Stability analysis of a nonscalar curvature singularity

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The behavior of test scalar waves on a dust-filled type-V locally rotationally symmetric spacetime is used to probe the nonscalar curvature singularity present and its associated Cauchy horizon. It is argued that the divergence of the stress-energy scalars for most wave modes makes the nonscalar curvature singularity unstable in general. However, a special subset of modes does not lead to divergence of the stress-energy scalars at the nonscalar singularity. These modes would leave the nonscalar curvature singularity unchanged. Furthermore, examination of the stress-energy tensor in a parallel-propagated orthonormal frame and stress-energy scalars show that the Cauchy horizon is left unchanged.

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I. INTRODUCTION

In this paper we investigate the stability of nonscalar curvature singularities and Cauchy horizons in a class of Bianchi type-V spacetimes, which are dust filled and locally rotationally symmetric. Our technique is to extend to these spacetimes a conjecture we have previously applied only to spacetimes containing quasiregular singularities [1–5].

A great deal of work has been done to understand the nature of singularities in classical general relativity [6]. In this paper we use a classification scheme based on that devised by Ellis and Schmidt [7], who classified singularities in maximal spacetimes into three basic types: quasiregular, nonscalar curvature, and scalar curvature. The mildest singularity is quasiregular, and the strongest is scalar curvature. At a scalar curvature singularity, physical quantities such as energy density and tidal forces diverge for some observers who approach the singularity. For a quasiregular singularity, no observers see physical quantities diverge, even though their world lines end at the singularity in a finite proper time. Here we are interested in nonscalar curvature singularities, at which some observers feel infinite tidal forces, even though for no observers do physical scalars diverge.

The classification scheme can be expressed mathematically. Start with a maximal spacetime with incomplete geodesics. In the scheme a singular point q is a C^k (or C^{k-}) quasiregular singularity ($k \geq 0$) if all components and appropriate derivatives of the Riemann tensor $R_{abcd;e_1 e_2 \cdots e_k}$ evaluated in an orthonormal (ON) frame parallel propagated (PP) along an incomplete geodesic ending at q are C^0 (or C^{0-}). In other words, the Riemann-tensor components and derivatives tend to finite limits (or are bounded) in every PPON frame. On the other hand, a singular point q is a C^k (or C^{k-}) curvature singularity if some components or derivatives are not bounded in this way. If all scalars in g_{ab}, the antisymmetric tensor $\eta_{abcd'}$ and $R_{abcd;e_i \cdots e_k}$, nevertheless tend to a finite limit (or are bounded), the singularity is nonscalar, but if any scalar is unbounded, the point q is a scalar curvature singularity.

In this paper we study dust-filled Bianchi type-V locally rotationally symmetric (LRS) spacetimes. Shepley [8] has shown that these spacetimes have an unusual singularity. King [9] has shown that the singularity is a nonscalar curvature singularity; in fact, he called it an "intermediate singularity," using an earlier classification scheme. In addition to this singularity, the spacetimes possess a Cauchy horizon associated with the nonscalar curvature singularity and, also, a scalar curvature singularity.

We have previously studied the stability of quasiregular singularities in Taub-Newman-Unti-Tamburino-(Taub-NUT-) type cosmologies [1–3] and colliding plane-wave spacetimes [4,5] using test scalar and electromagnetic waves. We conjectured that if one introduces a test field whose stress-energy tensor evaluated in a PPON frame mimics the behavior of the Riemann-tensor components which indicate a particular type of singularity (quasiregular, nonscalar curvature, or scalar curvature), then a complete nonlinear back-reaction calculation would show that this type of singularity actually occurs. For example, if a scalar quality such as $T_{\mu\nu}T^{\mu\nu}$ constructed from a test field's stress-energy tensor diverges as a quasiregular singularity is approached, the conjecture is that a scalar curvature singularity will actually develop if the field is allowed to influence the geometry. Evidence for this conjecture was presented from a few known exact solutions [2,4]. The evidence also showed that most test-field wave modes do in fact mimic scalar curvature singularities, but that very special wave modes can mimic nonscalar or quasiregular singularities. Therefore, if general fields are added to Taub-NUT-type cosmologies or colliding plane-wave spacetimes, one expects that their quasiregular singularities will be converted into scalar curvature singularities.

