Generation of squeezed radiation from vacuum in the cosmos and the laboratory

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The close analogy between the generation of gravitons from vacuum fluctuations of the gravitational field and the generation of photons from vacuum fluctuations of the electromagnetic field is studied. Gravitons produced in the cosmos and photons produced in the laboratory are governed by similar physical principles and mathematical equations. Both are described by the so-called squeezed vacuum quantum states. Squeezed vacuum optical radiation has been generated experimentally by separating the pump light from the squeezed fluctuations in an interferometric geometry. It may prove possible that some predictions of gravitation theory can be modeled, or even tested, in the laboratory by quantum optics experiments.

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I. INTRODUCTION

The phenomenon of particle creation in intense electromagnetic and gravitational fields is an important area of current research. One of the most striking examples is the well-known process of particle creation in the gravitational field of a black hole, the process of quantum evaporation of black holes [1]. Another example is the generation of gravitational waves from vacuum (quantum) fluctuations with the energy supplied by macroscopic (classical) variations of the gravitational field of the expanding Universe [2]. The highly variable gravitational field of the early Universe coupled the annihilation operators of the quantized gravitational waves to the creation operators, thereby leading to the generation of gravitons. These gravitons of quantum origin (relic gravitons) are predicted to exist in squeezed vacuum quantum states and can, in principle, be detected [3,4].

These predictions of gravitational theory are purely astrophysical in nature; they cannot be subjected to laboratory tests. However, the equations developed for the gravitational case are in one-to-one correspondence with the equations of quantum optics [5,6]. Laboratory experiments in the field of nonlinear quantum optics are possible, and have already demonstrated nonclassical (squeezed) states of electromagnetic radiation [7–11]. Thus, laboratory generation and detection of squeezed electromagnetic radiation appears analogous to cosmic processes involving gravitational radiation. Mathematically, the theory of squeezed gravitons is very similar to the theory of squeezed light. Therefore, it is of interest to spell out the connection between the optics experiments and the processes in strong gravitational fields. It is quite possible that some predictions of gravitational theory may eventually be modeled, or even tested, in laboratory conditions using quantum optics experiments.

The well-known experiments aimed at squeezing of light with the help of degenerate parametric amplifiers, that couple a (signal) annihilation operator $\hat{a}_s$ to the creation operators $\hat{a}_s^\dagger$, are of this category. They use the first-order nonlinearity of optical media.

Other recent experiments [12] done in fibers use the second-order nonlinearity (Kerr effect). Both these experiments bear an analogy to the gravitational wave generation taking place "naturally" in a gravitational environment. The laser radiation generates a time-dependent index grating that couples $\hat{a}_s$ to $\hat{a}_s^\dagger$ and vice versa and thus can be considered to generate photons from vacuum. In the experiments, the pump producing the grating is separated either by a filter (first order nonlinearity) or in a ring interferometer geometry (second order nonlinearity). In this way one may model the stream of photons emerging from the output as photons generated from vacuum by a time-dependent grating.

In the context of gravity-wave research, the notion of squeezed optical quantum states has often been raised in a different sense, namely, as a means for improved detection of a classical gravitational wave. For instance, it was shown [13] that the sensitivity of a laser interferometer gravity-wave detector can be increased using squeezed light. It was also argued [14] that any detector oscillator can be specially "prepared" in a squeezed state and used for gravity-wave detection during some interval of time before the thermal noise destroys squeezing and degrades the detector's sensitivity.

Here, however, we shall discuss the squeezing of gravitational waves themselves. In laboratory conditions, it is difficult to achieve even a modest amount of squeezing of light, i.e., to obtain a squeeze parameter r of the order of unity. In contrast, in the cosmos, the squeezed quantum states of gravitational waves are produced inevitably and with a much larger squeeze parameter, as the result of the rapid expansion of the early Universe.

To make the analogy between the theories of relic graviton production and squeezing of light more transparent, we begin with a formulation of Einstein's general relativity in a form similar to the theory of classical electromagnetic fields. We believe that a reader, who is familiar with classical electrodynamics but may not be acquainted with the notion of curved space-time which cosmologists use, will appreciate the use of the concept of gravitational field in flat Minkowski space-time. (More details about this "field-theoretical" formulation of general relativity are given in Refs. [15,4]; it is important to emphasize that we are dealing with a different mathematical formulation of general relativity, not with a physical alternative, Ref. [16]). This approach leads to manifestly nonlinear field equations. In contrast with the equations of quantum optics that are linear in vacuum and need a nonlinear optical medium in order to couple signal waves to a "pump" field, the gravitational field does not require any material medium for this purpose. In the case of gravidynamics the coupling is achieved automatically due to the nonlinearity of the gravitational field itself. As is often done, we will present the total gravitational field in the form of an approximate sum of a large "classical" contribution and a small quantized perturbation. This approach will be applied to the cosmological gravitational field of the expanding Universe acting upon zero-point quantum fluctuations of the gravitational waves. Even though it may sound too technical, we shall call the variable gravitational field of the early expanding Universe a "pump" field. As a result of the action of this pump, the initial vacuum state of each mode of the gravitational waves evolves into strongly squeezed vacuum states with very specific statistical properties.

The paper is organized as follows. In Sec. II we briefly exhibit the "field-theoretical" approach to general relativity that is adequate for comparison of electrodynamics with gravidynamics. In Sec. III we set up the linearized equations for the graviton creation and annihilation operators in the presence of a time-dependent gravitational field and demonstrate, in Sec. IV, the inevitable appearance of strongly squeezed vacuum states. Section V derives analogous equations for the coupling of the creation and annihilation operators of the photon field in a time dependent dielectric of a Kerr medium. Section VI investigates the problem of pair generation by two sheets of Kerr material. Section VII investigates the propagation of an electromagnetic field in a Kerr medium, both classically and quantum mechanically. Section VIII describes the interferometric system that can separate out the pump radiation, so that the "unused port," which in the absence of the time dependent grating emits vacuum fluctuations, in the presence of the pump radiates photons (squeezed vacuum states) generated by the time dependent grating. Then the experiment is described. The optical detection takes advantage of the homodyne amplification that is capable of detecting even a small number of photons produced from vacuum by the time dependent grating. In Sec. IX we give a short discussion of the results.

