.6d) can be regarded as a definition of $e^{2\zeta}$. Eliminating the $e^{2\zeta}$ terms from (2.6a)–(2.6c) we therefore obtain the system[5]

$$X' = \frac{1}{m}\left(\lambda_1 Z^2 + \lambda_2 W^2 + \sum_{i=3}^{s}\lambda_i W^{2\alpha_i}Z^{2\beta_i}\right) - \frac{(m-1)P}{m}, \tag{2.8a}$$

$$Y' = \frac{-1}{m}\left(\left(g_1{}^2-1\right)\lambda_1 Z^2 + \left(g_1g_2-1\right)\lambda_2 W^2 + \sum_{i=3}^{s}\left(g_1g_i-1\right)\lambda_i W^{2\alpha_i}Z^{2\beta_i}\right) - P, \tag{2.8b}$$

$$V' = \frac{-1}{m}\left(\left(g_1g_2-1\right)\lambda_1 Z^2 + \left(g_2{}^2-1\right)\lambda_2 W^2 + \sum_{i=3}^{s}\left(g_2g_i-1\right)\lambda_i W^{2\alpha_i}Z^{2\beta_i}\right) - P, \tag{2.8c}$$

$$Z' = YZ, \tag{2.8d}$$

$$W' = VW, \tag{2.8e}$$

where

$$P \equiv m\left((m+1)X^2 + \frac{2mX(g_2Y-g_1V)}{g_1-g_2} + \frac{\left[1+(m-1)g_2{}^2\right]Y^2}{(g_1-g_2)^2} - \frac{2\left[1+(m-1)g_1g_2\right]YV}{(g_1-g_2)^2} + \frac{\left[1+(m-1)g_1{}^2\right]V^2}{(g_1-g_2)^2}\right). \tag{2.8f}$$

[5]We are using a different normalization here for Z and W compared with that used in [28] and [29].

The dynamical system of the Kaluza-Klein models dealt with in [29], with n_e extra dimensions and a higher-dimensional cosmological constant Λ, is retrieved by setting $s=2$ and

$$g_1 = {g_2}^{-1} = \frac{\sqrt{m+n_e}}{\sqrt{n_e}}, \quad \lambda_1 = n_e(n_e-1)\tilde{\lambda}, \quad \lambda_2 = -2\Lambda. \tag{2.9}$$

In the case of the $R + aR^2$ theory, and indeed any higher-order theory given by the potential (1.13), the appropriate dynamical system is obtained by taking Eqs. (2.8) and setting $s = 3$,

$$g_1 = \frac{m+2}{2\sqrt{m+1}}, \quad g_2 = \frac{2(n-1)-m}{2(n-1)\sqrt{m+1}}, \quad g_3 = \frac{1}{\sqrt{m+1}}, \tag{2.10a}$$

$$\alpha_3 = \frac{n-1}{n}, \quad \beta_3 = \frac{1}{n}, \tag{2.10b}$$

and

$$\lambda_1 = \frac{-\tilde{\varepsilon}^{n-1}(n-1)}{2na_2}, \quad \lambda_2 = \frac{-\tilde{\varepsilon}(n-1)^2}{2na_2}, \quad \lambda_3 = \frac{(n-1)}{2a_2}. \tag{2.10c}$$

As in the case of the Kaluza-Klein models the phase space has a great many symmetries which greatly simplify the analysis. Equations (2.8b) and (2.8e) ensure that trajectories cannot cross either the $W=0$ or the $Z=0$ subspaces. These two subspaces correspond physically to the cases in which $\lambda_2 = 0$ and $\lambda_1 = 0$, respectively, with $\lambda_i = 0$, $i \geq 3$, also in both cases. As we have written them, Eqs. (2.8) are valid for $W \geq 0$ and $Z \geq 0$. It is possible to make the equations valid for all W and for all Z by introducing modulus signs in the terms $W^{2\alpha_i}$ and $Z^{2\beta_i}$ which involve fractional powers of Z and W. However, this merely introduces a trivial symmetry between trajectories in the $W > 0$ and $W < 0$ portions of the phase space, and between trajectories in the $Z < 0$ and $Z > 0$ portions of the phase space. Thus we may restrict our attention to $Z \geq 0$ and $W \geq 0$ without loss of generality.

The hyperboloid defined by $\bar{\lambda} = 0$, or

$$P + \lambda_1 Z^2 + \lambda_2 W^2 + \sum_{i=3}^{s} \lambda_i W^{2\alpha_i} Z^{2\beta_i} = 0, \tag{2.11}$$

similarly forms a surface which trajectories cannot cross. It partitions the phase space into the two physically distinct regions with $\bar{\lambda} > 0$ and $\bar{\lambda} < 0$.

If $W=0$ and $g_1 \neq 0$ then

$$V = \frac{g_1(g_1-g_2)(mX+c_1) + [1+(m-1)g_1g_2]\,Y}{1+(m-1){g_1}^2}, \tag{2.12}$$

while if $Z=0$ and $g_2 \neq 0$ then

$$Y = \frac{-g_2(g_1-g_2)(mX+c_2) + [1+(m-1)g_1g_2]\,V}{1+(m-1){g_2}^2}, \tag{2.13}$$

where c_1 and c_2 are arbitrary constants. (If $g_1 = 0$ or $g_2 = 0$ we have instead $V = Y + \text{const.}$) Physically (2.12) is equivalent to $s=1$ (i.e., $\lambda_i = 0$, $i \geq 2$), while (2.13) gives the same system with $\lambda_1 \to \lambda_2$ and $g_1 \to g_2$. Thus in each case a further degree of freedom can be integrated out, giving rise to a three-dimensional autonomous system. The properties of such systems were studied in [28] and [29]. Further simplifications arise if in addition one of the constants $\bar{\lambda}$ or λ_1 (or λ_2 as appropriate) is zero. In these cases it is in fact possible to integrate the field equations exactly. For completeness we list these solutions in Appendix B.

B. The (anti–)de Sitter subspaces

In addition to the $W=0$, $Z=0$, and $\bar{\lambda}=0$ subspaces, there is at least one other three-dimensional subspace which exists for all $s \geq 2$, which was not studied in detail in [29]. These subspaces may be identified by noting that solutions for which σ is constant globally form a special class. The actual value which this constant takes may be determined from the field equations. In particular, we find that if $V=Y$ and $W=\gamma Z$ ($\gamma \geq 0$), where

$$\lambda_1 g_1 + \lambda_2 g_2 \gamma^2 + \sum_{i=3}^{s} \lambda_i g_i \gamma^{2\alpha_i} = 0, \tag{2.14}$$

then the field equations (2.8) reduce to the three-dimensional system

$$X' = -\Lambda Z^2 - \frac{(m-1)P}{m}, \tag{2.15a}$$

$$Y' = -\Lambda Z^2 - P, \tag{2.15b}$$

$$Z' = YZ, \tag{2.15c}$$

where

$$\Lambda = \frac{-1}{m}\left(\lambda_1 + \lambda_2\gamma^2 + \sum_{i=3}^{s}\lambda_i\gamma^{2\alpha_i}\right), \tag{2.15d}$$

and P now simplifies down to

$$P = m(X-Y)\left[(m+1)X-(m-1)Y\right]. \tag{2.15e}$$

In the case $s=2$, for example, (2.14) is satisfied by any g_1 and g_2 provided that

$$\gamma = \left(\frac{-g_1\lambda_1}{g_2\lambda_2}\right)^{1/2} \tag{2.16}$$

is real. For $s > 2$, (2.14) may have more than one solution.

If $s=1$ we of course obtain Eqs. (2.15) if $g_1 = 0$ and $\lambda_1 = -m\Lambda$. Consequently, Eqs. (2.15) are formally equivalent to those of the Schwarzschild–de Sitter solution in $m+2$ dimensions, and can be integrated explicitly in terms of the coordinate $\hat{r}$. We therefore obtain the solution

$$\hat{g}_{ab}d\hat{x}^a d\hat{x}^b = -\hat{\Delta}dt^2 + \hat{\Delta}^{-1}d\hat{r}^2 + \hat{r}^2\bar{g}_{\alpha\beta}d\bar{x}^\alpha d\bar{x}^\beta, \tag{2.17a}$$

with

$$\hat{\Delta} = \bar{\lambda} - \frac{C}{\hat{r}^{m-1}} - \frac{\Lambda\gamma^{-2g_1/(g_1-g_2)}\hat{r}^2}{m+1}, \tag{2.17b}$$

and constant scalar field

$$e^{2\kappa\sigma} = \gamma^{m/(g_1-g_2)}, \tag{2.17c}$$

C being an arbitrary constant. In the case of the higher-order theories with potentials (1.13), a conformal transformation back to the original metric yields the solution

$$ds^2 = -\Delta dt^2 + \Delta^{-1}dr^2 + r^2\bar{g}_{\alpha\beta}d\bar{x}^\alpha d\bar{x}^\beta, \tag{2.18a}$$

with

$$\Delta = \bar{\lambda} - \frac{2GM}{r^{m-1}} - \frac{\Lambda\gamma^{-2(n-1)/n}r^2}{m+1}, \tag{2.18b}$$

and $M = C/(2G\gamma^{2(m-1)(n-1)/(mn)})$. [We have rescaled t in obtaining (2.18).] The solutions are asymptotically de Sitter if $\Lambda > 0$ (or anti–de Sitter if $\Lambda < 0$). In the case of the $R + aR^2$ theory, we have

$$\Delta = \bar{\lambda} - \frac{2GM}{r^{m-1}} + \frac{mr^2}{(m+1)(m+2)(m-2)a}, \tag{2.19}$$

and consequently solutions of this type exist only for $m > 2$, i.e., for $D > 4$. These solutions are asymptotically anti–de Sitter if $a > 0$. For $n > 2$ solutions of the type (2.18) also exist for $D = 4$. In general, it is possible to find more than one branch of solutions—some of which are asymptotically de Sitter, and some of which are asymptotically anti–de Sitter.

If $M = 0$ we retrieve cosmological de Sitter solutions, the existence of which has been discussed previously by Barrow and Ottewill [8] for an arbitrary $f(R)$ Lagrangian in four dimensions, and by Madsen and Barrow [31] for more generalized higher-derivative Lagrangians in arbitrary dimensions.

The solutions (2.17) do not exhaust all possible solutions to Eqs. (2.15). In particular, a special class of solutions for which $\hat{r}(\xi)$ is everywhere constant are also admitted. These solutions may be determined by direct integration of (2.15) in the case that $Y = X$. Since $P = 0$ in this instance we have $Y' = X'$ and (2.15) reduces to a two-dimensional autonomous system. The equations can be readily integrated using the coordinate $Z = e^\eta$, and we find

$$\hat{g}_{ab}d\hat{x}^a d\hat{x}^b = \gamma^{2g_1/(g_1-g_2)} \times \left[-Z^2dt^2 + \frac{dZ^2}{C-\Lambda Z^2} + \left(\frac{(m-1)\bar{\lambda}}{\Lambda}\right)\bar{g}_{\alpha\beta}d\bar{x}^\alpha d\bar{x}^\beta\right], \tag{2.20}$$

where C is an arbitrary constant, and $Z > 0$. (We have used the freedom of rescaling t to remove an unphysical constant.) Thus we find a solution with $\bar{\lambda} > 0$ provided that $\Lambda > 0$. In the case of spherical symmetry these solutions are topologically a product of two-dimensional anti–de Sitter space with an m sphere—a type of Robinson-Bertotti solution. Solutions with $\bar{\lambda} < 0$ and $\Lambda < 0$ similarly include a product of two-dimensional de Sitter space with m-dimensional hyperbolic space.

To compare these results with our later studies of the five-dimensional phase space it is useful also to give a description of the three-dimensional phase space here.

Provided that $\Lambda \neq 0$ then the only critical points at a finite distance from the origin are the lines

$$Z = 0, \qquad Y = X, \tag{2.21}$$

and

$$Z = 0, \qquad Y = \frac{(m+1)X}{m-1}. \tag{2.22}$$

These solutions have $\bar{\lambda} = 0$ as well as $W = Z = 0$. Solutions lying in the $Z = 0$ plane, as depicted in Fig. 1, are Schwarzschild solutions. Equations (2.15) can be integrated directly in this case since

$$Y = \frac{m}{m-1}(X + k), \tag{2.23}$$

where k is an arbitrary constant. The critical points in the first quadrant are found to correspond to the limit $\xi \to -\infty$, while the critical points in the third quadrant correspond to the limit $\xi \to \infty$. We find that end points of trajectories on the line $Y = X$ correspond to horizons with $\hat{r} \to$ const, while end points of trajectories on the line $Y = (m+1)X/(m-1)$ corre-

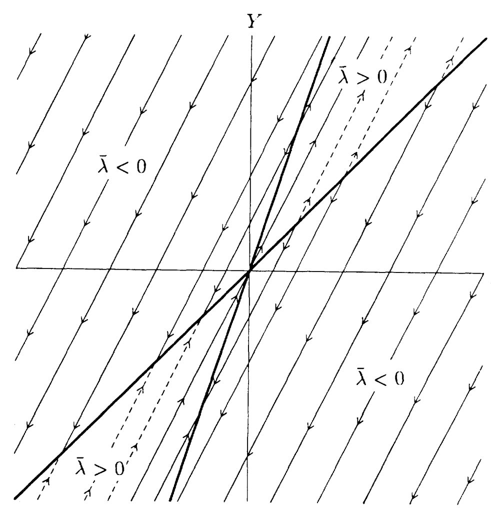

FIG. 1. Trajectories in the $W = 0$, $Z = 0$, $V = Y$ plane. The bold lines $Y = X$ and $Y = (m+1)X/(m-1)$ represent critical points which respectively correspond to the horizons of the positive-mass Schwarzschild solutions (dashed lines), and the singularities of the negative-mass Schwarzschild solutions. The trajectory through the origin is flat space.

spond to $\hat{r} \to 0$ singularities. The $Z = 0$ trajectories ending on the line $Y = X$ thus represent positive-mass Schwarzschild solutions, while those ending on the line $Y = (m+1)X/(m-1)$ represent negative-mass Schwarzschild solutions with naked singularities. The constant k in (2.23) is related to the Schwarzschild mass: the $k = 0$ trajectories represent flat space. Furthermore, if a trajectory which corresponds to a positive-mass Schwarzschild solution in one quadrant is traced back to the opposite quadrant, then one obtains the negative-mass Schwarzschild solution of the same absolute mass in the $\bar{\lambda} > 0$ region.

If $\Lambda = 0$ then additional critical points exist near the origin. Furthermore, (2.15) reduces to a two-dimensional autonomous system for all X, Y, and Z. We will defer discussion of this special subspace until Sec. III.

Small perturbations about the critical points (2.21) and (2.22) yield the eigenvalues $\lambda = 0, Y_0, 2X_0$ in the three-dimensional subspace [where $Y_0 = X_0$ or $Y_0 = (m+1)X_0/(m-1)$ as appropriate], and additional eigenvalues $\lambda = 0, Y_0$ for perturbations in the extra directions in the full five-dimensional phase space. The pattern of trajectories is therefore identical to that of the corresponding trajectories in [28] and [29]. In the three-dimensional subspace each critical point $(X_0, Y_0, 0)$ in the first (third) quadrant repels (attracts) a two-dimensional set of trajectories which lie approximately in the plane $Y = (mX \mp X_0)/(m-1)$. Since the trajectories are approximately planar the one zero eigenvalue corresponds to the degenerate direction perpendicular to this plane. In the five-dimensional phase space there is an extra degenerate direction, and the dimension of the set of trajectories repelled (attracted) for first (third) quadrant trajectories increases by one.

To complete the description of the phase space it is necessary to describe the critical points at infinity. Following [28] and [29], we introduce spherical polar coordinates

$$X = \rho \sin\theta \cos\phi, \qquad Y = \rho \sin\theta \sin\phi, \qquad Z = \rho\cos\theta. \tag{2.24}$$

The surface at infinity is then brought to a finite distance from the origin by the transformation

$$\rho = \bar{\rho}(1-\bar{\rho})^{-1}, \qquad 0 \le \bar{\rho} \le 1. \tag{2.25}$$

If we define a coordinate τ by $d\tau = \rho\, d\xi = \bar{\rho}(1-\bar{\rho})^{-1} d\xi$, then on the sphere at infinity, i.e., at $\bar{\rho} = 1$, $d\bar{\rho}/d\tau = 0$ identically while

$$\frac{d\theta}{d\tau} = \cos\theta\left\{-\Lambda\cos^2\theta\,(\cos\phi + \sin\phi) - \sin^2\theta\left[\sin\phi + \bar{P}_1\left(\frac{m-1}{m}\cos\phi + \sin\phi\right)\right]\right\}, \tag{2.26a}$$

$$\frac{d\phi}{d\tau} = \frac{1}{\sin\theta}\left[-\Lambda\cos^2\theta\,(\cos\phi - \sin\phi) + \sin^2\theta \bar{P}_1\left(\frac{m-1}{m}\sin\phi - \cos\phi\right)\right], \tag{2.26b}$$

where

$$\bar{P}_1 = m\left[(m+1)\cos^2\phi - 2m\cos\phi\sin\phi + (m-1)\sin^2\phi\right]. \tag{2.26c}$$

Four sets of critical points are found.

(i) First of all we obtain the end points of the lines of critical points $Y = X$ and $Y = (m+1)X/(m-1)$, for which $\bar{\lambda} = W = Z = 0$. The points located at

$$\theta = \frac{\pi}{2}, \qquad \phi = \frac{\pi}{4}, \frac{5\pi}{4} \tag{2.27a}$$

or

$$X = \pm\infty, \qquad Y = X, \qquad Z = 0, \tag{2.27b}$$

will be denoted L_1 and L_3. The points located at

$$\theta = \frac{\pi}{2}, \qquad \phi = \arctan\left(\frac{m+1}{m-1}\right) \tag{2.28a}$$

or

$$X = \pm\infty, \qquad Y = \left(\frac{m+1}{m-1}\right)X, \qquad Z = 0, \tag{2.28b}$$

will be denoted L_2 and L_4. As for points at a finite distance from the origin, the points L_1 and L_3 correspond to horizons, and L_2 and L_4 to $\hat{r} \to 0$ singularities. The points $L_{1,2}$ ($L_{3,4}$) repel (attract) a two-dimensional set of trajectories, which are unphysical, however, since they are confined to the sphere at infinity.