In this paper we extend our conjecture to include the nonscalar curvature singularity and Cauchy horizon in dust-filled type-V LRS spacetimes by examining the behavior of a test massless minimally coupled scalar field. In 1975, King [9] showed that such a field generally diverges at the nonscalar curvature singularity, but that it converges at the Cauchy horizon. He therefore concluded that the nonscalar curvature singularity is likely to be unstable, but that the Cauchy horizon is likely to be stable. However, it has been shown that in the Reissner-Nordström (RN) solution of a charged black hole, the divergence or nondivergence of the field itself is an unreliable guide to the stability of a Cauchy horizon [10,11]. The Cauchy horizon in this spacetime is the inner horizon of the black hole. Scalar fields arising from finite initial data converge at this horizon, but some of the field's derivatives, and consequently the stress-energy of the field, do *not* converge. The inner horizon of the RN spacetime is therefore likely to be unstable, even though the field itself does not diverge there.

Because of the unreliability of the field test alone, we need to investigate the behavior of the stress-energy tensor. Also, an extension of our previous test of singularities to this spacetime will not only indicate stability or instability, but the type of singularity into which the nonscalar curvature singularities or the Cauchy horizon will be converted. Therefore we conjecture that if a stress-energy scalar diverges at the nonscalar curvature singularity and/or Cauchy horizon, then the nonscalar curvature singularity and/or Cauchy horizon will be turned into a scalar curvature singularity. However, if the scalars are finite at the Cauchy horizon, we must test whether it is turned into a nonscalar curvature singularity by calculating $T_{(\mu\nu)}$ in a PPON frame. If $T_{(\mu\nu)}$ diverges, a nonscalar curvature singularity results. There is no way to convert the singularity and/or Cauchy horizon into a quasiregular singularity, since such singularities are topological in nature.

II. DESCRIPTION OF SPACETIME

Following King [9], we consider dust-filled type-V LRS spacetimes. Using a coordinate system intrinsically defined by matter, the metric is

$$ds^2=-(dt+a\,dz)^2+a^2Z^2dz^2+X^2e^{-2z}(dx^2+dy^2)\ , \tag{1}$$

where both $X=X(t)$ and $Z=Z(t)$ are positive functions and a is a positive constant. Sometimes it is convenient to switch into double-null coordinates u and v:

$$ds^2=-\Delta du\,dv+Q(dx^2+dy^2)\ , \tag{2}$$

where $\Delta=Z^2-1$ and $Q=X^2e^{-2z}$. The null coordinates are defined by

$$u=az+\int^t\frac{dt'}{Z(t')+1}\ ,\quad v=-az+\int^t\frac{dt'}{Z(t')-1}\ . \tag{3}$$

The field equations and their solution are given by Farnsworth [12]. In the (t,x,y,z) metric of Eq. (1), the content of the equations is completely summarized by [9]

$$\rho=\rho_0/X^2Z\ , \tag{4a}$$

$$Z=\frac{1}{m}\left[\dot{X}+\frac{X}{a}\right]\ , \tag{4b}$$

$$\dot{X}^2=m^2+\frac{\rho_0 am}{3X}\ , \tag{4c}$$

where ρ is the matter density, and ρ_0 and m are constants.

$$X(t)=2ma(X_0+X_1t+X_2t^2+\cdots)$$

and

$$Z(t)=1+Z_1t+Z_2t^2+\cdots\ ,$$

Eqs. (4) show that X_0 is the solution of $X_0^2(X_0-1)=\rho_0/24m^2$, so that $X_0>1$; also, $Z_1=-2(X_0-\frac{3}{4})/a$, so that $Z_1<-\frac{1}{2}a$.

A Penrose diagram of the spacetime is given in Fig. 1. The surface $t=0$ corresponds to a nonscalar curvature singularity and its associated Cauchy horizon. The nonscalar curvature singularity occurs as $u\to\infty$, $z\to\infty$, with v bounded, and the Cauchy horizon occurs when $v\to\infty$, with u and z bounded [9]. Matter flows from this spatially homogeneous region across the Cauchy horizon into a stationary inhomogeneous region.

The geodesic equations in (u,v,x,y) coordinates are

$$\ddot{u}+\frac{\Delta_{,u}}{\Delta}\dot{u}^2+\frac{Q_{,v}}{\Delta}(\dot{x}^2+\dot{y}^2)=0\ , \tag{5a}$$

$$\ddot{v}+\frac{\Delta_{,u}}{\Delta}\dot{v}^2+\frac{Q_{,u}}{\Delta}(\dot{x}^2+\dot{y}^2)=0\ , \tag{5b}$$