II. FIELD-THEORETICAL APPROACH TO GENERAL RELATIVITY

It is known that the electromagnetic fields are expressed in terms of components of a four-vector A_μ, the four-vector potential. In contrast, the gravitational field is expressed in terms of components of a symmetric second-rank tensor $h_{\mu\nu}$, the gravitational potentials. Our goal is to derive the field equations from the total action S consisting of the action of the gravitational field S^g and the action S^m describing sources of the gravitational field and their interaction with gravity, $S=S^g+S^m$. It turns out to be convenient to use, in addition to the gravitational potentials $h_{\mu\nu}$, another set of field variables $P^\alpha{}_{\mu\nu}$, which compose a third-rank tensor symmetric with respect to its lower indices. The $P^\alpha{}_{\mu\nu}$'s are not new physical fields, but rather combinations of the first derivatives of the $h_{\mu\nu}$'s. The relationship between the gravitational potentials $h_{\mu\nu}$ and the $P^\alpha{}_{\mu\nu}$'s will be established by the action principle. In formulating the action principle, we shall treat the $h_{\mu\nu}$'s and the $P^\alpha{}_{\mu\nu}$'s as independent.

The gravitational field is assumed to exist in ordinary Minkowski space-time with a four-dimensional interval $d\sigma$:

$$d\sigma^2=c^2dt^2-dx^2-dy^2-dz^2 \ . \tag{2.1}$$

This four-dimensional interval is written in Lorentzian coordinates. In arbitrary curvilinear coordinates the four-dimensional interval $d\sigma$ has a more complicated form:

$$d\sigma^2=\gamma_{\mu\nu}dx^\mu dx^\nu \ , \tag{2.2}$$

where $\gamma_{\mu\nu}$ is the metric tensor of Minkowski space-time written in the curvilinear coordinates. In Lorentzian coordinates, $\gamma_{\mu\nu}$ acquires the simplest form: $\gamma_{00}=1$, $\gamma_{11}=\gamma_{22}=\gamma_{33}=-1$, the rest of $\gamma_{\mu\nu}$ being equal to zero.

In Lorentzian coordinates, used in (2.1), the distinction between covariant and contravariant components of a tensor is very simple; they may differ, at most, by a sign. Yet, in order to make the formalism applicable to arbitrary curvilinear coordinates, and prepare it for the standard derivation of stress tensors, we shall make such a distinction. We shall use the subscript symbol "$;\alpha$" for covariant differentiations and note that the operation of covariant differentiation as well as the lowering or raising of indices is to be performed via the metric tensor $\gamma_{\mu\nu}$.

The gravitational part of the total action is

$$S^g=-\frac{1}{2c\kappa}\int d^4x\, L^g, \quad \kappa=\frac{8\pi G}{c^4} \ , \tag{2.3}$$

where the gravitational Lagrangian L^g has the form

$$L^g=(-\gamma)^{1/2}[h^{\mu\nu}{}_{;\alpha}P^\alpha{}_{\mu\nu}-(h^{\mu\nu}+\gamma^{\mu\nu})(P^\alpha{}_{\mu\beta}P^\beta{}_{\nu\alpha}-\tfrac{1}{3}P^\sigma{}_{\sigma\mu}P^\rho{}_{\rho\nu})] \tag{2.4}$$

and γ is the determinant of the matrix $\gamma_{\mu\nu}$. The nongravitational sources and fields and their interaction with the gravitational field are described by

$$S^m = \frac{1}{c}\int d^4x\, L^m \ , \tag{2.5}$$

where L^m is supposed to contain matter variables and gravitational field variables. Moreover, in order to arrive at the proper universal coupling of all material fields to gravity (Einstein's equivalence principle) we must assume that L^m is of the form

$$L^m[\phi^a, \sqrt{-\gamma}(\gamma^{\mu\nu}+h^{\mu\nu})] \ , \tag{2.6}$$

where ϕ_a is a symbolic representation of matter fields. The gravitational potentials $h^{\mu\nu}$ enter L^m only in sum with $\gamma^{\mu\nu}$.

The energy-momentum tensor $t_{\mu\nu}$ of the gravitational field itself and the energy-momentum tensor $\tau_{\mu\nu}$ of the nongravitational matter and fields interacting with gravity, are defined in the usual manner:

$$\kappa t_{\mu\nu} = -\frac{1}{\sqrt{-\gamma}}\frac{\delta L^g}{\delta\gamma^{\mu\nu}} \ , \quad \tau_{\mu\nu} = \frac{2}{\sqrt{-\gamma}}\frac{\delta L^m}{\delta\gamma^{\mu\nu}} \ , \tag{2.7}$$

where $\delta/\delta\gamma^{\mu\nu}$ is a variational derivative. The explicit expression for $\kappa t_{\mu\nu}$ that can be derived from L^g is

$$\kappa t_{\mu\nu} = P^\alpha{}_{\mu\beta}P^\beta{}_{\nu\alpha} - \tfrac{1}{2}\gamma_{\mu\nu}\gamma^{\sigma\rho}P^\alpha{}_{\sigma\beta}P^\beta{}_{\rho\alpha} - \tfrac{1}{3}(P_\mu P_\nu - \tfrac{1}{2}\gamma_{\mu\nu}P_\sigma P^\sigma) + Q_{\mu\nu} \ , \tag{2.8}$$

where $P_\mu \equiv P^\alpha{}_{\alpha\mu}$, $Q_{\mu\nu} \equiv Q^\alpha{}_{\mu\nu;\alpha}$ and

$$Q^\tau{}_{\mu\nu} \equiv \tfrac{1}{2}P^\sigma{}_{\alpha\beta}[\gamma_{\mu\nu}h^{\alpha\beta}\delta^\tau{}_\sigma + \gamma^{\tau\alpha}(\gamma_{\sigma\mu}h^\beta{}_\nu + \gamma_{\sigma\nu}h^\beta{}_\mu) - h^{\tau\beta}(\delta^\alpha{}_\mu\gamma_{\sigma\nu} + \delta^\alpha{}_\nu\gamma_{\sigma\mu}) - \delta^\tau{}_\sigma(\delta^\alpha{}_\mu h^\beta{}_\nu + \delta^\alpha{}_\nu h^\beta{}_\mu)] \ . \tag{2.9}$$