(ii) Two critical points, which we will denote M_1 and M_2, are located at

$$\theta = \frac{\pi}{2}, \qquad \phi = \arctan\left(\frac{m}{m-1}\right) \tag{2.29a}$$

or

$$X = \pm\infty, \qquad Y = \left(\frac{m}{m-1}\right)X, \qquad Z = 0, \tag{2.29b}$$

in the $\bar{\lambda} > 0$ portion of the phase space. These points correspond to the asymptotic region ($\hat{r} \to \infty$) of the Schwarzschild solutions which lie in the $Z = 0$ plane. These are the only trajectories which end on these points: they are found to be saddle points with respect to other directions in the phase space.

(iii) If $\Lambda < 0$ then there are two critical points, which we will denote S_1 and S_2, which are located at

$$\theta = \arctan\sqrt{-2\Lambda}, \qquad \phi = \frac{\pi}{4}, \frac{5\pi}{4} \tag{2.30a}$$

or

$$X = \pm\infty, \qquad Y = X, \qquad Z = \frac{X}{\sqrt{-\Lambda}}, \tag{2.30b}$$

in the $\bar{\lambda} < 0$ portion of the phase space. At these points we find that $\hat{r} \to$ const. These points correspond to the $Z \to \infty$ region of the Robinson-Bertotti-like solutions (2.20). An analysis of small perturbations in the three-dimensional subspace reveals that the point S_1 (S_2) in the first (third) quadrant is an attractor (repellor) for a two-dimensional set of trajectories: these are of course the solutions (2.20) with $\Lambda < 0$ and $\bar{\lambda} < 0$, which lie in the plane $Y = X$. This clarifies the nature of the corresponding points S_{1-8} in [29], which were not discussed in detail there.[6]

If $\Lambda > 0$ (and $\bar{\lambda} > 0$) then no $Z \to \infty$ region is defined since the maximum value Z can take is $(C/\Lambda)^{1/2}$. Thus instead of ending on the points $S_{1,2}$ trajectories move from one point on the $Z = 0$, $Y = X$ line to another point on the same line in the opposite quadrant. This pattern is made clear by Fig. 2 where we plot the $Y = X$ plane through the three-dimensional subspace.

(iv) If $\Lambda < 0$ then there are two critical points, which we will denote T_1 and T_2, which are located at

$$\theta = \arctan\left(\frac{-(2m^2+2m+1)\Lambda}{m+1}\right)^{1/2}, \tag{2.31a}$$

$$\phi = \arctan\left(\frac{m+1}{m}\right),$$

or

$$X = \pm\infty, \quad Y = \frac{(m+1)X}{m}, \quad Z = \frac{X}{m}\left(\frac{m+1}{-\Lambda}\right)^{1/2}, \tag{2.31b}$$

in the portion of the phase space with $\bar{\lambda} = 0$. Point T_1 (T_2) in the first (third) quadrant is found to attract (repel) a three-dimensional set of trajectories in the three-dimensional subspace. These points of course correspond to the asymptotic ($\hat{r} \to \infty$) region of the Schwarzschild–anti–de Sitter solutions (2.17) if $\bar{\lambda} > 0$. $T_{1,2}$ are end points for trajectories both in the $\bar{\lambda} > 0$ and $\bar{\lambda} < 0$ portions of the phase space.

If $\Lambda > 0$ then no asymptotic region is defined. Instead the trajectories move from the $Y = X$ or $Y = (m+1)X/(m-1)$ line in one quadrant to one of these two lines in the opposite quadrant. Trajectories starting and finishing on the $Y = X$ line with $\bar{\lambda} > 0$ correspond to the region of the positive-mass Schwarzschild–de Sitter solution between the Schwarzschild and de Sitter horizons. Trajectories with end points on both the $Y = X$ and $Y = (m+1)X/(m-1)$ lines represent the region of the negative-mass Schwarzschild–de Sitter solution between $\hat{r} = 0$ and the de Sitter horizon. Trajectories cannot have two end points on the $Y = (m+1)X/(m-1)$ line since they cannot cross the $Y = X$ plane.

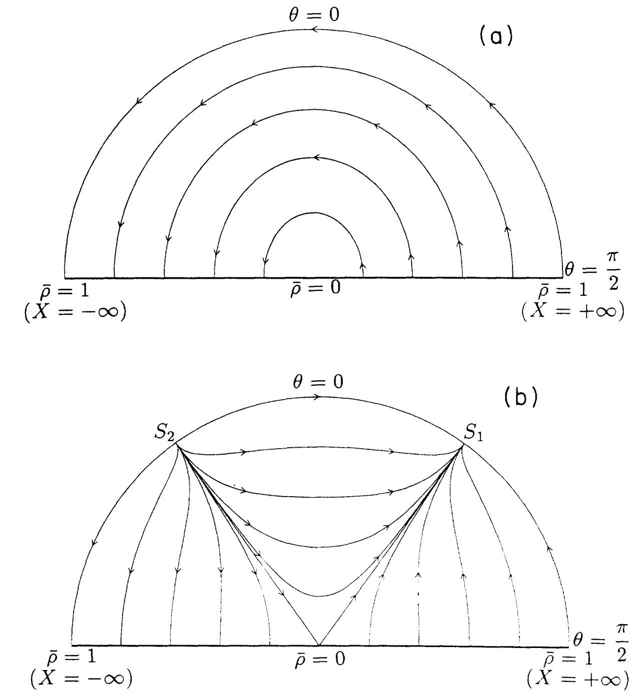

FIG. 2. The plane $V = Y = X$, $W = \gamma Z$, with radial coordinate $\bar{\rho}$ and angular coordinate θ: (a) $\Lambda > 0$, $\bar{\lambda} > 0$; (b) $\Lambda < 0$, $\bar{\lambda} < 0$. The solutions are given in (2.20) in terms of $Z = \bar{\rho}\cos\theta/(1-\bar{\rho})$.

The pattern of trajectories on the hemisphere at infinity of the three-dimensional subspace is sketched in Fig. 3. Although these trajectories are unphysical it is helpful to sketch them since by continuity arguments they will determine the behavior of the physical integral curves which lie within the hemisphere at infinity but near its surface.

C. The $W = 0$ and $Z = 0$ subspaces

These subspaces were discussed in [28] and [29] for particular values of g_1 and g_2. The properties for general g_1 and g_2 are similar. If $\Lambda \neq 0$ the only critical points at a finite distance from the origin in the full five-dimensional phase space have both $W = 0$ and $Z = 0$, and so are common to both subspaces. The critical points are located at $X = X_0$ and $Y = Y_0$, where

$$|X_0| \geq \left(\frac{m-1}{1+(m-1){g_1}^2}\right)^{1/2} |g_1 c_1|, \tag{2.32}$$

and Y_0 is given by solving the quadratic equation

$$(m-1){Y_0}^2 - 2mX_0Y_0 + (m+1-{g_1}^2){X_0}^2 + {g_1}^2{c_1}^2 = 0, \tag{2.33}$$

with V given by (2.12) with $X = X_0$.

[6]Since we are restricting the analysis to $W \geq 0$ and $Z \geq 0$ here, we of course obtain one-half or one-quarter as many critical points as in [28] and [29] for those points with $W \neq 0$ or $Z \neq 0$.

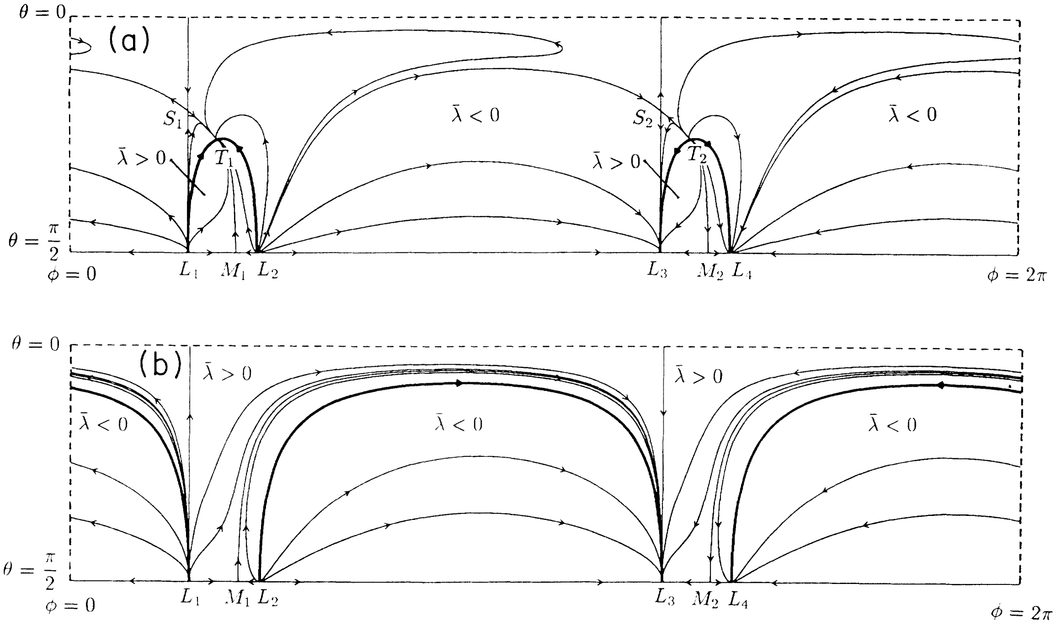

FIG. 3. The hemisphere at infinity for the three-dimensional (anti-)de Sitter subspace: (a) $\Lambda < 0$; (b) $\Lambda > 0$.

Consider the $W = 0$ subspace. If ${g_1}^2 \le m+1$ then the pattern of the trajectories is the same as in [28] and [29] since all critical points lie in the first and third quadrants. Points with $Y_0 > 0$ (first quadrant) correspond to the limit $\xi \to -\infty$, and those with $Y_0 < 0$ (third quadrant) to the limit $\xi \to +\infty$. Each point in the first (third) quadrant repels (attracts) a two-dimensional set of trajectories which lie approximately in the plane

$$Y = \left(\frac{m}{m-1}\right)(X \pm \{\,[1+(m-1){g_1}^2]{X_0}^2 -(m-1){g_1}^2{c_1}^2\}^{1/2}).$$

There is a zero eigenvalue corresponding to the degenerate direction perpendicular to this plane. In terms of the coordinate $\hat{r}$ one finds that $\hat{r} \to 0$ at all critical points except those for which $c_1 = mk$, which correspond to horizons. These special critical points are of course those lying in the $W = 0$, $Z = 0$, $V = Y$ plane (cf. Sec. II B). Figure 1 thus represents the plane which bisects the $W = 0$, $Z = 0$ subspace to pick out the Schwarzschild solutions.

If ${g_1}^2 > m+1$ then points for which $|X_0| < X_1$, where

$$X_1 = \frac{|g_1 c_1|}{\sqrt{1+(m-1){g_1}^2}}, \tag{2.34}$$

lie in the first and third quadrants, and have the same properties as for ${g_1}^2 < m+1$. If $|X_0| > X_1$, on the other hand, then one critical point lies in each quadrant. The two points in the first and third quadrants have the same properties as before. At the critical points in the second and fourth quadrants we find that $\hat{r} \to \infty$. As before there is one degenerate direction corresponding to a zero eigenvalue. However, both points are now saddle points with respect to the remaining two directions. Each critical point in the second (fourth) quadrant repels (attracts) one trajectory from the $\bar{\lambda} > 0$ region and one trajectory from the $\bar{\lambda} < 0$ region of the phase space: these are the $Z = 0$ solutions discussed in Appendix B 1, and depicted in Fig. 4(b). Each point in the second (fourth) quadrant similarly attracts (repels) one trajectory for each sign of λ_1: these trajectories are in fact the $\bar{\lambda} = 0$ solutions discussed in Appendix B 2, and depicted in Fig. 5(c). Their asymptotic form is given by (B27). Thus trajectories in the $\bar{\lambda} > 0$ and $\bar{\lambda} < 0$ regions of the phase space for which Z is not identically zero do not have end points in the second and fourth quadrants. If such solutions have asymptotic regions, the limit $\hat{r} \to \infty$ must be approached at critical points at the phase space infinity.

The phase space infinity of the $W = 0$ subspace can be studied once again by introducing coordinates (2.24), (2.25). The following equations are obtained for the angular coordinates on the $\bar{\rho} = 1$ sphere at infinity:

$$\frac{d\theta}{d\tau} = \cos\theta\left\{\frac{\lambda_1}{m}\cos^2\theta\left[\cos\phi + (1-{g_1}^2)\sin\phi\right] - \sin^2\theta\left[\sin\phi + \bar{P}_2\left(\frac{m-1}{m}\cos\phi + \sin\phi\right)\right]\right\}, \tag{2.35a}$$

$$\frac{d\phi}{d\tau} = \frac{1}{\sin\theta}\left[\frac{\lambda_1}{m}\cos^2\theta[(1-{g_1}^2)\cos\phi - \sin\phi] + \sin^2\theta\,\bar{P}_2\left(\frac{m-1}{m}\sin\phi - \cos\phi\right)\right], \tag{2.35b}$$

where

$$\bar{P}_2 = \frac{m}{1+(m-1)g_1{}^2}\left[\left(m+1-g_1{}^2\right)\cos^2\phi - 2m\cos\phi\sin\phi + (m-1)\sin^2\phi\right]. \tag{2.35c}$$

Four sets of critical points are found.

(i) Once again we obtain the end points of the curves of critical points with $\bar{\lambda} = W = Z = 0$. These points, which we will denote L_{5-8}, are located at

$$\theta = \frac{\pi}{2}, \qquad \phi = \arctan\left(\frac{m \pm \sqrt{1+g_1{}^2}}{m-1}\right), \tag{2.36a}$$

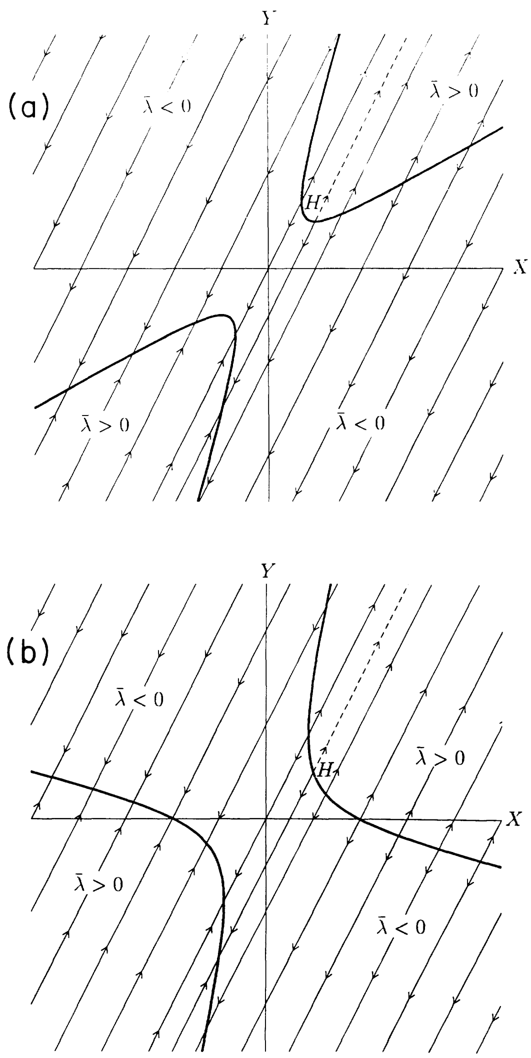

FIG. 4. The projection of trajectories in the $W = 0$, $Z = 0$ subspace onto the X,Y plane, with V given by (2.12), for nonzero c_1: (a) $g_1{}^2 < m+1$; (b) $g_1{}^2 > m+1$. The broken line corresponds to the Schwarzschild solution. The bold lines represent sections through the cone of critical points $\bar{\lambda} = 0$, $W = 0$, $Z = 0$. If $c_1 = 0$ we obtain the section which bisects the cone, so that instead of being hyperbolas the critical points fall on the lines $Y = [1/(m-1)]\{m \pm [1+(m-1)g_1{}^2]^{1/2}\}$.

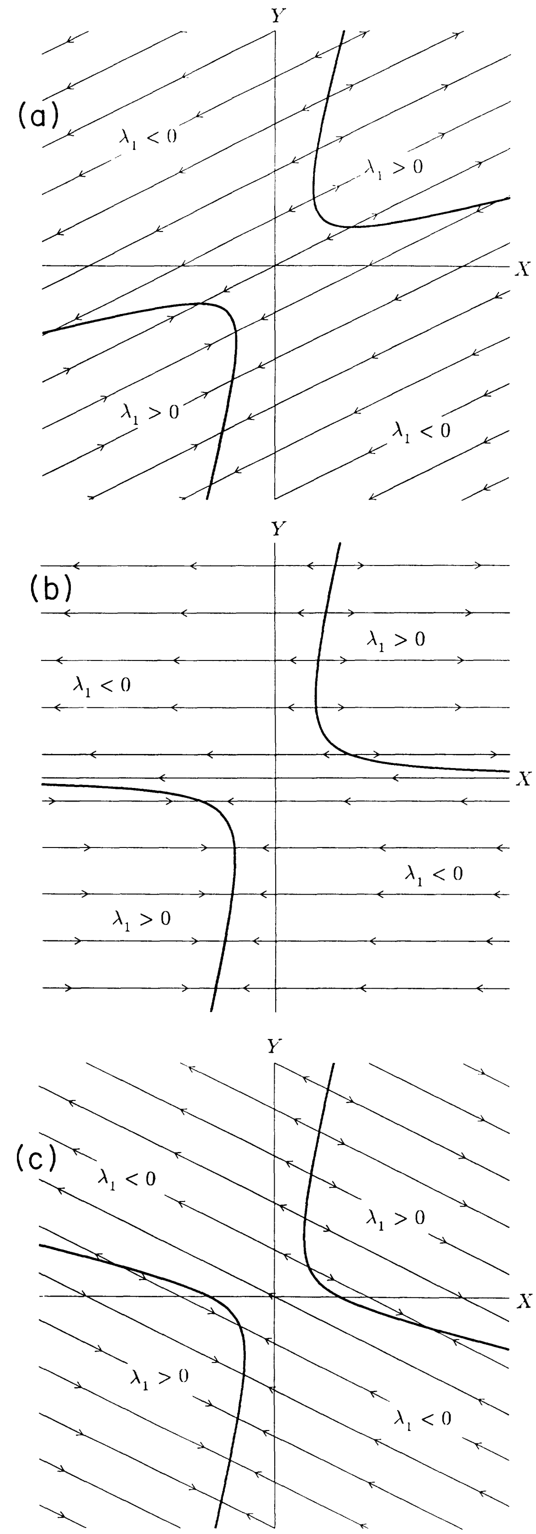

FIG. 5. The projection of trajectories in the $W = 0$, $\bar{\lambda} = 0$ subspace onto the X,Y plane, with V given by (2.12), for nonzero c_1: (a) $g_1{}^2 < m+1$; (b) $g_1{}^2 = m+1$; (c) $g_1{}^2 > m+1$. The bold lines represent the same critical points as in Fig. 4.

or

$$X = \pm\infty, \quad Y = X\left(\frac{m \pm \sqrt{1+{g_1}^2}}{m-1}\right), \quad Z = 0. \tag{2.36b}$$

These points have the same properties as those outlined above for the appropriate cases of the $\bar{\lambda} = W = Z = 0$ points at finite distances from the origin.