$$\dot{x}=c_1/Q\ , \tag{5c}$$

$$\dot{y}=c_2/Q\ , \tag{5d}$$

where $\dot{u}=du/ds$, etc. In the special case $c_1=0=c_2$,

$$\dot{u}=\frac{1}{2\Delta}[\beta\pm(\beta^2+4\Delta)^{1/2}]\ , \tag{6a}$$

$$\dot{v}=\frac{1}{2\Delta}[-\beta\pm(\beta^2+4\Delta)^{1/2}]\ , \tag{6b}$$

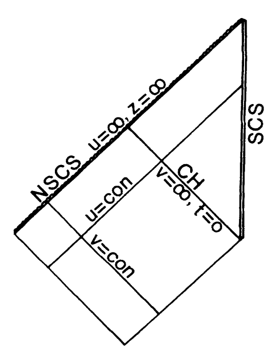

FIG. 1. Penrose diagram of an x=const, y=const two-surface of the type-V LRS spacetime. The Cauchy horizon (CH), nonscalar curvature singularity (NSCS), and scalar curvature singularity (SCS) are shown.

where β is a constant and where the upper sign corresponds to geodesics that approach the nonscalar curvature singularity and the lower sign corresponds to geodesics that approach the Cauchy horizon. On the latter geodesics, observers experience no divergence of tidal forces when they are in the neighborhood of the nonscalar curvature singularity [9].

III. SCALAR WAVES

The massless scalar-wave equation

$$\Box\phi=(-g)^{-1/2}[(-g)^{1/2}g^{ij}\phi_{,j}]_{,i}=0 \tag{7}$$

becomes

$$\frac{2}{\Delta}\phi_{,uv}+\frac{1}{\Delta Q}(Q_{,u}\phi_{,v}+Q_{,v}+Q_{,v}\phi_{,u})-\frac{1}{2Q}(\phi_{,xx}+\phi_{,yy})=0\ , \tag{8}$$

in (u,v,x,y) coordinates. Following King [9], we specify finite initial Cauchy data on a spacelike hypersurface which coincides with a $t=$const surface for $z_1\le z\le z_2$, but crosses the horizon $v=\infty$, as shown in Fig. 2. The development of this Cauchy surface contains both part of the nonscalar curvature singularity and part of the Cauchy horizon.

Solutions of the wave equation take the form

$$\phi=\sum_{b,c}\bar\phi_{bc}(u,v)\sin(bx+x_0)\sin(cy+y_0)+\sum_{b,c}\bar\phi'_{bc}(u,v)\sinh(bx+x_0)\sinh(cy+y_0)\ , \tag{9}$$

where x_0 and y_0 are constants; only the circular functions need be considered if ϕ is initially bounded. Substitution of this form into the wave equation yields

$$\frac{2}{\Delta}\phi_{,uv}+\frac{1}{\Delta Q}(Q_{,u}\phi_{,v}+Q_{,v}\phi_{,u})-\frac{E\phi}{2Q}=0\ , \tag{10}$$

where ϕ is either $\bar\phi_{bc}$ or $\bar\phi'_{bc}$ and E is either (b^2+c^2) for $\bar\phi'_{bc}$ or $-(b^2+c^2)$ for $\bar\phi_{bc}$. With the substitution

$$\phi=Q^{-1/2}\Psi=\frac{e^z}{X}\Psi\ , \tag{11}$$

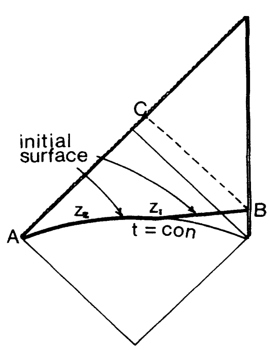

FIG. 2. Initial Cauchy surface. AC,BC forms the boundary of the Cauchy development of the surface AB. Nonzero initial data are restricted to a finite range of z, with $z_1\le z\le z_2$.

Eq. (10) takes on the self-adjoint form

$$\Psi_{,uv}+A(u,v)\Psi=0\ , \tag{12}$$

where

$$A(u,v)=-Q^{-1/2}(Q^{1/2})_{,uv}-\frac{\Delta E}{4Q}\ . \tag{13}$$

In terms of X and Z,

$$A(u,v)=-\frac{\Delta E e^{2z}}{4X^2}-\frac{\Delta}{4XZ^2}\left\{\Delta\ddot{X}+\left[\frac{Z^2+1}{Z}\right]\dot{X}\dot{Z}-\frac{2\dot{X}}{a}+\frac{X\dot{Z}}{aZ}-\frac{X}{a^2}\right\}\ , \tag{14}$$

where overdots represent derivatives with respect to time. Equations (10), (13), and (14) differ somewhat from the comparable results reported by King [9]. As $t\to 0$, that is, as one approaches either the Cauchy horizon $(v\to\infty)$ or the nonscalar curvature singularity $(u\to\infty)$,

$$A(u,v)\to(B_1e^{2u/a}+B_2)e^{-|Z_1|(u+v)}\ , \tag{15}$$

where B_1 and B_2 are constants independent of u and v.