By varying the action S with respect to $\sqrt{-\gamma}\,h^{\mu\nu}$ and $P^\alpha{}_{\mu\nu}$, one obtains a set of equations of motion that can be rearranged to read

$$h_{\mu\nu;\alpha}{}^{;\alpha} + \gamma_{\mu\nu}h^{\alpha\beta}{}_{;\alpha;\beta} - h^\alpha{}_{\nu;\alpha;\mu} - h^\alpha{}_{\mu;\alpha;\nu} = \frac{16\pi G}{c^4}(t_{\mu\nu}+\tau_{\mu\nu}) \ , \tag{2.10}$$

$$-h^{\mu\nu}{}_{;\rho} + (\gamma^{\mu\alpha}+h^{\mu\alpha})P^\nu{}_{\alpha\rho} + (\gamma^{\nu\alpha}+h^{\nu\alpha})P^\mu{}_{\alpha\rho} - \tfrac{1}{3}P_\alpha[(\gamma^{\mu\alpha}+h^{\mu\alpha})\delta^\nu{}_\rho + (\gamma^{\nu\alpha}+h^{\nu\alpha})\delta^\mu{}_\rho] = 0 \ . \tag{2.11}$$

It is clear from (2.11) that the field variables $h_{\mu\nu}$ and $P^\alpha{}_{\mu\nu}$ are indeed related, and thus only the $h_{\mu\nu}$'s can be considered to be the independent set of gravitational field potentials.

Equation (2.10) is a wave equation for the gravitational potentials in Minkowski space-time, similar to the wave equation of electromagnetic fields. The left-hand side is linear in $h_{\mu\nu}$; the right-hand side is the "driving" energy-momentum tensor, consisting of a matter energy momentum tensor $\tau_{\mu\nu}$, as well as nonlinear contributions of the gravitational field itself represented precisely in the form of a gravitational energy-momentum tensor $t_{\mu\nu}$.

The above is a view of gravitational fields different from that represented by curved space-time. We arrived at a nonlinear wave equation. The solution of this equation yields a distribution of $h_{\mu\nu}$ in Minkowski space-time. Connection with the geometrical formulation of general relativity is made when this solution is used to obtain the metric tensor $g_{\mu\nu}$ of curved space-time according to the formula

$$\sqrt{-g}\, g^{\mu\nu} = \sqrt{-\gamma}(\gamma^{\mu\nu}+h^{\mu\nu}) \ . \tag{2.12}$$

Einstein's differential equation for the metric tensor $g_{\mu\nu}$ is obtained when (2.12) is substituted into (2.10) and the matter energy momentum tensor $T_{\mu\nu}$ is defined with respect to $g_{\mu\nu}$ identified as the metric tensor of curved space-time: $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$.

The theory allows a freedom of choice of gauge similar to that of classical electrodynamics. One can apply the gauge transformations to the gravitational variables $h_{\mu\nu}$ and matter variables without changing the field equations. At the expense of gauge freedom one can impose some gauge conditions which are normally used for diminishing the number of variables and simplifying the field equations. A convenient choice, similar to the often used electrodynamical gauge condition $A^\alpha{}_{;\alpha}=0$, is $h^\alpha{}_{\mu;\alpha}=0$.

III. GRAVITATIONAL FIELD OF EXPANDING UNIVERSE

Let us apply the developed formalism to the description of the gravitational field of the homogeneous isotropic Universe. From our new point of view this is just a specific gravitational field $h_{\mu\nu}(t,x,y,z)$ given in Minkowski space-time (2.1). Since we are using the Lorentzian coordinates, all covariant derivatives $;\alpha$ in (2.10) and (2.11) reduce to ordinary derivatives $,\alpha$.

Let us take the nonvanishing gravitational potentials in the form

$$h_{00} = a^3(t) - 1 \ , \quad h_{11} = h_{22} = h_{33} = 1 - a(t) \ , \tag{3.1}$$

where $a(t)$ is, as yet, an unspecified function of time. This specific form of the potentials is needed, if one is to obtain the well-known Friedmann solutions of the Einstein equations for a spatially flat homogeneous isotropic expanding Universe via this new field theoretical approach. Note that $a(t)=1$ corresponds to the absence of all gravitational field potentials.

One can calculate the gravitational energy-momentum tensor $t_{\mu\nu}$, (2.8), and find that the nonvanishing components of $\kappa t_{\mu\nu}$ are

$$\kappa t_{00} = -\frac{3\ddot a}{2a}(a^2-1) - 3\dot a^2 \ ,$$
$$\kappa t_{11} = \kappa t_{22} = \kappa t_{33} = -\frac{\ddot a}{2a}(3a^3 - a^2 + a - 3) - \dot a^2(3a-1) \ , \tag{3.2}$$

where the overdot(s) indicates the time derivative (for simplicity we choose the units for which the velocity of light $c=1$).

The nongravitational sources are assumed to be those of a perfect fluid with the Lagrangian

$$L^m=\tfrac{1}{2}\sqrt{-g}\,[\epsilon+3p-(\epsilon+p)g_{\mu\nu}u^\mu u^\nu]\ , \tag{3.3}$$

where ϵ, p, and u^μ are variables characterizing the matter, and $g_{\mu\nu}$ is defined by (2.12). One can find the nonvanishing components of the energy-momentum tensor $\tau_{\mu\nu}$:

$$\begin{aligned}&\tau_{00}=\epsilon+\tfrac{3}{4}(a^2-1)(\epsilon-p)\ ,\\ &\tau_{11}=\tau_{22}=\tau_{33}=p-\tfrac{1}{4}(a^2-1)(\epsilon-p)\ .\end{aligned} \tag{3.4}$$

By substituting expressions (3.1), (3.2), and (3.4) into the field equations (2.10), one can derive equations governing the function $a(t)$ and, hence, the gravitational field (3.1):

$$\frac{\ddot a}{a}=-\frac{4\pi G}{3}(\epsilon+3p)\ ,\quad \left[\frac{\dot a}{a}\right]^2=\frac{8\pi G}{3}\epsilon\ . \tag{3.5}$$

[In "geometrical" language, these are, of course, the Einstein equations for a spatially flat cosmological model: $ds^2=dt^2-a^2(t)(dx^2+dy^2+dz^2)$.] By specifying the relationship between ϵ and p ("the equation of state") one can solve these equations and find a specific function $a(t)$.

IV. SQUEEZED QUANTUM STATES OF RELIC GRAVITONS

The gravitational field (3.1) is the leading term of a more complicated and realistic cosmological gravitational field which also includes the gravity-wave perturbations. Let us write the total field $h_{\mu\nu}$ in the form

$$h_{\mu\nu}=h^{(0)}_{\mu\nu}+h^{(1)}_{\mu\nu}(t,\mathbf{r})\ , \tag{4.1}$$

where $h^{(0)}_{\mu\nu}$ is given by (3.1). The three-dimensional vector $\mathbf{r}$ expresses the coordinates of a point in our Minkowski space-time. By choice of gauge, one may set $h^{(1)\mu\nu}{}_{,\nu}=0$.