(ii) We once again obtain the critical points M_1 and M_2 given by (2.29). As before, these points, corresponding to asymptotically flat solutions, are found to act as saddles with respect to all trajectories other than the two-dimensional bunch of $W = Z = 0$ Schwarzschild solutions.

(iii) If $\lambda_1 > 0$ and ${g_1}^2 \le m+1$ or if $\lambda_1 < 0$ and ${g_1}^2 \ge m+1$ then there are two critical points, which we will denote N_1 and N_2, which are located at

$$\theta = \arctan\left[\lambda_1\left(\frac{m+1-{g_1}^2}{m} + \frac{m}{m+1-{g_1}^2}\right)\right]^{1/2}, \tag{2.37a}$$

$$\phi = \arctan\left(\frac{m+1-{g_1}^2}{m}\right)$$

or

$$X = \pm\infty, \qquad Y = \left(\frac{m+1-{g_1}^2}{m}\right)X,$$
$$Z = \left(\frac{m+1-{g_1}^2}{m\lambda_1}\right)^{1/2} X, \tag{2.37b}$$

on the $\bar{\lambda} = 0$ surface. In the full five-dimensional phase space these points also have

$$W = 0, \qquad V = \left(\frac{m+1-g_1g_2}{m}\right)X. \tag{2.37c}$$

If ${g_1}^2 \le 1$ then the point N_1 (N_2) in the first (third) quadrant attracts (repels) a three-dimensional set of trajectories in the three-dimensional subspace. (N_1 attracts all trajectories in the $\bar{\lambda} > 0$ region, and some $\bar{\lambda} < 0$ trajectories if ${g_1}^2 < 1$.) If $1 < {g_1}^2 \le m+1$ then N_1 (N_2) only attracts (repels) the two-dimensional set of trajectories lying in the $\bar{\lambda} = 0$ surface, and acts as a saddle with respect to other directions. If ${g_1}^2 = m+1$ then N_1 and N_2 are degenerate with the points L_5 and L_7. If ${g_1}^2 > m+1$ then the point N_1 (N_2) lying in the fourth (second) quadrant repels (attracts) a three-dimensional set of trajectories in the subspace.

(iv) If $\lambda_1 > 0$ then there are two critical points, which we will denote P_1 and P_2, which are located at

$$\theta = \arctan\left(\frac{2\lambda_1}{m}\left[1+(m-1){g_1}^2\right]\right)^{1/2}, \quad \phi = \frac{\pi}{4}, \frac{5\pi}{4} \tag{2.38a}$$

or

$$X = \pm\infty, \qquad Y = X,$$
$$Z = \left(\frac{m}{\lambda_1\left[1+(m-1){g_1}^2\right]}\right)^{1/2} X. \tag{2.38b}$$

In the full five-dimensional phase space these points also have

$$W = 0, \qquad V = \left(\frac{m{g_1}^2 - g_1g_2 + 1}{1+(m-1){g_1}^2}\right)X. \tag{2.38c}$$

If ${g_1}^2 < 1$ these points lie in the $\bar{\lambda} < 0$ portion of the phase space, while if ${g_1}^2 > 1$ they have $\bar{\lambda} > 0$. If ${g_1}^2 = 1$ they are degenerate with points N_1 and N_2.

If ${g_1}^2 < 1$ the point P_1 (P_2) in the first (third) quadrant attracts (repels) a two-dimensional set of $\bar{\lambda} < 0$ trajectories, acting as a saddle with respect to other directions. The two-dimensional separatrix separates $\bar{\lambda} < 0$ trajectories with an end point on N_1 (N_2) from trajectories with two end points on the $\bar{\lambda} = W = Z = 0$ curve. If ${g_1}^2 \ge 1$ then P_1 (P_2) attracts (repels) a three-dimensional set of trajectories: all $\bar{\lambda} > 0$ solutions apart from those lying in the $Z = 0$ plane.

In Figs. 6 and 7 we sketch trajectories on the hemisphere at infinity of the $W = 0$ subspace for the various cases which give distinct behaviors.

The properties of the $Z = 0$ subspace follow by symmetry upon making the substitutions $Y \leftrightarrow V$, $Z \leftrightarrow W$, $\lambda_1 \to \lambda_2$, and $g_1 \leftrightarrow g_2$ in the above discussion. To set our notation, the critical points on the sphere at infinity located at

$$X = \pm\infty, \qquad V = \left(\frac{m+1-{g_2}^2}{m}\right)X,$$
$$W = \left(\frac{m+1-{g_2}^2}{m\lambda_2}\right)^{1/2} X, \tag{2.39}$$
$$Z = 0, \qquad Y = \left(\frac{(m+1-g_1g_2)}{m}\right)X,$$

will be denoted R_1 and R_2. The points located at

$$X = \pm\infty, \qquad V = X,$$
$$W = \left(\frac{m}{\lambda_2\left[1+(m-1){g_2}^2\right]}\right)^{1/2} X, \tag{2.40}$$
$$Z = 0, \qquad Y = \left(\frac{m{g_2}^2 - g_1g_2 + 1}{1+(m-1){g_2}^2}\right)X,$$

will be denoted Q_1 and Q_2.

D. Global properties of solutions

We turn now to the global properties of the solutions as deduced from the nature of trajectories in the full five-dimensional phase space with $\Lambda \neq 0$. The behavior of the trajectories may be pieced together in a relatively straightforward manner from the properties of trajectories in the subspaces already discussed, since if $\Lambda \neq 0$ the only critical points at infinity other than those already found are the extension of points L_{1-8} to the one-parameter family of critical points which coincide with the intersection of the $\bar{\lambda} = 0$, $Z = 0$, $W = 0$ surface and the sphere at infinity. We shall denote the whole set $\{L(y)\}$, where

$$\frac{m - \sqrt{1 + (m-1){g_1}^2}}{m-1} \leq y \leq \frac{m + \sqrt{1 + (m-1){g_1}^2}}{m-1} . \tag{2.41}$$

The points $L(y)$ are located at

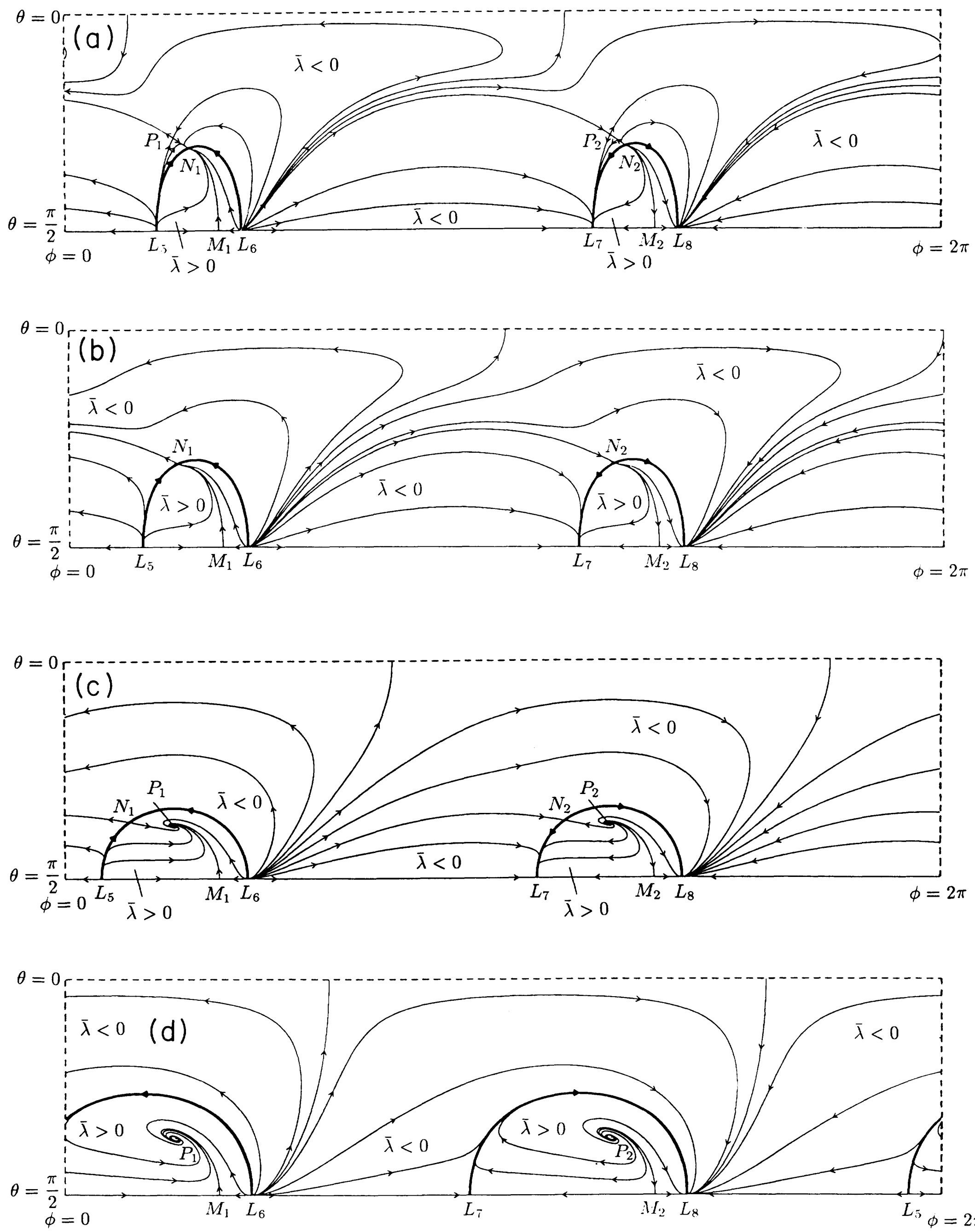

FIG. 6. The hemisphere at infinity for the three-dimensional $W = 0$ subspace with $\lambda_1 > 0$: (a) ${g_1}^2 < 1$; (b) ${g_1}^2 = 1$; (c) $1 < {g_1}^2 < m + 1$; (d) ${g_1}^2 \geq m + 1$.

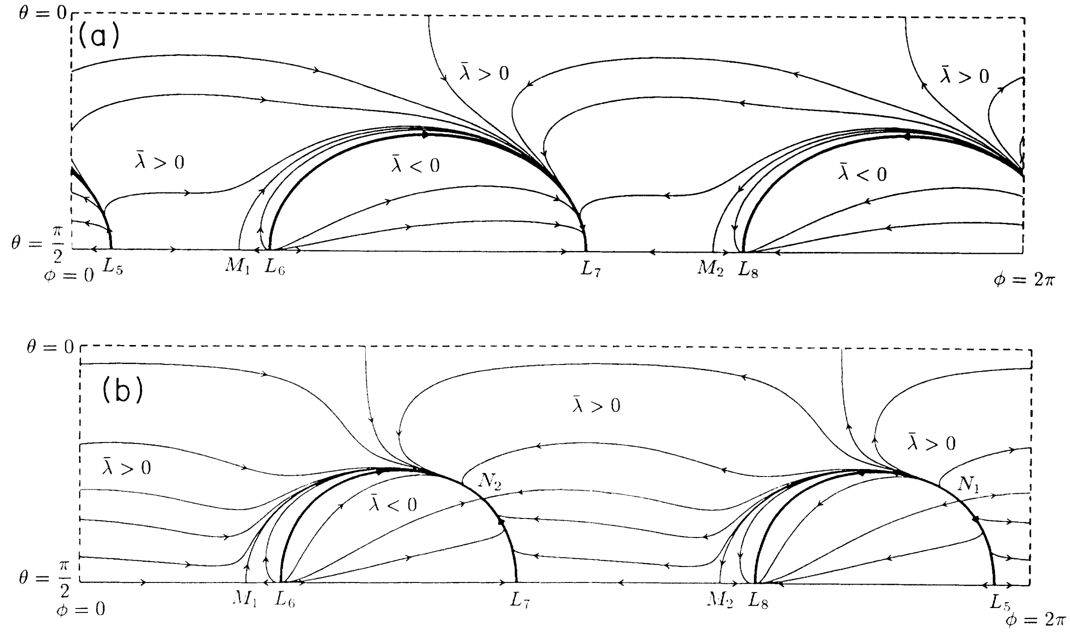

FIG. 7. The hemisphere at infinity for the three-dimensional $W = 0$ subspace with $\lambda_1 < 0$: (a) ${g_1}^2 \leq m+1$; (b) ${g_1}^2 > m+1$.

$$X = \pm\infty, \qquad Y = yX, \qquad Z = 0, \qquad W = 0,$$
$$V = \left(\frac{mg_1(g_1 - g_2) + \left[1 + (m-1)g_1 g_2\right] y \pm \sqrt{2my - (m-1)y^2 + {g_1}^2 - m - 1}}{1 + (m-1){g_1}^2} \right) X. \tag{2.42}$$

Generically, apart from a few exceptions[7] trajectories which are not confined to the sphere at infinity have at least one critical point on the $\bar{\lambda} = W = Z = 0$ curve. Provided that ${g_1}^2 \leq m+1$ and ${g_2}^2 \leq m+1$, then all such critical points take values X_0, Y_0, and V_0 which are either all positive, or all negative. They respectively either repel or attract a three-dimensional set of integral curves in the five-dimensional phase space, the remaining two directions being degenerate. The points all correspond to $\hat{r} = 0$ singularities with the exception of the points with $Y_0 = V_0$ which correspond to horizons.

If ${g_1}^2 > m+1$ or ${g_2}^2 > m+1$ then in addition to the critical points for which X_0, Y_0, and V_0 are all of the same sign, critical points of mixed signs also exist. These critical points are saddle points with respect to most trajectories in the phase space. The $W = Z = 0$ solutions form one separatrix of trajectories with end points at the saddle points; for these solutions we still have $\hat{r} \to 0$ as the critical points are approached. Other separatrices are formed by the $\bar{\lambda} = W = 0$ solutions, or the $\bar{\lambda} = Z = 0$ solutions, as appropriate. For these solutions $\hat{r} \to \infty$ as the saddle points are approached, as is discussed in Appendix B. For most trajectories, however, including the $\bar{\lambda} > 0$ ones which are of prime interest to us, it is necessary to examine the behavior of the solutions at the phase space infinity in order to determine the asymptotic ($\hat{r} \to \infty$) behavior of solutions for which an asymptotic region exists. Apart from such solutions there are also many trajectories with two end points on the $\bar{\lambda} = W = Z = 0$ curve which have no asymptotic region. Generally, they connect two points at which $\hat{r} \to 0$. However, a subset with two end points in the $V = Y$, $W = \gamma Z$ subspace describes Schwarzschild–de Sitter-like solutions, as was discussed in Sec. II B.

All critical points at infinity other than the points $L(y)$ and $S_{1,2}$ are found to correspond to $\hat{r} \to \infty$ provided that $0 < {g_1}^2 < m+1$ and $0 < {g_2}^2 < m+1$. If $g_1 = 0$ then the points $P_{1,2}$ correspond to $\hat{r} \to$ const (indicating the presence of Robinson-Bertotti-like solutions), while the points $N_{1,2}$ still correspond to $\hat{r} \to \infty$. The same is true for the points $Q_{1,2}$ and $R_{1,2}$, respectively, if $g_2 = 0$. If ${g_1}^2 > m+1$ then points $N_{1,2}$ correspond to $\hat{r} \to 0$, while

[7] The $W = 0$, $\bar{\lambda} = 0$, ${g_1}^2 > 1$ solutions given by Eqs. (B29)–(B35) in Appendix B have end points on N_1 and N_2 on the sphere at infinity, and have no asymptotic region. A similar class of solutions exists for $Z = 0$. Also, in the case of the Robinson-Bertotti-type solutions (2.20), some trajectories join the points S_1 and S_2 [cf. Fig. 2(b)].

TABLE I. Asymptotic form of solutions for trajectories approaching critical points at infinity from within the sphere at infinity.

