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Extrema of mass, stationarity, and staticity, and solutions to the Einstein-Yang-Mills equations

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A simple formula is derived for the variation of mass and other asymptotic conserved quantities in Einstein-Yang-Mills theory. For asymptotically flat initial data with a single asymptotic region and no interior boundary, it follows directly from our mass-variation formula that initial data for stationary solutions are extrema of mass at fixed electric charge. When generalized to include an interior boundary, this formula provides a simple derivation of a generalized form of the first law of black-hole mechanics. We also argue, but do not rigorously prove, that in the case of a single asymptotic region with no interior boundary stationarity is necessary for an extremum of mass at fixed charge; when an interior boundary is present, we argue that a necessary condition for an extremum of mass at fixed angular momentum, electric charge, and boundary area is that the solution be a stationary black hole, with the boundary serving as the bifurcation surface of the horizon. Then, by a completely different argument, we prove that if a foliation by maximal slices (i.e., slices with a vanishing trace of extrinsic curvature) exists, a necessary condition for an extremum of mass when no interior boundary is present is that the solution be static. A generalization of the argument to the case in which an interior boundary is present proves that a necessary condition for a solution of the Einstein-Yang-Mills equation to be an extremum of mass at fixed area of the boundary surface is that the solution be static. This enables us to prove (modulo the existence of a maximal slice) that if the stationary Killing field of a stationary black hole with bifurcate Killing horizon is normal to the horizon, and if the electrostatic potential asymptotically vanishes at infinity, then the black hole must be static. (This closes a significant gap in the black-hole uniqueness theorems.) Finally, by generalizing the type of argument used to predict the "sphaleron" solution of Yang-Mills-Higgs theory, we argue that the initial-data space for Einstein-Yang-Mills theory with a single asymptotic region should contain a countable infinity of saddle points of mass. Similarly in the case of an interior boundary, there should exist a countable infinity of saddle points of mass at fixed boundary area. We propose that this accounts for the existence and properties of the Bartnik-McKinnon and colored black-hole solutions. Similar arguments in the black-hole case indicate the presence of a countable infinity of extrema of mass at fixed area, electric charge, and angular momentum, thus suggesting the existence of colored generalizations of the charged Kerr solutions. A number of other conjectures concerning stationary solutions of the Einstein-Yang-Mills equations and related systems are formulated. Among these is the prediction of the existence of a countable infinity of new static solutions of the Yang-Mills-Higgs equations related to the sphaleron.

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I. INTRODUCTION

In the past few years, static, asymptotically flat solutions to the Einstein-Yang-Mills equations with gauge group G=SU(2) have been discovered. First, Bartnik and McKinnon [1] numerically obtained a discrete family of globally static, spherically symmetric, solutions on $\mathbb{R}^4$. (A rigorous proof of the existence of these solutions has recently been given [2].) Soon thereafter analogous families of static, spherically symmetric "colored black holes" were found [3–5]. The discovery of all of these solutions came as a surprise to most researchers, since there are no analogous solutions in Einstein-Maxwell theory, and the black hole uniqueness theorems, proven for other fields, generally had been viewed as indicating the unlikelihood of the existence of colored black holes.

The existence of these solutions raises a number of questions, among the most prominent of which are the following. (i) What is the essential feature (or features) of the Yang-Mills group G, which is responsible for the existence of the new solutions, i.e., why does one obtain new solutions with G=SU(2) but not with G =U(1)? (ii) What accounts for the features of these new solutions, such as their discreteness and instability? (iii) Would one expect any further new solutions of similar character, in particular, ones which are stationary but not spherically symmetric and, perhaps, not static?

One of the two principal goals of this paper is to propose answers to all of the above questions. We shall argue that the key feature of the Yang-Mills group G, which gives rise to the new solutions is the presence of so-called "large gauge transformations," i.e., cross sections of the (trivial) Yang-Mills bundle which cannot be continuously deformed into each other. Thus, we claim that it is essential that G have a nonvanishing third homotopy group, as occurs for SU(2), or, more generally, for any compact semisimple Lie group. The relevance of the large gauge transformations is that they cause the mass functional on the phase space of Einstein-Yang-Millls theory to possess a countably infinite set of disconnected minima—namely, flat spacetime with vanishing Yang-Mills potential and all large gauge transformations of this solution—rather than a single minimum. Consequently, we shall argue that additional saddle points of the mass function should exist. Indeed, we shall argue for the existence of an infinite sequence of saddle points, with each successive saddle point having larger mass and a larger number of unstable modes. We believe that these saddle points account for the existence of the Bartnik-McKinnon and colored black-hole solutions. In this manner, the discreteness and instability of these solutions are naturally explained. Similar arguments predict new solutions which are colored analogues of the charged Kerr black holes.

A crucial facet of our plausibility argument to account for the known Einstein-Yang-Mills solutions and to predict new ones concerns the relationship between extrema of the mass functional and stationary solutions. The second principal goal of this paper is to explore this relationship in detail. We consider the (constrained) phase space of Einstein-Yang-Mills theory, i.e., the set of initial data $(h_{ab}, \pi^{ab}, A_a{}^{\Gamma}, E^a{}_{\Gamma})$ on a hypersurface Σ satisfying the Einstein and Yang-Mills constraints. (Our notational conventions on indices are explained near the end of this section and the phase space and the constraints are introduced in Sec. II.) We focus attention on the following two cases: (a) The initial data is asymptotically flat with a single asymptotic region and no interior boundary; (b) in addition to the single asymptotically flat region, there is an interior boundary two-surface S. In Sec. II, we prove that in case (a), initial data for stationary solutions are extrema of mass at fixed electric charge (measured at infinity). Furthermore, if the electrostatic potential $A_0{}^{\Gamma}$ vanishes at infinity, then stationary solutions are extrema of mass even with respect to variations that change the electric charge. A generalization of these results to case (b) proves that stationary black holes always are extrema of mass at fixed electric charge, angular momentum, and horizon area. Furthermore, if $A_0{}^{\Gamma}$ vanishes at infinity and the angular velocity of the horizon vanishes, then stationary black holes are extrema of mass when merely the horizon area is held fixed. As a by-product of this analysis, we obtain a simple proof of a generalization of the first law of black-hole mechanics. Arguments are also presented in support of the converse of these results—specifically, that in case (a) extrema of mass at fixed charge are stationary solutions and in case (b) that extrema of mass at fixed electric charge, angular momentum, and boundary surface area are stationary black holes. However, we do not succeed in obtaining a rigorous proof of these converse results.

Nevertheless, in Sec. III we succeed in proving the following results: If a foliation of the spacetime by asymptotically flat, maximal (i.e., with vanishing trace of the extrinsic curvature) surfaces exists, then in case (a) extrema of mass always correspond to static solutions whereas in case (b) extrema of mass at fixed boundary area always correspond to static black holes. The hypotheses of these results are stronger than the converses described at the end of the previous paragraph in that they require in case (a) an extremum of mass with respect to all variations satisfying the constraints rather than merely those preserving electric charge, and in case (b) they require an extremum of mass with respect to variations which may change the charge and/or angular momentum. In addition, a maximal foliation is assumed. However, the conclusion also is stronger in that the solution must be static rather than merely stationary. (The precise definition of static used in this paper is given at the end of this section.)

By combining the results of Sec. III with those of Sec. II, we obtain as by-products of our analysis the following conclusions (modulo the existence of a maximal slice). In case (a) it follows that any stationary solution of the Einstein-Yang-Mills equation for which $A_0{}^{\Gamma}$ vanishes asymptotically (or for which the Yang-Mills electric charge Q vanishes) must be static. In case (b), it follows that any stationary black hole with bifurcate Killing horizon for which $A_0{}^{\Gamma}$ vanishes asymptotically (or $Q = 0$) and for which the stationary Killing field is normal to the horizon (or for which the canonical angular momentum $\mathcal{J}$ vanishes) must be static. This second result closes a significant gap that has existed in the proof of the black-hole uniqueness theorems.

In Sec. IV, we present our plausibility arguments for the existence of Einstein-Yang-Mills solutions. The argument is similar in character to the type of argument used to predict the existence of the sphaleron solution [6, 7] of the Yang-Mills-Higgs system but our argument predicts an infinite sequence of new solutions. For the purpose of making our arguments, we assume conditions on the mass function sufficient to ensure that a Riemannian metric G_{AB} on Γ can be chosen so that the flow generated by the vector field $M^A = -G^{AB}\nabla_B m$ carries each point of Γ to a critical point of m. (Here, upper case latin letters denote indices of tensor fields on Γ, viewed as an infinite dimensional manifold.) Our discussion is heuristic in that we do not attempt to precisely define a manifold structure on Γ nor prove, e.g., that m is a differentiable function on Γ. (Actually there is a difficulty with our assumption due to the behavior of m along certain "nonlinear scaling directions" in Γ, but our arguments in Sec. IV take account of this behavior.) As already mentioned above, on the phase space Γ corresponding to case (a), the mass functional has a countably infinite, disconnected set of absolute minima (with $m = 0$), corresponding to trivial initial data and all large gauge transformations of this trivial data. We argue that because the set of absolute minima of m is disconnected whereas Γ is connected, the flow defined by M^A cannot carry all points of phase space to the flat spacetime absolute minima (or other local minima, if any exist). By minimizing m over the set Γ_1 of points which do not flow to local minima, we argue for the existence of a saddle point of m, with $m = m_1 > 0$. By the results of Sec. III, this yields a static solution to the Einstein-Yang-Mills equations, which should be unstable because small perturbations can decrease m. We believe that this accounts for the $n = 1$ Bartnik-McKinnon solution.

Again, the minima of m on Γ_1 (which correspond to saddle points of m on Γ) will be composed of a countably infinite set of disconnected components on account of the presence of large gauge transformations. We now repeat the above argument, replacing Γ by Γ_1. (To do so, we must make the additional assumption that Γ_1 is connected.) We conclude that there should exist an additional saddle point, which implies the existence of another static solution with mass $m_2 > m_1$ and one additional unstable mode. We believe that this solution is precisely the $n = 2$ Bartnik-McKinnon solution and that further repetition of this argument accounts for all of the higher n Bartnik-McKinnon solutions. A similar argument explains the existence of the discrete family of colored black-hole solutions at any given value of horizon surface area.

The above argument depends only upon the qualitative features of phase space resulting from the presence of "large gauge transformations," the existence of (trivial) known minima of the mass functional, and the relationship between extrema of mass and stationary and static solutions developed in Secs. II and III. Hence, the same basic argument can be used to predict the existence of additional new stationary solutions to the Einstein-Yang-Mills equations as well as new stationary solutions for similar systems, such as the Yang-Mills-Higgs equations. In particular, on the basis of this argument, we conjecture that there exist new colored black-hole solutions which are related to the charged Kerr solutions in the same way as the known colored black holes are related to the Schwarzschild black hole. We also conjecture that there exist new static solutions to the Yang-Mills-Higgs system which are analogues of the $n > 1$ Bartnik-McKinnon solution. (The $n = 1$ Bartnik-McKinnon solution presumably corresponds to the known sphaleron solution.) These and other conjectures are formulated at the end of Sec. IV.

We shall adhere to the following conventions on index notation in this paper: As already mentioned above, we shall use upper case latin indices to denote tensor indices on phase space, e.g., $\nabla_A m$ denotes the gradient of the mass functional on phase space. Lower case greek indices will be used to denote tensors on (four-dimensional) spacetime, e.g., $g_{\mu\nu}$ denotes the (Lorentzian) spacetime metric. Lower case latin indices will denote tensors on the (three-dimensional, spacelike) Cauchy hypersurface Σ used to define initial data, e.g., h_{ab} denotes the (Riemannian) metric on Σ. Finally, we shall use upper case greek indices for the Yang-Mills Lie algebra. Thus, $A_\mu{}^\Gamma$ denotes the (Lie-algebra-valued, four-component) Yang-Mills potential on spacetime, and $A_a{}^\Gamma$ denotes the (Lie-algebra-valued, three-component) Yang-Mills potential on Σ. The structure tensor for the Lie algebra is denoted $c^\Gamma{}_{\Lambda\Delta}$ and Lie algebra indices are raised and lowered with the Killing metric $-\frac{1}{2}c^\Lambda{}_{\Gamma\Delta}c^\Delta{}_{\Sigma\Lambda}$. All indices are intended as abstract indices, though little harm will be done by viewing them as representing components.

We shall use units where $G = c = 1$, and we choose the Yang-Mills coupling constant to be unity. Thus, the Einstein-Yang-Mills action takes the form

$$S_{\rm EYM} = \int_M [R - F_{\mu\nu}{}^\Lambda F^{\mu\nu}{}_\Lambda]\sqrt{-g}, \tag{1}$$

where R is the (spacetime) scalar curvature, and $F_{\mu\nu}{}^\Lambda$ is the (spacetime) Yang-Mills field strength:

$$F_{\mu\nu}{}^\Lambda = 2\nabla_{[\mu}A_{\nu]}^\Lambda + c^\Lambda{}_{\Gamma\Delta}A_\mu{}^\Gamma A_\nu{}^\Delta. \tag{2}$$

We have omitted the usual overall factor of $1/16\pi$ in Eq. (1), since it would be a nuisance to carry that factor through our definitions of canonically conjugate momenta, etc. However, to agree with conventions, we will, in effect, reinsert that factor when we define mass and other quantities in Sec. II.

Finally, we explain the precise meaning of the terms "stationary" and "static" used throughout this paper. First, for an ordinary Hamiltonian system in particle mechanics, we say that a solution is *stationary* if it is time independent, i.e., if the time derivative of all q's and p's vanish. We say that a solution is *static* if in addition to being stationary, all of the momenta vanish. For the Einstein-Yang-Mills (EYM) system, these definitions are carried over as follows: Let $(M; g_{\mu\nu}, A_\mu{}^\Lambda)$ be an asymptotically flat solution to the EYM equations on spacetime manifold M. We say that this solution is *stationary* if there exists a Killing field t^μ of $g_{\mu\nu}$ which asymptotically approaches a time translation at infinity, and is such that for some choice of gauge for the Yang-Mills field we have $\pounds_t A_\mu^\Lambda = 0$. Note that it is not required that t^μ be timelike outside of a neighborhood of infinity. We say that this solution is *static* if the stationary Killing field t^μ is hypersurface orthogonal and the Yang-Mills electric field [see Eq. (8) below] vanishes on these orthogonal hypersurfaces. Since the extrinsic curvature of the orthogonal hypersurfaces vanishes, this corresponds to the vanishing of both the Einstein and Yang-Mills momenta [see Eqs. (6) and (7) below]. Note that our terminology differs slightly from usage common in general relativity where "static" normally would mean hypersurface orthogonality of t^μ but would not require the vanishing of the electric field.

II. STATIONARY SOLUTIONS AND EXTREMA OF MASS

In ordinary Hamiltonian particle mechanics on a phase space Γ the dynamical evolution equations take the form

$$\nabla_A H = \Omega_{AB} h^B, \tag{3}$$

where $H : \Gamma \to \mathbb{R}$ is the Hamiltonian, h^A is the Hamiltonian vector field on Γ (i.e., the possible dynamical trajectories are the integral curves of h^A), and Ω_{AB} is the symplectic form on Γ. Since Ω_{AB} is nondegenerate, we can invert Eq. (3) to obtain

$$h^A = \Omega^{AB}\nabla_B H \tag{4}$$

where Ω^{AB} denotes the inverse symplectic form. (This yields the usual form of Hamilton's equation of motion with $\Omega = \sum dp \wedge dq$.) From Eqs. (3) and (4), we immediately obtain the following key conclusion: *A point $X \in \Gamma$ is an extremum of H (i.e., $\nabla_A H = 0$ at X) if and only if X corresponds to a stationary solution (i.e., $h^A = 0$ at X).* (Note: We shall use the words "extremum" and "critical point" interchangeably throughout this paper.)