The gravitational wave perturbations which we are about to study do not couple to perturbations of ϵ and p. For this reason we set the perturbations of ϵ and p equal to zero and proceed solely with the perturbations $h^{(1)}_{\mu\nu}$. Using an appropriate gauge, one may impose the additional constraints $h^{(1)}_{0\mu}=0$, $h^{(1)}_{\mu\nu}\gamma^{\mu\nu}=0$, so that one is left with only two independent polarization components (designated by $s=1,2$) of $h^{(1)}_{\mu\nu}$. For a wave with the wave vector $\mathbf{n}$ one can write down the nonzero components of the field:

$$h^{(1)}_{ij}(t,x,y,z)=[u^s_n(t)e^{i\mathbf{n}\cdot\mathbf{r}}+u^{s\dagger}_n(t)e^{-i\mathbf{n}\cdot\mathbf{r}}]p^s_{ij}\ ,\qquad i,j=1,2,3, \tag{4.2}$$

where the constant polarization tensors p^s_{ij} satisfy the conditions $p^s_{ik}\gamma^{ik}=0$, $p^s_{ik}n^k=0$ and the dagger indicates the Hermitian conjugate. In this way $h^{(1)}_{ij}$ is a Hermitian operator.

Now one substitutes (4.1) into the field equations (2.10) and linearizes them with respect to $h^{(1)}_{ij}$. It is clear that the left-hand side of (2.10) is simply the usual d'Alembert differential operator applied to $h^{(1)}_{ij}$. The right-hand side of (2.10) contains products of $h^{(0)}_{\mu\nu}$ and $h^{(1)}_{\mu\nu}$ since the nonlinearities are collected on this side. These nonlinear terms govern the interaction of the (linearized) waves with the external gravitational field (3.1).

For a given perturbation with the wave vector $\mathbf{n}$ and for each of the two polarization components, the field equations reduce to a single equation for the time-dependent function $u(t)$ (indices n and s are omitted henceforth for simplicity):

$$\ddot u+n^2u=\left[\frac{n^2(a^2-1)}{a^2}+\frac{\ddot a}{a}+\left[\frac{\dot a}{a}\right]^2\right]u-\frac{\dot a}{a}\dot u\ , \tag{4.3}$$

where $n^2=(n^1)^2+(n^2)^2+(n^3)^2$. If there is no pump field (3.1), i.e., $a(t)=1$, the right-hand side of (4.3) vanishes. The gravitational potentials are dimensionless, and so is the pump parameter $a(t)$. Therefore, the coupling to the gravitational wave perturbations must involve derivatives of a with respect to time without involving any dimensional coupling constants. This is in contrast with electrodynamics in which coupling is introduced by a nonlinear medium, and the coupling involves medium parameters.

It is convenient to introduce a new time coordinate η related to t by $d\eta=a(t)^{-1}dt$, and to denote the η time derivative by a prime. Equation (4.3) becomes especially simple [2]:

$$u''+\left[n^2-\frac{a''}{a}\right]u=0 \tag{4.4}$$

which reduces the problem to a problem of a parametrically excited oscillator.[1] Associated with the equation of motion (4.4) is the Hamiltonian [3]

$$H=n\hat A^\dagger\hat A+\sigma(\eta)\hat A^{\dagger 2}+\sigma^*(\eta)\hat A^2\ , \tag{4.5}$$

where A is the gravitational wave complex amplitude:

$$\hat A=\left[\frac{n}{2}\right]^{1/2}\left[u+\frac{i}{n}\left[u'-\frac{a'}{a}u\right]\right] \tag{4.6}$$

and $\sigma(\eta)$ is the coupling function, $\sigma(\eta)=ia'/2a$. The coupling is provided by the nonlinearity of the gravitational field equations and occurs in vacuum. Note that H belongs to the class of Hamiltonians that characterize a number of physical processes in quantum optics. We needed a nonstandard derivation of the basic equation (4.4) in order to show explicitly how a varying cosmological gravitational field plays the role of a "pump" field that couples to the gravitational wave excitations. As a consequence, the common language of cosmologists ("gravitational wavelength crosses the Hubble radius") can be replaced with a language more understandable to experimenters in optics ("the frequency of the pump is of the order of twice the frequency of the wave"). The Hamiltonian (4.5) is shared by both the cosmological phe-

[1]Electromagnetic waves do not couple to the gravitational field pump $a(t)$ in the same way as gravitational waves. In Maxwell's equations for the same problem, the variable frequency term a''/a does not appear and they reduce to $u''+n^2u=0$.

nomena and quantum optics and thus connects the experiments with squeezed light to processes occurring in the cosmos (see Sec. IX).

Now we will analyze (4.4) in more detail. In some interesting and realistic cosmological situations, e.g., the inflationary Universe, the function a''/a approaches zero asymptotically for $\eta\rightarrow-\infty$ and $\eta\rightarrow+\infty$. In the asymptotic regions $\eta\rightarrow-\infty$ and $\eta\rightarrow+\infty$, solutions to the classical Eq. (4.4) are very simple: $u(\eta)\sim e^{\pm in\eta}$. The general complex solution to (4.4) can be written in the form

$$u(\eta)=a\xi(\eta)+b^{\dagger}\xi^{*}(\eta)\ , \tag{4.7}$$

where $\xi(\eta)$ is a normalized basis function and $\xi^{*}(\eta)$ is its complex conjugate. The same general solution can be decomposed into other basis functions $\chi(\eta)$ and $\chi^{*}(\eta)$:

$$u(\eta)=c\chi(\eta)+d^{\dagger}\chi^{*}(\eta)\ . \tag{4.8}$$

One can choose the basis functions in such a way that

$$\xi(\eta)\rightarrow\frac{1}{\sqrt{2n}}e^{-in\eta}\ \text{ for }\eta\rightarrow-\infty \tag{4.9a}$$

and

$$\chi(\eta)\rightarrow\frac{1}{\sqrt{2n}}e^{-in\eta}\ \text{ for }\eta\rightarrow+\infty\ . \tag{4.9b}$$

Since (4.7) and (4.8) describe the same solution, their coefficients are related:

$$a=vc+wd^{\dagger},\quad b^{\dagger}=w^{*}c+v^{*}d^{\dagger}\ , \tag{4.10}$$

where

$$|v|^2-|w|^2=1\ . \tag{4.11}$$

For quantized fields, the coefficients $a,b^{\dagger},c,d^{\dagger}$ have the meaning of creation and annihilation operators and the relations (4.10) are called Bogoliubov transformations.