	Values of constants	$e^{2\hat{u}}$	$e^{2\hat{v}}$	$e^{2\kappa\sigma}$
$M_{1,2}$	$\bar{\lambda} > 0$	const	const	const
$N_{1,2}$	${g_1}^2 < m+1,\ \lambda_1 > 0,\ \bar{\lambda} = 0$	$\hat{r}^2$	$\hat{r}^{2({g_1}^2-1)}$	$\hat{r}^{mg_1}$
$P_{1,2}$	$g_1 \neq 0,\ \lambda_1 > 0,\ \text{sgn}\,\bar{\lambda} = \text{sgn}\,({g_1}^2 - 1)$	$\hat{r}^{2/{g_1}^2}$	const	$\hat{r}^{m/g_1}$
$Q_{1,2}$	$g_2 \neq 0,\ \lambda_2 > 0,\ \text{sgn}\,\bar{\lambda} = \text{sgn}\,({g_2}^2 - 1)$	$\hat{r}^{2/{g_2}^2}$	const	$\hat{r}^{m/g_2}$
$R_{1,2}$	${g_2}^2 < m+1,\ \lambda_2 > 0,\ \bar{\lambda} = 0$	$\hat{r}^2$	$\hat{r}^{2({g_2}^2-1)}$	$\hat{r}^{mg_2}$
$T_{1,2}$	$\Lambda < 0,\ \bar{\lambda} = 0$	$\hat{r}^2$	$\hat{r}^{-2}$	const

if ${g_2}^2 > m+1$ then the points $R_{1,2}$ similarly correspond to $\hat{r} \to 0$. To discuss the asymptotic form of the solutions it is perhaps more convenient to examine the behavior of the metric functions of the more usual Schwarzschild-type coordinates

$$\hat{g}_{ab}d\hat{x}^a d\hat{x}^b = -e^{2\hat{u}}dt^2 + e^{2\hat{v}}d\hat{r}^2 + \hat{r}^2\bar{g}_{\alpha\beta}d\bar{x}^\alpha d\bar{x}^\beta, \quad (2.43)$$

rather than (2.1a). In Table I we display the asymptotic form of the metric functions (2.43), and of the scalar field, for integral curves from regions of the phase space at a finite distance from the origin which approach each of the critical points on the sphere at infinity for which $\hat{r} \to \infty$.[8]

In order to classify the various solutions we must first of all determine the nature of the various critical points at infinity. It is straightforward but laborious to evaluate the eigenvalue spectrum for small perturbations about the points. In Table II we summarize the results for such an analysis. We display the eigenvalues for the points with $X > 0$. For the corresponding points with $X < 0$ the sign of the eigenvalues is simply reversed.

The qualitative behavior of the trajectories is largely dependent on the values of g_1 and g_2, apart from the case of points $M_{1,2}$. These points, the only ones which correspond to solutions asymptotically flat in terms of $\hat{r}$, are end points for a three-dimensional set of solutions for all values of g_1 and g_2. These solutions are just those lying in the $W = Z = 0$ subspace, which is physically equivalent to $\lambda_1 = \lambda_2 = \cdots = \lambda_s = 0$. Thus models with nonzero λ_i possess no solutions which are asymptotically flat in terms of $\hat{r}$ if $\Lambda \neq 0$.

The eigenvalues for small perturbations near the points $N_{1,2}$, $P_{1,2}$, $Q_{1,2}$, and $R_{1,2}$ are essentially independent of the constants g_i, $i \geq 3$. The only exception is one eigenvalue at each point in the case that at least one of the α_i or β_i is less than one-half. On account of the ordering (2.2) we also have

$$0 < \alpha_3 < \alpha_4 < \cdots < \alpha_{s-1} < \alpha_s < 1, \quad (2.44)$$

and

$$1 > \beta_3 > \beta_4 > \cdots > \beta_{s-1} > \beta_s > 0. \quad (2.45)$$

Thus either α_3 or β_s has the smallest value of the α_i and β_i—this value being important in defining coordinates at the phase space infinity which lead to a well-defined spectrum of linearized perturbations. If we define

$$g_\alpha = \begin{cases} g_3 & \text{if } \alpha_3 < \beta_s, \\ g_2 & \text{otherwise,} \end{cases} \quad (2.46)$$

and

$$g_\beta = \begin{cases} g_s & \text{if } \beta_s < \alpha_3, \\ g_2 & \text{otherwise,} \end{cases} \quad (2.47)$$

then we find that one of the eigenvalues at the points $N_{1,2}$, $P_{1,2}$ depends on the factor $(g_1 - g_\alpha)$, while one of the eigenvalues at the points $Q_{1,2}$, $R_{1,2}$ depends on the factor $(g_\beta - g_2)$. However, since both these factors are positive for each choice of α and β there is no qualitative difference between the alternatives.

In Table III we summarize the nature of the set of solutions with end points at $N_{1,2}$ and $P_{1,2}$. The corresponding results for $R_{1,2}$ and $Q_{1,2}$, respectively, may be obtained by substituting $g_1 \to g_2$, $W \to Z$. We display the dimension, $d_\mathcal{A}$, of the maximal set, $\mathcal{A}$, of trajectories with end points at each point—for ${g_1}^2 \leq m+1$ this means the dimension of the set of trajectories attracted to (repelled from) the point N_1 (N_2), and vice versa if ${g_1}^2 > m+1$. In the case of the point P_1 (P_2) it means the dimension of the set of trajectories attracted (repelled). Each point with $d_\mathcal{A} < 5$ will also be the end point for a $(5 - d_\mathcal{A})$-dimensional separatrix of saddle-point trajectories. Of most interest are the points which are end points for a five-dimensional set of trajectories, as they represent solutions with the most typical behavior. If $g_1 > 0$ then such points ($N_{1,2}$) exist only when ${g_1}^2 > m+1$ and $\lambda_1 < 0$. Such points exist for all $g_1 < 0$ if $\lambda_1 > 0$: for ${g_1}^2 \leq 1$ they are the points $N_{1,2}$, and for ${g_1}^2 > 1$ the points $P_{1,2}$. For the higher-derivative theories, however, $g_1 > 0$ and ${g_1}^2 < m+1$ and thus the points $N_{1,2}$ or $P_{1,2}$ are end points for at most a four-dimensional set of solutions.

The nature of the set of solutions with end points at $S_{1,2}$ and $T_{1,2}$ is dependent on both the constants g_i and

[8] If ${g_1}^2 > m+1$ or ${g_2}^2 > m+1$ then we will also have $\hat{r} \to \infty$ for $\bar{\lambda} = 0$ trajectories which approach points $L(y)$ with $y < 0$. However, such trajectories are confined to the sphere at infinity and thus do not represent physical integral curves, and so we omit them.

TABLE II. Eigenvalues of critical points at infinity. The eigenvalues for small perturbations which are degenerate have the degeneracy listed in parentheses. The values of y and v listed are defined by (2.41) and $V = vX$ in (2.42).

	Eigenvalues (with degeneracies)
$L(y)$	$0,\ (2);\ 2;\ y;\ v.$
M_1	$-1,\ (3);\ \frac{1}{m-1},\ (2).$
N_1	$\frac{-1}{m}\left(m+1-g_1{}^2\right),\ (3);\ \frac{2}{m}\left(g_1{}^2-1\right);\ \frac{g_1}{m}\left(g_1-g_\alpha\right).$
P_1	$-1,\ (2);\ \frac{-1}{2}\left(1\pm\sqrt{\frac{9+(m-9)g_1{}^2}{1+(m-1)g_1{}^2}}\right);\ \frac{g_1\left(g_1-g_\alpha\right)}{1+(m-1)g_1{}^2}.$
Q_1	$-1,\ (2);\ \frac{-1}{2}\left(1\pm\sqrt{\frac{9+(m-9)g_2{}^2}{1+(m-1)g_2{}^2}}\right);\ \frac{-g_2\left(g_\beta-g_2\right)}{1+(m-1)g_2{}^2}.$
R_1	$\frac{-1}{m}\left(m+1-g_2{}^2\right),\ (3);\ \frac{2}{m}\left(g_2{}^2-1\right);\ \frac{-g_2}{m}\left(g_\beta-g_2\right).$
S_1	$-2;\ -1;\ 1;\ \frac{1}{2m}\left(-m\pm\sqrt{m^2-\frac{8m(g_1-g_2)\Lambda_g}{\Lambda}}\right).$
T_1	$\frac{-(m+1)}{m},\ (2);\ \frac{-2}{m};$
	$\frac{1}{2m}\left(-m-1\pm\sqrt{(m+1)^2-\frac{8(m+1)(g_1-g_2)\Lambda_g}{\Lambda}}\right).$

the constants λ_i. However, the three eigenvalues corresponding to directions which lie within the anti-de Sitter subspace $V = Y$, $W = \gamma Z$ are independent of the g_i and λ_i, and so any differences are determined by the remaining two eigenvalues, which are given by the solutions of the equations

$$\lambda_s{}^2 + \lambda_s + \frac{2(g_1 - g_2)\Lambda_g}{m\Lambda} = 0, \qquad (2.48a)$$

for the points $S_{1,2}$, and

$$\lambda_T{}^2 + \left(\frac{m+1}{m}\right)\lambda_T + \frac{2(m+1)(g_1-g_2)\Lambda_g}{m^2\Lambda} = 0, \qquad (2.48b)$$

for the points $T_{1,2}$, where

$$\Lambda_g = \lambda_2 g_2 \gamma^2 + \sum_{i=3}^{s} \lambda_i g_i \alpha_i \gamma^{2\alpha_i} = -\lambda_1 g_1 - \sum_{i=3}^{s} \lambda_i g_i \beta_i \gamma^{2\alpha_i}. \qquad (2.48c)$$

TABLE III. Nature of trajectories that approach points $N_{1,2}$ and $P_{1,2}$.

$N_{1,2}$	$d_{\mathcal{A}}$	Nature of solutions $\{\mathcal{A}\}$
$0 \le g_1 < 1$	4	$W=0,\ \bar\lambda \ge 0;\ W=0,\ \bar\lambda < 0$
$-1 < g_1 < 0$	5	$\bar\lambda \ge 0;\ \bar\lambda < 0$
$g_1 = 1$	4	$W=0,\ \bar\lambda \ge 0$; 3-dim. $W=0,\ \bar\lambda < 0$ separatrix
$g_1 = -1$	5	$\bar\lambda \ge 0$; 4-dim. $\bar\lambda < 0$ separatrix
$g_1 > 0,\ 1 < g_1{}^2 \le m+1$	3	$W=0,\ \bar\lambda = 0$
$g_1 < 0,\ 1 < g_1{}^2 \le m+1$	4	$\bar\lambda = 0$
$g_1 > 0,\ g_1{}^2 > m+1$	5	$\bar\lambda \ge 0;\ \bar\lambda < 0$
$g_1 < 0,\ g_1{}^2 > m+1$	4	$W=0,\ \bar\lambda \ge 0;\ W=0,\ \bar\lambda < 0$

$P_{1,2}$	$d_{\mathcal{A}}$	Nature of solutions $\{\mathcal{A}\}$
$0 \le g_1 < 1$	3	$W=0,\ \bar\lambda < 0$ separatrix
$-1 < g_1 < 0$	4	$\bar\lambda < 0$ separatrix
$g_1 > 1$	4	$W=0,\ \bar\lambda > 0$
$g_1 < -1$	5	$\bar\lambda > 0$

For all choices we find only two possibilities: the dimension of the maximal set of trajectories with end points at $S_{1,2}$ is either three or four; while the dimension of the maximal set of trajectories with end points at $T_{1,2}$ is either four or five. The latter was true in the case of the Kaluza-Klein models studied in [29].

This completes our classification of the solutions for Einstein gravity coupled to a scalar field with potential (1.10) for which the constant Λ, defined by (2.14) and (2.15d), is nonzero. For nonzero λ_i solutions which are asymptotically flat in terms of $\hat{r}$ do not exist. This is not the case if $\Lambda = 0$, however, as we shall see in the next section.

III. SOLUTIONS WITH A SCHWARZSCHILD SUBSPACE (SPECIAL POLYNOMIAL R THEORIES)

If there exist solutions γ of (2.14) such that the constant Λ defined by (2.15d) vanishes, then the structure of the phase space is significantly altered. No such solutions exist if $s = 2$. However, if $s \geq 3$, which applies in the case of the $R + aR^2$ theory and other higher-derivative models with potential (1.13), then such solutions are possible. We see from (2.17) that if $\Lambda = 0$ we immediately obtain the Schwarzschild solution—thus all solutions lying in the three-dimensional subspace $V = Y$, $W = \gamma Z$ (for appropriate γ), and not just those in the $Z = 0$ plane, are Schwarzschild solutions. This may be verified by direct integration since Eqs. (2.15) are now equivalent to the equations which lead to the solutions of Appendix B1, with the added restriction that $c_1 = mk$.

If we compare the phase-space structure with that of the subspace of Sec. II B, we observe that the points $S_{1,2}$ and $T_{1,2}$ do not exist in the Schwarzschild subspace. Instead, we have an additional one-parameter family of critical points at a finite distance from the origin. These may be parametrized in terms of their arbitrary value of $Z = Z_0 > 0$, and are located at

$$X = Y = V = 0, \qquad Z = Z_0, \qquad W = \gamma Z_0. \quad (3.1)$$

We shall denote these points $O(Z_0)$, and the point at infinity ($Z_0 \to \infty$) will be denoted O_1. These are the only additional critical points.

Small perturbations about the points $O(Z_0)$ yield three zero eigenvalues, indicating a high degree of degeneracy. However, their properties with regard to the structure of the phase space may be ascertained from the following observations. Firstly, if $Y = X$ or $Y = (m+1)X/(m-1)$, then $P = 0$, $X' = 0$, and $Y' = 0$, so that the motion of the trajectories is entirely in the Z direction, as is depicted in Fig. 8. Furthermore, since (2.23) is now true for all solutions we may foliate the three-dimensional subspace by a stack of planes $Y = m(X + k)/(m - 1)$ to which trajectories are confined. Each plane is described by a two-dimensional autonomous system

$$X' = X^2 - m^2k^2, \quad (3.2a)$$

$$Z' = \frac{m}{m-1}(X + k)\, Z, \quad (3.2b)$$

with solution

$$Z = C_0\, |X + mk|^{1/2}\, |X - mk|^{(m+1)/[2(m-1)]} \quad (3.3a)$$

if $k \neq 0$, or

$$Z = C_0 X^{m/(m-1)}, \quad (3.3b)$$

if $k = 0$, where C_0 is an arbitrary constant in both cases. The pattern of these trajectories is depicted in Fig. 9. The critical points at $\pm mk$ are of course those that lie on the $\bar{\lambda} = W = Z = 0$ curve, and as expected they either attract or repel a two-dimensional bunch of solutions in the subspace. From Figs. 8 and 9 we can see that for finite Z_0 the points $O(Z_0)$ neither attract nor repel any trajectories, and so there are no solutions for which they are end points. Clearly, however, all trajectories which reach infinity other than those in the planes depicted in Fig. 8 approach the point O_1. This is borne out by the plot of the hemisphere at infinity with coordinates θ and ϕ defined by (2.24), which is shown in Fig. 10(a).

The fact that O_1 appears to be an end point for many solutions is in fact something of a misnomer, which arises from the fact that $Z \sim X^{m/(m-1)}$ as $X \to \infty$ for all solutions with an asymptotic region, so that Z grows more rapidly than either X or Y. Thus although the solutions (3.3) do not have $X = 0$ or $Y = 0$ as $Z \to \infty$, they are nonetheless projected onto the north pole, O_1, if coordinates (2.24) are used. This degeneracy can be lifted if instead of (2.24) we use coordinates (ρ, θ, ϕ) defined by

$$X = \rho \sin\theta \cos\phi\,, \qquad Y = \rho \sin\theta \sin\phi\,,$$
$$Z^{(m-1)/m} = \rho\cos\theta\,, \quad (3.4)$$

and perform the same analysis as before. This yields the plot Fig. 10(b) for the sphere at infinity. In these coordinates we obtain a line of critical points on the sphere at infinity with arbitrary angle θ:

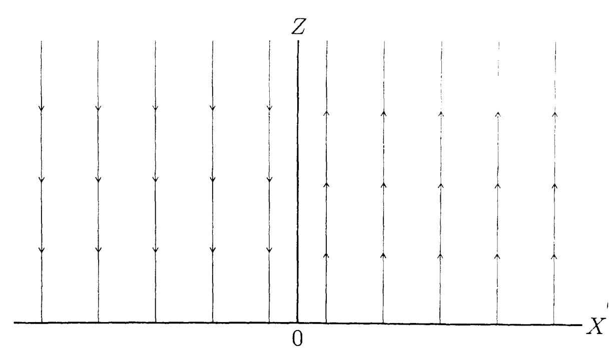

FIG. 8. The plane $Y = X$ or $Y = (m + 1)X/(m - 1)$ in the Schwarzschild subspace. Both axes are one-parameter families of critical points. The solution in the $Y = X$ plane corresponds to the $\bar{\lambda} = 0$, $\Lambda = 0$ limit of (2.20), while the solution in the $Y = (m + 1)X/(m - 1)$ plane corresponds to the limit $g_1 = 0$, $\bar{\Lambda}_1 = 0$ of the solutions (B13)–(B21) in Appendix B.

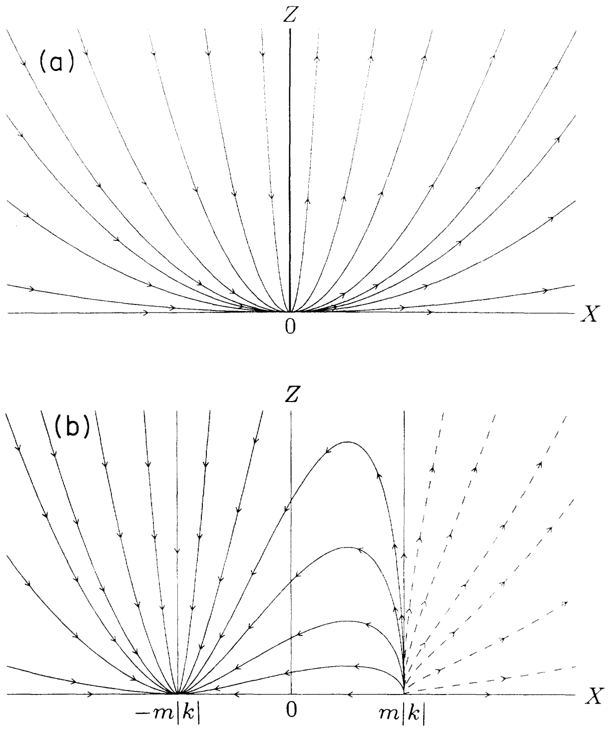

FIG. 9. The planes $Y = m(X+k)/(m-1)$ in the Schwarzschild subspace: (a) $k = 0$; (b) $k < 0$. The dashed trajectories are positive-mass Schwarzschild solutions. If $k > 0$ these trajectories are located in the $X < -m|k|$ region.

$$0 < \theta < \frac{\pi}{2}, \qquad \phi = \arctan\left(\frac{m}{m-1}\right) . \tag{3.5}$$

Perturbations about these critical points still yield one zero eigenvalue, corresponding to the degenerate direction θ, but each point in the first (third) quadrant is found to attract (repel) a two-dimensional set of solutions from regions of the phase space at a finite distance from the origin.