The above result holds even when constraints are present on Γ, i.e., when the allowed points of phase space are restricted to a submanifold $\bar{\Gamma} \subset \Gamma$. However, the "only if" direction of this result is of less interest in this case because H is required to be an extremum with respect to variations which violate the constraints and thus are unphysical. A more relevant result is obtained by pulling back Eq. (3) to $\bar{\Gamma}$:

$$\bar{\nabla}_A H = \bar{\Omega}_{AB} h^B, \tag{5}$$

where $\bar{\Omega}_{AB}$ denotes the pullback of Ω_{AB} to $\bar{\Gamma}$ and $\bar{\nabla}_A$ denotes the gradient operator on $\bar{\Gamma}$. (It is assumed here that h^A on $\bar{\Gamma}$ is tangent to $\bar{\Gamma}$, i.e., that dynamical evolution preserves the constraints.) From Eq. (5) we immediately obtain the following result: *A point $X \in \bar{\Gamma}$ is an extremum of H on $\bar{\Gamma}$ (i.e., $\bar{\nabla}_A H = 0$ at X so that H is an extremum with respect to all variations which satisfy the constraints) if and only if h^A is a degeneracy direction of $\bar{\Omega}_{AB}$ (i.e., $\bar{\Omega}_{AB} h^B = 0$).* (By definition, $\bar{\Omega}_{AB}$ will fail to be nondegenerate if some of the constraints are first class.) In the case where $\bar{\Omega}_{AB}$ is degenerate, the degeneracy vectors normally are interpreted as representing pure gauge variations. Thus, the above result may be interpreted as saying that $X \in \bar{\Gamma}$ is an extremum of H under constrained variations if and only if it is gauge equivalent to a stationary solution.

The main purpose of this section is to extend the above results to general relativity. Of course, these results would be immediately applicable if we could rigorously define an infinite dimensional manifold structure on the phase space Γ of general relativity and its constraint submanifold $\bar{\Gamma}$, define a symplectic form Ω_{AB} on Γ, define the infinitesimal time evolution operator as a vector field h^A on $\bar{\Gamma}$ and find a differentiable function H on Γ such that Eq. (3) holds on Γ. However, there are formidable technical difficulties in proceeding in this manner and we shall not attempt to follow this route. Nevertheless, as we shall see, it is quite simple to derive a formula corresponding to Eq. (3) for the variation of the Hamiltonian of general relativity, which allows us to prove (rigorously) that initial data corresponding to stationary solutions are extrema of H on $\bar{\Gamma}$. In the Einstein-Yang-Mills (EYM) case, this shows that initial data for stationary solutions are extrema of the Arnowitt-Deser-Misner (ADM) mass at fixed electric charge. A generalization of this derivation to include an interior boundary surface enables us to easily obtain a strengthened version of the first law of black-hole mechanics and prove that stationary black holes are extrema of ADM mass at fixed electric charge, angular momentum, and horizon area. We are not able to give a rigorous proof of the converse results—i.e., in the first case, that extrema of mass at fixed electric charge are initial data for solutions which are stationary and, in the second case, that extrema of mass at fixed electric charge, angular momentum, and boundary area correspond to stationary black holes. However, we sketch an argument for this which we believe could be made into a rigorous proof with further analysis. In addition, rigorous results along these lines will be proven in Sec. III.

As we shall see more explicitly below, it is essential that the proper "boundary terms" be included in the definition of H in order that Eq. (3) will hold. The precise form of these boundary terms will depend upon the theory under consideration. In order to be explicit and definite about these terms, we shall restrict attention here to the case of SU(2) Einstein-Yang-Mills (EYM) theory, which, of course, is the case of prime interest for our applications in Sec. IV. (For simplicity, we also shall restrict attention to the case when the Yang-Mills principal bundle over spacetime is trivial, so that a globally smooth Yang-Mills potential $A_\mu{}^\Lambda$ exists.) Note, however, that all of the analysis of this section and the next also applies straightforwardly to Einstein-Maxwell theory. Indeed, the only place in this section and the next where the SU(2) structure of the Yang-Mills gauge group G is used is in the derivation of Eq. (41), which is used to obtain Eqs. (47) and (49). [For G=U(1) Eq. (41) holds trivially.] Thus the results of this section and the next could be generalized to the case of an arbitrary compact Yang-Mills gauge group G by making appropriate changes in Eqs. (47) and (49). (Compactness of G is needed to ensure the existence of a positive definite invariant metric for the Lie algebra.) Furthermore, the arguments of Sec. IV apply to the case of an arbitrary compact G which is semisimple. Finally, we emphasize that most of the considerations of this section are extremely general; indeed, they should be applicable to any diffeomorphism invariant theory which can be given a Hamiltonian formulation.

Some of the analysis of this section is closely related to the results of Schutz and Sorkin [8]. Indeed, our proof that stationary solutions are extrema of H can be viewed, in essence, as a Hamiltonian version of their Lagrangian argument. The main difference in the two analyses is that we keep explicit track of the boundary terms which occur, and we generalize the results to include an interior boundary. In addition, the issue of whether an extremum of H with respect to only constrained variations implies stationarity was not considered by Schutz and Sorkin.

The Hamiltonian formulation of Einstein Yang-Mills theory given below is closely related to the work of Benguria, Cordero, and Teitelboim [9]. An important difference between these two analysis occurs in the choice of the variables which are identified as "nondynamical." We identify the lapse and shift vector N^μ and the normal component $n^\mu A_\mu{}^\Lambda$ of the Yang-Mills potential as nondynamical [see discussion below Eq. (2.29)], whereas Benguria *et al.* [9], in effect, identify N^μ and $N^\mu A_\mu{}^\Lambda$ as nondynamical. This difference is of no consequence when the lapse is nonzero, but when the lapse vanishes, it is not consistent to view $N^\mu A_\mu{}^\Lambda$ as nondynamical. This causes some differences in the analysis of angular momentum.

A point in the phase space Γ of EYM theory corresponds to the specification of the fields $(h_{ab}, \pi^{ab}, A_a{}^\Lambda, E^a{}_\Lambda)$ on a three-dimensional manifold Σ. Here h_{ab} is a Riemannian metric on Σ and $A_a{}^\Lambda$ is a Lie-algebra-valued one-form on Σ . (Our index notational conventions were stated near the end of Sec. I above.) The momentum π^{ab} canonically conjugate to h_{ab} is related to the extrinsic curvature K_{ab} of Σ in the spacetime obtained by evolving this initial data by

$$\pi^{ab} = \sqrt{h}(K^{ab} - h^{ab}K). \tag{6}$$

The momentum $\Pi^a{}_\Lambda$, canonically conjugate to $A_a{}^\Lambda$, is simply

$$\Pi^a{}_\Lambda = 4E^a{}_\Lambda, \tag{7}$$

where $E^a{}_\Lambda$ is the electric field in the evolved spacetime, viewed as a tensor density of weight, 1/2, i.e.,

$$E^a{}_\Lambda = \sqrt{h}F_\mu{}^a{}_\Lambda n^\mu, \tag{8}$$

where n^μ is the unit normal to Σ in the evolved spacetime and the (spacetime) Yang-Mills field strength $F_{\mu\nu}{}^\Lambda$ is given by Eq. (2). (Here spatial indices are raised and lowered with the spatial metric h_{ab} and the Lie algebra indices are lowered and raised with the Killing metric $-\frac{1}{2}c^\Lambda{}_{\Gamma\Delta}c^\Delta{}_{\Sigma\Lambda}$.) In Eqs. (6) and (8), the quantity "$\sqrt{h}$" is defined by

$$\epsilon_{abc} = \sqrt{h}\eta_{abc}, \tag{9}$$

where ϵ_{abc} is the volume element on Σ associated with h_{ab} and η_{abc} is an arbitrary fixed volume element (e.g., associated with a local coordinate system) on Σ.

We shall be concerned in this paper only with asymptotically flat initial data. We shall take "asymptotically flat" to mean the following: There is an "asymptotic region" $U \subset \Sigma$ which is diffeomorphic to $(\mathbb{R}^3 - B)$ with B compact and U is such that, in U,

$$h_{ab} = e_{ab} + \tilde{h}_{ab}(\theta,\varphi)/r + o(r^{-1}), \tag{10}$$

$$\pi^{ab} = \tilde{\pi}^{ab}(\theta,\varphi)/r^2 + o(r^{-2}), \tag{11}$$

$$A_a{}^\Lambda = \tilde{A}_a{}^\Lambda(\theta,\varphi)/r + o(r^{-1}), \tag{12}$$

$$E^a{}_\Lambda = \tilde{E}^a{}_\Lambda(\theta,\varphi)/r^2 + o(r^{-2}), \tag{13}$$

where e_{ab} is the pullback to Σ (under the above diffeomorphism) of the flat metric on $\mathbb{R}^3$, and (r,θ,φ) are standard spherical coordinates with respect to e_{ab} . [For these asymptotic conditions, the fixed volume element η_{abc} implicit in the expressions (6) and (8) for π^{ab} and $E^a{}_\Lambda$ is chosen in U to be the volume element associated with e_{ab}.] We also require that any kth-order derivatives of these quantities (with respect to the flat derivative operator associated with e_{ab}) fall off k powers of r faster than specified in Eqs. (10)–(13). In addition, since we shall wish to employ the notion of angular momentum for certain purposes below, we impose the further restrictions on the initial data which are needed to ensure that angular momentum is well defined (see below). For many of our results, it appears likely that the relatively strong asymptotic conditions we use could be weakened to the Sobolev space conditions which ensure that ADM mass is well defined [10], but we shall not investigate this issue further here.

As already mentioned in the Introduction, we shall focus attention on the following two cases.

Case (a): Σ is a manifold without boundary and can be written as the disjoint union of an "asymptotic region" U and a compact set C. The phase space $\Gamma^{(a)}$ is taken to consist of all smooth initial data on Σ which are asymptotically flat on U in the above sense. (Since we shall not attempt to define a manifold structure on $\Gamma^{(a)}$, there is no harm done in restricting consideration to smooth initial data.)

Case (b): Σ is a manifold with boundary and the boundary is comprised by a compact two-surface S; furthermore, Σ is the union of a compact set C (containing S) and an asymptotic region U. The phase space $\Gamma^{(b)}$ consists of all smooth initial data on Σ which are asymptotically flat on U.

As is well known, constraints are present in Einstein-Yang-Mills theory. The allowed initial data is restricted to the constraint submanifold $\bar{\Gamma}$ defined by the vanishing at each point $x \in \Sigma$ of the quantities

$$\begin{aligned} 0 = \mathcal{C}_\Lambda &= 4\sqrt{h}\mathcal{D}_a(E^a{}_\Lambda/\sqrt{h}) \\ &= 4[\sqrt{h}D_a(E^a{}_\Lambda/\sqrt{h}) + c_{\Lambda\Gamma}{}^\Delta A_a{}^\Gamma E^a{}_\Delta], \end{aligned} \tag{14}$$

$$\begin{aligned} 0 = \mathcal{C}_0 &= \sqrt{h}[-R + (1/h)(\pi_{ab}\pi^{ab} - \tfrac{1}{2}\pi^2)] \\ &\quad + (2/\sqrt{h})E_a{}^\Lambda E^a{}_\Lambda + \sqrt{h}F_{ab}{}^\Lambda F^{ab}_\Lambda, \end{aligned} \tag{15}$$

$$0 = \mathcal{C}_a = -2\sqrt{h}D_b(\pi_a{}^b/\sqrt{h}) + 4F_{ab}{}^\Lambda E^b{}_\Lambda, \tag{16}$$

where D_a is the derivative operator on Σ compatible with the metric h_{ab} , $\mathcal{D}_a$ denotes the (metric compatible) gauge-covariant derivative operator, and R denotes the scalar curvature of h_{ab}.

The equations of motion of EYM theory can be formally derived from a Hamiltonian H. Indeed, we shall see below that in case (a), a true Hamiltonian satisfying Eq. (3) can be found by including certain "surface integral contributions" to H. In any case, the "volume integral contribution," H_V to H has the "pure constraint" form

$$H_V = \int_\Sigma (N^\mu \mathcal{C}_\mu + N^\mu A_\mu{}^\Lambda \mathcal{C}_\Lambda), \tag{17}$$

where the fixed volume element η_{abc} [see Eq. (9)] is understood here and below in all volume integrals over Σ. As we shall explain further below, N^μ has the interpretation of being the "lapse" and "shift" functions and $A_0{}^\Lambda$ has the interpretation of being the component of the spacetime Yang-Mills potential normal to Σ. Neither N^μ nor $A_0{}^\Lambda$ are to be viewed as dynamical variables and they may be prescribed arbitrarily. The "pure constraint" form of H_V is not special to EYM theory; in fact, the "volume integral contribution" to any Hamiltonian arising from a diffeomorphism-invariant Lagrangian always takes such a form (see the appendix of [11]).

If we compute the change in H_V caused by arbitrary infinitesimal variations $(\delta h_{ab}, \delta\pi^{ab}, \delta A_a{}^\Lambda, \delta E^a{}_\Lambda)$ of compact support, we obtain, after integration by parts, the formula

$$\delta H_V = \int_\Sigma (\mathcal{P}^{ab}\delta h_{ab} + \mathcal{Q}_{ab}\delta\pi^{ab} + \mathcal{R}^a{}_\Lambda \delta A_a{}^\Lambda + \mathcal{S}_a{}^\Lambda \delta E^a{}_\Lambda), \tag{18}$$

where $\mathcal{P}^{ab}$, $\mathcal{Q}_{ab}$, $\mathcal{R}^a{}_\Lambda$, and $\mathcal{S}_a{}^\Lambda$ are given by

$$\mathcal{P}^{ab} = \sqrt{h}N^0 a^{ab} + \sqrt{h}[h^{ab}D^cD_c(N^0) - D^aD^b(N^0)] - \pounds_{N^i}\pi^{ab}, \quad (19)$$

$$\mathcal{Q}_{ab} = \frac{N^0}{\sqrt{h}}(2\pi_{ab} - \pi h_{ab}) + \pounds_{N^i}h_{ab}, \quad (20)$$

$$\mathcal{R}^a{}_\Lambda = -4[\sqrt{h}\mathcal{D}_b(N^0F^{ab}_\Lambda) + N^0 c_{\Lambda\Gamma}^{\ \ \Delta}A_0{}^\Gamma E^a{}_\Delta - \pounds_{N^i}E^a{}_\Lambda], \quad (21)$$

$$\mathcal{S}_a{}^\Lambda = 4[N^0E_a{}^\Lambda/\sqrt{h} + \mathcal{D}_a(N^0A_0{}^\Lambda) + \pounds_{N^i}A_a{}^\Lambda], \quad (22)$$

with

$$a^{ab} \equiv (2/h)(E^a{}_\Lambda E^{b\Lambda} - \tfrac{1}{2}h^{ab}E_c{}^\Lambda E^c{}_\Lambda) + 2(F^{ac}_\Lambda F_c{}^{b\Lambda} + \tfrac{1}{4}h^{ab}F^{cd}_\Lambda F_{cd}^\Lambda) + (R^{ab} - \tfrac{1}{2}h^{ab}R) + (1/h)[(2\pi^a{}_c\pi^{bc} - \pi\pi^{ab}) - \tfrac{1}{2}h^{ab}(\pi^{cd}\pi_{cd} - \tfrac{1}{2}\pi^2)]. \quad (23)$$

In Eqs. (19)–(22) we have inserted the index i on $\pounds_{N^i}$ to emphasize that the Lie derivatives here are to be taken on the manifold Σ with respect to the vector field N^a. (By contrast, the symbol $\pounds_{N^\mu}$ used below denotes the Lie derivative on the spacetime manifold with respect to N^μ.) The Lie derivatives of h_{ab} and $A_a{}^\Lambda$ are the ordinary Lie derivatives of these tensor fields, i.e.,

$$\pounds_{N^i}h_{ab} = 2D_{(a}N_{b)},$$
$$\pounds_{N^i}A_a{}^\Lambda = N^bD_bA_a{}^\Lambda + A_b{}^\Lambda D_aN^b, \quad (24)$$

but the Lie derivatives of the tensor densities π^{ab} and $E^a{}_\Lambda$ are to be understood throughout this paper as the natural notion of Lie derivative for tensor densities, i.e.,

$$\pounds_{N^i}\pi^{ab} \equiv \frac{1}{3!}\eta^{cde}\pounds_{N^i}(\pi^{ab}\eta_{cde}) = \sqrt{h}N^cD_c(\pi^{ab}/\sqrt{h}) - 2\pi^{c(a}D_cN^{b)} + \pi^{ab}D_cN^c, \quad (25)$$

$$\pounds_{N^i}E^a{}_\Lambda \equiv \frac{1}{3!}\eta^{bcd}\pounds_{N^i}(E^a{}_\Lambda\eta_{bcd}) = \sqrt{h}N^cD_c(E^a{}_\Lambda/\sqrt{h}) - E^c{}_\Lambda D_cN^a + E^a{}_\Lambda D_cN^c, \quad (26)$$

where the ordinary expression for the Lie derivatives of tensor fields is understood on the right-hand side of the first line of (25) and (26), and $\eta^{abc} = \eta^{[abc]}$ is defined by $\eta^{abc}\eta_{abc} = 3!$

The Einstein-Yang-Mills equations are precisely equivalent to the constraint equations (14)–(16) (which can be obtained by extremizing H_V with respect to N^μ and $A_0{}^\Lambda$) together with the evolution equations obtained from Eq. (18),

$$\dot{\pi}^{ab} = -\mathcal{P}^{ab}, \quad (27)$$

$$\dot{h}_{ab} = \mathcal{Q}_{ab}, \quad (28)$$

$$\dot{E}^a{}_\Lambda = -\mathcal{R}^a{}_\Lambda/4, \quad (29)$$

$$\dot{A}_a{}^\Lambda = \mathcal{S}_a{}^\Lambda/4. \quad (30)$$

Indeed for a given (arbitrarily prescribed) choice of N^μ and $A_0{}^\Lambda$ (with $N^0 \neq 0$), if we are given a solution $(h_{ab}(t), \pi^{ab}(t), A_a{}^\Lambda(t), E^a{}_\Lambda(t))$ to Eqs. (14)–(16) and Eqs. (27)–(30) on Σ, we obtain a spacetime solution $(g_{\mu\nu}, A_\mu{}^\Lambda)$ to the EYM equations on the spacetime manifold $M = \mathbb{R}\times\Sigma$ as follows: If we let (x_1, x_2, x_3) denote local coordinates on Σ, the components of the spacetime metric $g_{\mu\nu}$ in coordinates (t, x_1, x_2, x_3) on M are given by

$$ds^2 = -[(N^0)^2 - N^aN_a]dt^2 + \sum_i^3 N_i dt\,dx^i + \sum_{i,j}^3 h_{ij}dx^idx^j \quad (31)$$

[In Eq. (31)—and only in this equation—the indices i, j must be viewed as coordinate components rather than abstract indices.] This gives N^0 the interpretation of being the "lapse function" and N^a the interpretation of being the "shift vector" for the t = const hypersurfaces on M. The spacetime Yang-Mills potential $A_\mu{}^\Lambda$ then is determined by the conditions that its pullback to any t = const hypersurface be $A_a{}^\Lambda(t)$ and that its normal component $n^\mu A_\mu{}^\Lambda$ be $A_0{}^\Lambda$, where n^μ is the unit normal to the t =const hypersurface in the metric (31). The overdots appearing in Eqs. (27)–(30) then correspond to Lie derivatives with respect to N^μ in the spacetime constructed in this manner.