The complex numbers v,w can be parametrized by the three real numbers r, θ, and $\varphi(r\geq 0)$:

$$v=e^{-i\theta}\cosh r,\quad w=-e^{i(\theta+2\varphi)}\sinh r\ , \tag{4.12}$$

r, θ, and φ are the usual squeeze parameters (see, for instance, Refs. [17,5,18]). The coordinate vector $\mathbf{r}$ should not be confused with the squeeze parameter r, because the former always appears boldface. It can be shown (see, for example, Ref. [3]) that the transformation (4.10) is always associated with the squeeze operators so that the notion of squeezed quantum states arises inevitably in processes of this kind. One deals with two-mode or one-mode squeezed states depending on the choice of modes.

The modes in (4.2) are traveling waves in the asymptotic limit (4.9). Within the interval of coupling, $-\infty<\eta<+\infty$, the pump field couples this set of waves. The pump field also couples creation operators (with dagger) to annihilation operators (without dagger). In general [see (4.10)] one finds that the evolved operators c,d are linear combinations of the initial creation and annihilation operators of both forward and backward waves.

If one considers classical waves, one can show [2] that a traveling wave will be always amplified and a backward wave will be generated. The amplitude A of the forward wave and the amplitude B of the backward wave obey the relation $|A|^2-|B|^2=1$. Standing waves evolve into standing waves with an amplification or decay factor depending upon their initial phase. After averaging over initial phase, one always obtains amplification. Quantum mechanically, one deals with the process of particle pair creation. The mean number of particles $\langle N\rangle$, equal to zero initially, goes to $\langle N\rangle=\sinh^2 r\gg 1$ finally.

The time-dependent parameters $r(\eta),\theta(\eta),\varphi(\eta)$ obey the differential equations

$$r'=-\frac{a'}{a}\cos 2\varphi\ , \tag{4.13}$$

$$\theta'=n-\frac{a'}{a}\sin 2\varphi\ \tanh r\ , \tag{4.14}$$

$$\varphi'=-n+\frac{a'}{a}\sin 2\varphi\ \coth 2r\ , \tag{4.15}$$

that link them to the pump $a(t)$. These equations allow one to find definite values of the squeeze parameters for a given cosmological model, i.e., for a given pump field $a(t)$.

It can be shown [3] that in the case of inflationary cosmological models, the squeeze parameter r is very large, it ranges from $r\approx 1$ to $r\approx 120$ for relic gravitational waves of different frequencies. In the limit of large r, the Gaussian distribution for the phase ϕ is very narrow, like a δ function. It is concentrated near the values

$$\phi=\varphi_0+\pi l\ , \tag{4.16}$$

where φ_0 is a constant, the same for all unit vectors $\mathbf{n}/n$, and $l=0,\pm,\ldots$.

The negligibly small variance of the phase distribution leads to an important conclusion: the amplification of zero-point quantum fluctuations results in the production of standing waves. Indeed, let us consider a given $\mathbf{n}$. The terms, contributing to the total resulting wave field $h(\eta,x,y,z)$, can be written in the general form [4]

$$h_{\mathbf{n}}=A_1\sin(-n\eta+\phi_1)\cos\mathbf{n}\cdot\mathbf{r}+A_2\sin(-n\eta+\phi_2)\sin\mathbf{n}\cdot\mathbf{r}\ . \tag{4.17}$$

The amplitudes A_1 and A_2 form a broad Gaussian distribution and are, in general, different. (The rms values of A_1 and A_2 are proportional to $\sinh r$.) However, the phases ϕ_1 and ϕ_2 form a very narrow Gaussian distribution and are essentially fixed and equal, or differ by $\pm\pi$. Therefore, expression (4.17) can be written as a product of a function of time and a function of spatial coordinates:

$$h_n=\pm\sin(-n\eta+\varphi_0)(A_1\cos\mathbf{n}\cdot\mathbf{r}\pm A_2\sin\mathbf{n}\cdot\mathbf{r})\ . \tag{4.18}$$

Expression (4.18) describes a standing wave. A characteristic feature of a standing wave pattern is that every field component h_n vanishes over all space at every half period. The randomness of the wave field is displayed in its spatial functions $A_1\cos\mathbf{n}\cdot\mathbf{r}\pm A_2\sin\mathbf{n}\cdot\mathbf{r}$. The total field $h(\eta,\mathbf{r})$ is obtained by summing over all $\mathbf{n}$-mode contributions (4.18). This is why we say that the relic gravitational waves are now present in the cosmos as a stochastic ensemble of standing waves.

V. AN ELECTROMAGNETIC MODEL

In the preceding sections we developed the equations for the parametric coupling of gravitational waves, starting from the basic nonlinearity of the equations of general relativity. We shall now develop an analogous equation for the electromagnetic field, starting with a model that is in one-to-one correspondence with the gravitational model of a uniform expanding universe. Consider an infinite space containing a dielectric medium that varies in time as

$$\epsilon(t)=\langle\epsilon\rangle+\Delta\epsilon(t)\ . \tag{5.1}$$

For convenience one may assume at the outset that the variation of the dielectric constant is independently controlled by, e.g., Maxwell demons who modulate the "spring constant" of the springs tying negative charges to positive charges. The electromagnetic field in the medium obeys the equations

$$\nabla\times\mathbf{E}=-\mu_0\frac{\partial\mathbf{H}}{\partial t} \tag{5.2}$$

and

$$\nabla\times\mathbf{H}=\epsilon\frac{\partial\mathbf{E}}{\partial t} \tag{5.3}$$

which can be combined into

$$\nabla\times(\nabla\times\mathbf{E})=\nabla(\nabla\cdot\mathbf{E})-\nabla^2\mathbf{E}$$
$$=-\mu_0\frac{\partial}{\partial t}\left[\epsilon\frac{\partial\mathbf{E}}{\partial t}\right]\ . \tag{5.4}$$

If the field is purely transverse, its divergence vanishes (which can be adopted as a gauge condition) and one obtains the wave equation

$$\nabla^2 E=\mu_0\frac{\partial}{\partial t}\left[\epsilon\frac{\partial E}{\partial t}\right]\ . \tag{5.5}$$

We decompose the field into the modes with spatial dependence $\cos(\mathbf{k}\cdot\mathbf{r})$and $\sin(\mathbf{k}\cdot\mathbf{r})$ where

$$(k^1)^2+(k^2)^2+(k^3)^2=k^2 \tag{5.6}$$

and substitute Eq. (5.1) for ϵ into (5.5).