The crucial question now is: do any trajectories which lie outside the Schwarzschild subspace have an end point at O_1? If such trajectories do exist, and if any of them curve back to another end point in the Schwarzschild subspace corresponding to a horizon, then we would have solutions with nontrivial scalar fields which violate the no-hair theorems. The corresponding Schwarzschild solution in any equivalent higher-derivative theory would be nonunique. The answer is that the eigenvalues for perturbations in the two extra directions are given by

$$\lambda_O^{\,2} = \frac{-2}{m}(g_1 - g_2)\Lambda_g , \tag{3.6}$$

where Λ_g is given by (2.48c). Provided that $\Lambda_g > 0$ then O_1 will be a "center" with respect to the extra directions, and the only solutions with an end point there will indeed be just the Schwarzschild solutions. If $\Lambda_g < 0$, however, then O_1 will be a saddle point with respect to the extra two directions, and a four-dimensional separatrix of solutions will have an end point at O_1 within the full five-dimensional phase space. Unfortunately, it is not

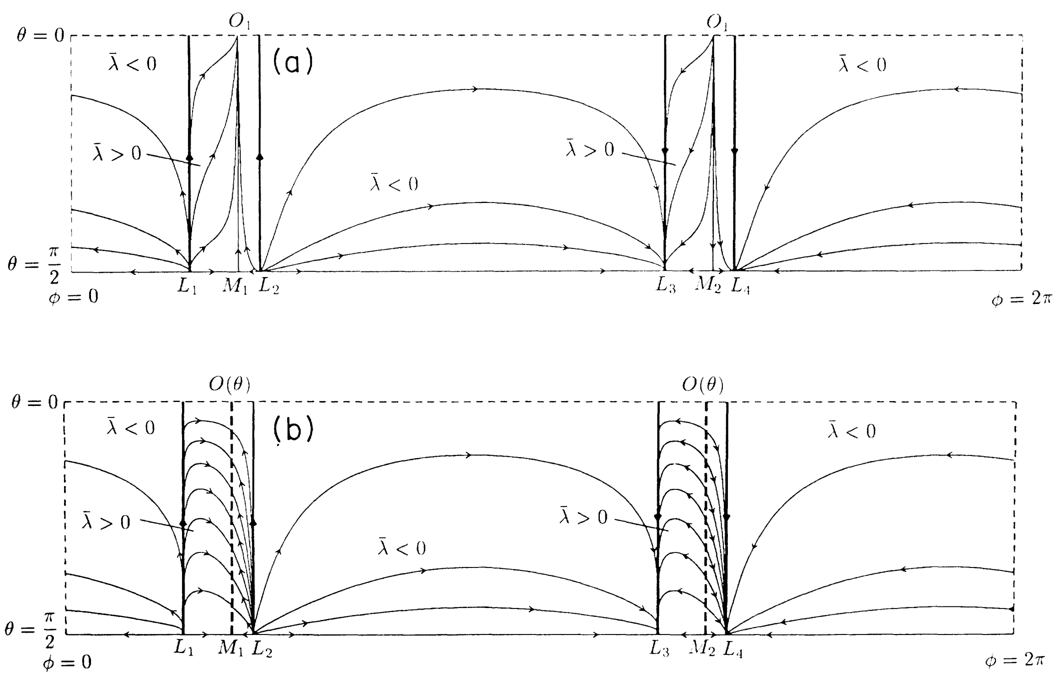

FIG. 10. The hemisphere at infinity for the three-dimensional Schwarzschild subspace: (a) with θ, ϕ defined by (2.24); (b) with θ, ϕ defined by (3.4).

immediately obvious whether the solutions with an end point at O_1 which lie outside the Schwarzschild subspace have a second end point on the $\bar{\lambda} = W = Z = 0$ curve at a point corresponding to a horizon or to a singularity. Thus we cannot extend the no-hair theorem, or the corresponding uniqueness theorem for higher-derivative theories, to the case in which $\Lambda_g < 0$.

For the $R + aR^2$ theory, or indeed for any action (1.7) with coefficients (1.11), which gives rise to an equivalent potential (1.13), we find that a Schwarzschild subspace is obtained only for $\tilde{\varepsilon} = 1$ and $\gamma = 1$. Consequently, $\Lambda_g = m(n-1)/(4na\sqrt{m+1})$ and

$$\lambda_O^{\ 2} = \frac{-m}{4a(m+1)}, \tag{3.7}$$

so that O_1 is an end point for Schwarzschild solutions only provided $a > 0$. This is of course the same condition required for Whitt's proof in the case of the $R + aR^2$ theory (cf. Appendix A).

To complete our proof of the uniqueness theorem for higher-derivative black holes it is still necessary to check the asymptotic behavior of the metric functions near the critical points at infinity for which $e^{2\kappa\sigma}$ is not asymptotically constant in terms of the physical radial coordinate r instead of the coordinate $\hat{r}$ used in Table I. The results for higher-derivative actions with equivalent potential (1.13) are listed in Table IV, in terms of metric functions for the physical metric in Schwarzschild-type coordinates:

$$\exp\left(\frac{-4\kappa\sigma}{m\sqrt{m+1}}\right)\hat{g}_{ab}\,d\hat{x}^a d\hat{x}^b = -e^{2u}dt^2 + e^{2v}dr^2 + r^2\bar{g}_{\alpha\beta}\,d\bar{x}^\alpha d\bar{x}^\beta. \tag{3.8}$$

(Near the points $T_{1,2}$ the solutions are of course still asymptotically anti–de Sitter in terms of the coordinate r as well as $\hat{r}$.) It is evident that, as required, none of these solutions are asymptotically flat.

From Table III, we see that for the higher-derivative theories points $N_{1,2}$ are end points for the three-dimensional set of $W = 0$, $\bar{\lambda} = 0$ solutions, while points $P_{1,2}$ are end points for the four-dimensional set of $W = 0$, $\bar{\lambda} > 0$ solutions. The constant g_2 can take either sign depending on the relative values of m and n, and its absolute value is only restricted by $g_2^{\ 2} < m + 1$. Thus the dimension of the maximal set of solutions with end points at $R_{1,2}$ and $Q_{1,2}$ is three, four, or five depending on the particular values of m and n. For $m = 2$ (i.e., the four-dimensional theory) and $n \geq 3$, points $R_{1,2}$ are end points for a five-dimensional set of solutions of either sign of $\bar{\lambda}$, while points $Q_{1,2}$ are similarly end points for a four-dimensional separatrix. If $m = 2$ and $n = 2$, the dimension of these sets is reduced by one with the additional restriction that $Z = 0$. In this case, or for any m and n such that $m = 2(n-1)$, the points $R_{1,2}$ correspond to asymptotically anti–de Sitter solutions. These solutions appear to form a special class unrelated to those of Sec. II B.

The points $S_{1,2}$ and $T_{1,2}$ exist provided that $m \neq 2(n-1)$, γ being given by roots of the polynomial (2.14):

$$[2(n-1)-m]\left(\tilde{\varepsilon}\gamma^{2/n}\right)^n - 2n\left(\tilde{\varepsilon}\gamma^{2/n}\right)^{n-1} + m + 2 = 0, \tag{3.9}$$

for which $\tilde{\varepsilon}\gamma^{2/n} \neq 1$ (and $\gamma > 0$). If we factor out the Schwarzschild root (3.9) reduces to the polynomial

$$[2(n-1)-m]\left(\tilde{\varepsilon}\gamma^{2/n}\right)^{n-1} - (m+2)\sum_{j=0}^{n-2}\left(\tilde{\varepsilon}\gamma^{2/n}\right)^j = 0. \tag{3.10}$$

For the $R + aR^2$ theory ($n = 2$), for example, the only solution (with $\gamma > 0$) is

$$\tilde{\varepsilon} = -1, \qquad \gamma = \left(\frac{m+2}{m-2}\right), \qquad m \geq 2, \tag{3.11}$$

which gives

$$\gamma^{-1}\Lambda = \frac{-m}{(m+2)(m-2)a}, \tag{3.12}$$

as is appropriate for the exact solution (2.19). Thus the points $S_{1,2}$ and $T_{1,2}$ exist only for $a < 0$, $m > 2$. We find that

$$\Lambda_g = \frac{-m}{8a\sqrt{m+1}}\left(\frac{m+2}{m-2}\right) \tag{3.13}$$

and consequently a four-dimensional set of trajectories

TABLE IV. Asymptotic form of the higher-derivative theory metric for polynomial R actions (1.7) with coefficients (1.11) in terms of the physical radial coordinate r, for the critical points at infinity at which r is not proportional to $\hat{r}$.

	Values of constants	e^{2u}	e^{2v}
$N_{1,2}$	$\frac{\tilde{\varepsilon}^{n-1}}{a} < 0,\ \bar{\lambda} = 0$	r^2	r^m
$P_{1,2}$	$\frac{\tilde{\varepsilon}^{n-1}}{a} < 0,\ \bar{\lambda} > 0$	$r^{4/(m+2)}$	const
$Q_{1,2}$	$m \neq 2(n-1),\ \frac{\tilde{\varepsilon}}{a} < 0,$ sgn $\bar{\lambda}$ = sgn $[m - 4n(n-1)]$	$r^{4(n-1)(2n-1)/[m-2(n-1)]}$	const
$R_{1,2}$	$\frac{\tilde{\varepsilon}}{a} < 0,\ \bar{\lambda} = 0$	r^2	$r^{[m-4n(n-1)]/[(n-1)(2n-1)]}$

have end points at $S_{1,2}$, and a five-dimensional set of trajectories have end points at the points $T_{1,2}$, with anti–de Sitter asymptotics. If $a > 0$, the structure of the phase space is completely different: the points $S_{1,2}$ and $T_{1,2}$ do not exist. Instead we find a class of solutions with two end points lying in the $V = Y$, $W = (m+2)Z/(m-2)$ subspace—these are asymptotically de Sitter. In the case of four dimensions ($m = 2$) no subspaces of purely Schwarzschild–(anti-)de Sitter solutions exist in the $R + aR^2$ theory. However, the points $R_{1,2}$ (which are end points for a four-dimensional set of solutions) nonetheless have anti–de Sitter asymptotics.

IV. GENERAL POLYNOMIAL R THEORIES

The analysis of the previous two sections applies exactly only to those polynomial R theories with coefficients (1.11). For Lagrangians of degree $n \geq 3$ in R we shall see, however, that in the general case the field equations become equivalent to those given by one exponential sum potential or another near all the critical points. Consequently, the global properties of all theories polynomial in R can be obtained from the results of Secs. II and III. As an independent check on this argument in one case, however, we shall first treat the general cubic theory in detail.

A. The $R + aR^2 + bR^3$ theory

The general cubic theory with $f = R + aR^2 + bR^3$ is described by a potential (1.6) which contains terms nonpolynomial in $\exp[\kappa\sigma/\sqrt{D-1}]$. The potential (1.6) may be conveniently written as

$$\mathcal{V} = \frac{-1}{4\kappa^2}\left\{\lambda_1 \exp\left(\frac{-4g_1\kappa\sigma}{m}\right) + \lambda_2 \exp\left(\frac{-4g_2\kappa\sigma}{m}\right)\left[\frac{b\varepsilon}{|b\varepsilon|} + \frac{\lambda_4}{\lambda_2}\exp\left(\frac{-8(g_1-g_2)\kappa\sigma}{3m}\right)\right]^{3/2} + \lambda_3 \exp\left(\frac{-4g_3\kappa\sigma}{m}\right)\right\}, \tag{4.1a}$$

where

$$g_1 = \frac{m+2}{2\sqrt{m+1}}, \qquad g_2 = \frac{-(m-4)}{4\sqrt{m+1}}, \qquad g_3 = \frac{1}{\sqrt{m+1}}, \tag{4.1b}$$

$$\lambda_1 = \frac{2a\varepsilon}{27b^2}\left(a^2 - \frac{9}{2}b\right), \qquad \lambda_2 = \frac{-2\varepsilon\tilde{\varepsilon}a}{3\sqrt{3}\,|a||b|^{1/2}}, \qquad \lambda_3 = \frac{a}{3b}, \qquad \lambda_4 = \frac{\lambda_2}{3b}\left(a^2 - 3b\right), \tag{4.1c}$$

and $\tilde{\varepsilon}$ is now defined by

$$\tilde{\varepsilon} = \begin{cases} 1 & \text{if } 1 + 3bR/a > 0, \\ -1 & \text{if } 1 + 3bR/a < 0. \end{cases} \tag{4.1d}$$

As before, ε is defined by (1.2b). The quantities g_1, g_2, and g_3 are still identical to those given by (2.10a) for $n = 3$. However, the coefficients λ_i now differ in general from those given by (2.10c)—the new coefficients λ_i reduce to the values (2.10c), with $\lambda_4 = 0$, in the special case $a^2 = 3b$, when the potential is an exponential sum. Our new definition for $\tilde{\varepsilon}$ also reduces to the former definition (1.13b) if $a^2 = 3b$. The Einstein-scalar field equations with the potential (4.1) are given by

$$\zeta'' = (m-1)^2\bar{\lambda}e^{2\zeta} + \lambda_1 e^{2\eta} + \lambda_2 e^{2\chi}\left[\frac{b\varepsilon}{|b\varepsilon|} + \frac{\lambda_4}{\lambda_2}e^{2(\eta-\chi)}\right]^{3/2} + \lambda_3 e^{2(\eta+2\chi)/3}, \tag{4.2a}$$

$$\begin{aligned}\eta'' = {} & m(m-1)\bar{\lambda}e^{2\zeta} + \frac{1}{m}\left(m+1-{g_1}^2\right)\lambda_1 e^{2\eta} + \frac{1}{m}\left(m+1-g_1g_2\right)\lambda_2 e^{2\chi}\left[\frac{b\varepsilon}{|b\varepsilon|} + \frac{\lambda_4}{\lambda_2}e^{2(\eta-\chi)}\right]^{3/2} \\ & + \frac{1}{m}(m+1-g_1g_3)\lambda_3 e^{2(\eta+2\chi)/3} - \frac{g_1}{m}(g_1-g_2)\lambda_4 e^{2(2\eta+\chi)/3}\left[\frac{b\varepsilon}{|b\varepsilon|} + \frac{\lambda_4}{\lambda_2}e^{2(\eta-\chi)}\right]^{1/2},\end{aligned} \tag{4.2b}$$

$$\begin{aligned}\chi'' = {} & m(m-1)\bar{\lambda}e^{2\zeta} + \frac{1}{m}\left(m+1-g_1g_2\right)\lambda_1 e^{2\eta} + \frac{1}{m}\left(m+1-{g_2}^2\right)\lambda_2 e^{2\chi}\left[\frac{b\varepsilon}{|b\varepsilon|} + \frac{\lambda_4}{\lambda_2}e^{2(\eta-\chi)}\right]^{3/2} \\ & + \frac{1}{m}\left(m+1-g_2g_3\right)\lambda_3 e^{2(\eta+2\chi)/3} - \frac{g_2}{m}(g_1-g_2)\lambda_4 e^{2(2\eta+\chi)/3}\left[\frac{b\varepsilon}{|b\varepsilon|} + \frac{\lambda_4}{\lambda_2}e^{2(\eta-\chi)}\right]^{1/2},\end{aligned} \tag{4.2c}$$

with the constraint

$$(m+1)\zeta'^2+\frac{2m\zeta'(g_2\eta'-g_1\chi')}{g_1-g_2}+\frac{1+(m-1){g_2}^2}{(g_1-g_2)^2}\eta'^2-2\frac{1+(m-1)g_1g_2}{(g_1-g_2)^2}\eta'\chi'+\frac{1+(m-1){g_1}^2}{(g_1-g_2)^2}\chi'^2$$
$$+(m-1)\bar{\lambda}e^{2\zeta}+\frac{\lambda_1e^{2\eta}}{m}+\frac{\lambda_2e^{2\chi}}{m}e^{2\chi}\left[\frac{b\varepsilon}{|b\varepsilon|}+\frac{\lambda_4}{\lambda_2}e^{2(\eta-\chi)}\right]^{3/2}+\frac{\lambda_3}{m}e^{2(\eta+2\chi)/3}=0, \quad (4.2d)$$

where the variables ζ, η, and χ are defined as before by (2.3)–(2.5). If we now define the variables X, Y, Z, V, and W by (2.7) once again, and use the constraint (4.2d) to eliminate the $e^{2\zeta}$ terms from (4.2a)–(4.2c) we obtain the first-order system

$$X'=\frac{1}{m}(\lambda_1Z^2+\lambda_2\Gamma^2+\lambda_3W^{4/3}Z^{2/3})-\frac{(m-1)P}{m}, \quad (4.3a)$$

$$Y'=\frac{-1}{m}\left[({g_1}^2-1)\lambda_1Z^2+(g_1g_2-1)\lambda_2\Gamma^2+(g_1g_3-1)\lambda_3W^{4/3}Z^{2/3}+g_1(g_1-g_2)\lambda_4Z^{4/3}\Gamma^{2/3}\right]-P, \quad (4.3b)$$

$$V'=\frac{-1}{m}\left[(g_1g_2-1)\lambda_1Z^2+({g_2}^2-1)\lambda_2\Gamma^2+(g_2g_3-1)\lambda_3W^{4/3}Z^{2/3}+g_2(g_1-g_2)\lambda_4Z^{4/3}\Gamma^{2/3}\right]-P, \quad (4.3c)$$

$$Z'=YZ, \quad (4.3d)$$

$$W'=VW, \quad (4.3e)$$

where

$$\Gamma\equiv\left[\frac{b\varepsilon}{|b\varepsilon|}W^{4/3}+\frac{\lambda_4}{\lambda_2}Z^{4/3}\right]^{3/4}, \quad (4.3f)$$

and P is still given by (2.8f).