As mentioned above, N^μ and $A_0{}^\Lambda$ are not viewed as dynamical variables; i.e., they are not represented in the phase space Γ. We may choose N^μ and $A_0{}^\Lambda$ arbitrarily and, by solving Eqs. (14)–(16) and (27)–(30), obtain a solution of the EYM equations on $\mathbb{R}\times\Sigma$ as described above. Our choice of N^μ is dictated by the type of "time evolution" we seek. We shall be interested only in the case where N^μ corresponds asymptotically to a time translation or rotation, so we shall restrict attention to those cases. With regard to $A_0{}^\Lambda$, we would be free to choose $A_0{}^\Lambda = 0$, but this would, in certain cases, exclude the possibility of choosing a gauge in which $A_a{}^\Lambda$ is stationary (i.e., time independent). For this reason, we will restrict $A_0{}^\Lambda$ only by requiring that it approach an (angle-dependent) limit $\tilde{A}_0{}^\Lambda(\theta,\varphi)$ as $r \to \infty$ and that the kth derivatives of $A_0{}^\Lambda$ are $O(1/r^k)$ as $r \to \infty$.

Equation (18) is precisely of the form of Eq. (3) contracted into a vector on Γ corresponding to a perturbation $(\delta h_{ab}, \delta\pi^{ab}, \delta A_a{}^\Lambda, \delta E^a{}_\Lambda)$ of compact support on Σ. However, as pointed out by Regge and Teitelboim [12] in the vacuum case, Eq. (18) fails to hold for H_V when we perturb towards arbitrary nearby initial data, i.e., when $(\delta h_{ab}, \delta\pi^{ab})$ merely satisfy asymptotic flatness boundary conditions at infinity (i.e., rather than being of compact support). The reason is that nonvanishing "surface terms" will arise when one integrates by parts to try to put δH_V in the form of Eq. (18). However, in case (a) these surface terms can be canceled by adding additional surface terms to H_V to obtain H. Hence, generalizing the Regge-Teitelboim Hamiltonian to the EYM case, we obtain, as the Hamiltonian in case (a) (i.e., a single asymptotic region with no interior boundary),

$$H = H_V + \oint_\infty dS^a \{ N^0[\partial^b h_{ab} - \partial_a h_b{}^b] + 2N^b \pi_{ab}/\sqrt{h} + 4(N^0 A_0{}^\Lambda + N^b A_b{}^\Lambda) E_{a\Lambda}/\sqrt{h}\}. \tag{32}$$

Here the surface integrals are taken over two-spheres of coordinate radius r (with dS^a representing the proper volume element on these surfaces), with the limit as $r \to \infty$ then taken. (The derivative operator ∂_a appearing in the first term is the one associated with the flat metric e_{ab} and the index of h_{ab} in that term is raised with e^{ab}. The term "$\sqrt{h}$" is the square root of the determinant of h_{ab} with respect to the volume element associated with e_{ab}, i.e., near infinity we choose η_{abc} [see Eq. (9)] to coincide with the volume element associated with e_{ab}.) The first two terms are the same as obtained by Regge and Teitelboim [12] in the vacuum case. By a direct calculation, it can be verified that for all asymptotically flat perturbations, i.e., not merely ones of compact support, and for N^μ and $A_0{}^\Lambda$ satisfying the asymptotic conditions of the previous paragraph, we have

$$\delta H = \int_\Sigma (\mathcal{P}^{ab}\delta h_{ab} + \mathcal{Q}_{ab}\delta\pi^{ab} + \mathcal{R}^a{}_\Lambda \delta A_a{}^\Lambda + \mathcal{S}_a{}^\Lambda \delta E^a{}_\Lambda); \tag{33}$$

i.e., Eq. (3) is satisfied.

We define the canonical energy $\mathcal{E}$ on the constraint submanifold $\bar{\Gamma}$ of phase space to be the Hamiltonian function H (multiplied by $1/16\pi$) corresponding to the case where N^μ is an asymptotic time translation, i.e., $N^0 \to 1$ and $N^a \to 0$ at infinity. Recalling that H_V is pure constraint and hence vanishes on $\bar{\Gamma}$, we obtain, from Eq. (32),

$$\mathcal{E} = m + (1/4\pi) \oint_\infty dS^a A_0{}^\Lambda E_{a\Lambda}, \tag{34}$$

where m is the *ADM mass*, defined by

$$m = (1/16\pi) \oint_\infty dS^a [\partial^b h_{ab} - \partial_a h_b{}^b]. \tag{35}$$

Thus, the canonical energy differs from the ADM mass by the term

$$\mathcal{E}_{\rm YM} = (1/4\pi) \oint_\infty dS^a A_0{}^\Lambda E_{a\Lambda}, \tag{36}$$

which is highly "gauge dependent" since $A_0{}^\Lambda$ can be chosen arbitrarily. However, if we consider a stationary solution of the EYM equations, then $A_0{}^\Lambda$ is uniquely determined—up to addition of a gauge covariantly constant Lie-algebra-valued scalar field which commutes with $E^a{}_\Lambda$ (if any exist) and up to a time-independent gauge transformation—by the condition that $\dot{A}_a{}^\Lambda = 0$ and $\dot{E}^a{}_\Lambda = 0$ for all time when N^μ is chosen to be the stationary Killing field. If we make this choice of $A_0{}^\Lambda$ for a stationary solution, then $\mathcal{E}_{YM}$ becomes invariantly defined, since it is manifestly invariant under a time-independent gauge transformation and addition of a gauge-covariantly constant scalar field also does not change $\mathcal{E}_{YM}$ on account of the constraint (14). Furthermore, for this choice of $A_0{}^\Lambda$, we have, by Eq. (30),

$$0 = \dot{A}_a{}^\Lambda = N^0 E_a{}^\Lambda/\sqrt{h} + \mathcal{D}_a(N^0 A_0{}^\Lambda) + \pounds_{N^i} A_a{}^\Lambda. \tag{37}$$

Contracting this equation with $A_0{}^\Lambda$, we find

$$0 = \partial_a(A_0{}^\Lambda A_{0\Lambda}) + O(r^{-2}), \tag{38}$$

which shows that the magnitude of $A_0{}^\Lambda$ is asymptotically constant. We define V by

$$V = \lim_{r\to\infty} (A_0{}^\Lambda A_{0\Lambda})^{1/2}. \tag{39}$$

In addition, Eq. (29) yields

$$0 = -\dot{E}^a{}_\Lambda = \sqrt{h}\mathcal{D}_b(N^0 F^{ab}_\Lambda) + N^0 c_{\Lambda\Gamma}{}^\Delta A_0{}^\Gamma E^a{}_\Delta - \pounds_{N^i} E^a{}_\Lambda, \tag{40}$$

which shows that, asymptotically, $A_0{}^\Lambda$ and $E_a{}^\Lambda$ point in the same Lie algebra direction, i.e.,

$$A_0{}^{[\Lambda} E_a{}^{\Gamma]} = O(r^{-3}). \tag{41}$$

[Here, the fact that we are considering G =SU(2) has been used.] Finally, by contracting (37) with $N^0 A_0{}^\Lambda$ and taking the divergence of the resulting equation, we find

$$D^a D_a \chi^2 = O(1/r^4), \tag{42}$$

where $\chi^\Lambda \equiv N^0 A_0{}^\Lambda$ and $\chi^2 = \chi^\Lambda \chi_\Lambda$, and where we have made use of (14), (41) and of the fact that $N^0 \to 1$ and $N^a = O(1/r)$ as $r \to \infty$. From this we conclude that

$$\chi^2 = V^2 + C/r + o(1/r) \tag{43}$$

with C a constant. Applying $\mathcal{D}_r$ to (43) we obtain

$$2N^0 A_{0\Lambda} \mathcal{D}_r(N^0 A_0{}^\Lambda) = -C/r^2 + o(1/r^2). \tag{44}$$

But, by (37) we have

$$\mathcal{D}_r(N^0 A_0{}^\Lambda) = -E_r{}^\Lambda + o(1/r^2). \tag{45}$$

Substituting this equation in (44), we obtain

$$N^0 A_0{}^\Lambda E_{r\Lambda} = C/2r^2 + o(1/r^2). \tag{46}$$

Thus, taking Eq. (41) into account, we find that, in the stationary case,

$$\mathcal{E}_{YM} = VQ, \tag{47}$$

where the Yang-Mills charge Q is defined by [13]

$$Q = \pm(1/4\pi) \oint_\infty |E^a{}_\Lambda r_a| dS, \tag{48}$$

where r^a denotes the unit radial vector in the metric e_{ab}, vertical bars denote the Lie-algebra norm, and the $\pm$ choice is made depending upon whether $\lim_{r\to\infty} r^2 A_0{}^\Lambda E^a{}_\Lambda r_a = C/2$ is positive or negative. Thus, in the stationary case, $\mathcal{E}_{YM}$ has the interpretation of being the electrostatic contribution to the energy due to the presence of a charge with a nonzero potential at infinity.

A similar analysis establishes that for an arbitrary (nonstationary) perturbation of a stationary solution, we have

$$\delta\mathcal{E}_{YM} = (1/4\pi) \oint_\infty A_0{}^\Lambda \delta E_{a\Lambda} = V\delta Q, \tag{49}$$

where

$$\delta Q = \pm(1/4\pi)\delta \oint_\infty |E^a{}_\Lambda r_a| dS. \tag{50}$$

By construction, H satisfies Eq. (33) for perturbations of any initial data set in $\bar{\Gamma}^a$ for any choice of $A_0{}^\Lambda$ approaching a direction-dependent limit at infinity and for any choice of N^μ which asymptotically approaches a time translation (or rotation) at infinity. Consider, now, initial data in $\bar{\Gamma}^a$ corresponding to a stationary solution to the EYM equations. Suppose we choose N^μ to be the stationary Killing field and choose $A_0{}^\Lambda$ so that $A_a{}^\Lambda$ and $E^a{}_\Lambda$ are time independent. Then, for this choice of N^μ and $A_a{}^\Lambda$ it follows from Eqs. (27)–(30) that the right-hand side of Eq. (33) vanishes. Taking into account Eqs. (34), (36), and (49), we obtain the following theorem.

Theorem 2.1. Let $(h_{ab}, \pi^{ab}, A_a{}^\Lambda, E^a{}_\Lambda)$ be smooth data for a stationary, asymptotically flat solution of the EYM equations for case (a) defined above (i.e., the initial data surface Σ has only one asymptotic region and no interior boundary). Let $(\delta h_{ab}, \delta\pi^{ab}, \delta A_a{}^\Lambda, \delta E^a{}_\Lambda)$ denote an arbitrary (not necessarily stationary) smooth, asymptotically flat solution of the linearized constraint equations. Then, we have

$$0 = \delta\mathcal{E} = \delta m + V\delta Q. \tag{51}$$

Thus, in particular, every stationary solution is an extremum of ADM mass m [defined by Eq. (35)] at fixed Yang-Mills electric charge Q [defined by Eq. (48)].

We emphasize that the proof of this theorem consists of a straightforward computation starting from the definition of $\mathcal{E}$, Eq. (34). Our discussion above concerning the existence of a Hamiltonian formulation of EYM theory merely serves to explain why a simple result such as Eq. (51) should be expected. The issue of whether the converse of this theorem is valid, i.e., whether extrema of mass at fixed electric charge correspond to stationary solutions, will be addressed at the end of this section.

In an exactly similar manner, we define the canonical angular momentum $\mathcal{J}$ on the constraint submanifold $\bar{\Gamma}$ of phase space to be the Hamiltonian function H (multiplied by $-1/16\pi$) corresponding to the case where N^μ is an asymptotic rotation at infinity, i.e., $N^0 \to 0$ and $N^a \to \phi^a$, where ϕ^a is a rotational Killing field of e_{ab} . From Eq. (32), we obtain,

$$\mathcal{J} = -(1/16\pi) \oint_\infty (2\phi_b \pi^{ab} + 4\phi^b A_b{}^\Lambda E^a{}_\Lambda) dS_a. \tag{52}$$

The first term is just the Regge-Teitelboim [12] expression for angular momentum in the vacuum case. The second term does not appear to have been considered previously (see, however, Eq. (2.11) of [9] and the footnote to Eq. (4) of [14]), although it would arise (as a term to be added to the usual volume integral expression for angular momentum) even for the case of Maxwell fields in flat spacetime. In particular, as indicated above, this term does not appear in the definition of angular momentum given in [9].

For initial data merely satisfying the asymptotic conditions (10)–(13), the first surface integral in the expression for $\mathcal{J}$ need not have a limit as $r \to \infty$. (The limit as $r \to \infty$ of the second surface integral always exists, but need not be gauge invariant.) Thus, additional restrictions on the initial data must be imposed in order that $\mathcal{J}$ be well defined. One possible set of additional restrictions is suggested by the following formula for $\mathcal{J}$ which is obtained by converting the surface integral expression (52) to a volume integral and then using the constraints (14) and (16):

$$\mathcal{J} = -(1/16\pi) \int_\Sigma (\pi^{ab} \pounds_{\phi^i} h_{ab} + 4E^a{}_\Lambda \pounds_{\phi^i} A_a{}^\Lambda), \tag{53}$$

where ϕ^a is any vector field on Σ which asymptotically approaches a rotational Killing field of e_{ab}. From Eq. (53) we see that if ϕ^a can be chosen so that $\pounds_{\phi^i} h_{ab}$ and $\pounds_{\phi^i} A_a{}^\Lambda$ fall off slightly faster than required by our asymptotic conditions, i.e., if

$$\pounds_{\phi^i} h_{ab} = O(1/r^{1+\epsilon}), \tag{54}$$

$$\pounds_{\phi^i} A_a{}^\Lambda = O(1/r^{1+\epsilon}), \tag{55}$$

then the integral (53) converges, and $\mathcal{J}$ is well defined. An alternative set of restrictions which also should ensure that $\mathcal{J}$ is well defined is that the magnetic part of the Weyl tensor [15] and the Yang-Mills magnetic field fall off slightly faster than automatically required by our asymptotic conditions. We shall not attempt here to investigate the relationship between these and other possible restrictions [16, 17], but shall merely assume below in any context in which $\mathcal{J}$ is used, that we have imposed suitable asymptotic restrictions to make $\mathcal{J}$ well defined.