The wave equation reduces to

$$\frac{\partial}{\partial t}\left[\frac{\epsilon}{\langle\epsilon\rangle}\frac{\partial E}{\partial t}\right]+(kc)^2E=0 \tag{5.7}$$

with $c^2=1/\mu_0\langle\epsilon\rangle$. If we now introduce the new variable

$$u\equiv\frac{\epsilon}{\langle\epsilon\rangle}\frac{\partial E}{\partial t} \tag{5.8}$$

we obtain, after taking the second time derivative,

$$\ddot{u}+(kc)^2\frac{\langle\epsilon\rangle}{\epsilon(t)}u=0\ . \tag{5.9}$$

This equation is analogous to the equation for gravitons, except that now it is not even a linearized approximation of the equation, but a description of the evolution of the total field. Of course, the field can be quantized and then (5.9) describes a generation of photons from vacuum fluctuations by parametric pumping of the dielectric constant. In agreement with the cosmological model, the field generated from vacuum forms independent standing waves.

VI. RADIATION FROM A NONLINEAR SHEET

We shall now address a more realistic model in which the modulation of the dielectric is not by Maxwell demons, but by virtue of the fact that the dielectric is nonlinear. We shall concentrate on the production of waves by a thin nonlinear dielectric sheet placed at $z=$const. The sheet exhibits the Kerr effect

$$\Delta\epsilon=\epsilon_0\chi^{(3)}E^2 \tag{6.1}$$

over a distance $\Delta z'$, driven by an electric field $E(z,t)$:

$$E(z,t)=\underline{E}e^{-i\omega t}+\underline{E}^*e^{i\omega t}\ , \tag{6.2}$$

where $\underline{E}$ is the complex amplitude. We consider again the wave equation (5.5). One can rearrange it so that the dielectric modulation appears on the right-hand side as a source term:

$$-\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}+\frac{\partial^2}{\partial z^2}E=\frac{1}{c^2}\frac{\partial}{\partial t}\left[\frac{\Delta\epsilon}{\langle\epsilon\rangle}\frac{\partial E}{\partial t}\right]\ . \tag{6.3}$$

An incident field of the form (6.2) induces polarization currents in the dielectric that radiate to the left and the right. If the sheet of thickness $\Delta z'$ is differential, and the polarization current density is finite, the scattered field E_s is small and its reaction on the sheet can be ignored. The source term on the right-hand side of (6.3) in the range $z',z'+\Delta z'$ evaluates to

$$\frac{1}{c^2}\frac{\partial}{\partial t}\left[\frac{\Delta\epsilon}{\langle\epsilon\rangle}\frac{\partial E}{\partial t}\right]$$
$$=-\frac{\omega^2}{c^2}\frac{\epsilon_0}{\langle\epsilon\rangle}\chi^{(3)}[3\underline{E}^3e^{-3i\omega t}+\underline{E}^2\underline{E}^*e^{-i\omega t}+\text{c.c.}]\ . \tag{6.4}$$

One treats the sheet as a delta function discontinuity that causes a jump of $\partial E_s/\partial z$. The waves that travel off to the left and right also obey the wave equation. $\partial E_s/\partial z$ is equal to $-ikE_s$ on the left, ikE_s on the right. Connecting the scattered fields on the two sides by integrating (6.3), one finds

$$E_s=i\frac{\omega}{2c}\frac{\epsilon_0}{\langle\epsilon\rangle}\chi^{(3)}\Delta z'[3\underline{E}^3e^{-3i\omega t}+\underline{E}^2\underline{E}^*e^{-i\omega t}]+\text{c.c.} \tag{6.5}$$

The scattered field consists of a component at the frequency ω, and one at 3ω. If the sheet wave were replaced by one "pumped" by Maxwell demons at 2ω, one would have obtained essentially the same result. The localized pump now produces pairs of traveling waves, not standing waves.

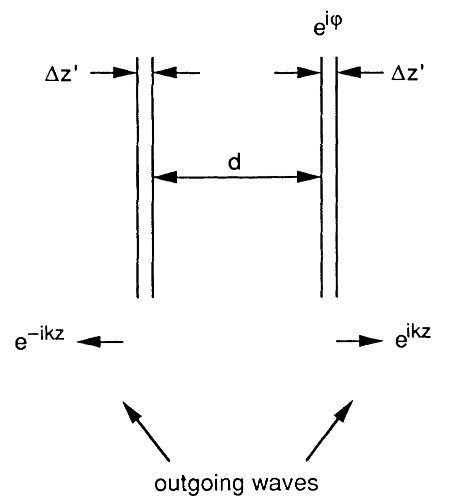

FIG. 1. Two Kerr sheets, spaced a distance d apart.

If we have two sheets of dielectric, the second sheet producing an induced current delayed by the phase angle φ, then cancellation of the wave at frequency ω traveling to the left can be achieved (see Fig. 1), if the sheet spacing d is adjusted to be equal to

$$\varphi - kd = 0 \ , \quad \varphi + kd = \pi \ . \tag{6.6}$$

Similarly, the backward traveling wave at frequency zero can be canceled. Thus, properly spaced sheets may generate a wave traveling in one direction. This is the case realized in the experiment with the moving index grating (see below). In other words, the violation of the properties of homogeneity and isotropy, which were characteristic for the cosmological problem, make it possible to generate traveling waves in the electromagnetic case.

VII. TRAVELING WAVE EXCITATION

If the Kerr effect is small, as is usually the case, it produces a small change per wavelength. In evaluating the source term, one may use the form of the field $E(z,t)$ that is proportional to $\exp[-i(\omega t - kz)]$ plus its complex conjugate

$$E(z,t) = \underline{E}(z) e^{-i(\omega t - kz)} + \underline{E}^*(z) e^{i(\omega t - kz)} \tag{7.1}$$

modified by a function $\underline{E}(z)$ that is slowly varying with distance. Furthermore, all Kerr media are dispersive, so that the wave at 3ω with propagation constant $k(3\omega)$ drops quickly out of synchronism with the source that has the spatial dependence of $\exp[3ik(\omega)z]$. Hence one may ignore the excitation of the third harmonic and only retain the part of the source that drives the fundamental. This phenomenon is known in nonlinear optics as "phase matching" or the lack thereof.