The differences between Eqs. (4.3) and (2.8) do not give rise to any significant changes to the analysis of Secs. II and III. The position of all critical points $W=0$ and $Z=0$, and in particular the $\bar{\lambda}=W=Z=0$ surface, is unchanged, as are the eigenvalues for small perturbations about them. The number of critical points at infinity with $Z\neq 0$ or $W\neq 0$ is also the same, and their location is more or less the same. For points $R_{1,2}$ and $Q_{1,2}$ the definitions (2.39) and (2.40) are the same, if we now use λ_2 as defined in (4.2c). These points now exist if $\varepsilon\tilde{\varepsilon}/a<0$. In the case of points $N_{1,2}$ and $P_{1,2}$, we must make the replacement $\lambda_1\to\lambda_1+\lambda_4\left[\lambda_4/\lambda_2\right]^{1/2}$ in the definitions (2.37) and (2.38), with λ_1, λ_2, and λ_4 given by (4.2c). These points now exist if

$$b>0,\quad a^2\geq 3b,\quad \tilde{\varepsilon}=+1,\quad \varepsilon a<0, \quad (4.4)$$

or if

$$b>0,\quad a^2\geq 3b,\quad \tilde{\varepsilon}=-1,\quad (a^2-4b)\,\varepsilon a>0. \quad (4.5)$$

In the case of points $S_{1,2}$, $T_{1,2}$, and O_1, our previous definitions are once again valid if we replace Eqs. (2.14) and (2.15d), which respectively define γ and Λ, by

$$\lambda_1g_1+\lambda_2g_2\left[\frac{\lambda_4}{\lambda_2}\pm\gamma^{4/3}\right]^{3/2}+\lambda_3g_3\gamma^{4/3}+\lambda_4(g_1-g_2)\left[\frac{\lambda_4}{\lambda_2}\pm\gamma^{4/3}\right]^{1/2}=0, \quad (4.6)$$

and

$$\Lambda=\frac{-1}{m}\left\{\lambda_1+\lambda_2\left[\frac{\lambda_4}{\lambda_2}\pm\gamma^{4/3}\right]^{3/2}+\lambda_3\gamma^{4/3}\right\}. \quad (4.7)$$

Here the plus (minus) sign corresponds to $b\varepsilon>0$ ($b\varepsilon<0$). The eigenvalues about the various critical points are unchanged from those given in Table II, if Λ_g is defined by

$$\Lambda_g=\gamma^{4/3}\left\{\lambda_2g_2\left[\frac{\lambda_4}{\lambda_2}\pm\gamma^{4/3}\right]^{1/2}+\frac{2}{3}\lambda_3g_3+\frac{1}{3}\lambda_4(g_1-g_2)\left[\frac{\lambda_4}{\lambda_2}\pm\gamma^{4/3}\right]^{-1/2}\right\}$$
$$=-\lambda_1g_1-\frac{1}{3}\lambda_3g_3\gamma^{4/3}-\left[\frac{\lambda_4}{\lambda_2}\pm\gamma^{4/3}\right]^{-1/2}\left[\frac{1}{3}(2g_1+g_2)\gamma^{4/3}+\frac{\lambda_4}{\lambda_2}g_1\right], \quad (4.8)$$

instead of (2.48c).

A Schwarzschild subspace ($\Lambda=0$) is obtained only for $\varepsilon=+1$ and $\tilde{\varepsilon}=+1$, the appropriate solution then being given by $\gamma=1$. Eigenvalues for small perturbations within the subspace give the same results as previously, and furthermore we find that for perturbations in the additional two directions the eigenvalues are once again

$$\lambda_O^2 = \frac{-m}{4a(m+1)}\,. \tag{4.9}$$

Thus just as in the special case of polynomial actions with coefficients (1.11), the black hole uniqueness theorem applies to solutions with $a > 0$. Most significantly, this result applies independently of the coefficient b of the R^3 term.

With regard to the (anti–)de Sitter subspaces, we find that (4.6) has the solution

$$\gamma^{4/3} = \begin{cases} \dfrac{(m+2)\left[(m-2)a^2 - 4(m-4)b \pm a\sqrt{(m-2)^2a^2 - 4m(m-4)b}\right]}{2(m-4)^2 b\varepsilon} & \text{if } m \neq 4, \\ \dfrac{3\left(4b - a^2\right)}{a^2\varepsilon} & \text{if } m = 4, \end{cases} \tag{4.10}$$

where the sign of ε is taken so as to make the right-hand side (RHS) positive. Consequently,

$$\gamma^{-4/3}\Lambda = \begin{cases} \dfrac{-(m-2)a \mp \sqrt{(m-2)^2a^2 - 4m(m-4)b}}{2(m+2)(m-4)b} & \text{if } m \neq 4, \\ \dfrac{-1}{3a} & \text{if } m = 4. \end{cases} \tag{4.11}$$

Thus exact Schwarzschild–de Sitter-type solutions of the form (2.18) are also obtained for $m = 2$, in contrast with the $R + aR^2$ theory. Furthermore, although in the case of the $R + aR^2$ theory only anti–de Sitter solutions are admissable if $a > 0$, for the present theory both de Sitter and anti–de Sitter solutions are obtained for $a > 0$ if $m \neq 4$ (for either sign of b).

B. Theories of arbitrary degree

At the next order with $f = R + aR^2 + bR^3 + cR^4$ the polynomial giving R in terms of f' has three branches, giving rise to potentials such as

$$\mathcal{V} = \frac{\varepsilon}{4\kappa^2 c^3}\exp\left(\frac{-2D\kappa\sigma}{(D-2)\sqrt{D-1}}\right)\left\{\frac{b^4}{256} + ac^3 q - 2bc^3\Sigma_0 - 6c^4 q\left[\Sigma_+ + \Sigma_-\right] + 3c^4\left[\Sigma_+^2 + \Sigma_-^2\right]\right\}, \tag{4.12a}$$

where

$$q = \frac{1}{16c^2}\left(\frac{8}{3}ac - b^2\right), \tag{4.12b}$$

$$\Sigma_0 \equiv \frac{1}{64c^3}\left\{4abc - b^3 - 8c^2\left[1 - \varepsilon\exp\left(\frac{2\kappa\sigma}{\sqrt{D-1}}\right)\right]\right\}, \tag{4.12c}$$

and

$$\Sigma_\pm \equiv \left(\Sigma_0 \pm \sqrt{q^3 + \Sigma_0^2}\right)^{2/3}. \tag{4.12d}$$

Such a potential would lead once again to equations similar to (4.3), if the g_i of (2.10a) with $n = 4$ are used, but now with a further level of nesting of terms involving irrational roots. In the special case that $3b^2 = 8ac$ (i.e., $q = 0$), $\Sigma_+ = 2\Sigma_0$ and $\Sigma_- = 0$, so that the potential (4.12) and the resulting differential equations become almost identical to those of the $R + aR^2 + bR^3$ theory, the only differences being the values of the λ_i and the factors α_3 and β_3 [which appear implicitly in (4.3) in the fractional powers].

We will not study the theory generated by the potential (4.12) in detail, but remark that it should not lead to any differences from our former analysis any more substantial than, say, the differences between the cubic and quadratic theories. In particular, the uniqueness of the Schwarzschild solutions should still apply to theories with $a > 0$.

We will now show that the uniqueness theorem does indeed apply to theories with a polynomial R action of arbitrary degree n, if $a_2 > 0$. We note, first of all, that the Einstein-scalar field equations with a general potential $\mathcal{V}$, derived from the action (1.4a), are given by

$$\zeta'' = (m-1)^2\bar{\lambda}e^{2\zeta} - 4\kappa^2\mathcal{V}\exp\left[\frac{2(g_1\chi - g_2\eta)}{g_1 - g_2}\right], \tag{4.13a}$$

$$\eta'' = m(m-1)\bar{\lambda}e^{2\zeta} - \left[\frac{(m+1)4\kappa^2\mathcal{V}}{m} + \kappa g_1\frac{d\mathcal{V}}{d\sigma}\right]\exp\left[\frac{2(g_1\chi - g_2\eta)}{g_1 - g_2}\right], \tag{4.13b}$$

$$\chi'' = m(m-1)\bar{\lambda}e^{2\zeta} - \left[\frac{(m+1)4\kappa^2\mathcal{V}}{m} + \kappa g_2\frac{d\mathcal{V}}{d\sigma}\right]\exp\left[\frac{2(g_1\chi - g_2\eta)}{g_1 - g_2}\right], \tag{4.13c}$$

with the constraint

$$(m+1)\zeta'^2 + \frac{2m\zeta'(g_2\eta' - g_1\chi')}{g_1 - g_2} + \frac{1+(m-1){g_2}^2}{(g_1-g_2)^2}\eta'^2 - 2\frac{1+(m-1)g_1g_2}{(g_1-g_2)^2}\eta'\chi'$$
$$+\frac{1+(m-1){g_1}^2}{(g_1-g_2)^2}\chi'^2 + (m-1)\bar{\lambda}e^{2\zeta} - \frac{4\kappa^2\mathcal{V}}{m}\exp\left[\frac{2(g_1\chi - g_2\eta)}{g_1 - g_2}\right] = 0, \tag{4.13d}$$

where we have used the coordinates (2.1), and functions ζ, η, and χ defined by (2.3)–(2.5) with g_1 and g_2 as yet undetermined (apart from the assumption that $g_1 \neq g_2$). In (4.13) σ is assumed to be defined implicitly by the inverse relation to (2.4) and (2.5), viz.,

$$\kappa\sigma = \frac{m}{2}\left(\frac{\chi - \eta}{g_1 - g_2}\right). \tag{4.14}$$

For a general polynomial R action (1.1), (1.7) this implicit definition becomes

$$\varepsilon\exp\left(\frac{2\kappa\sigma}{\sqrt{m+1}}\right) = \varepsilon\exp\left(\frac{2(n-1)(\chi-\eta)}{n}\right) = 1 + \sum_{p=2}^{n} pa_pR^{p-1}, \tag{4.15}$$

on account of (1.2), while

$$\mathcal{V} = \frac{\varepsilon}{4\kappa^2}\exp\left(\frac{-4g_1\kappa\sigma}{m}\right)\sum_{p=2}^{n}(p-1)a_pR^p, \tag{4.16}$$

and

$$\frac{d\mathcal{V}}{d\sigma} = \frac{\varepsilon}{m\kappa}\exp\left(\frac{-4g_1\kappa\sigma}{m}\right)\left[(g_1-g_3)R + \sum_{p=2}^{n}(g_1 - pg_3)a_pR^p\right], \tag{4.17}$$

if we take g_1, g_2, and g_3 to be given by (2.10a) as before. Thus if we define variables X, Y, Z, V and W, by (2.7) we obtain the five-dimensional autonomous system of first-order differential equations

$$X' = \frac{-\varepsilon Z^2}{m}\sum_{p=2}^{n}(p-1)a_pR^p - \frac{(m-1)P}{m}, \tag{4.18a}$$

$$Y' = \frac{-\varepsilon Z^2}{m}\left\{g_1(g_1-g_3)R + \sum_{p=2}^{n}\left[\left({g_1}^2 - 1\right) - p\left(g_1g_3 - 1\right)\right]a_pR^p\right\} - P$$
$$= \frac{-\varepsilon Z^2}{4(m+1)}\left\{(m+2)R + \sum_{p=2}^{n}[m+2p]\,a_pR^p\right\} - P, \tag{4.18b}$$

$$V' = \frac{-\varepsilon Z^2}{m}\left\{g_2(g_1-g_3)R + \sum_{p=2}^{n}\left[(g_1g_2 - 1) - p\left(g_2g_3 - 1\right)\right]a_pR^p\right\} - P$$
$$= \frac{-\varepsilon Z^2}{4(n-1)(m+1)}\left\{[2(n-1)-m]\,R + \sum_{p=2}^{n}[-(2n+m) + 2(2n-1)p]\,a_pR^p\right\} - P, \tag{4.18c}$$

$$Z' = YZ, \tag{4.18d}$$

$$W' = VW, \tag{4.18e}$$

where R is now defined implicitly in terms of W and Z by

$$\varepsilon\left(\frac{W}{Z}\right)^{2(n-1)/n}=1+\sum_{p=2}^{n}pa_pR^{p-1}, \tag{4.18f}$$

and P is still defined by (2.8f).

The properties of the phase space defined by (4.18) differ little in their essentials from the examples studied earlier. We first note that an (anti-)de Sitter subspace is defined by the (constant) values of R given by roots of the polynomial

$$m+\sum_{p=2}^{n}[m-2(p-1)]\,a_pR^{p-1}=0. \tag{4.19}$$

The three-dimensional subspace is given by $V=Y$ and $W=\gamma Z$, where, by (4.18f),

$$\varepsilon\gamma^{2(n-1)/n}=1+\sum_{p=2}^{n}pa_pR^{p-1}, \tag{4.20}$$

R being a solution of (4.19). The field equations of the subspace are once again given by (2.15) with Λ now defined by

$$\Lambda=\frac{\varepsilon}{m}\sum_{p=2}^{n}(p-1)a_pR^p. \tag{4.21}$$

Our earlier results concerning the (anti-)de Sitter subspace follow through. In particular, we are lead to the Schwarzschild–de Sitter–type solutions (2.18), which in terms of the original physical metric are given by

$$ds^2=-\Delta dt^2+\Delta^{-1}dr^2+r^2\bar{g}_{\alpha\beta}d\bar{x}^\alpha d\bar{x}^\beta, \tag{4.22a}$$

with

$$\Delta=\bar{\lambda}-\frac{2GM}{r^{m-1}}-\frac{\left[\sum_{p=2}^{n}(p-1)a_pR^p\right]r^2}{m(m+1)\left[1+\sum_{p=2}^{n}pa_pR^{p-1}\right]}, \tag{4.22b}$$

R being a solution of (4.19). Furthermore, we once again obtain Robinson-Bertotti solutions of the form (2.20) for our new definitions of γ and Λ.

Let us now consider the critical points of the system (4.18). It can be readily seen from (4.18a)–(4.18c) that the only critical points at finite values of X, Y, V, Z, and W must all have $P=0$, and also either (i) $W=Z=0$;[9] or (ii) $R+\sum_{p=2}^{n}pa_pR^p=0$ and $R+\sum_{p=2}^{n}a_pR^p=0$. The only value of R which simultaneously satisfies both of the conditions (ii) is $R=0$. We therefore retrieve the same critical points as were found in Secs. II and III, viz., (i) the familiar $W=0$, $Z=0$, $\bar{\lambda}=0$ surface discussed in Sec. II; (ii) $X=Y=V=0$, $W=Z=Z_0$ ($R=0$). These are the points $O(Z_0)$ which lie in the Schwarzschild subspace, discussed in Sec. III.

The phase space at infinity may be studied either by a direct analysis of the four-sphere at infinity, using an approach similar to that described in Secs. II B and Sec. II C, or by the computationally more simple method of defining new variables δ, y, v, z, and w by

$$X=\frac{\pm1}{\delta},\qquad Y=\frac{\pm y}{\delta},\qquad V=\frac{\pm v}{\delta},$$
$$Z=\frac{\pm z}{\delta},\qquad W=\frac{\pm w}{\delta}, \tag{4.23}$$

and classifying the $\delta=0$ critical points of the resulting field equations

$$\pm\frac{d\delta}{d\tau}=\frac{\delta}{m}\left[z^2f_1+(m-1)P_\delta\right], \tag{4.24a}$$

$$\pm\frac{dy}{d\tau}=\frac{1}{m}\left\{(y-1)z^2f_1-g_1z^2f_2+P_\delta[(m-1)y-m]\right\}, \tag{4.24b}$$

$$\pm\frac{dv}{d\tau}=\frac{1}{m}\left\{(v-1)z^2f_1-g_2z^2f_2+P_\delta[(m-1)v-m]\right\}, \tag{4.24c}$$

$$\pm\frac{dz}{d\tau}=\frac{z}{m}\left[my+z^2f_1+(m-1)P_\delta\right], \tag{4.24d}$$

$$\pm\frac{dw}{d\tau}=\frac{w}{m}\left[mv+z^2f_1+(m-1)P_\delta\right], \tag{4.24e}$$

where $d\tau=\delta^{-1}d\xi$, $P_\delta\equiv\delta^2P$, and

$$f_1\equiv\varepsilon\sum_{p=2}^{n}(p-1)a_pR^p, \tag{4.24f}$$

$$f_2\equiv\varepsilon\left[(g_1-g_3)R+\sum_{p=2}^{n}(g_1-pg_3)\,a_pR^p\right], \tag{4.24g}$$

with R now defined implicity in terms of w and z. This method picks out all critical points at infinity apart from those with $X=0$. To obtain all critical points we must repeat the calculation with each of the phase-space coordinates in turn defined as $\pm1/\delta$.

It is convenient to divide the critical points at infinity arising from this analysis into three classes according to the behavior of the scalar curvature R at the points: (i) $R=0$; (ii) $R=\text{const}\neq0$; or (iii) $R\to\infty$.

(i) Critical points with $R=0$ have $f_1=0$ and $f_2=0$. Consequently the only possibilities admitted by the equations (4.24) are the points $L(y)$, defined by (2.41),

[9]Ostensibly it would seem that we can simply take $Z=0$ here. However, since R is defined implicity in terms of W/Z the summation terms in (4.18a)–(4.18c) will not vanish simply if the overall factor of Z^2 vanishes. The leading order R^n term within the summations is of order W^2Z^{-2}, and so for consistency one must also require that $W=0$ when setting $Z=0$.

(2.42), and the points $M_{1,2}$, defined by (2.29) (with $V = Y$ and $W = 0$ also). If we repeat the above analysis for the system of equations defined by putting $Z = 1/\delta$, with the other variables proportional to Z, then for $R = 0$ we also obtain the point O_1 defined by the $Z_0 \to \infty$ limit of (3.1).

(ii) If $R = \text{const} \neq 0$ then we must have $z \neq 0$ and $w \neq 0$ on account of (4.18f). Equations (4.24) then imply that $v = y$ and $f_2 = 0$. This latter condition is equivalent to (4.19) for $R \neq 0$. Thus such critical points must lie in the anti–de Sitter subspace, which as we have already discussed is well defined. By the analysis of Sec. II B, these critical points are therefore $S_{1,2}$ and $T_{1,2}$, where the γ and Λ of the defining relation (2.30b) and (2.31b) are now taken to be given by (4.20) and (4.21), respectively.