In parallel with Theorem 2.1 above, Eq. (33) immediately yields the following result: For the phase space $\Gamma^{(a)}$ of case (a), any axisymmetric data satisfying the constraints is an extremum of $\mathcal{J}$ on the constraint submanifold $\bar{\Gamma}^{(a)}$, i.e., $\delta\mathcal{J} = 0$ for any perturbation off of an axisymmetric background. In addition, Eq. (53) also shows that in case (a), we have $\mathcal{J} = 0$ for any axisymmetric solution.

We turn now to case (b), where, again, Σ has one asymptotic region U, but now Σ has a smooth "interior boundary" S. Again, the initial data is required to be asymptotically flat on U. In this case, however, it does not appear that we can obtain a Hamiltonian function H on $\Gamma^{(b)}$ which satisfies Eq. (33) [i.e., the version of Eq. (3) appropriate to EYM theory] unless we put strong restrictions on N^μ at S. Nevertheless, we can obtain a generalization of Theorem 2.1 in the following manner. Again, we define H_V by Eq. (17). Then H_V vanishes identically on the constraint submanifold. Hence, if we consider any initial data satisfying the constraints and perturb it towards another solution of the constraints, the linearized perturbation obviously satisfies

$$0 = \delta H_V = \int_\Sigma N^\mu \delta\mathcal{C}_\mu + \int_\Sigma N^\mu A_\mu{}^\Lambda \delta\mathcal{C}_\Lambda. \qquad (56)$$

We integrate this equation by parts to remove all the derivatives from the perturbations as in the derivation of Eq. (18) above. When we do so, we obtain a volume integral which is precisely the same as the right-hand side of Eq. (18). In addition, we obtain surface integral terms arising both from infinity and from the interior boundary S. The surface terms at infinity are identical to those occurring in case (a), which were computed above. The surface terms at S can be computed in an entirely straightforward manner. The case of prime interest for our purposes occurs when N^μ asymptotically approaches a linear combination of a time translation and rotation at infinity, i.e., more precisely, as $r \to \infty$ we have $N^0 \to 1$ and $N^a \to \Omega\phi^a$ where ϕ^a is an axial Killing field of e_{ab} and Ω is a constant. In that case, we obtain

$$\begin{aligned}16\pi(\delta\mathcal{E} - \Omega\delta\mathcal{J}) = &\int_\Sigma (\mathcal{P}^{ab}\delta h_{ab} + \mathcal{Q}_{ab}\delta\pi^{ab} + \mathcal{R}^a{}_\Lambda \delta A_a{}^\Lambda + \mathcal{S}_a{}^\Lambda \delta E^a{}_\Lambda) + \oint_S dS_a D_b N^0 (h^{ac}h^{bd} - h^{ab}h^{cd})\delta h_{cd} \\ &- \oint_S dS_a [N^0 (h^{ac}h^{bd} - h^{ab}h^{cd}) D_b(\delta h_{cd}) + (2/\sqrt{h}) N^b \delta(\pi^a{}_b) - (1/\sqrt{h}) N^a \pi^{bc}\delta h_{bc} \\ &+ (4/\sqrt{h})(N^0 A_0{}^\Lambda + N^b A_b{}^\Lambda)\delta E^a{}_\Lambda - 4N^0 F^a{}_b{}^\Lambda \delta A^b{}_\Lambda],\end{aligned} \qquad (57)$$

where $\mathcal{E}$ and $\mathcal{J}$ again are defined by Eqs. (34) and (52). We emphasize that Eq. (57) is an identity which is satisfied whenever $(h_{ab}, \pi^{ab}, A_a{}^\Lambda, E^a{}_\Lambda)$ is an asymptotically flat solution of the constraints, $(\delta h_{ab}, \delta\pi^{ab}, \delta A_a{}^\Lambda, \delta E^a{}_\Lambda)$ is an asymptotically flat solution of the linearized constraints, $A_0{}^\Lambda$ has a (direction-dependent) limit as $r \to \infty$, and N^μ asymptotically approaches the linear combination $t^\mu + \Omega\phi^\mu$ of a time translation and rotation at infinity. ($A_0{}^\Lambda$ and N^μ otherwise can be chosen arbitrarily.) Note that no boundary conditions whatsoever have been imposed on any quantities at S. We also emphasize that the proof of Eq. (57) consists of the straightforward calculation outlined above.

Suppose, now, that we are given a solution of the EYM equations describing a stationary black hole with bifurcate Killing horizon. By a theorem of Hawking [18], there exists a Killing field χ^μ which vanishes on the bifurcation 2-surface. One possibility is that χ^μ coincides with the given stationary Killing field t^μ . If not, then Hawking's theorem states that the black hole is axisymmetric and χ^μ is a linear combination of t^μ and the axial Killing field ϕ^μ. Thus, in either case there is a constant Ω (possibly zero) such that

$$\chi^\mu = t^\mu + \Omega\phi^\mu \qquad (58)$$

and Ω then has the interpretation of being the angular velocity of the horizon.

We now choose Σ to be an asymptotically flat hypersurface which intersects (and terminates at) the bifurcation 2-surface S of the stationary black hole. We choose $N^\mu = \chi^\mu$ and we choose $A_0{}^\Lambda$ so that $\dot{A}_a{}^\Lambda = \dot{E}^a{}_\Lambda = 0$. Then the volume integral in Eq. (57) vanishes by Eqs. (27)–(30) (since the background solution is stationary) as does the second surface integral contribution from S (since $N^\mu = 0$ on S). The derivative of N^0 normal to S is proportional to the surface gravity κ of the horizon. Since κ is constant over S (see, e.g., [19]), this term can be pulled out of the integral, and we obtain

$$\oint_S dS_a D_b(N^0)\delta h_{cd}(h^{ac}h^{bd} - h^{ab}h^{cd}) = 2\kappa\delta A, \qquad (59)$$

where A is the area of S. Taking into account Eq. (57) and the second equality of Eq. (51), we thus obtain the following theorem.

Theorem 2.2. Let $(h_{ab}, \pi^{ab}, A_a{}^\Lambda, E^a{}_\Lambda)$ on a hypersurface Σ be smooth, asymptotically flat initial data for a stationary black hole, whose bifurcation 2-surface, S, lies on Σ. Let $(\delta h_{ab}, \delta\pi^{ab}, \delta A_a{}^\Lambda, \delta E^a{}_\Lambda)$ be an arbitrary smooth asymptotically flat solution of the linearized constraints on Σ. Then, we have

$$\delta m + V\delta Q - \Omega\delta\mathcal{J} = (1/8\pi)\kappa\delta A, \qquad (60)$$

where m, V, Q, $\mathcal{J}$, and Ω are defined by Eqs. (35), (39), (48), (52), and (58), respectively. Thus, in particular, any stationary black hole with bifurcate Killing horizon is an extremum of mass at fixed electric charge, canonical angular momentum, and horizon area.

Theorem 2.2 is, in essence, an extension of the first law of black-hole mechanics [20] to the EYM case. Several points, however, are worthy of emphasis. First, our derivation proves that Eq. (60) actually holds for arbitrary asymptotically flat perturbations of a stationary black hole, not merely for perturbations to other stationary black holes as required in the usual formulation and proof of the first law [20]. A proof of the first law under a weakening of the hypothesis that the perturbation take one to a nearby stationary black hole was previously given by Hawking [21], but his derivation still required that the perturbation be axisymmetric and "$(t\text{-}\phi)$ symmetric."

Second, as discussed above, our definition of canonical angular momentum differs from the usual Regge-Teitelboim surface integral by the second surface integral appearing in Eq. (52). In the Maxwell case with a globally smooth A_a [i.e., a trivial U(1) bundle], if one considers only stationary perturbations of a stationary black hole, this term would not contribute since both A_a and δA_a will fall off faster than $1/r$ at infinity. However, even in the Maxwell case, this term can contribute to $\delta\mathcal{J}$ for nonstationary perturbations. Our derivation shows that its contribution must be included in Eq. (60). In the EYM case, this term can, in principle, contribute to $\delta\mathcal{J}$ even for stationary perturbations.

Third, a term of the form $V\delta Q$ could be expected to appear in Eq. (60) by analogy with the form of the first law in Einstein-Maxwell theory. However, in EYM theory, it does not seem obvious, *a priori*, whether one should expect to find the δQ term to represent the change in electric charge at infinity or at the horizon, since these two quantities need not be equal. Our formula (60) shows that it is the change in charge at infinity which is relevant to the first law.

Finally, it is worth noting that although the term $V\delta Q$ is present in Eq. (60), there is no corresponding term present for magnetic charge. In Einstein-Maxwell theory, the absence of such a term is not surprising, since for a globally smooth A_a the magnetic charge automatically vanishes, and even if one generalizes the theory to consider nontrivial U(1) bundles, the magnetic charge is quantized by the Dirac quantization condition and thus cannot vary under a perturbation. However, in EYM theory magnetic charge is not quantized and can vary under perturbations. Magnetically charged Reissner-Nordström black holes exist in EYM theory at the discrete values P_n of magnetic charge given by the Dirac condition and at all values of horizon surface area $A \geq 4\pi P_n^2$. Our formula (60) already shows that two Reissner-Nordström solutions of different magnetic charge and the same area A cannot be connected by a sequence of stationary black-hole solutions of area A and $Q = \mathcal{J} = 0$ since those solutions have different masses. It strongly suggests that although magnetic charge is not quantized in EYM theory, the possible values it can take for stationary black holes probably are quantized.

Theorem 2.1 establishes that in case (a) a stationary solution is an extremum of m at fixed Q with respect to all first-order variations in the initial data which satisfy the linearized constraints. Theorem 2.2 similarly establishes that in case (b), stationary black holes are extrema of m at fixed Q and $\mathcal{J}$ and at fixed area A of the boundary surface S. We turn our attention, now, to the issue of whether the converse of these results hold; specifically, whether, in case (a), initial data which are an extremum of m at fixed Q are necessarily initial data for a stationary solution and whether, in case (b), initial data which are an extremum of m at fixed Q, $\mathcal{J}$, and A are necessarily initial data for a stationary black hole.

Consider case (a), and, for simplicity, we analyze, first, the pure vacuum case; i.e., we set to zero the Yang-Mills initial data and their variations in all of our formulas. Then, $\mathcal{E} = m$ and Eq. (33) reduces to

$$16\pi\delta m = \int_\Sigma (\mathcal{P}^{ab}\delta h_{ab} + \mathcal{Q}_{ab}\delta\pi^{ab}), \tag{61}$$

where $\mathcal{P}^{ab}$ and $\mathcal{Q}_{ab}$ are given by Eqs. (19) and (20) with $A_a{}^\Lambda = E^a{}_\Lambda = 0$. Now, if $\delta m = 0$ for all $(\delta h_{ab}, \delta\pi^{ab})$, it would follow immediately from Eq. (61) that $\mathcal{P}^{ab} = 0$, $\mathcal{Q}_{ab} = 0$, and thus that the solution determined by the initial data is stationary. This is, in essence, the result obtained by Schutz and Sorkin [8]. However, the situation of interest for us is the case where $\delta m = 0$ only when $(\delta h_{ab}, \delta\pi^{ab})$ satisfy the linearized constraints, in which case stationarity (up to gauge) does not follow immediately.

An argument that extrema of mass with respect to properly constrained variations implies stationarity (in the vacuum case) was given by Brill, Deser, and Fadeev [22]. They start with the surface integral formula (35) for m and propose to treat the constraints on the variations $(\delta h_{ab}, \delta\pi^{ab})$ by the method of "Lagrange multipliers." This leads them to extremize the quantity

$$m' = m + (1/16\pi)\int_\Sigma L^\mu \mathcal{C}_\mu \tag{62}$$

with respect to unconstrained variations, where the vector field L^μ on Σ is a Lagrange multiplier. Now, let $(\delta h_{ab}, \delta\pi^{ab})$ be a variation (not necessarily satisfying the constraints) of compact support on Σ. Then δm will vanish (since it is given by a surface integral at infinity) and since, in the vacuum case under consideration here, the second term in Eq. (62) is of the same form as H_V [see Eq. (17)], the requirement that m' be an extremum with respect to all such variations implies that L^μ must be a Killing field [see Eq. (18) and Eqs. (27)–(28)]. Further arguments (not explicitly made in [22]) would then show that L^μ must asymptotically approach a time translation, so that the initial data correspond to a stationary solution.

The above argument is deficient in several respects. One difficulty is that m, being given by a surface integral expression, is not likely to be suitably differentiable off of $\bar{\Gamma}$, though this could probably be remedied by defining m off of $\bar{\Gamma}$ by adding H_V to the surface integral expression. However, the most serious deficiency involves the lack of justification for the above form of the Lagrange multiplier method in the case relevant here where infinitely many constraints are present [since Eqs. (14)–(16) must hold at each point of Σ]. This method would be justified if we could introduce a metric G_{AB} on the unconstrained phase space Γ such that for points that lie on the constraint submanifold $\bar{\Gamma}$ every tangent vector (i.e., linearized perturbation) X^A can be written uniquely as the sum of a vector Y^A tangent to $\bar{\Gamma}$ (i.e., a perturbation satisfying the linearized constraints) plus a vector of the form $G^{AB}\nabla_B(\int_\Sigma L^\mu \mathcal{C}_\mu)$ for some L^μ. Indeed, it is the existence of such a decomposition which would ensure that extrema of m with respect to constrained variations coincide with extrema of m' with respect to unconstrained variations. (In finite dimensions, any metric gives rise to such a decomposition with respect to the constraints, so there is no difficulty with the use of the Lagrange multiplier method in finite dimensions or in infinite dimensions when only finitely many constraints are present.)

If one chooses an (appropriately weighted) L^2 inner product to define G_{AB}, one could come close to obtaining the desired decomposition. However, at best, the second term in the decomposition would be expressible as a limit of terms of the form $G^{AB}\nabla_B(\int_\Sigma L^\mu \mathcal{C}_\mu)$ and this limit need not itself have the desired form. Thus, although suggestive, the argument of [22] certainly falls short of providing a proof that, in the vacuum case, extrema of mass correspond to stationary solutions.

The following appears to us to be the most promising line of argument toward proving the validity of the above converses to theorems 2.1 and 2.2. We consider, again, case (a) and, for simplicity, we again initially consider the vacuum case. We claim, first, that given any initial data (h_{ab}, π^{ab}) satisfying the constraints there always exists a perturbation of the initial data $(\bar{\delta} h_{ab}, \bar{\delta}\pi^{ab})$ which satisfies the linearized constraints in a neighborhood of infinity (but not necessarily on all of Σ) and which yields a nonvanishing change in mass, $\bar{\delta} m \neq 0$. Indeed, if the initial data (h_{ab}, π^{ab}) satisfies $\pi^a{}_a = 0$ in a neighborhood of infinity, we can obtain the desired perturbed initial data by using the conformal method (described further in the next section), taking the perturbed conformal factor to be a monopole solution of the linearized Lichnerowicz equation in a neighborhood of infinity and smoothly extending it to the interior in an arbitrary fashion. If $\pi^a{}_a \neq 0$ in a neighborhood of infinity, we can "time evolve" the initial data to make $\pi^a{}_a = 0$ in a neighborhood of infinity, obtain the desired perturbed initial data on this new time slice, and then evolve the perturbed initial data backward in time to obtain the desired perturbed data on the original slice.

Given the above perturbation $(\bar{\delta} h_{ab}, \bar{\delta}\pi^{ab})$ to the initial data, the key issue becomes the following: Can we find a new asymptotically flat perturbation $(\delta h_{ab}, \delta\pi^{ab})$ to the initial data which satisfies

$$\delta \mathcal{C}_\mu = S_\mu \tag{63}$$

with

$$S_\mu = -\bar{\delta}\mathcal{C}_\mu \tag{64}$$

such that δh_{ab} falls off at infinity sufficiently rapidly that $\delta m = 0$? If the answer is "yes," then the combined perturbation to the initial data $(\bar{\delta} h_{ab} + \delta h_{ab}, \bar{\delta}\pi^{ab} + \delta\pi^{ab})$ will satisfy the linearized constraints with mass variation $\bar{\delta} m \neq 0$. Thus, a necessary condition for an extremum of mass is that Eq. (63) have no solution with δh_{ab} falling off at infinity sufficiently rapidly that $\delta m = 0$.