One may make the "slowly varying envelope approximation" by setting

$$\frac{\partial^2}{\partial z^2} E \simeq \left\{ 2ik \frac{\partial \underline{E}}{\partial z} - k^2 \underline{E} \right\} e^{-i(\omega t - kz)} + \text{c.c.} \tag{7.2}$$

When this approximation is entered into (6.3) we find

$$-2ik \frac{\partial \underline{E}}{\partial z} = \frac{\omega^2}{c^2} \frac{\epsilon_0}{\langle \epsilon \rangle} \chi^{(3)} \underline{E}^2 \underline{E}^* \ . \tag{7.3}$$

The field can be quantized by introducing the operator variable $\hat{U}$ with expectation value proportional to $\underline{E}$ so normalized that $\hat{U}^\dagger \hat{U}$ is the photon flux. The following commutation relation is obeyed:

$$[\hat{U}(z,t), \hat{U}^\dagger(z',t)] = \delta(z - z') \ . \tag{7.4}$$

The interaction Hamiltonian associated with this third order nonlinearity is

$$H' = \frac{\kappa}{2} \hat{U}^\dagger \hat{U}^\dagger \hat{U} \hat{U} \ , \tag{7.5}$$

where the coupling parameter κ [6] is proportional to $\chi^{(3)}$. The above Hamiltonian leads to the equation of motion

$$\frac{\partial}{\partial z} \hat{U} = i\kappa \hat{U}^\dagger \hat{U} \hat{U} \ , \tag{7.6}$$

the meaning of $\kappa \langle \hat{U}^\dagger \hat{U} \rangle$ being the Kerr phase shift per unit length. Because $\hat{U}^\dagger \hat{U}$ is an invariant of (7.6), the full nonlinear operator equation can be integrated to give

$$\hat{U}(z,t) = e^{i\kappa z \hat{U}^\dagger(0,t) \hat{U}(0,t)} \hat{U}(0,t) \ . \tag{7.7}$$

One can now introduce a linearization by setting

$$\hat{U}(z,t) = U_0 + \hat{A}(z,t) \ , \tag{7.8}$$

where the operator properties of $\hat{U}$ are now carried by $\hat{A}$, which obeys the same commutation relations as $\hat{U}$. After elimination of nonresonant terms, the Hamiltonian governing the evolution of $\hat{A}$ becomes

$$H' = \kappa [2|U_0|^2 A^\dagger A + \tfrac{1}{2} U_0^{*2} \hat{A}^2 + \tfrac{1}{2} U_0^2 A^{\dagger 2}] \tag{7.9}$$

which compares closely with (4.5), the Hamiltonian derived from a linearized version of the gravitational field equations. Ignoring second-order terms in $\hat{A}$, one may linearize the solution obtaining

$$\hat{A}(z,t) = v \hat{A}(0,t) + w \hat{A}^\dagger(0,t) \ , \tag{7.10}$$

where v and w are given by (4.12) with

$$\theta = -\arctan \Phi \ ,$$

$$\varphi = \frac{\pi}{4} - \theta \ ,$$

$$r = \ln[\Phi + \sqrt{1 + \Phi^2}] \ ,$$

and Φ is defined as the pump induced phase shift within the distance z:

$$\Phi \equiv \kappa |A_0|^2 z \ .$$

VIII. THE SEPARATION OF THE PUMP RADIATION

The generation of photons from zero-point fluctuations, the interpretation of the solution to (7.6) and (7.8), can also be viewed as a diversion of photons of the laser pump in a nonlinear optical medium. In experiments in which the pump is not separated from this radiation, this

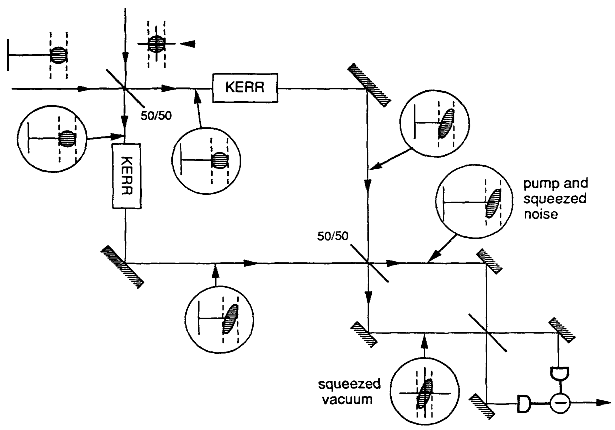

FIG. 2. The progress of coherent state inputs through squeezer to balanced detector.

interpretation is the more natural one. However, it is possible to separate the pump from this radiation in a Mach-Zehnder configuration as shown in Fig. 2 (after Ref. [19]). The pump radiation emerges in one output port of the interferometer, the photons generated by the zero-point fluctuations interacting with the index grating appear in the other output port (see Fig. 2). Since the radiation emerging from the port (squeezed vacuum) is fully described by (7.6) and (7.8) we may interpret the experiment as generating radiation from zero-point fluctuations interacting with a moving index grating. (This is an effect analogous to the particle production by moving boundaries, reflectors, mirrors, etc. A clear physical picture of the effects of this kind is given in Unruh's paper [20].)

The actual experimental arrangement is shown in Fig. 3. If pulses are used, the balanced nonlinear Mach-Zehnder interferometer can be replaced by a fiber ring with which it becomes generically equivalent. The two beam splitters of the Mach-Zehnder are replaced by one 50/50 fiber coupler which is entered by the pulse and exited after the two portions of the pulse have traveled through the fiber ring. If the pulses are much shorter

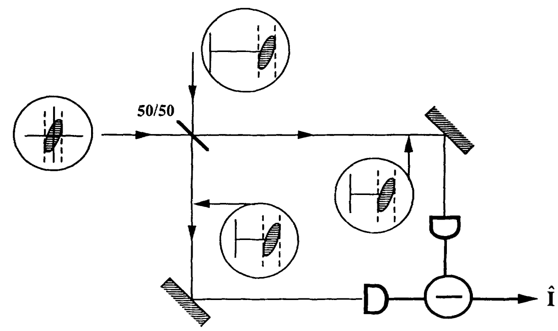

FIG. 4. The squeezed vacuum and the local oscillator input to balanced detector.

than the fiber length (30 mm versus 50 m) one may ignore the brief interaction of the pulses when they meet halfway around the loop. The remaining components used in the setup have been discussed in Ref. [12].