(iii) Obviously points at which $R \to \infty$ cannot correspond to regular solutions. However, it is nevertheless interesting to check whether the structure of the phase space is preserved when compared to our earlier examples. If $R \to \infty$, then, on account of (4.15),

$$R \sim \pm \left\{ \frac{1}{na_n} \left[\varepsilon \exp\left(\frac{2\kappa\sigma}{\sqrt{m+1}} \right) - 1 \right] \right\}^{1/(n-1)}. \qquad (4.25)$$

In the limit $R \to \infty$ the field equations are therefore equivalent to those derived from an exponential sum potential (1.10) consisting of a $n+1$ term, with g_1, g_2, and g_3 given by (2.10a) and

$$g_j = \frac{2(n-1) + m(j-3)}{2(n-1)\sqrt{m+1}}, \qquad 4 \leq j \leq n+1. \qquad (4.26)$$

We could also derive the values of the appropriate coefficients λ_i; however, they are not of much concern to us here. We merely note that since the field equations are equivalent to those of an exponential sum potential, the only critical points at which $R \to \infty$ are precisely the points $N_{1,2}$, $P_{1,2}$, $Q_{1,2}$, and $R_{1,2}$ (with appropriate λ_i in the definitions). The asymptotic form of the solutions is therefore given by Table I, or in terms of the physical metric, by Table IV. In particular, all of these critical points correspond to asymptotic regions in which the physical metric is not asymptotically flat.

Thus the only integral curves which join a critical point corresponding to a horizon to a critical point corresponding to an asymptotically flat region must be the trajectories with one end point on the curve formed by the intersection of the $W = 0$, $Z = 0$, $\bar{\lambda} = 0$ surface with the Schwarzschild subspace, and with a second end point at $M_{1,2}$ or O_1. It therefore only remains to determine the dimension of the set of such solutions. As before, this may be found by a linearized analysis of small perturbations about the points. This analysis will be unchanged from that of Secs. II and III since by (4.15) we have, to leading order in R,

$$\varepsilon \exp\left(\frac{2\kappa\sigma}{\sqrt{m+1}} \right) = 1 + 2aR + O(R^2). \qquad (4.27)$$

Consequently, the linearized perturbation equations will be identical to those obtained in the $R + aR^2$ theory. In particular, if $a_2 > 0$ then no trajectories in directions orthogonal to the Schwarzschild subspace will have end points at $M_{1,2}$ or O_1. This completes our proof of the uniqueness theorem. We have of course assumed that $a_2 \neq 0$ throughout. If $a_2 = 0$ then it seems likely that the properties of the asymptotically flat solutions will be essentially determined by the value of a_p, where p is the least value such that $a_p \neq 0$.

V. DISCUSSION

To conclude, we have shown that in theories with an action polynomial in the Ricci scalar, the only static spherically symmetric solution with a regular horizon is the Schwarzschild solution, provided that the coefficient $a_2 = a$ of the quadratic term is positive. In fact, if we drop the condition of regularity on the horizon then the only asymptotically flat solutions are still the positive- and negative-mass Schwarzschild solutions. The generalised scalar potential model of Sec. II does possess non-Schwarzschild $W = 0$, $Z = 0$, $\bar{\lambda} > 0$ solutions with naked singularities which approach flat space asymptotically near the points $M_{1,2}$. However, these solutions correspond physically to $\lambda_1 = \lambda_2 = \cdots = \lambda_s = 0$, a possibility which is not admitted in the higher-derivative theories. We have not considered the theories for which the quadratic terms vanish but which have nonzero terms at higher order. In such cases the condition $a_2 > 0$ presumably translates into a condition on the coefficients of higher terms in the series. The question of solutions with $a_2 < 0$ is also not fully clear, and could perhaps be resolved by a numerical study.

The fact that we have been able to prove a uniqueness theorem for static spherically symmetric black holes in the case that $a_2 > 0$ has immediate important physical implications when considered in conjunction with the earlier work of Pechlaner and Sexl [19] and Michel [20]. These authors observed that in the case of the $R + aR^2$ theory in four dimensions the solutions with $R = 0$ are *not* the solutions which correspond to the weak-field limit about any physical body such as a star or point particle. This is also the case for the general theory with a polynomial R action which we are considering here. Suppose, for example, that we wish to match our solutions onto a star with energy-momentum tensor T_{ab} in its interior. If we add such a term to (1.8) and trace the result we find that

$$\frac{-1}{2}(D-2)R + \sum_{p=2}^{n} a_p \left\{ \left(p - \frac{D}{2} \right) R^p + (D-1)p(p-1)R^{p-3}[R\Box R + (p-2)R^{;c}R_{;c}] \right\} = 2\kappa^2 T. \qquad (5.1)$$

Thus if we set $T = 0$ at the surface of the star we cannot conclude, as in the case of the Schwarzschild solution, that $R = 0$. Our results show, however, that if $a_2 > 0$ and $R \neq 0$ anywhere in the domain of outer communications, then the solutions are *not asymptotically flat.*

Pechlaner and Sexl, on the contrary, assumed the existence of non-Schwarzschild solutions which are asymptotically flat. Our full nonlinear analysis seems to invalidate this assumption in the $a_2 > 0$ case. Consequently, their weak-field analysis of the $R + aR^2$ theory in four dimensions, in which they derived experimental bounds on the parameter a from resulting fifth-force-type effects, needs to be reexamined with appropriately changed boundary conditions.

If the Schwarzschild solutions are not the solutions of physical interest in these models, it would seem that the large class of asymptotically anti–de Sitter and de Sitter solutions hold more promise. These include both the exact solutions (2.18) [with Λ given, for example, by (3.12) and (4.11) in the quadratic and cubic order theories, respectively], and also other solutions asymptotic to them at infinity, or at the de Sitter cosmological horizon. Asymptotically (anti–)de Sitter solutions have also been found in a number of models in $D > 4$ dimensions which incorporate a Gauss-Bonnet term [32–34] and other dimensionally continued Euler densities [35], and thus appear to be a generic feature of higher-derivative theories. In fact, maximally symmetric solutions have been found to exist in a much wider class of higher-derivative models [31], so presumably such models also possess asymptotically (anti–)de Sitter black hole solutions.

Returning to the models with dimensionally continued Euler densities, we note that similarly to the solutions found here, only asymptotically anti–de Sitter solutions are found if the coefficient of the Gauss-Bonnet term is positive in the quadratic theory, while a de Sitter branch can be obtained at higher order [34]. One important difference between our solutions and those of Boulware and Deser [32], for example, is that the asymptotically anti–de Sitter branch of the Boulware-Deser solutions has a negative gravitational mass, giving rise to an instability. Our solutions have a positive gravitational mass, and so the question of their stability is still an open problem. The issue of the weak-field limit has not been addressed in Refs. [32–34], since of course it is really only relevant in compactified models.

If the de Sitter sector is to be treated as a serious physical model it is clear that the effective cosmological term must be small to be consistent with observation. It is interesting to note that recent astronomical evidence actually favors a small[10] positive cosmological constant [36]. If higher-order terms in R are obtained from the dimensional reduction of an action corresponding to the low-energy limit of a higher-dimensional string theory, then the status of the effective cosmological term is at best uncertain. If we ignore the values of the dilaton and compacton and assume that the compactification scale is approximately of order $\sqrt{\alpha'}$ (in units in which $c = \hbar = 1$, as used throughout this paper), where α' is the Regge slope parameter, then the coefficients a_p are of order $(\ell_{\text{Planck}})^{2(p-1)}$, and Λ is of order $(\ell_{\text{Planck}})^{-2}$, to within a few orders of magnitude. Such a colossally large effective cosmological term would of course spell disaster for these models. However, no definite statements can be made without some knowledge of the expectation values of the dilaton and compacton fields, both of which couple nontrivially to the higher-order curvature terms in four dimensions. In fact, these scalar fields should really be treated dynamically, which would necessitate a complete reexamination of the model. In the $D > 4$ uncompactified quadratic order theory with a Gauss-Bonnet term the dilaton has the effect of removing the anti–de Sitter branch [34], but it is not clear how the dilaton would affect compactified models.

In view of these problems it would also be interesting, especially in the quadratic order case in four dimensions, to consider the addition to the Lagrangian of a term comprising the square of the Ricci tensor. In that case the effective theory contains a massive spin-2 field with a nontrivial coupling to gravity in addition to the scalar field. The effective energy-momentum tensor of the extra excitations does not satisfy criteria usually required in the proof of the no-hair theorems, and thus it seems plausible that the no-hair theorems could be circumvented in such a model. However, the dynamical system arising from a static spherically symmetric ansatz for the metric and the other fields is considerably more complicated than in the models we have studied in this paper, and it is not clear to us whether it can be reduced to a tractable form.

We remark parenthetically that the results of this paper seem to hold up some hope for the problem of finding black hole solutions in dimensionally reduced theories in which the internal space is non-Ricci flat, since such models can also be treated by the formalism of Secs. II and III. In [28] and [29] the potential corresponding to (1.10) was limited to two terms at most. However, we have seen that at least three exponential terms are required in the potential in order to obtain a Schwarzschild subspace, and thus no asymptotically flat solutions were found in [28] and [29] for internal spaces of nonzero curvature. More complicated models with $s \geq 3$ (or some equivalent condition if more than one scalar field is present) may yield more interesting results.

[10]The "best-fit" value favored in Ref. [36] is $\Lambda = 3.1 \times 10^{-52}$ m^{-2}, or $\Lambda = 8.1 \times 10^{-122}$ in dimensionless Planck units.

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APPENDIX A

In the case of the quadratic action $R + aR^2$ in four dimensions with $a > 0$, Whitt [21] has shown that, provided the energy momentum of any additional matter fields satisfies certain conditions, then the usual no-hair theorem for stationary, axisymmetric, asymptotically flat black holes [37] is still valid. He was able to obtain this result by proving that the curvature scalar for such solutions vanishes in the domain of outer communications, and therefore the effective theory coincides with the usual Einstein-Hilbert theory. Unfortunately his proof cannot be generalized to more general polynomial actions of the form (1.7). For completeness we will present an account of his proof here, in order to show why the theorem cannot be directly extended to more general polynomial R actions, and also to correct some errors present in the original paper.

Whitt's proof is based on a study of the four-dimensional Killing bivector [38]:

$$\rho_{ab} = 2m_{[a}k_{b]}, \tag{A1}$$

where m^a is the spacelike Killing vector with period 2π and k^a is the timelike Killing vector normalized to unity at infinity. These vector fields commute [39]:

$$k^b m^a{}_{;b} - m^b k^a{}_{;b} = 0. \tag{A2}$$

We will take l^a to denote either of the Killing vectors k^a and m^a. Making use of (A2) and the identity

$$(l_{[a;b}l_{c]})^{;c} = \tfrac{2}{3} l^c R_{c[a}l_{b]}\,, \tag{A3}$$

which is valid for any Killing vector [40], we find

$$(l_{[a;b}k_c m_{d]})^{;d} = -\tfrac{1}{2} l^d R_{d[a}k_b m_{c]}. \tag{A4}$$

We can now use the vacuum field equations of the higher-derivative theory to evaluate the right-hand side of this expression. For the $R + aR^2$ case one finds

$$(l_{[a;b}k_c m_{d]})^{;d} = \frac{-a\, l^d R_{;d[a}k_b m_{c]}}{1 + 2aR}, \tag{A5a}$$

or in the general case, with field equations (1.8),

$$(l_{[a;b}k_c m_{d]})^{;d} = \frac{-\left(\sum_{p=2}^{n} a_p p\, l^d R_{;d[a}k_b m_{c]}\right)}{2\left(1 + \sum_{p=2}^{k} a_p p R^{p-1}\right)}. \tag{A5b}$$

Using the orthogonality of the Killing vectors to $R_{;a}$ and the antisymmetry of the derivatives of the Killing vectors, these expressions become, respectively,

$$(l_{[a;b}k_c m_{d]})^{;d} = \frac{-aR^{;d} l_{[d;a}k_b m_{c]}}{1 + 2aR} \tag{A6a}$$

and

$$(l_{[a;b}k_c m_{d]})^{;d} = \frac{-\sum_{p=2}^{n} a_p p (R^{p-1})^{;d} l_{[d;a}k_b m_{c]}}{2\left(1 + \sum_{p=2}^{n} a_p p R^{p-1}\right)}. \tag{A6b}$$

The solution of Eqs. (A6) is given, respectively, by

$$l_{[a;b}k_c m_{d]} = C(1 + 2aR)^{-1/2}\epsilon_{abcd} \tag{A7a}$$

and

$$l_{[a;b}k_c m_{d]} = C\left(1 + \sum_{p=2}^{n} a_p p R^{p-1}\right)^{-1/2} \epsilon_{abcd}\,, \tag{A7b}$$

where C is an arbitrary constant. Now we know that the left-hand side of (A7) vanishes on the axis of rotation since $m^a = 0$ there. On account of the asymptotic flatness of the solutions R cannot be singular everywhere on the axis, and hence C must vanish giving

$$k_{[a;b}\rho_{cd]} = 0, \qquad m_{[a;b}\rho_{cd]} = 0, \tag{A8}$$

throughout the domain of outer communications.

From (A8) it follows [40] that if the domain of outer communications is simply connected and admits no closed timelike curves, then ρ_{ab} is timelike throughout the domain of outer communications, becoming null on its inner boundary, which is a null hypersurface.

At this point Whitt [21] considers the trace of the field equations in the absence of matter, which for the quadratic theory yield

$$-6aR^{;a}{}_{;a} + R = 0. \tag{A9}$$

One can now multiply by R and integrate over the domain of outer communications to obtain

$$6a\int RR_{;a}\,d\Sigma^a = 6a\int R^{;a}R_{;a}\sqrt{-g}\,d^4x + \int R^2\sqrt{-g}\,d^4x\,. \tag{A10}$$

The left-hand side vanishes because $R_{;a}$ is orthogonal to the inner boundary and is zero on the outer boundary due to asymptotic flatness. Moreover, the right-hand side is positive definite, provided $a > 0$, since $R_{;a}$ cannot be timelike anywhere in the domain of outer communications by virtue of being orthogonal to ρ_{ab}. Hence R must be identically zero there.

The conformally rescaled theory coincides, therefore, with the usual Einstein theory since the scalar field σ is clearly zero if R vanishes and it follows that the uniqueness theorem for stationary, axisymmetric, asymptotically flat solutions still holds. Moreover, it is easy to see that the result remains valid when additional matter fields are included in the action, provided that the energy-momentum tensor is traceless, $T^a_{\ a} = 0$, and satisfies the matter circularity condition [40] $T_{d[a}k_b m_{c]} = 0$. In such cases the usual no-hair theorems are still valid. These conditions are always satisfied, in particular, by a stationary, axisymmetric electromagnetic field. Unfortunately, however, Whitt's result cannot be extended to the general polynomial R action since in that case the argument for R being zero in the domain of outer communications breaks down. Specifically, the trace of the field equation now becomes

$$-R + \sum_{p=2}^{n} a_p\left[(p-2)R^p + 3p(R^{p-1})^{;a}{}_{;a}\right] = 0 \tag{A11}$$

and therefore, integrating by parts and discarding the boundary term as before we find

$$-\int R^2\sqrt{-g}\,d^4x+\sum_{p=2}^{n}(p-2)a_p\int R^{p+1}\sqrt{-g}\,d^4x$$
$$=\sum_{p=2}^{n}3p(p-1)a_p\int R^{p-2}R_{;a}R^{;a}\sqrt{-g}\,d^4x. \quad (A12)$$

The integrals in the sum on the left-hand side do not have a definite sign for even p, while the integrals in the sum on the right-hand side do not have a definite sign for odd p. Thus it is impossible to conclude from this relation that R must vanish in the general case, or indeed in any special case other than $n = 2$, $a_2 > 0$.

APPENDIX B

We list here the exact solutions obtained in the cases (i) $\mathcal{V} \equiv 0$, i.e., $\lambda_1 = \lambda_2 = \cdots = \lambda_s$ appropriate to the $W = 0$, $Z = 0$ subspace, and (ii) $s = 1$, $\bar{\lambda} = 0$, appropriate to the $\bar{\lambda} = 0$ surface in the $W = 0$ subspace. Corresponding solutions for the $\bar{\lambda} = 0$ surface in the $Z = 0$ subspace may be obtained by substituting $g_1 \to g_2$, $\lambda_1 \to \lambda_2$.

1. Solutions with $W = 0$ and $Z = 0$

The one-parameter family of solutions was derived in [28]. In that case solutions were parametrized in terms of the number n_e of extra dimensions in the Kaluza-Klein model. Here a parametrization in terms of g_1 (or g_2) is more natural. We will use the former parametrization. We find

$$e^{\zeta}=\frac{CA_1}{\Delta_1}\exp\left[\frac{1}{2}(m-1)C\xi\right], \quad (B1)$$

$$\eta=\frac{m}{m-1}(\zeta+k\xi)+\text{const}, \quad (B2)$$

$$e^{\hat{u}}=A_0e^{-a_0\xi}, \quad (B3)$$

$$e^{\hat{v}}=\frac{(m-1)CA_1\exp\left[\frac{1}{2}(m-1)C\xi\right]}{(m-1)CA_1{}^2\exp\left[(m-1)C\zeta\right]+\left[a_0+\frac{1}{2}(m-1)C\right]\Delta_1}, \quad (B4)$$

$$\hat{r}^{m-1}=\frac{CA_1}{A_0\Delta_1}\exp\left[\left(a_0+\frac{1}{2}(m-1)C\right)\xi\right], \quad (B5)$$

$$\frac{2g_1\kappa\sigma}{m}=\frac{(m-1)g_1{}^2(c_1-mk)}{1+(m-1)g_1{}^2}+\text{const}, \quad (B6)$$

where

$$\Delta_1\equiv\bar{\lambda}-A_1{}^2\exp\left[(m-1)C\zeta\right], \quad (B7)$$

while k, A_0, and A_1 are arbitrary constants, C is a nonzero constant given by

$$\tfrac{1}{4}(m-1)^2\left[1+(m-1)g_1{}^2\right]C^2=m^2k^2+(m-1)c_1{}^2g_1{}^2, \quad (B8)$$

and the constant a_0 is defined by

$$a_0=\frac{mk+(m-1)g_1{}^2c_1}{1+(m-1)g_1{}^2}, \quad (B9)$$

and lies in the range $-\frac{1}{2}(m-1)|C| \le a_0 \le \frac{1}{2}(m-1)|C|$. Solutions have an asymptotic ($\hat{r} \to \infty$) region only if $\bar{\lambda} > 0$.