The question of whether Eq. (63) can be solved is essentially the same question as arises when analyzing the issue of linearization stability [23], i.e., the issue of whether, given a solution of the linearized constraint equations, one can find a corresponding one-parameter family of exact solutions. In that case one again encounters an equation of the form (63), except that now the left-hand side represents the linearized change in $\mathcal{C}_\mu$ due to the second-order perturbation and the source term S_μ now represents the contribution to the second-order perturbation equations arising from the terms quadratic in the first-order perturbation. If one can solve Eq. (63) with an arbitrary S_μ, then the implicit function theorem guarantees linearization stability. On the other hand, if there is a source term S_μ which can be constructed from a first-order perturbation for which Eq. (63) cannot be solved, then linearization stability fails for that perturbation.

The issue of linearization stability has been extensively analyzed in the case where the Cauchy surface Σ is compact (without boundary), under the additional hypothesis that $K = \text{const}$ [where $K = -\pi^a{}_a/(2\sqrt{h})$ denotes the trace of the extrinsic curvature of Σ in the spacetime]. It has been shown [23] that in the compact case, Eq. (63) can be solved for a given smooth source S_μ if and only if S_μ is orthogonal (in the L^2 sense) to the kernel of the L^2 adjoint $\mathcal{A}^\dagger$ of the linear operator $\mathcal{A}$ defined by

$$\mathcal{A}(\delta h_{ab}, \delta\pi^{ab}) = \delta\mathcal{C}_\mu . \tag{65}$$

Now, the adjoint operator $\mathcal{A}^\dagger$ can be computed by considering the integral

$$\int_\Sigma M^\mu \delta\mathcal{C}_\mu \tag{66}$$

and integrating by parts to remove all derivatives from δh_{ab} and $\delta\pi^{ab}$. (No boundary terms arise here since Σ is compact.) However, in the vacuum case under consideration here, the integral (66) is precisely of the form δH_V, where H_V was defined by Eq. (17). Equation (18) thus effectively computes $\mathcal{A}^\dagger$, and Eqs. (27) and (28) show that M^μ lies in the kernel of $\mathcal{A}^\dagger$, if and only if M^μ is a Killing field. Hence, we recover Moncrief's result [24] that linearization instability can occur only when the spacetime admits a Killing field.

In the asymptotically flat case (a), with no additional restrictions on the perturbations, if we multiply Eq. (63) by M^μ and integrate over Σ, we find that a necessary condition for solving (63) is that S_μ must be orthogonal to any M^μ for which $\mathcal{P}^{ab} = 0$ and $\mathcal{Q}_{ab} = 0$ and for which the additional surface term at infinity obtained in this case vanishes identically. These conditions require M^μ to be a Killing field which vanishes asymptotically as $r \to \infty$. In analogy with the results proven in the compact case, it seems reasonable to conjecture that this condition also is sufficient to be able to solve Eq. (63) for any smooth S_μ of sufficiently rapid falloff at infinity. If so, this strongly suggests that all asymptotically flat spacetimes are linearization stable, since it should not be difficult to prove that there cannot exist Killing fields which vanish asymptotically.

If we now apply this argument to the case of interest for us, namely, solving Eq. (63) for S_μ of compact support where we additionally require the perturbation to fall off sufficiently rapidly that it satisfy $\delta m = 0$, then we find, as the corresponding necessary condition to solve Eq. (63), that S_μ must be orthogonal to any M^μ which is a Killing field for which $M^a \to 0$ at infinity, but M^0 now may go to a constant. We conjecture that this condition also is sufficient to solve Eq. (63) for any smooth S_μ of compact support. If so, then a necessary condition for a vacuum spacetime to be an extremum of mass with respect to vacuum perturbations is that it be stationary. In fact, in the next section under the additional hypothesis that the spacetime admits a foliation by maximal slices, we will give a rigorous proof of the stronger result that the spacetime must be static.

The above argument can be generalized in a straightforward manner to the EYM equation for case (a). Again, given any EYM initial data which satisfies the EYM constraints, there should be no difficulty in finding smooth perturbed initial data which satisfies the linearized EYM constraints in a neighborhood of infinity, and satisfies $\bar{\delta}Q = 0$ but $\bar{\delta}m \neq 0$. Our task then is to solve linearized constraints

$$\delta\mathcal{C}_\mu = S_\mu, \tag{67}$$

$$\delta\mathcal{C}_\Lambda = S_\Lambda \tag{68}$$

(with $S_\mu = -\bar{\delta}\mathcal{C}_\mu$ and $S_\Lambda = -\bar{\delta}\mathcal{C}_\Lambda$ smooth and of compact support) for a perturbation $(\delta h_{ab}, \delta\pi^{ab}, \delta A_a{}^\Lambda, \delta E^a{}_\Lambda)$ for which δh_{ab} and $\delta E^a{}_\Lambda$ fall off at infinity sufficiently rapidly that $\delta m = \delta Q = 0$. The left-hand sides of Eqs. (67) and (68) define an operator $\mathcal{B}$ mapping a perturbation of EYM initial data into a covariant vector field and Lie-algebra-valued scalar field on Σ. To compute the adjoint of $\mathcal{B}$ we multiply Eq. (67) by a vector field M^μ, multiply Eq. (68) by a Lie-algebra-valued function a^Λ, and integrate the sum over Σ. As in the vacuum case, in order that (M^μ, a^Λ) be in the kernel of $\mathcal{B}^\dagger$, M^μ must be a Killing field for the background initial data; in addition, we now find that a^Λ must be such that $\dot{A}_a{}^\Lambda = \pounds_{M^\mu} A_a{}^\Lambda = 0$ and $\dot{E}^a{}_\Lambda = \pounds_{M^\mu} E^a{}_\Lambda = 0$ for the background spacetime in a gauge where $M^\mu A_\mu{}^\Lambda = a^\Lambda$. If our perturbed initial data were restricted only by asymptotic flatness conditions, then in order not to generate surface terms at infinity we also must have $M^\mu \to 0$ and $a^\Lambda \to 0$ as $r \to \infty$. However, with the additional restriction $\delta m = \delta Q = 0$, M^0 may approach a constant and a^Λ may approach an angle-dependent constant (proportional to the $1/r^2$ part of the radial component of the background electric field) as $r \to \infty$. In parallel with the vacuum case, we conjecture that Eqs. (67) and (68) can be solved for arbitrary smooth (S_μ, S_Λ) of compact support unless there exist (M^μ, a^Λ) satisfying those asymptotic conditions where M^μ is a Killing field for the background spacetime and a^Λ is such that $\pounds_{M^\mu} A_a{}^\Lambda = 0$ and $\pounds_{M^\mu} E^a{}_\Lambda = 0$ in a gauge with $M^\mu A_\mu{}^\Lambda = a^\Lambda$. If so, this means that a necessary condition for an extremum of mass at fixed charge is that the background EYM data correspond to a stationary solution.

In order to extend these arguments to case (b), we again start with a perturbation $(\bar{\delta}h_{ab}, \bar{\delta}\pi^{ab}, \bar{\delta}A_a{}^\Lambda, \bar{\delta}E^a{}_\Lambda)$ which satisfies the constraints near infinity and also satisfies $\bar{\delta}Q = 0$, $\bar{\delta}\mathcal{J} = 0$ but $\bar{\delta}m \neq 0$. In addition, we require $\bar{\delta}A = 0$ at the boundary S, which easily can be achieved by setting the perturbation equal to zero in a neighborhood of S, since we do not attempt to solve the constraints near S. Again we attempt to solve Eqs. (67) and (68), but now with the additional restriction at infinity that $\delta\mathcal{J} = 0$ as well as $\delta m = \delta Q = 0$; furthermore at the boundary surface S, we require $\delta A = 0$. As in the previous case, in order to have a nontrivial element of the kernel of $\mathcal{B}^\dagger$, the background spacetime must admit a Killing field M^μ and a Lie-algebra-valued scalar field a^Λ such that $\pounds_{M^\mu} A_a{}^\Lambda = 0$ and $\pounds_{M^\mu} E^a{}_\Lambda = 0$ in a gauge where $M^\mu A_\mu{}^\Lambda = a^\Lambda$. However, on account of our stronger asymptotic conditions on the perturbation resulting from the requirement that $\delta\mathcal{J} = 0$, in this case no surface terms at infinity will be generated if M^0 approaches a constant, M^a approaches a rotational Killing field, and a^Λ approaches a (direction-dependent) constant as $r \to \infty$. Furthermore, in order not to generate surface terms at S, we must have $M^\mu = 0$ at S and $D_a M^0$ must have constant magnitude on S. The fact that the Killing field M^μ vanishes at S implies that the null geodesics orthogonal to S generate a bifurcate Killing horizon (see [25]). These null geodesics generating the Killing horizon cannot reach infinity (since $M^\mu M_\mu = 0$ on the horizon but $M^\mu M_\mu \neq 0$ near infinity), and this implies that the boundary of the future of S cannot intersect $\mathcal{I}^+$. This implies, in turn, that the causal future of S comprises (or is contained within) a black hole. Furthermore, M^0 cannot asymptotically vanish at infinity, since otherwise the orbits of M^μ would have to be closed, but the orbits of M^μ on the Killing horizon cannot be closed. [In any case, if M^0 vanished at infinity, then the source terms occurring in Eqs. (67) and (68) would be orthogonal to (M^μ, a^Λ); i.e., they would be orthogonal to the kernel of $\mathcal{B}^\dagger$.] It presumably follows that the spacetime must admit a Killing field t^μ which asymptotically approaches a time translation at infinity; i.e., the spacetime must be stationary (and if $M^\mu \neq t^\mu$, it also must be axisymmetric). Thus, if an (M^μ, a^Λ) satisfying the above conditions exists, the initial data for the background spacetime corresponds to a stationary black hole.

The nonexistence of (M^μ, a^Λ) satisfying the above asymptotic conditions at infinity and boundary conditions at S is necessary in order to be able to solve Eqs. (67) and (68) with arbitrary smooth source terms of compact support for a perturbation satisfying $\delta\mathcal{J} = \delta Q = \delta m = \delta A = 0$. We conjecture that the nonexistence of such an (M^μ, a^Λ) also is sufficient to solve Eqs. (67) and (68) for such a perturbation. If so, then the desired converse of theorem 2.2 holds: In order for EYM initial data for case (b) to be an extremum of mass at fixed charge, canonical angular momentum, and area S, the initial data must correspond to a stationary black hole, with S being the bifurcation surface of a bifurcate Killing horizon.

The above arguments, of course, fall short of providing complete proofs of the converses of theorems 2.1 and 2.2. The main missing ingredient is a generalization to the asymptotically flat, EYM case of the theorems, used in the analysis of linearization stability, giving necessary and sufficient conditions for solving Eq. (63) for vacuum perturbations on a compact manifold. Many of the standard theorems concerning elliptic operators on compact manifolds have been generalized to the asymptotically flat case [26], but it appears that considerable further work would have to be done to obtain the precise form of the results which we require here. Fortunately, in the next section we shall give a simple, rigorous proof of a closely related result: Under the additional hypothesis of the existence of a maximal ($K = 0$) foliation, in case (a) any extremum of mass must be static and, in case (b), any extremum of mass at fixed area of S must be a static black hole. Our plausibility arguments concerning the existence and properties of the known Bartnik-McKinnon and colored black-hole solutions will rely only on the results of the next section. However, our plausibility arguments concerning the existence of new solutions will rely on the converses of theorems 2.1 and 2.2 for which we have given plausibility arguments above.

III. EXTREMA OF MASS AND STATIC SOLUTIONS

In this section we shall show that a necessary condition for an EYM solution to be an extremum of mass is that the solution be static. As explained at the end of the Introduction, by static, we mean precisely that there is a hypersurface orthogonal Killing field N^μ which approaches a time translation at infinity, and, in addition, that the Yang-Mills electric field $E^a{}_\Lambda$ vanishes on the hypersurfaces orthogonal to N^μ. Our results apply to both cases (a) and (b) of the preceding section. Note that our hypotheses are stronger than those of the plausibility arguments given at the end of the preceding section in that in case (a) we require m to be an extremum with respect to all variations (not just variations which preserve Q) and in case (b) we require m to be extremum with respect to all variations which keep A fixed (not just variations which keep A, Q, and $\mathcal{J}$ fixed). In addition, we require the existence of a foliation by asymptotically flat maximal ($K = 0$) slices. On the other hand, the conclusion of our theorems are correspondingly stronger, since we prove that the solution must be static, not merely stationary.

The results of this section will be used in our plausibility arguments in Sec. IV for the existence and properties of the Bartnik-McKinnon and colored black-hole solutions. In addition, by combining the theorems of this section with those of the preceding section, we obtain a number of results of interest in their own right. These will be described at the end of this section.

Before we begin, it is worth pointing out that if we were to consider a theory where the matter fields ψ_i are such that the dominant energy condition is satisfied and the Einstein-matter equations are scale invariant, it would follow immediately that the only possible extremum of mass in case (a) is flat spacetime (with no matter fields present). Here, by scale invariant we mean that there is a transformation of the form $g_{\mu\nu} \to \lambda^2 g_{\mu\nu}$, $\psi_i \to \lambda^{s_i}\psi_i$ with λ a constant which takes solutions to solutions. [Thus, scale invariance will hold in most cases where the matter fields obey linear equations, such as Einstein-Maxwell theory (or vacuum general relativity).] Under such a scale transformation, we have $m \to \lambda m$, so a necessary condition for a solution to be an extremum of mass is that $m = 0$. The positive energy theorem [27, 28] then implies that the solution is flat spacetime. [Note that for Einstein-Maxwell theory, we automatically have $Q = 0$ in case (a), so theorem 2.1 then establishes that in case (a) there cannot exist stationary solutions of the Einstein-Maxwell equations apart from flat spacetime.] However, since the EYM equations (for a non-Abelian Yang-Mills group) are not scale invariant, this line of argument is inapplicable here, as is manifestly demonstrated by the existence of the Bartnik-McKinnon solutions.

Nevertheless, our approach in this section will be similar to the argument of the preceding paragraph in that we will simply obtain explicitly a solution of the linearized constraints for which $\delta m \neq 0$, noting that the procedure will fail only when the background solution is static. This linearized solution will be obtained via arguments similar to those given by O'Murchadha and York [52] and will use a generalization to the EYM case of the conformal approach for solving the constraint equations, which has been extensively developed by York and collaborators (see, e.g., [29]). The key observation upon which this approach relies is that for EYM initial data with $\pi_a{}^a = 0$, the Einstein and Yang-Mills momentum constraint operators $\mathcal{C}_a$ and $\mathcal{C}_\Lambda$ [defined by Eqs. (14) and (16)], are conformally invariant. Specifically, it can be verified directly that if $(\tilde{h}_{ab}, \tilde{\pi}^{ab}, \tilde{A}_a{}^\Lambda, \tilde{E}^a{}_\Lambda)$ satisfy the constraints (14) and (16) with $\tilde{\pi}_a{}^a = 0$, and if ϕ is any smooth, positive function, then

$$h_{ab} = \phi^4 \tilde{h}_{ab}, \tag{69}$$

$$\pi^{ab} = \phi^{-4}\tilde{\pi}^{ab}, \tag{70}$$

$$A_a{}^\Lambda = \tilde{A}_a{}^\Lambda, \tag{71}$$

$$E^a{}_\Lambda = \tilde{E}^a{}_\Lambda, \tag{72}$$

also satisfy Eqs. (14) and (16). Hence, in order to obtain a solution of the full set of constraint equations (14)–(16), we need only solve the Lichnerowicz equation

$$\tilde{R} - 8\phi^{-1}\tilde{D}^a\tilde{D}_a\phi = (1/\tilde{h})\tilde{\pi}^{ab}\tilde{\pi}_{ab}\phi^{-8} + (2/\tilde{h})\tilde{E}^a{}_\Lambda \tilde{E}_a{}^\Lambda\phi^{-4} + \tilde{F}_{ab}{}^\Lambda \tilde{F}^{ab}{}_\Lambda\phi^{-4} \tag{73}$$

for the conformal factor ϕ which ensures that $(h_{ab}, \pi^{ab}, A_a{}^\Lambda, E^a{}_\Lambda)$ solves (15), where all indices in Eq. (73) are raised and lowered with $\tilde{h}_{ab}$. [Of course we are not guaranteed, in general, that a positive solution to Eq. (73) will exist globally.] Similarly, in order to solve the linearized constraint equations off of a background solution $(h_{ab}, \pi^{ab}, A_a{}^\Lambda, E^a{}_\Lambda)$ with $\pi^a{}_a = 0$, we need only find a solution $(\bar{\delta} h_{ab}, \bar{\delta}\pi^{ab}, \bar{\delta} A_a{}^\Lambda, \bar{\delta} E^a{}_\Lambda)$ to the linearized momentum constraints satisfying $h_{ab}\bar{\delta}\pi^{ab} + \bar{\delta} h_{ab}\pi^{ab} = 0$ and then solve the linearized Lichnerowicz equation.