The simplest way to confirm the generation of squeezed vacuum radiation would be to place a detector at the "vacuum port" of the fiber ring interferometer. This is not a practical solution, however, because it is impossible to balance the fiber coupler so perfectly that no "pump" photons would exit through the vacuum port. These pump photons would be mistaken for squeezed vacuum photons which may even be swamped out by the pump photons. There is a better way, however, which utilizes the fact that the squeezed vacuum radiation has a very definite phase relation with respect to laser photons that generated the grating. This is done in the experiment by homodyning the squeezed vacuum radiation with the pump radiation "emerging" from the ring after one roundtrip. We show in Fig. 4 how the local oscillator pump and the vacuum mix in the homodyne detection. If the pump by itself is detected by the balanced detector (with the vacuum port blocked) the balanced detector output is pure shot noise. This follows from the theory of Yuen and Shapiro [21] which shows that in a homodyne detection with a balanced receiver the shot noise must be

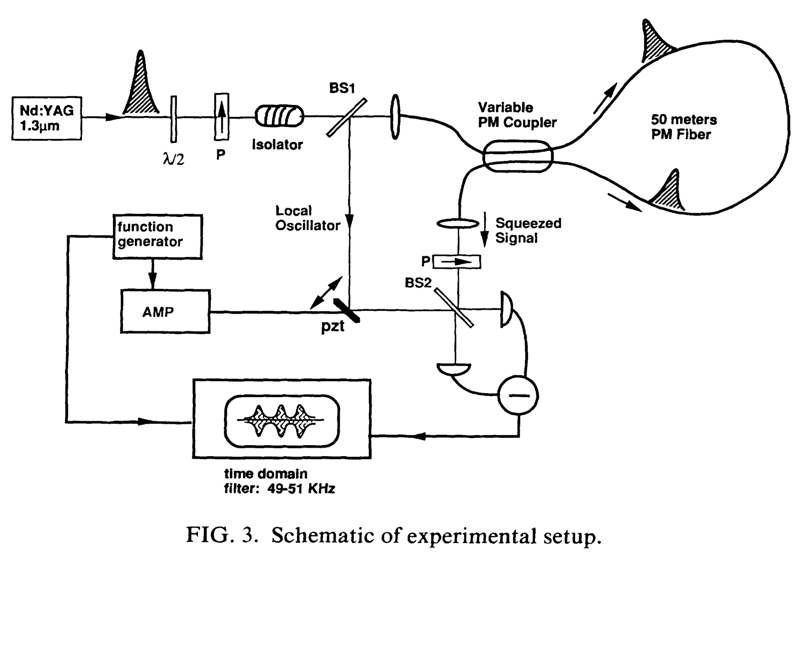

FIG. 3. Schematic of experimental setup.

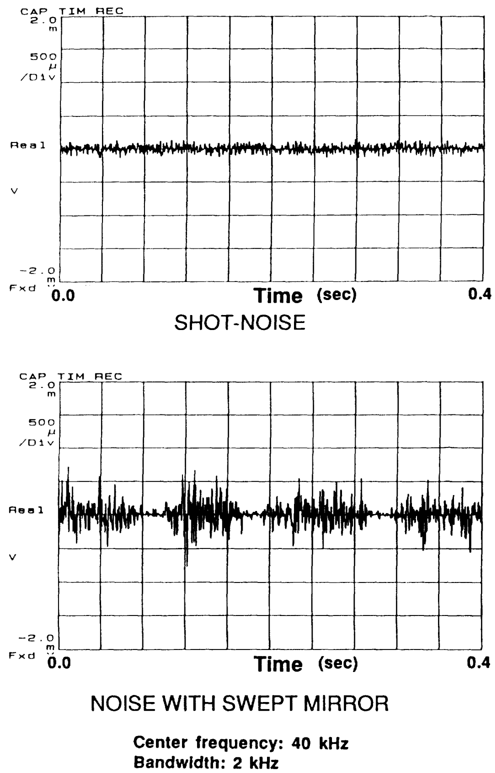

FIG. 5. Time domain observation of noise reduction in the difference current below the shot noise level; (a) shot noise level, (b) squeezing measurement.

interpreted as caused by the zero-point fluctuations entering through the signal port (the vacuum port in Fig. 4). When the squeezed vacuum radiation impinges upon the vacuum port of the balanced detector, the detector shows less than shot noise when the relative phases of the two fields are properly adjusted. If the phase is modulated faster than the drift, the difference current exhibits below shot noise and above shot noise behavior as a function of time. This is what was observed experimentally. Figure 5 shows a time domain capture of the difference current shot noise [part (a)], taken with the squeezed vacuum arm blocked. In Fig. 5(b) we show the difference current noise with the squeezed vacuum port unblocked. At the correct relative phase between the local oscillator and squeezed vacuum signal, the noise is periodically reduced below the shot noise level. The data were taken with a 2-kHz filter centered about 40 kHz. With more precise power spectrum measurements, noise suppression of 5 dB below the shot noise level was confirmed at the highest pump peak power level. One can see that the greater grating strengths give larger reduction below, and larger increases above, the shot noise level. This is as predicted by theory.

IX. DISCUSSION

We have demonstrated the analogy of the linearized equations for light propagation in fibers with the linearized equations for gravitons in the presence of an external gravitational field. In the case of gravitons, we have taken into account the nonlinear terms of the fundamental gravitational equations, which describe the interaction of the pump gravitational field with gravitational waves, i.e., the terms of order $h^{(0)}h^{(1)}$, but we neglected the nonlinearities of the gravitational waves themselves, i.e., the terms of order $h^{(1)}h^{(1)}$. A possible way of taking these latter nonlinearities into account would be to let them play the role of correcting terms in the equations governing the pump field function $a(t)$, in a similar fashion to the approach suggested in Ref. [22].

We hope that the quantum optics experiments can be of some importance for a better understanding of some subtle predictions of the quantum field theory in presence of gravitational fields. It would be of interest to find and study an experimental optical analog to the spatial autocorrelation functions of relic gravitational waves (and primordial density perturbations). These correlation functions should have very specific statistical properties and play a significant role in the current cosmological research [23]. It is hard to say, at present, how useful the quantum optics laboratory experiments may be for getting insight into the nature of similar cosmological processes. We hope that this paper, which established the analogy, will stimulate optics experiments that are designed to mimic cosmological processes.

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