In these asymptotic limits, the wave equation (12) becomes separable, with $\Psi(u,v)=U(u)V(v)$. Let k be the separation constant and define $\alpha=2/a-|Z_1|$. Then

$$\Psi_k(u,v)=\exp\left[\frac{kB_1}{\alpha}e^{\alpha u}-\frac{kB_2}{|Z_1|}e^{-|Z_1|u}+\frac{1}{k|Z_1|}e^{-|Z_1|v}\right]\ , \tag{16a}$$

if $\alpha\neq 0$, and

$$\Psi_k(u,v)=\exp\left[kB_1u-\frac{kB_2}{|Z_1|}e^{-|Z_1|u}+\frac{1}{k|Z_1|}e^{-|Z_1|v}\right]\ , \tag{16b}$$

if $\alpha=0$. Mode solutions of Eq. (7) then become, asymptotically,

$$\phi_{kbc}(u,v)=C_{kbc}e^{u/a}\Psi_k(u,v)\sin(bx+x_0)\times\sin(cy+y_0)\ . \tag{17}$$

At the Cauchy horizon $(v\to\infty)$, ϕ_{kbc} converges. At the nonscalar curvature singularity $(u\to\infty)$, ϕ_{kbc} converges if both $\alpha>0$ and $kB_1<0$, or if both $\alpha=0$ and $(1/a+kB_1)\le 0$; otherwise, ϕ_{kbc} diverges. The field therefore converges at the Cauchy horizon and diverges in general at the nonscalar curvature singularity, in agreement with the results of King [9].

IV. STRESS-ENERGY CALCULATIONS

For a minimally coupled scalar field

$$T_{\mu\nu}=\phi_{,\mu}\phi_{,\nu}-\tfrac{1}{2}g_{\mu\nu}S \tag{18}$$

where

$$S=g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} \; . \tag{19}$$

It is straightforward to show that the scalars $T\equiv T^{\mu}{}_{\mu}=-S$ and $T_{\mu\nu}T^{\mu\nu}=S^2$, where

$$S=-\left[\frac{4}{\Delta}\phi_{,\mu}\phi_{,\nu}-\frac{1}{Q}(\phi_{,x}^2+\phi_{,y}^2)\right] . \tag{20}$$

Thus the divergence or finiteness of the scalars depends solely on the behavior of S.

Asymptotically, the quantities Δ and Q become

$$\Delta=\Delta_0 e^{-|Z_1|(u+v)} , \tag{21a}$$

$$Q=Q_0 e^{-2u/a} , \tag{21b}$$

where Δ_0 and Q_0 are constants. The quantity S_{kbc} derived from the mode ϕ_{kbc} then becomes

$$S_{kbc}=C_{kbc}^2\{D_k f_\alpha(u)\sin^2(bx+x_0)\sin^2(cy+y_0) +D_k' g_\alpha(u)[b^2\cos^2(bx+x_0)\sin^2(cy+y_0)+c^2\sin^2(bx+x_0)\cos^2(cy+y_0)]\}\exp\left[\frac{2}{k|Z_1|}e^{-|Z_1|v}\right] . \tag{22}$$

Here

$$D_k=-\frac{2}{|Z_1||\varepsilon|k}\exp\left[2|Z_1|\int_\varepsilon^{t_0}\frac{Zdt'}{Z^2-1}\right] \tag{23a}$$

and

$$D_k'=\frac{1}{X^2(0)}\exp\left[\frac{2}{a}\int_0^{t_0}\frac{dt'}{Z+1}\right] \tag{23b}$$

are constants, with t_0 arbitrary and $\varepsilon \ll 1$. Also,

$$f_\alpha(u)=\begin{cases}\left[\frac{1}{a}e^{|Z_1|u}+kB_1e^{2u/a}+kB_2\right]\exp\left[\frac{2u}{a}+\frac{2kB_1}{\alpha}e^{\alpha u}-\frac{2kB_2}{|Z_1|}e^{-|Z_1|u}\right] & (\alpha\neq 0) ,\\ \left[\left[\frac{1}{a}+kB_1\right]e^{2u/a}+kB_2\right]\exp\left[\frac{2u}{a}+2kB_1u-\frac{2kB_2}{|Z_1|}e^{-|Z_1|u}\right] & (\alpha=0) ,\end{cases} \tag{24}$$

and

$$g_\alpha(u)=\begin{cases}\exp\left[\frac{4u}{a}+\frac{2kB_1}{\alpha}e^{\alpha u}-\frac{2kB_2}{|Z_1|}e^{-|Z_1|u}\right] & (\alpha\neq 0) ,\\ \exp\left[\frac{4u}{a}+2kB_1u-\frac{2kB_2}{|Z_1|}e^{-|Z_1|u}\right] & (\alpha=0) .\end{cases} \tag{25}$$