Consider, now, case (a) of the previous section [i.e., as defined precisely following Eq. (13) above, Σ has one asymptotic region and no interior boundary] and let $(h_{ab}, \pi^{ab}, A_a{}^\Lambda, E^a{}_\Lambda)$ be asymptotically flat initial data with $\pi^a{}_a = 0$ which satisfies the constraints (14)–(16). Then, since (14) and (16) are linear in the momenta π^{ab} and $E^a{}_\Lambda$, it is easy to verify that we obtain a solution to the linearized momentum constraints (14) and (16) by simply scaling down the momenta, keeping h_{ab} and $A_a{}^\Lambda$ fixed, i.e., by choosing

$$\bar{\delta} h_{ab} = \bar{\delta} A_a{}^\Lambda = 0, \tag{74}$$

$$\bar{\delta} \pi^{ab} = -\pi^{ab}, \tag{75}$$

$$\bar{\delta} E^a{}_\Lambda = -E^a{}_\Lambda. \tag{76}$$

Hence, we will obtain a solution to the full set of linearized constraints by solving the linearized Lichnerowicz equation

$$D^a D_a(\delta\phi) - \mu\delta\phi = \rho, \tag{77}$$

where

$$\mu = (1/h)\pi^{ab}\pi_{ab} + (1/h)E^a{}_\Lambda E_a{}^\Lambda + \tfrac{1}{2}F_{ab}{}^\Lambda F^{ab}_\Lambda, \tag{78}$$

$$\rho = (1/4h)\pi^{ab}\pi_{ab} + (1/2h)E^a{}_\Lambda E_a{}^\Lambda. \tag{79}$$

Thus, we have, $\mu \geq 0$, $\rho \geq 0$, and as $r \to \infty$, we have

$$\mu = O(1/r^4), \tag{80}$$

$$\rho = O(1/r^4). \tag{81}$$

We note, first, that since both μ and ρ are non-negative and fall off to zero sufficiently rapidly as $r \to \infty$, there exists a (unique) solution to Eq. (77) with $\delta\phi \to 0$ as $r \to \infty$ [see theorem (1.5) of [30] for the case $\Sigma = \mathbb{R}^3$ and the results of [31] for the general case]. Second, by rewriting Eq. (77) as a flat space Laplace equation with source term, it can be shown [32] that as $r \to \infty$, $\delta\phi$ is of the form

$$\delta\phi = c/r + o(1/r), \tag{82}$$

where c is a constant. Derivatives of $\delta\phi$ go to zero corresponding powers of r faster. Thus, the perturbation constructed by this procedure is asymptotically flat. The change in ADM mass, Eq. (35), associated with this perturbation, is easily computed to be

$$\delta m = 2c. \tag{83}$$

Next, we note that Eq. (77) is of the form for which the Hopf maximum principle applies (see, e.g. [33]), which implies that $\delta\phi$ cannot achieve a local maximum at any point $x \in \Sigma$. Thus, if we consider the (compact) region of Σ enclosed by a sphere of coordinate radius r near infinity, the maximum of $\delta\phi$ in this region must be achieved on the boundary. If we let $r \to \infty$ and take Eq. (82) into account, this implies that $\delta\phi \leq 0$ everywhere on Σ .

Finally, we note that the following lemma holds for solutions to Eq. (77).

Lemma: Let $\delta\phi$ be a solution of solution to Eq. (77) on an asymptotically flat three-manifold Σ, with $\mu \geq 0$, $\rho \geq 0$ and $\mu = O(1/r^{2+\epsilon})$ as $r \to \infty$ for some $\epsilon > 0$. Suppose further that $\delta\phi \leq 0$ everywhere on Σ, and $\delta\phi = o(1/r)$ as $r \to \infty$, i.e., the asymptotic form (82) holds with $c = 0$. Then $\delta\phi = 0$ on Σ, which, of course, is possible only if $\rho = 0$ everywhere on Σ. In other words, if $\rho \neq 0$ at some point of Σ, then every solution, $\delta\phi$, of Eq. (77) having the asymptotic form (82) and satisfying $\delta\phi \leq 0$ on Σ has $c \neq 0$.

A proof of this lemma is given in the appendix. As a direct consequence of this lemma and the preceding remarks, we obtain the following theorem.

Theorem 3.1. Let $(h_{ab}, \pi^{ab}, A_a{}^\Lambda, E^a{}_\Lambda)$ be an initial data set in $\bar{\Gamma}^{(a)}$ – i.e., Σ has one asymptotic region and no interior boundary, the data are asymptotically flat, and the EYM constraints (14)–(16) are satisfied. Suppose furthermore that $\pi^a{}_a = 0$. Then, if $\pi^{ab} \neq 0$ or $E^a{}_\Lambda \neq 0$ at some point $x \in \Sigma$, there exists an asymptotically flat perturbation $(\delta h_{ab}, \delta\pi^{ab}, \delta A_a{}^\Lambda, \delta E^a{}_\Lambda)$ satisfying the linearized EYM constraints such that $\delta m \neq 0$. Furthermore, if the background initial data satisfies $Q = 0$, there exists a perturbation satisfying $\delta m \neq 0$ and $\delta Q = 0$. In other words, if $\pi^a{}_a = 0$, a necessary condition for an extremum of mass is that $\pi^{ab} = E^a{}_\Lambda = 0$ everywhere on Σ; in addition if $Q = 0$ this condition is also necessary for an extremum of mass at fixed charge.

Proof: If $\pi^{ab} \neq 0$ or $E^a{}_\Lambda \neq 0$ at $x \in \Sigma$, then by Eq. (79), we have $\rho \neq 0$ at x. Hence, by Eq. (83) and the above lemma, the asymptotically flat perturbation constructed from Eqs. (74)–(77) satisfies $\delta m < 0$. If $Q = 0$, this perturbation also satisfies $\delta Q = 0$. □

As an immediate consequence of this theorem, we obtain the following result.

Corollary: Let $(M; g_{\mu\nu}, F_{\mu\nu}{}^\Lambda)$ be a solution of the EYM equations which can be foliated by maximal ($K = 0$) slices Σ_t such that the induced initial data on each Σ_t is in $\bar{\Gamma}^{(a)}$. Suppose that this solution is an extremum of ADM mass on each slice [or an extremum of ADM mass on each slice at (fixed) zero charge]. Then this solution is static. Specifically the maximal slices are orthogonal to a Killing field and $E^a{}_\Lambda = 0$ on these slices.

Proof: By theorem 3.1 above we have $\pi^{ab} = 0$ and $E^a{}_\Lambda = 0$ on each Σ_t. Let N^0 denote the lapse function for the maximal slices, let n^μ denote the unit normal field for these slices, and choose $N^\mu = N^0 n^\mu$ as the time evolution vector field for these slices. Then, we obviously have

$$\pounds_{N^\mu}\pi^{ab} = \dot{\pi}^{ab} = 0 \tag{84}$$

and by Eqs. (20) and (28) we also have

$$\pounds_{N^\mu} h_{ab} = \dot{h}_{ab} = 0 \tag{85}$$

since $\pi^{ab} = 0$ and $N^a = 0$. Equations (84) and (85) imply that N^μ is a Killing field for the spacetime metric $g_{\mu\nu} = h_{\mu\nu} - n_\mu n_\nu$. This Killing field is orthogonal to each Σ_t, and thus is manifestly hypersurface orthogonal. Finally it is obvious that

$$\pounds_{N^\mu} E^a{}_\Lambda = \dot{E}^a{}_\Lambda = 0 \tag{86}$$

and taking $A_0{}^\Lambda = 0$ we obtain, from Eqs. (22) and (30),

$$\pounds_{N^\mu} A_a{}^\Lambda = \dot{A}_a{}^\Lambda = 0, \tag{87}$$

so in this gauge, the Yang-Mills field is stationary. Thus the solution is static. □

The above results are easily extended to case (b) (where Σ has one asymptotic region and an interior boundary S). We again construct a solution to the linearized constraints via Eqs. (74)–(77). The only change is that, in addition to the boundary condition $\delta\phi \to 0$ at infinity, we also impose the Dirichlet boundary condition $\delta\phi = 0$ on S. Note that since $\delta\phi = 0$ on S and $\bar{\delta}h_{ab} = 0$ everywhere, we automatically obtain a solution to the linearized constraints with $\delta A = 0$ on S. Repetition of the arguments leading to theorem 3.1 then yields the following result.

Theorem 3.2. Let $(h_{ab}, \pi^{ab}, A_a{}^\Lambda, E^a{}_\Lambda)$ be an initial data set in $\bar{\Gamma}^{(b)}$–i.e., Σ has one asymptotic region and an interior boundary S, the data are asymptotically flat, and the EYM constraints (14)–(16) are satisfied. Suppose further that $\pi^a{}_a = 0$. Then, if $\pi^{ab} \neq 0$ or $E^a{}_\Lambda \neq 0$ at some point $x \in \Sigma$, there exists an asymptotically flat perturbation $(\delta h_{ab}, \delta\pi^{ab}, \delta A_a{}^\Lambda, \delta E^a{}_\Lambda)$ satisfying the linearized EYM constraints such that $\delta m \neq 0$ and $\delta A = 0$. Furthermore, if the background initial data satisfies $Q = 0$ or $\mathcal{J} = 0$, then there exists a perturbation satisfying $\delta m \neq 0$, $\delta A = 0$, and respectively, $\delta Q = 0$ or $\delta\mathcal{J} = 0$. In other words, if $\pi^a{}_a = 0$, a necessary condition for an extremum of mass at fixed area A of S is that $\pi^{ab} = E^a{}_\Lambda = 0$ everywhere on Σ; if in addition, $Q = 0$ or $\mathcal{J} = 0$ this condition is necessary for an extremum of m at fixed Q or $\mathcal{J}$, respectively.

In parallel with the corollary to theorem 3.1, we have the following additional result.

Corollary: Let $(M; g_{\mu\nu}, F_{\mu\nu}{}^\Lambda)$ be a solution of the EYM equations which is asymptotically flat at spatial and null infinity and is strongly asymptotically predictable (see, e.g., [18] or [19]). Suppose that a region of M can be foliated by maximal slices Σ_t such that each Σ_t terminates on the same boundary compact two-surface S and the induced initial data on each Σ_t is in $\bar{\Gamma}^{(b)}$. Suppose that on each slice this solution is an extremum of m at fixed area of S [or it is an extremum of mass at fixed area of S, and (fixed) zero charge and/or (fixed) zero angular momentum]. Then the solution describes a static black hole, and S is the bifurcation surface of a bifurcate Killing horizon.

Proof: That the region foliated by Σ_t is static follows exactly as in the proof of the corollary to theorem 3.1. Since the lapse function N^0 of the foliation vanishes at S, the static Killing field $N^\mu = N^0 n^\mu$ vanishes at S, which implies that the null geodesics orthogonal to S generate a bifurcate Killing horizon (see [25]). As in the argument given at the end of Sec. II, the causal future of S must comprise (or be contained within) a black hole. □

Remark: In the corollaries to theorems 3.1 and 3.2, it should be possible to replace the assumption of a maximal foliation by the assumption that there is a single maximal slice satisfying the stated conditions. Given such a maximal slice, we expect that existence of a maximal foliation in case (a) should follow from the arguments of proposition 3.3 of [37]. In case (b), we expect that either the existence of a maximal foliation could also be proven directly or that it could be proven that $N^0 n^\mu$ is a Killing field, where N^0 is a solution to the linear equation for the lapse function which preserves $K = 0$ to first order. However, we shall not attempt to investigate this issue further here.

Although our original intention in obtaining theorems 3.1 and 3.2 and their corollaries was for their use in the plausibility arguments of Sec. IV, these results yield some new theorems of interest in their own right when combined with theorems 2.1 and 2.2 of the preceding section. In particular, for case (a), we have the following theorem.

Theorem 3.3. Let $(M; g_{\mu\nu}, F_{\mu\nu}{}^\Lambda)$ be a stationary solution of the EYM equations possessing a maximal ($K = 0$) slice Σ, which is asymptotically orthogonal to the stationary Killing field, and on which the induced initial data lies in $\bar{\Gamma}^{(a)}$. Suppose, further, that $VQ = 0$, where V and Q were defined by Eqs. (39) and (48), respectively. Then the solution is static. In particular, this means that the electric field vanishes whenever V or Q vanishes.

Proof : If $VQ = 0$, then either $V = 0$ or $Q = 0$. By theorem 2.1 it follows that the solution is either an extremum of m or an extremum of m at fixed (zero) charge. By applying the stationary isometries to Σ, we obtain a family Σ_t of maximal slices with induced data in $\bar{\Gamma}^{(a)}$. Hence, by the argument given in the proof of the corollary to theorem 3.1, the solution is static. □

Note that this theorem generalizes corollary 1 of [34] (see also [35, 36]) to the nonspherically symmetric case and also shows that in the stationary case we can choose $V = 0$ if and only if $Q = 0$.

Similarly, for case (b) we have the following theorem.

Theorem 3.4. Let $(M; g_{\mu\nu}, F_{\mu\nu}{}^\Lambda)$ be a solution of the EYM equations describing a stationary black hole with a bifurcate Killing horizon. Suppose there exists a maximal ($K = 0$) slice Σ asymptotically orthogonal to the stationary Killing field, passing through the bifurcation surface S, and such that the induced initial data on Σ lies in $\bar{\Gamma}^{(b)}$. Suppose further that $VQ = \Omega\mathcal{J} = 0$ [see Eqs. (39), (48), (52), and (58)]. Then the solution is static.

Proof: By applying the stationary isometries to Σ, we obtain a family Σ_t of hypersurfaces satisfying the conditions of the corollary to theorem 3.2. The remainder of the proof exactly parallels the proof of theorem 3.3.□

Apart from the unwanted hypothesis of the existence of a maximal slice (see the remark below), theorem 3.4 applied to the vacuum case closes a gap that has existed for many years in the black-hole uniqueness theorems. A theorem of Hawking (see propositions 9.3.5 and 9.3.6 of [18]) is often quoted as having proved that a stationary black hole must either be static or axisymmetric. In fact, however, the results actually establish the following: First, modulo an analyticity assumption, the event horizon of a stationary black hole (with Killing field t^μ timelike at infinity) must be a Killing horizon. Second, if the Killing field χ^μ generating the horizon differs from t^μ, then the black hole is axisymmetric; this corresponds to the case $\Omega \neq 0$ [see Eq. (58)]. However, if $\chi^\mu = t^\mu$ (i.e., $\Omega = 0$), then it has been proven that the black hole must be static only when the additional hypothesis is made that t^μ is strictly timelike outside of the horizon. In other words, Hawking's theorem does not treat the case where $\Omega = 0$ but there exists an ergoregion outside the black hole – although a plausibility argument was given that if any such black hole exists, it should be unstable (see pp. 327–328 of [18] and see also [53]). In the vacuum case, where $V = 0$ holds automatically, theorem 3.4 considers precisely the case of interest, namely, $\Omega = 0$, and proves that the black hole must be static. Note, however, that in the EYM case, theorem 3.4 leaves open the possibility of nonstatic black holes with $\Omega = 0$. Indeed, in Sec. IV we will conjecture that such solutions exist, although, in accord with the plausibility arguments of [18], we expect any such solutions to be unstable.