As $\xi \to -\infty$ we find $\hat{r} \to 0$, giving rise to a singularity except in the special case $c_1 = mk = a_0 = -\frac{1}{2}(m-1)|C|$, when $\hat{r} \to$ const, suggesting the possible presence of a horizon. This can be immediately verified to be true since (B5) can be inverted. If we choose[11]

$$A_1=\begin{cases}\bar{\lambda}A_0/|\bar{\lambda}|, & C>0,\\ |\bar{\lambda}|/A_0, & C<0,\end{cases} \quad (B10)$$

we find

$$e^{2\hat{u}}=e^{-2\hat{v}}=\bar{\lambda}\left(1-\frac{|C|}{|\bar{\lambda}|\hat{r}^{m-1}}\right), \quad (B11)$$

which correspond to the domain of outer communications of the positive-mass Schwarzschild solution for $\bar{\lambda} > 0$. Similarly, as $\xi \to +\infty$ we find $\hat{r} \to 0$, giving rise to a singularity, unless $c_1 = mk = a_0 = \frac{1}{2}(m-1)|C|$. In that

[11]Equations (B10)–(B12) correct a sign discrepancy in [28] for the $\bar{\lambda} < 0$ and $\xi \to +\infty$ cases.

case we once again obtain (B11) if we choose

$$A_1 = \begin{cases} -|\bar{\lambda}|/A_0, & C > 0, \\ -\bar{\lambda}A_0/|\bar{\lambda}|, & C < 0. \end{cases} \tag{B12}$$

$$e^{\eta} = \frac{\bar{C}B_1}{\Delta_2} \exp\left[\frac{1}{2}\bar{C}\xi\right], \tag{B13}$$

$$e^{\zeta} = B_0 \exp\left[\frac{m(\eta + \bar{k}\xi)}{m+1-g_1^{\,2}}\right], \tag{B14}$$

$$e^{\hat{u}} = B_0^{\,m} B_2^{\,(m-1)} \left\{\frac{\bar{C}B_1}{\Delta_2} \exp\left[\left(\frac{1}{2}\bar{C} + m\bar{k} - (m-1)b_0\right)\xi\right]\right\}^{1/(m+1-g_1^{\,2})}, \tag{B15}$$

$$e^{\hat{v}} = \frac{\left(m+1-g_1^{\,2}\right) B_0 \left(\Delta_2^{1-g_1^{\,2}} \{\bar{C}B_1 \exp[(\frac{1}{2}\bar{C} + \bar{k})\xi]\}^m\right)^{1/(m+1-g_1^{\,2})}}{\bar{C}B_1^{\,2} \exp\left[\bar{C}\xi\right] + \left(\frac{1}{2}\bar{C} + b_0\right)\Delta_2}, \tag{B16}$$

$$\hat{r} = \frac{1}{B_0 B_2} \left\{\frac{\bar{C}B_1}{\Delta_2} \exp\left[\left(\frac{1}{2}\bar{C} + b_0\right)\xi\right]\right\}^{1/(m+1-g_1^{\,2})}, \tag{B17}$$

$$\exp\left(\frac{2g_1\kappa\sigma}{m}\right) = \frac{1}{B_2} \left\{\frac{\bar{C}B_1}{\Delta_2} \exp\left[\left(\frac{1}{2}\bar{C} + \frac{1}{g_1^{\,2}}(m\bar{k} + b_0)\right)\xi\right]\right\}^{g_1^{\,2}/(m+1-g_1^{\,2})}, \tag{B18}$$

where

$$\Delta_2 \equiv \bar{\Lambda}_1 - B_1^{\,2} \exp\left[\bar{C}\xi\right], \tag{B19a}$$

with

$$\bar{\Lambda}_1 \equiv \frac{1}{m}\left(m+1-g_1^{\,2}\right)\lambda_1, \tag{B19b}$$

while B_0, B_1, B_2, and $\bar{k}$ are arbitrary constants, $\bar{C}$ is a nonzero constant given by

$$\tfrac{1}{4}[1 + (m-1)g_1^{\,2}]\bar{C}^2 = m^2\bar{k}^2 + \left(m+1-g_1^{\,2}\right) g_1^{\,2} c_1^{\,2}, \tag{B20}$$

and the constant b_0 is defined by

$$b_0 = \frac{(g_1^{\,2}-1)m\bar{k} + \left(m+1-g_1^{\,2}\right) g_1^{\,2} c_1}{1 + (m-1)g_1^{\,2}} \tag{B21}$$

and lies in the range $-\frac{1}{2}|\bar{C}| \leq b_0 \leq \frac{1}{2}|\bar{C}|$.

We now find that the limit $\xi \to -\infty$ corresponds to $\hat{r} \to 0$ except in the special instances when $m\bar{k}/(g_1^{\,2} - 1) = c_1 = b_0 = -\frac{1}{2}|\bar{C}|$, for which $\hat{r} \to$ const, suggesting the possible presence of a horizon. This indeed the case: (B17) can be inverted and if we make the choice

$$B_0^{\,g_1^{\,2}} B_2^{\,g_1^{\,2}-1} = \frac{1}{m+1-g_1^{\,2}}, \tag{B22}$$

$$B_1 = \begin{cases} B_0^{\,m+1} B_2^{\,m}, & \bar{C} > 0, \\ \bar{\Lambda}_1/(B_0^{\,m+1} B_2^{\,m}), & \bar{C} < 0, \end{cases} \tag{B23}$$

then we find the solution

$$ds^2 = -\hat{r}^2\bar{\Delta}\, dt^2 + \hat{r}^{2(g_1^{\,2}-1)} \frac{d\hat{r}^2}{\bar{\Delta}} + \hat{r}^2 \bar{g}_{\alpha\beta} d\bar{x}^\alpha d\bar{x}^\beta, \tag{B24a}$$

where

$$\bar{\Delta} = \bar{\Lambda}_1 \left(1 - \frac{|\bar{C}|}{|\bar{\Lambda}_1|\left(m+1-g_1^{\,2}\right)\hat{r}^{m+1-g_1^{\,2}}}\right), \tag{B24b}$$

while the scalar field is given by

$$\exp\left(\frac{2g_1\kappa\sigma}{m}\right) = \frac{\hat{r}^{g_1^{\,2}}}{m+1-g_1^{\,2}}. \tag{B24c}$$

Similarly the limit $\xi \to +\infty$ also corresponds to $\hat{r} \to 0$ except for the special cases when $m\bar{k}/(g_1^{\,2} - 1) = c_1 = b_0 = \frac{1}{2}|\bar{C}|$ which $\hat{r} \to$ const. Equation (B17) can once again be inverted, and we retrieve the solution (2.18) if we now make the choice (B22) and[12]

2. Solutions with $W = 0$ and $\bar{\lambda} = 0$

These solutions may be derived as in [28]. If $g_1^{\,2} < m+1$ we find

[12]There are some sign discrepancies in [28] and [29] for the expressions corresponding to (B23), (B24b), and (B25).

$$B_1 = \begin{cases} -\bar{\Lambda}_1/(B_0{}^{m+1}B_2{}^m), & \bar{C} > 0, \\ -B_0{}^{m+1}B_2{}^m, & \bar{C} < 0. \end{cases} \tag{B25}$$

The spacetimes thus have naked singularities except in the special cases above.

An asymptotic region is defined only for $\lambda_1 > 0$: $\hat{r} \to \infty$ when $\bar{C}\xi = \ln|\bar{\Lambda}_1/B_1^2|$. This limit is approached at the points $N_{1,2}$ at infinity [cf. (2.37)]. All $\lambda_1 > 0$ solutions have one end point at N_1 or N_2, and another end point on the $\lambda_1 = 0$ curve of critical points. The $\lambda_1 < 0$ solutions have one end point on the $\lambda_1 = 0$ curve in the first quadrant, and another end point on the $\lambda_1 = 0$ curve in the third quadrant. The asymptotic form of all the $\lambda_1 > 0$ solutions (B15)–(B18) is given by

$$e^{2\hat{u}} \sim \hat{r}^2, \qquad e^{2\hat{v}} \sim \hat{r}^{2(g_1{}^2-1)}, \qquad e^{2\kappa\sigma} \sim \hat{r}^{mg_1}. \tag{B26}$$

Thus none of the solutions is asymptotically flat. The general solution holds for $\bar{C} \neq 0$. If one sets $\bar{C} = 0$ while integrating the differential equations one finds a solution which corresponds to the $\bar{C} = 0$ limit of (B24).

If $g_1{}^2 > m + 1$ then the solutions given by expressions (B13)–(B21) are still valid but b_0 now lies in the range $b_0 \leq -\frac{1}{2}|\bar{C}|$, $b_0 \geq \frac{1}{2}|\bar{C}|$. Furthermore, the behavior of the solutions and the direction of trajectories near the critical points is greatly changed in some instances. We now find that the limit $\xi \to -\infty$ corresponds to $\hat{r} \to 0$ as before if $b_0 < -\frac{1}{2}|\bar{C}|$. However, if $b_0 \geq \frac{1}{2}|\bar{C}|$ we find that $\hat{r} \to \infty$. Similarly, the limit $\xi \to +\infty$ corresponds to $\hat{r} \to 0$ if $b_0 > \frac{1}{2}|\bar{C}|$, and $\hat{r} \to \infty$ if $b_0 \leq -\frac{1}{2}|\bar{C}|$. Critical points in the first and third quadrants have $\hat{r} \to 0$, while those in the fourth and second quadrants have $\hat{r} \to \infty$. Since each solution has a different end point in the $\hat{r} \to \infty$ regime, the asymptotic behaviors vary. We find

$$e^{2\hat{u}} \sim \hat{r}^{2\hat{u}_0}, \qquad e^{2\hat{v}} \sim \hat{r}^{2\hat{v}_0}, \qquad \exp\left(\frac{2g_1\kappa\sigma}{m}\right) \sim \hat{r}^{2\sigma_0}, \tag{B27a}$$

where

$$\hat{u}_0 = \frac{\pm\frac{1}{2}|\bar{C}| + m\bar{k} - (m-1)b_0}{b_0 \pm \frac{1}{2}|\bar{C}|}, \tag{B27b}$$

$$\hat{v}_0 = \frac{m\left(\bar{k} \pm \frac{1}{2}|\bar{C}|\right)}{b_0 \pm \frac{1}{2}|\bar{C}|}, \tag{B27c}$$

$$\sigma_0 = \frac{\pm\frac{1}{2}|\bar{C}| + \frac{1}{g_1^2}\left(m\bar{k} + b_0\right)}{b_0 \pm \frac{1}{2}|\bar{C}|}, \tag{B27d}$$

and the upper (lower) sign refers to $\xi \to -\infty$ ($\xi \to +\infty$). For the limiting case solutions with $b_0 = \frac{1}{2}|\bar{C}|$, if $\xi \to -\infty$, or $b_0 = -\frac{1}{2}|\bar{C}|$, if $\xi \to +\infty$ we obtain the asymptotic behavior appropriate to (B24), viz.,

$$e^{2\hat{u}} \sim \hat{r}^{g_1{}^2-m+1}, \qquad e^{2\hat{v}} \sim \hat{r}^{g_1{}^2+m-1}, \qquad e^{2\kappa\sigma} \sim \hat{r}^{mg_1}. \tag{B28}$$

The limit $\bar{C}\xi = \ln|\bar{\Lambda}_1/B_1{}^2|$ for the $\lambda_1 < 0$ solutions, which is reached at the points $N_{1,2}$, now corresponds to $\hat{r} \to 0$. The $\lambda_1 < 0$ solutions given by (B13)–(B21) all have one end point at N_1 or N_2 and another end point on the $\lambda_1 = 0$ curve. If this second end point is in the fourth or second quadrant the solutions have an asymptotic region; otherwise they do not. The $\lambda_1 > 0$ solutions, on the other hand, all have asymptotic regions: they now describe trajectories with one end point on the $\bar{\lambda} = 0$ curve in the first (or third) quadrant, and a second end point on the same curve in the fourth (or second) quadrant.

In addition to the above solutions, a second group of solutions also exist if $g_1{}^2 > m + 1$ and $\lambda_1 < 0$. These solutions are given by

$$e^{\eta} = \frac{|\tilde{C}|}{\bar{\Lambda}_1{}^{1/2}\cos\left[\tilde{C}(\xi - \xi_0)\right]}, \tag{B29}$$

$$e^{\zeta} = B_0 \exp\left[\frac{-m(\eta + \bar{k}\xi)}{g_1{}^2 - m - 1}\right], \tag{B30}$$

$$e^{\hat{u}} = B_0{}^m B_2{}^{(m-1)}\left(\bar{\Lambda}_1{}^{1/2}|\tilde{C}|^{-1}\cos\left[\tilde{C}(\xi - \xi_0)\right]\exp\left\{\left[(m-1)b_0 - m\bar{k}\right]\xi\right\}\right)^{1/(g_1{}^2-m-1)}, \tag{B31}$$

$$e^{\hat{v}} = \frac{-\left(g_1{}^2 - m - 1\right)B_0}{b_0 + \tilde{C}\tan\left[\tilde{C}(\xi - \xi_0)\right]}\left\{\bar{\Lambda}_1{}^{1/2}|\tilde{C}|^{-1}\cos\left[\tilde{C}(\xi - \xi_0)\right]\exp(-\bar{k}\xi)\right\}^{m/(g_1{}^2-m-1)}, \tag{B32}$$

$$\hat{r} = \frac{1}{B_0B_2}\left\{\bar{\Lambda}_1{}^{1/2}|\tilde{C}|^{-1}\cos\left[\tilde{C}(\xi - \xi_0)\right]\exp(-b_0\xi)\right\}^{1/(g_1{}^2-m-1)}, \tag{B33}$$

$$\exp\left(\frac{2g_1\kappa\sigma}{m}\right) = \frac{1}{B_2}\left(\left\{\bar{\Lambda}_1^{1/2}|\tilde{C}|^{-1}\cos\left[\tilde{C}(\xi-\xi_0)\right]\right\}^{g_1^2}\exp\left[-(m\bar{k}+b_0)\xi\right]\right)^{1/(g_1^2-m-1)}, \tag{B34}$$

where $\bar{\Lambda}_1$ and b_0 are defined as before, ξ_0, B_0, B_2, and $\bar{k}$ are arbitrary constants, and $\tilde{C}$ is a nonzero constant given by

$$[1+(m-1)g_1^2]\tilde{C}^2 = (g_1^2-m-1)g_1^2c_1^2 - m^2\bar{k}^2. \tag{B35}$$

These solutions have no asymptotic region, and correspond to trajectories with one end point at N_1 and a second end point at N_2. These end points are reached when $\xi = \xi_0 + (2j+1)/(2\pi|\tilde{C}|)$, for integer j.

If $g_1^2 = m+1$ we find the solutions

$$e^{\eta} = B_1\hat{C}^2e^{\hat{C}\xi/2}, \tag{B36}$$

$$e^{\zeta} = B_0\exp\left(\bar{k}\xi + \lambda_1 B_1 e^{\hat{C}\xi}\right), \tag{B37}$$

$$e^{\hat{u}} = B_0^{\,m}B_2^{\,(m-1)}\left(\exp\left\{\left[\bar{k}-(m-1)b_0\right]\xi + \lambda_1 B_1 e^{\hat{C}\xi}\right\}\right)^{1/m}, \tag{B38}$$

$$e^{\hat{v}} = \frac{mB_0\exp\left(\bar{k}\xi + \lambda_1 B_1 e^{\hat{C}\xi}\right)}{\bar{k}+b_0+\lambda_1 B_1\hat{C}e^{\hat{C}\xi}}, \tag{B39}$$

$$\hat{r} = \frac{1}{B_0B_2}\left\{\exp\left[(\bar{k}+b_0)\xi + \lambda_1 B_1 e^{\hat{C}\xi}\right]\right\}^{1/m}, \tag{B40}$$

$$\exp\left(\frac{2g_1\kappa\sigma}{m}\right) = \frac{1}{B_2}\left(\exp\left\{\left[\bar{k}+\frac{1}{m+1}\left(b_0-\frac{m\hat{C}}{2}\right)\right]\xi + \lambda_1 B_1 e^{\hat{C}\xi}\right\}\right)^{(m+1)/m}, \tag{B41}$$

where B_0, B_1, and B_2 are arbitrary constants,

$$b_0 = \frac{1}{m}\left(\frac{\hat{C}}{2}+(m+1)c_1\right), \tag{B42}$$

$\bar{k}$ is a constant which lies in the range $|\bar{k}| > \sqrt{m^2-1}\,|c_1|/m$, and $\hat{C}$ is a nonzero constant given by

$$\hat{C} = \left(\frac{2}{m-1}\right)\left[m\bar{k} \pm \sqrt{m^2\bar{k}^2-(m^2-1)c_1^2}\right], \tag{B43}$$

and has the same sign as $\bar{k}$. We now find that for $\bar{k}>0$ as $\xi \to -\infty$ $\hat{r}\to 0$ except in the case that $\bar{k} = -c_1 = -b_0 = \hat{C}/2$, when $\hat{r}\to$ const. The same is true for $\bar{k}<0$ in the limit $\xi\to+\infty$. On the other hand, if $\bar{k}<0$ and $\xi\to-\infty$, or if $\bar{k}>0$ and $\xi\to+\infty$, then $\hat{r}\to\infty$. All solutions have an asymptotic region. In terms of the phase space, all trajectories approach the points $N_{1,2}$ at infinity which coincide with points $L_{5,7}$. In the case that $\bar{k} = -c_1 = -b_0 = \hat{C}/2$ we can invert (B40), and if we make the choice

$$B_2 = B_0^{-1}, \qquad B_1 = \frac{mB_0^2}{\lambda_1\hat{C}}, \tag{B44}$$

we obtain the solution

$$ds^2 = -\hat{r}^2\hat{C}\ln\hat{r}\,dt^2 + \frac{\hat{r}^{2m}\,d\hat{r}^2}{\hat{C}\ln\hat{r}} + \hat{r}^2\bar{g}_{\alpha\beta}d\bar{x}^\alpha d\bar{x}^\beta, \tag{B45a}$$

with scalar field

$$\exp\left(\frac{2g_1\kappa\sigma}{m}\right) = B_0\hat{r}^{m+1}. \tag{B45b}$$

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