At the nonscalar curvature singularity ($u\to\infty$), if $\alpha=0$ the quantity S_{kbc} remains bounded if $(4/a+2kB_1)\le 0$, but diverges if $(4/a+2kB_1)>0$, unless $1/a+kB_1=0$ and $b=c=0$. If $\alpha>0$, S_{kbc} remains bounded if $kB_1<0$, but diverges if $kB_1\ge 0$. If $\alpha<0$, S_{kbc} always diverges. Thus the scalars generally diverge, and according to our conjecture, the nonscalar curvature singularity is transformed in general into a scalar curvature singularity. Only special wave modes leave the singularity unchanged.

At the Cauchy horizon ($v\to\infty$), all S_{kbc} remain bounded, and so, by our conjecture, the horizon is not converted into a scalar curvature singularity. We still need to test, however, whether the horizon turns into a nonscalar curvature singularity. In order to do this, we need to calculatè the stress energy in a PPON frame. It is straightforward to calculate the stress-energy tensor $T_{\alpha\beta}$ in a coordinate frame using Eq. (18). All components are finite at the Cauchy horizon. Using the PPON frame given in the Appendix for x,y constant motion, we can compute $T_{(\mu\nu)}=E^{\alpha}_{(\mu)}E^{\beta}_{(\nu)}T_{\alpha\beta}$ along geodesics through the Cauchy horizon: All components remain finite. Therefore the addition of a minimally coupled scalar field does not turn the Cauchy horizon into a curvature singularity of any type. Since quasiregular singularities are topological, no field perturbation can turn a horizon into a quasiregular singularity. Hence the Cauchy horizon remains a nonsingular Cauchy horizon.

V. CONCLUSIONS

The results of test perturbations are easy to summarize. In general, wave modes are expected to change the nonscalar curvature singularity of the dust-filled type-V LRS spacetime into a scalar curvature singularity, but a small class of wave modes leaves the nonscalar curvature singularity as it was. The Cauchy horizon is left unchanged under arbitrary test perturbations.

When we considered Taub-NUT-type cosmologies [1,2], examination of the behavior of wave modes alone predicted different results than the behavior of $T_{(ab)}$ in a PPON frame and the behavior of stress-energy scalars. In the type-V LRS spacetime, however, examination of $T_{(ab)}$ and stress-energy scalars led to agreement with the conclusions of King [9], who looked at the behavior of the wave amplitudes alone.

To investigate and test our conjecture further, we need to examine other spacetimes with nonscalar curvature singularities and any associated Cauchy horizon. We also need to compare with related exact solutions in which fields have been added. Such work is underway.

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APPENDIX: PARALLEL-PROPAGATED ORTHONORMAL-FRAME VECTORS

From the geodesic equation (7), it is straightforward to derive x,y-constant frame vectors which satisfy the parallel-propagation condition $E^{\mu}_{(0);\nu}E^{\nu}_{(a)}=0$ and the orthogonality condition $E_{(a)\mu}E^{\mu}_{(b)}=\delta_{(ab)}$. The frame vectors are

$$E^{\mu}_{(0)}=\begin{bmatrix}\dfrac{\beta\pm\sqrt{w}}{2\Delta}\\ \dfrac{-\beta\pm\sqrt{w}}{2\Delta}\\ 0\\ 0\end{bmatrix},\quad E^{\mu}_{(1)}=\begin{bmatrix}\dfrac{\beta\pm\sqrt{w}}{2\Delta}\\ \dfrac{\beta\mp\sqrt{w}}{2\Delta}\\ 0\\ 0\end{bmatrix},$$

$$E^{\mu}_{(2)}=\begin{bmatrix}0\\ 0\\ 1/\sqrt{Q}\\ 0\end{bmatrix},\quad E^{\mu}_{(3)}=\begin{bmatrix}0\\ 0\\ 0\\ 1/\sqrt{Q}\end{bmatrix},$$

where $w=\beta^2+4\Delta$, $\Delta=Z^2-1$, and $Q=X^2e^{-2z}$. The upper signs refer to frame vectors carried by geodesics which approach the nonscalar curvature singularity, and the lower signs refer to frame vectors carried by geodesics which approach the Cauchy horizon.

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