It is worth pointing that in the Einstein-Maxwell case with trivial bundle we also can prove (subject to the $\pi_a{}^a = 0$ assumption) that in case (b) an extremum of m at fixed Q and boundary surface area A must be static in the restricted sense of being stationary with a hypersurface orthogonal Killing field (i.e., without the additional demand that $E^a = 0$). [In case (a) we already noted that $Q = 0$ automatically in the Einstein-Maxwell theory and we proved at the beginning of this section that there do not exist any extrema of m apart from flat spacetime.] This result is obtained in complete analogy with the discussion at the beginning of this section by choosing the following conformal perturbation of the initial data

$$\bar{\delta} h_{ab} = \bar{\delta} E^a = 0, \tag{88}$$

$$\bar{\delta} \pi^{ab} = -\pi^{ab}, \tag{89}$$

$$\bar{\delta} A_a = -A_a, \tag{90}$$

in place of (74)–(76). Here we have made explicit use of the fact that in the Abelian case the constraints (14) and (16) are linear in A_a, and hence Eqs. (88)–(90) solve the linearized momentum constraints. The only remaining requirement is that $\delta\phi$ solve the linearized Lichnerowicz equation, which will again have the form (77) with $F_{ab}F^{ab}$ replacing $(1/2h)E^a E_a$ in (79). Therefore in the Abelian case with trivial bundle, a repetition of the argument leading to theorem 3.2 proves that (if $\pi_a{}^a = 0$) a necessary condition for an extremum of mass at fixed area, A, of S, and electric charge, Q, is that $\pi^{ab} = 0$, $F_{ab} = 0$. The analogous corollary states that a solution which extremizes m at fixed Q and A is static (in the restricted sense) and also has vanishing magnetic field (i.e., $F_{ab} = 0$). Hence, combining these results with theorem 2.2, we obtain the following modification of theorem 3.4: The condition that $VQ = 0$ can be dropped from the hypothesis, and the conclusion is altered by interpreting *static* in the restricted sense (i.e., not requiring $E^a = 0$) and adding the condition that $F_{ab} = 0$. Thus, the gap in the black-hole uniqueness theorems discussed above is also closed (under the maximal slice assumption) in the Einstein-Maxwell case (with trivial bundle). Of course, the original form of theorem 3.4 also remains valid in the Einstein-Maxwell case.

Remark: It appears likely that the existence of a maximal slice can be proven for stationary solutions corresponding to cases (a) and (b), i.e., that the results of [37] can be generalized (under suitable global hypotheses concerning the spacetime) to treat the case where the stationary Killing field is known to be timelike only in a neighborhood of infinity. If so, the word "maximal" can be removed from the hypotheses of theorems 3.3 and 3.4. This issue is presently being investigated.

Note added. The proof of the existence of maximal slices has now been completed, so the gap discussed above that had existed in the black-hole uniqueness theorems is now completely closed. A paper presenting this proof is currently in preparation [38].

IV. STATIONARY EYM SOLUTIONS

In this section, we will give plausibility arguments for the existence of stationary EYM solutions in the case where the Yang-Mills gauge group is SU(2). The arguments involve the behavior of the ADM mass m as a function on the constraint submanifolds $\bar{\Gamma}^{(a)}$ and $\bar{\Gamma}^{(b)}$. [See Sec. II below Eq. (13) for the definitions of $\bar{\Gamma}^{(a)}$ and $\bar{\Gamma}^{(b)}$.] In short, we shall argue that mass function possesses a sequence of saddle points, which correspond to the known Bartnik-McKinnon and colored black-hole solutions as well as to new (i.e., as yet undiscovered) black-hole solutions. By the results of Secs. II and III, these extrema of mass correspond to stationary solutions.

Although there is a very well developed mathematical theory proving the existence of saddle points in situations similar to ours—namely, the "mountain pass lemma" and related results (see, e.g., [39] and [40])—we see little hope that the mass function on $\bar{\Gamma}^{(a)}$ or $\bar{\Gamma}^{(b)}$ could be proven to satisfy the Palais-Smale or other similar conditions required in the hypotheses of these theorems. To keep our discussion as simple as possible, we shall proceed by making some assumptions about the mass function which are convenient for making our arguments. Thus, needless to say, we see no realistic prospect that the plausibility arguments which we are about to give could be converted into a genuine existence proof for EYM solutions. In our view, the main value of our arguments lies in their accounting for the existence and certain properties of the known EYM solutions and in the conjectures that they spawn regarding the existence and properties of other solutions to the EYM equations and related systems. Some conjectures suggested by our arguments will be listed at the end of this section.

The key fact upon which our arguments are based is the existence of "large gauge transformations" in SU(2) Yang-Mills theory. To explain this concept, consider, first, the case where "space" Σ is the 3-sphere S^3. Since SU(2) also has the manifold structure S^3, the trivial Yang-Mills bundle has manifold structure $S^3 \times S^3$. [There do not exist nontrivial SU(2) bundles over S^3.] The spatial Yang-Mills vector potential is described gauge invariantly by a connection on this principal bundle, and a choice of gauge, which allows us to represent the connection (as we have been doing above) by a Lie-algebra-valued one-form field $A_a{}^\Lambda$ on Σ, corresponds to a choice of cross section of this bundle. Since the bundle is trivial, cross sections exist globally and are specified by giving a smooth map $f : \Sigma \to$SU(2), i.e., a map from S^3 to S^3. However, there exist a countable infinity of such maps, characterized by the winding number k, which are homotopically inequivalent. This means that, for any Yang-Mills connection, there exists a set of vector potential representatives $\{(A_a{}^\Lambda)_k\}$ which are gauge equivalent but cannot be continuously deformed to each other via a one-parameter family of gauge equivalent vector potentials. (However, they can be continuously deformed to each other via a sequence of gauge inequivalent vector potentials.) Note that for $\Sigma = S^3$ the necessary and sufficient condition for the existence of "large gauge transformations" (i.e., homotopically inequivalent cross sections) for an arbitrary Yang-Mills group G is that the third homotopy group π_3 of G be nontrivial. This is the case for any compact, simple (or semisimple) Lie group [41], so our arguments below apply more generally. However, it seems likely that the Einstein-Yang-Mills solutions we would predict in this more general case will be the solutions associated with an SU(2) [or SO(3)] subgroup of G.

In the case $\Sigma = \mathbb{R}^3$ with asymptotic flatness conditions on ${A_a}^\Lambda$, the situation with respect to "large gauge transformations" is the same as for $\Sigma = S^3$, provided that we also impose suitable asymptotic gauge conditions on ${A_a}^\Lambda$ at infinity to restrict the asymptotic behavior of the cross section. An example of such a gauge condition in the case where the Yang-Mills magnetic field ${B_a}^\Lambda$ is $o(1/r^2)$ is that ${A_a}^\Lambda$ be $o(1/r)$ as $r \to \infty$. We shall not attempt to select a suitable gauge condition here for the general behavior of initial data allowed by our asymptotic conditions, but merely will assume below that this has been done.

Our analysis of case (a) in Secs. II and III allows many topologies for Σ other than $\mathbb{R}^3$. However, assuming the validity of the Poincaré conjecture, all of these other topologies are nonsimply connected. Theorems of Gannon [42] then establish that the evolved spacetime must be singular. Hence, globally nonsingular solutions to the EYM equations in case (a) can occur only for $\Sigma = \mathbb{R}^3$, and we shall restrict attention to that case.

Consider, now, the constraint submanifold $\bar{\Gamma}^{(a)}$ of EYM phase space for case (a) (see Sec. II), with $\Sigma = \mathbb{R}^3$. Consider the ADM mass m as a function on this phase space, i.e., $m : \bar{\Gamma}^{(a)} \to \mathbb{R}$. Although we know very little about the function m, the positive mass theorem [27, 28] gives us one vital fact: The absolute minimum of m is $m = 0$ and this minimum is achieved precisely when the initial data correspond to Minkowski spacetime with vanishing Yang-Mills field strength, ${F_{\mu\nu}}^\Lambda = 0$. It is possible that other local minima of m exist. If so, then by the corollary to theorem 3.1, these minima would correspond to static solutions to the EYM equations. Furthermore, these solutions should be stable to linearized perturbations. However, we see no reason to expect the existence of any such additional local minima of m. For this reason, and for simplicity, we shall assume that the data corresponding to the trivial solution (i.e., flat spacetime with ${F_{\mu\nu}}^\Lambda = 0$) are the only local minimum of m, although our arguments below are applicable even if additional local minima exist.

The key observation is that, on account of the presence of large gauge transformations, there are a countable number of disconnected regions of $\bar{\Gamma}^{(a)}$ corresponding to the trivial solution. In addition to the initial data ($h_{ab} = e_{ab}, \pi^{ab} = 0, {A_a}^\Lambda = 0, {E^a}_\Lambda = 0$), together with all initial data obtained by applying spacetime diffeomorphisms and "small" Yang-Mills gauge transformations to it, we also have, for each integer k, the data ($h_{ab} = e_{ab}, \pi^{ab} = 0, ({A_a}^\Lambda)_k, {E^a}_\Lambda = 0$), and all diffeomorphisms and "small" gauge transformations of it, where $({A_a}^\Lambda)_k$ differs from ${A_a}^\Lambda = 0$ by a gauge transformation in the kth homotopy class. This means that rather than having a single minimum, the mass function m actually has a countable infinity of disconnected minima.

It will be convenient, at this stage, to assume that we can pass from $\bar{\Gamma}^{(a)}$ to a manifold $\tilde{\Gamma}^{(a)}$ whose points consist of equivalence classes of points of $\bar{\Gamma}^{(a)}$ under transformations corresponding to spacetime diffeomorphisms and "small" Yang-Mills gauge transformations. (Since m is invariant under such transformations, it is well defined on $\tilde{\Gamma}^{(a)}$.) The absolute minima of m on $\tilde{\Gamma}^{(a)}$ then comprise a countably infinite set of points.

For the purposes of making our argument, we shall assume, further, that a complete Riemannian metric G_{AB} can be put on $\tilde{\Gamma}^{(a)}$ such that (i) for any $\epsilon > 0$, $G^{AB}\nabla_A m \nabla_B m$ is bounded away from zero in the region outside the balls of radius ϵ around each critical point, and (ii) the critical points of m are isolated in the sense that there exists an $\epsilon > 0$ such that the distance between any pair of critical points is greater than ϵ. Actually, as we shall explain further below, these assumptions are not plausibly consistent with the fact that m must go to zero along certain "nonlinear scaling directions" in $\tilde{\Gamma}^{(a)}$; however, as we also shall explain below, the argument can be modified to take this fact into account. In addition, when we treat case (b) below, assumption (ii) probably will not be strictly valid because the lowest magnetically charged (Abelian) Reissner-Nordström black hole appears to be an accumulation point of the critical points corresponding to the colored black holes, but this violation of that assumption will not affect our arguments. Apart from these apparent violations, the above assumptions do not appear to be blatantly implausible, although we also see little reason to expect that they would hold in the precise form we have stated. Nevertheless, we believe that the argument we are about to give is quite robust, and, hence, not critically dependent upon the precise, specific choice of assumptions we make.

Consider, now, the integral curves of the vector field $M^A \equiv -G^{AB}\nabla_B m$ on $\tilde{\Gamma}^{(a)}$. It follows from the assumptions of the preceding paragraph, and the fact that $m \geq 0$, that each point $X \in \tilde{\Gamma}^{(a)}$ must be carried by the flow of these integral curves towards a critical point of m. Furthermore, if a point X flows to a critical point which is a local minimum of m, then so do all other points in a sufficiently small open neighborhood of X, i.e., the inverse image (under the flow of M^A) of a local minimum comprises an open subset of $\tilde{\Gamma}^{(a)}$. Now, as discussed above, there exists a countably infinite set of discrete global minima of m on $\tilde{\Gamma}^{(a)}$. If there were no other critical points of m other than these global minima (and possibly additional local minima) we could thereby express $\tilde{\Gamma}^{(a)}$ as a disjoint union of open sets. However, this contradicts the connectedness of $\tilde{\Gamma}^{(a)}$. Thus, there must exist at least one additional extremum of m which is not a local minimum. By the corollary to theorem 3.1, this should correspond to a static solution of the EYM equations. Since m is not a local minimum, this solution should be unstable.

The above argument corresponds, in essence, to the argument given for the existence of the sphaleron solution [6, 7] of Yang-Mills-Higgs theory in flat spacetime. However, we now shall show that by refining the above argument further (and, of course, making some additional assumptions) a discrete infinity of new solutions actually are predicted, which possess certain characteristic features. We conjecture that a similar discrete infinity of new solutions also should exist in Yang-Mills-Higgs theory.

As argued above, not all points of $\tilde{\Gamma}^{(a)}$ can approach local minima of m under the flow generated by M^A. Let $\Gamma_1 \subset \tilde{\Gamma}^{(a)}$ denote the set of points which do not flow to local minima. Since $\tilde{\Gamma}^{(a)} - \Gamma_1$ is composed of disjoint open sets, it seems reasonable to suppose that Γ_1 is a surface (possibly with cusps, bifurcations, etc.) of codimension 1 in $\tilde{\Gamma}^{(a)}$. Clearly M^A must be everywhere tangent to Γ_1, which implies that any critical point of m restricted to Γ_1 must be a critical point of m on $\tilde{\Gamma}^{(a)}$. Furthermore, by construction, Γ_1 must contain all critical points of m on $\tilde{\Gamma}^{(a)}$ which are not local minima on $\tilde{\Gamma}^{(a)}$. Hence, by the above argument, there must be at least one critical point of m on Γ_1. We assume, now, that a critical point X_1 can be found whose mass m_1 is less than or equal to that of any other critical point of m on Γ_1. Then m_1 must be an absolute minimum of m restricted to Γ_1, since any point of Γ_1 with mass less than m_1 would have "nowhere to go" under the flow generated by M^A. (Of course, X_1 cannot locally minimize m on $\tilde{\Gamma}^{(a)}$ with respect to the direction orthogonal to Γ_1 since no local minimum of m on $\tilde{\Gamma}^{(a)}$ can lie on Γ_1.) By the corollary to theorem 3.1, X_1 should correspond to a static EYM solution. We believe that this solution is precisely the $n = 1$ Bartnik-McKinnon solution.

In fact, however, there actually must be a discrete infinity of local minima of m restricted to Γ_1: namely X_1 and all large gauge transformations of X_1. (It is conceivable that other local minima of m restricted to Γ_1 also exist, but we see no reason to expect this and, for simplicity, will assume that no other local minima are present.) Hence, the situation with respect to the mass function m restricted to Γ_1 is essentially the same as for the mass function m on $\tilde{\Gamma}^{(a)}$. Under the additional hypothesis that Γ_1 is connected, a repetition of the argument of the previous paragraph predicts the existence of a submanifold $\Gamma_2 \subset \Gamma_1$ of codimension 1 in Γ_1 (and, hence, codimension 2 in $\tilde{\Gamma}^{(a)}$) and a point X_2 whose mass m_2 minimizes m restricted to Γ_2. Then X_2 is an extremum of m on $\tilde{\Gamma}^{(a)}$ which is not a local minimum with respect to any direction in $\tilde{\Gamma}^{(a)}$ lying in the two-dimensional subspace orthogonal to Γ_2. We believe that X_2 corresponds to the $n = 2$ Bartnik-McKinnon solution. Continued repetition of this argument should account for all the higher n Bartnik-McKinnon solutions.

Before discussing some properties of the solutions predicted by the above argument, we point out a problem with the initial assumptions and explain how to modify the argument to take this into account. Although the EYM equations are not scale invariant (see the third paragraph of Sec. III), the conformal invariance of the momentum constraints allows us to define a notion of "nonlinear scaling" of initial data as follows: Given initial data $(h_{ab}, \pi^{ab}, A_a{}^{\Lambda}, E^a{}_{\Lambda})$ in $\tilde{\Gamma}^{(a)}$, we obtain a one-parameter family of new initial data by choosing $(h_{ab}, \pi^{ab}, A_a{}^{\Lambda}, E^a{}_{\Lambda})$ as "conformal initial data" (see Sec. III) and then solving the Lichnerowicz equation (73) with the asymptotic behavior $\phi \to \lambda$ at infinity (rather than $\phi \to 1$), where λ is a constant. (We then must perform a scaling diffeomorphism $x \to x' = \lambda^2 x$ near infinity so that the new data will satisfy our asymptotic flatness conditions in the form stated in Sec. II.) The limit $\lambda \to \infty$ corresponds to "(nonlinearly) scaling the configurations to large size" but decreasing the energy density at a sufficiently rapid rate that we would expect that $m(\lambda) \to 0$ in this limit. On the other hand, unless the original initial data was that of flat spacetime, as λ is decreased we would expect a tiny "bag of gold" configuration [43] to be produced and that, as $\lambda \to 0$, this "bag" will "pinch off." We also would expect $m(\lambda) \to 0$ as $\lambda \to 0$. We have verified numerically that this "pinch-off" behavior does occur in some simple examples, and that in these examples $m \to 0$ as $\lambda \to \infty$ and as $\lambda \to 0$. However, there is no limit critical point in $\tilde{\Gamma}^{(a)}$ corresponding to $\lambda \to \infty$ or $\lambda \to 0$ so (since the critical points are assumed to be isolated) it is not plausible that the minimum distance along this curve to all critical points approaches zero as $\lambda \to \infty$ or $\lambda \to 0$. In that case, assumption (i) combined with $m \to 0$ as $\lambda \to \infty$ or $\lambda \to 0$, and with $m \geq 0$ everywhere on $\tilde{\Gamma}^{(a)}$ yields a contradiction.

However, this difficulty may be remedied by passing to yet a new space $\tilde{\Gamma}'^{(a)}$ whose points consist of equivalence classes of points of $\tilde{\Gamma}^{(a)}$ under the above (nonlinear) scaling. We define the mass function m' on $\tilde{\Gamma}'^{(a)}$ to be the maximum mass occurring in the scaling equivalence class in $\tilde{\Gamma}^{(a)}$. If a unique maximum of m' occurs in all cases (as it does in our numerical examples), then m' should be a well-behaved function on $\tilde{\Gamma}'^{(a)}$, and the above arguments should apply to it. (Even if this is not strictly the case, we believe that the argument is sufficiently robust that the conclusions remain plausible.) If $X' \in \tilde{\Gamma}'^{(a)}$ is an extremum of m' on $\tilde{\Gamma}'^{(a)}$, then the corresponding point $X \in \tilde{\Gamma}^{(a)}$ which maximizes m in the scaling direction will be an extremum of m on $\tilde{\Gamma}^{(a)}$, and, hence, should correspond to a static solution. Thus, the only notable change brought about by this modification of our argument is that at each stationary solution (except flat spacetime) there is an extra "unstable" (i.e., mass decreasing) direction corresponding to this scaling transformation.

We return, now, to consider the properties of the solutions predicted by the above argument and compare them with the properties of the Bartnik-McKinnon solutions. First, our argument predicts a countable infinity of nonsingular static solutions (with vanishing electric field) to the EYM equations. Our argument also predicts an increasing sequence of masses for these solutions, $0 < m_1 < m_2 < \cdots$. Furthermore, these solutions all are unstable. Indeed, if we identify the number of unstable modes with the dimension of the tangent subspace in $\tilde{\Gamma}^{(a)}$ along which m decreases (and, if we take into account the "scaling direction" discussed above) we predict that the

nth solution should have precisely $n+1$ unstable modes. Finally, we note that our argument also could be made by restricting consideration to spherically symmetric initial data. (The large gauge transformations of the trivial initial data can be connected by a sequence of spherically symmetric configurations, so the relevant portion of the spherically symmetric phase space is connected.) Hence, the solutions predicted by our argument should be spherically symmetric. (In view of Israel's theorem [44] for the Einstein-Maxwell case, the fact that the predicted solutions are static also suggests that they should be spherically symmetric.) All of the above properties except the prediction concerning the precise number of unstable modes are well-known features of the Bartnik-McKinnon solutions. It would be interesting to check this latter prediction. In any case, the evidence in favor of identifying the solutions predicted by our argument with the Bartnik-McKinnon solutions seems quite strong.

Our arguments can be adapted to case (b) to predict static black-hole solutions of the EYM equations by making the following modifications: (1) We restrict attention to the case where the boundary surface S is topologically S^2 and Σ has topology $S^2 \times \mathbb{R}$. (The first restriction is necessary for stationary black holes by proposition 9.3.2 of [18].) (2) In addition to imposing a Yang-Mills gauge condition at infinity, we also must impose a gauge condition at S in order to ensure the presence of "large gauge transformations." (3) Instead of working with the entire space $\tilde{\Gamma}^{(b)}$, we work on the submanifold consisting of initial data for which S is an extremal surface with fixed area A (and A minimizes the area of two-spheres homotopic to S in Σ.) (4) On account of the requirement that A be fixed, the "scaling behavior" for case (a) does not occur here, so we do not need to pass to an analogue of the space $\tilde{\Gamma}'^{(a)}$.

On the submanifold of $\tilde{\Gamma}^{(b)}$ corresponding to a fixed A, the mass function m possesses a countable infinity of absolute minima [45], namely, the Schwarzschild solution of mass $m_S = (A/16\pi)^{1/2}$ and all large gauge transformations of it. Our previous argument then predicts that for each value of A, there should exist a countably infinite sequence $X_i \in \tilde{\Gamma}^{(b)}$ of extrema of m at fixed A, with $m_S < m_1 < m_2 < \cdots$. By the corollary to theorem 3.2, each X_i should correspond to a static black hole (with vanishing electric field). Again, these solutions should be spherically symmetric since the argument could be made restricting consideration to the spherically symmetric initial data. In addition, these solutions should be unstable. The colored black-hole solutions [3–5] have all these properties [46], so we identify the solutions predicted by our argument with the colored black holes. (The Abelian, magnetically charged Reissner-Nordström black holes are not candidates for being the solutions predicted by our arguments, since our arguments predict infinitely many solutions, but, at fixed A, there are at most finitely many such Reissner-Nordström solutions. The fact that, at given A, the mass of the lowest magnetically charged Reissner-Nordström black hole is higher than the mass of all the colored black holes confirms that they are not relevant to our argument.) Furthermore, our argument predicts that the nth colored black-hole solution should have precisely n unstable modes. This prediction has very recently been verified numerically by Bizon [47] up to $n = 4$.

Our arguments can be extended readily to analyze the possibility of stationary, nonstatic EYM solutions. For case (a), theorem 2.1 shows that stationary solutions must be extrema of m at fixed Q, and the arguments given at the end of Sec. II strongly suggest that the converse also holds. However, there are no known minima of m in $\tilde{\Gamma}^{(a)}$ at any fixed $Q \neq 0$, and we see no reason to expect any to exist. Hence, we also see no reason to expect any new extrema of m at fixed Q which are not local minima. Thus, we would expect the known Bartnik-McKinnon solutions to be the only stationary, nonsingular solutions of the EYM equations in case (a).

On the other hand, for case (b), by theorem 2.2 and the arguments of Sec. II, we expect stationary black-hole solutions to correspond to extrema of mass at fixed A, Q, and $\mathcal{J}$. Now, for each Q and $\mathcal{J}$ and each $A \geq 4\pi(Q^4 + \mathcal{J}^2)^{1/2}$, there are known stationary EYM black-hole solutions, namely, the (Abelian) electrically charged Kerr solutions (see, e.g., [19]). These solutions presumably minimize m at fixed A, Q, and $\mathcal{J}$. Thus, for each A, Q, and $\mathcal{J}$ with $A \geq 4\pi(Q^4 + \mathcal{J}^2)^{1/2}$, our arguments predict the existence of a countably infinite family of new, non-Abelian EYM stationary black holes. Except for staticity and spherical symmetry, this family should share the properties described above for the known colored black holes, namely, the masses in the sequence should be strictly increasing, and the solutions should be unstable, with the nth solution in the sequence possessing precisely n unstable modes.

It is worth noting that if we choose A sufficiently large and Q sufficiently small, there should be no difficulty in adjusting $\mathcal{J}$ so that for, say, the new $n = 1$ solution we have $\Omega = 0$, where Ω denotes the angular velocity of the horizon. If $\mathcal{J} \neq 0$ when $\Omega = 0$, the stationary Killing field t^μ cannot be hypersurface orthogonal. However, even if $\mathcal{J} = 0$, we do not expect t^μ to be hypersurface orthogonal since Israel's theorem [44], proven in the Einstein-Maxwell case, suggests that the solution should then be spherically symmetric and Bizon and Popp [34] have proven that no non-Abelian spherically symmetric EYM black-hole solutions exist with $Q \neq 0$. Hence, this predicted solution would be an excellent candidate for a stationary black hole which is nonrotating (i.e., $\Omega = 0$) but possesses a non-hypersurface-orthogonal stationary Killing field. [The possible existence of such solutions in the vacuum and Einstein-Maxwell cases was ruled out in Sec. III (see the discussion following theorem 3.4 above).] The existence of such a solution would not violate the physical arguments given in [18], since this solution would be unstable.

Finally, we note that the arguments of this section also should be applicable to other Hamiltonian theories involving an SU(2) Yang-Mills field, provided that the following two conditions are satisfied. (1) The theory should not possess a scale invariance. [A scale invariance would make our assumptions about the mass function blatantly implausible and the type of argument given near the beginning of Sec. III would likely rule out the existence of any stationary solutions in case (a).] (2) The theory should admit a stable solution; i.e., our argument relies on the existence of a local minimum of the total energy. In particular, our arguments are not applicable to pure Yang-Mills theory (since that theory possesses a scale invariance), but they should be applicable to Yang-Mills-Higgs theory in flat spacetime. As already indicated above, we believe that the sphaleron solution [6] corresponds to the $n = 1$ solution of our argument. Hence, we believe that there should exist an additional countable infinity of stationary Yang-Mills-Higgs solutions, which, together with the sphaleron, would comprise a family analogous to the Bartnik-McKinnon solutions of EYM theory. The "new sphaleron" of Klinkhamer [48] (see also [49]–[51]), plausibly could be a member of this family.

For the convenience of the reader, we conclude this section by listing some of the key conjectures which were suggested by the arguments of this section.

(1) The nth Bartnik-McKinnon solution has precisely $n+1$ unstable modes. The nth colored black-hole solution has precisely n unstable modes. (As mentioned above, for the colored black holes, this conjecture has been verified by Bizon [47] up to $n = 4$.)

(2) In case (a) (see Sec. II), the Bartnik-McKinnon solutions are the only stationary nonsingular EYM solutions.

(3) In case (b), in addition to the known colored black-hole solutions, there exist similar discrete families of new colored black-hole solutions at each value of A, Q, and $\mathcal{J}$ satisfying $A \geq 4\pi(Q^4 + \mathcal{J}^2)^{1/2}$.

(4) In the Yang-Mills-Higgs theory there should exist a countably infinite family of new unstable solutions analogous to the Bartnik-McKinnon solutions, with the sphaleron solution comprising the $n = 1$ member of this family.

Note added. After submission of this paper, Bizon [47] numerically investigated the instability of the lowest-lying Bartnik-McKinnon solutions and found the nth solution to have n unstable modes, rather than $n + 1$ as we had conjectured. The most straightforward interpretation of this discrepancy is simply that the $n = 1$ Bartnik-McKinnon solution actually corresponds to a local minimum of mass on the space $\tilde{\Gamma}'^{(a)}$. In that case, the first solution predicted by our argument would be the $n = 2$ Bartnik-McKinnon solution, which should have $1 + 1 = 2$ unstable modes, and, similarly, the higher n Bartnik-McKinnon solutions should have n unstable modes.

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APPENDIX

In this appendix we give a proof of the lemma mentioned in Sec. III. For simplicity of notation we use ϕ instead of $\delta\phi$.

Lemma: Let ϕ be a solution of

$$D^a D_a \phi - \mu\phi = \rho \tag{A1}$$

on a connected, asymptotically flat three-manifold (Σ, h_{ab}), with $\mu \geq 0$, $\rho \geq 0$ and $\mu = O(1/r^{2+\epsilon})$ as $r \to \infty$ for some $\epsilon > 0$. Suppose further that $\phi \leq 0$ everywhere on Σ, and $\phi = o(1/r)$ as $r \to \infty$; i.e., the asymptotic form (82) holds with $c = 0$. Then $\phi = 0$ on Σ, which, of course, is possible only if $\rho = 0$ everywhere on Σ.

Proof: We note, first, that by the Hopf maximum principle (see, e.g., [33]) ϕ cannot attain a globally maximum value at any interior point of Σ unless ϕ is constant. Hence, in order to prove the lemma, it suffices to show that there exists a point $x \in \Sigma$ such that $\phi(x) = 0$. We shall establish this by showing that ϕ vanishes in a neighborhood of infinity.

Let (r, θ, φ) denote a spherical coordinate system associated with the flat metric e_{ab}, with coordinate range $r_0 < r < \infty$, so that the coordinate system covers a neighborhood of infinity on Σ. Our asymptotic flatness condition (10) on h_{ab} then implies that as $r \to \infty$ we have

$$h_{rr} = 1 + O(1/r), \tag{A2}$$

$$h_{r\theta},\ h_{r\varphi} = O(1), \tag{A3}$$

$$h_{\theta\theta} = r^2 + O(r), \tag{A4}$$

$$h_{\theta\varphi} = O(r), \tag{A5}$$

$$h_{\varphi\varphi} = r^2 \sin^2\theta[1 + O(1/r)], \tag{A6}$$

$$\sqrt{h} = r^2 \sin\theta[1 + O(1/r)]. \tag{A7}$$

In addition the first derivative with respect to r of these metric components falls off an additional power of r faster than specified in Eqs. (A2)–(A6). (In fact for our arguments below, much weaker asymptotic conditions on h_{ab} would suffice.)

We define

$$G(r) = -(1/r)\int_{S_r} \sqrt{h}h^{rr}\phi\, d\theta\, d\varphi, \tag{A8}$$

where S_r denotes the two-sphere of coordinate radius r. Since $\phi \leq 0$ we have $G(r) \geq 0$ for all r. Furthermore, if at some r we have $G(r) = 0$, then ϕ vanishes on S_r and, hence, by the maximum principle, ϕ vanishes on Σ. In addition, our asymptotic conditions on ϕ and the components of h_{ab} ensure that $G(r) \to 0$ and $G'(r) \to 0$ as $r \to \infty$, where the primes denote derivatives with respect to r.

Integrating (A1) over S_r we obtain an "evolution equation" for G as a function of r;

$$G'' = F(r) + \frac{1}{r^2}H + (H/r)' - (1/r)\int_{S_r} \rho\sqrt{h}\, d\theta\, d\varphi, \tag{A9}$$

where

$$F(r) = -(1/r)\int_{S_r} \sqrt{h}\mu\phi\, d\theta\, d\varphi, \tag{A10}$$

$$H(r) = (1/r)\int_{S_r} \sqrt{h}h^{rr}\eta\phi\, d\theta\, d\varphi, \tag{A11}$$

$$\eta = \frac{rh_{rr}}{\sqrt{h}}\left[r^2 \frac{\partial}{\partial r}\left(\frac{\sqrt{h}h^{rr}}{r^2}\right) - \left(\frac{\partial}{\partial\theta}(\sqrt{h}h^{r\theta}) + \frac{\partial}{\partial r}(\sqrt{h}h^{r\varphi})\right)\right]. \tag{A12}$$

The following bounds then follow immediately from the definitions (A8), (A10), (A11):

$$|H(r)| \le \eta_m(r)G(r), \tag{A13}$$

$$0 \le F(r) \le \mu_m(r)G(r), \tag{A14}$$

where

$$\mu_m(r) = \mathrm{Max}_{S_r}\{\mu h^{rr}\} = O(1/r^{2+\epsilon}), \tag{A15}$$

$$\eta_m(r) = \mathrm{Max}_{S_r}\{|\eta|\} = O(1/r). \tag{A16}$$

Combining these inequalities with (A9) we obtain the key inequality,

$$G'' \le \mu_m G + (1/r^2)\eta_m G + (H/r)' - (1/r)\int_{S_r} \rho\sqrt{h}\, d\theta\, d\varphi$$
$$\le \mu_m G + (1/r^2)\eta_m G + (H/r)'. \tag{A17}$$

Now choose $r_1 > r_0$. From Eqs. (A15)–(A17) we find that there exists a constant C_1 such that $\forall r \ge r_1$:

$$G''(r) \le (C_1/r^{2+\epsilon})G_M(r_1) + (H/r)', \tag{A18}$$

where

$$G_M(r) \equiv \max_{r' \ge r} G(r'). \tag{A19}$$

(Since $G \ge 0$ and $G \to 0$ as $r \to \infty$, it follows that G attains a maximum value on any interval $[r, \infty)$.) Integrating (A18) from r to ∞, we obtain $\forall r \ge r_1$:

$$-G'(r) \le (C_2/r^{1+\epsilon})G_M(r_1) - H/r$$
$$\le (C_3/r^{1+\epsilon})G_M(r_1) \tag{A20}$$

where (A13) and (A16) were used in the second inequality. Integrating (A20) from r_1 to ∞ we obtain

$$G(r) \le (C_4/r^\epsilon)G_M(r_1). \tag{A21}$$

It follows upon maximization of both sides of Eq. (A21) over the interval $[r_1, \infty)$ that

$$G_M(r_1) \le (C_4/r_1^\epsilon)G_M(r_1). \tag{A22}$$

However, if we choose r_1 sufficiently large that $r_1 > C_4^{1/\epsilon}$, Eq. (A22) yields a contradiction unless $G_M(r_1) = 0$. Thus, there exists an $r_1 \in \mathbb{R}$ such that $G_M(r_1) = 0$, which in turn implies that $G(r) = 0$ for all $r > r_1$. Hence by the maximum principle, $\phi = 0$ everywhere on Σ. □

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