Evolution of the density parameter in inflationary cosmology reexamined

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The evolution of the cosmological density parameter is examined with particular reference to the inflationary phase of the early Universe. The standard treatments of the Friedmann-Lemâitre equations as a dynamical system are considerably extended, and are generalized to allow the presence of an arbitrary mixture of perfect fluids with different equations of state. Phase-plane diagrams are constructed which show the evolution of the density parameter with the expansion of the Universe in both inflationary and noninflationary cases. More detailed models are constructed from sequences of simple models, and the relation between the results presented here and those previously obtained is shown explicitly, using analytical and graphical methods. The treatment is then extended to deal with the inflationary case directly, namely, when the energy density is dominated by a scalar inflaton field. It is shown in this case that a number of special regions of parameter space can be projected onto plane systems so as to permit a useful phase-plane representation, and some of these cases are displayed graphically.

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I. INTRODUCTION

A previous paper by two of the present authors [1] was devoted to the problem of understanding how the cosmological density parameter, defined by $\Omega \equiv 8\pi G\rho/3H^2$, where ρ is the total cosmic energy density and H is the Hubble parameter, evolves with the expansion of the Universe. In that study, phase-plane diagrams illustrating the relationship between Ω and S, the cosmic scale factor, were constructed under the approximation that the Universe could be well described as being dominated by a single matter component during each epoch of its evolution.

This assumption makes the drawing of the phase plane easier, albeit at the price of introducing points in the (S,Ω) plane where there are discontinuities in $d\Omega/dS$. Since the matching conditions on all the other physically relevant quantities can be satisfied (for details see Ref. [1]) the qualitative picture produced in the phase planes suffers no serious damage from this procedure. What does cause a problem, however, is the fact that in the process of dividing the evolution of the Universe into arbitrary epochs labeled by the initial and final values of S, an implicit assumption was introduced that the mechanism of inflation produces exactly the same number of e-foldings of expansion in universes having different values of Ω at the start of inflation.

This paper will accordingly devote itself to two aims: the first is to show how to deal with the study of the phase diagrams for the evolution of Ω without making any such restrictive assumption as that described above, while the second is to use the methods and approximations commonly used in the study of inflationary models in order explicitly to discover the dynamical effects of the inflaton field on the behavior of Ω.

It is appropriate to mention the earlier work on this topic here: the problem was first examined thoroughly for a dust-filled universe with a cosmological constant in the paper by Stabell and Refsdal [2]. A different approach to the same problem was adopted by Harrison [4], who utilized the classification due to Robertson [3]. McVittie [5] also briefly discussed universes filled with matter described by a polytropic equation of state (see [6]). Most of the content of these papers is contained in the review by Felten and Isaacman [7]. Szydlowski, Heller, and Golda [8] have examined the homogeneous isotropic cosmological models from the point of view of

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their structural stability, and have also treated the case in which the dynamical system is nonconservative. It should be noted here that the study by Ehlers and Rindler [9] provides an entirely different formulation of the problem of following the phase flow of the Friedmann-Lemâitre universes. That work should be seen as complementary to the present paper. Finally, Lidsey [10] has treated the Ω problem within the context of a specific scalar field model.

The next two sections, on pure fluids and combinations of fluids, respectively, review the work contained in the earlier paper [1]. We make a number of important improvements to the treatment given there, and correct some minor errors. For the sake of comparison with the most significant results of the earlier work just mentioned, we generalize and extend the results of [2] and provide analogous phase portraits to those in that paper for some cases in which the fluid pressure does not vanish.

The equations for the phase portraits of those models in which the inflation is driven by the energy density of a scalar field potential are derived in Sec. IV, without the necessity for any restrictive assumptions about whether the field is rolling slowly or rapidly down the potential. These equations form, in the most general of the cases under consideration, a nonautonomous system of dimension higher than 2, so that the phase plane approach of [1] is here restricted in its usefulness. This forces us to consider alternative methods of approaching the problem, and the latter parts of that section are therefore devoted mainly to developing useful ways of looking at the evolution of the density parameter in these situations. Finally we draw some conclusions and discuss the implications of our results for relativistic cosmological theories.

Some of the possible types of behaviors of the dynamics of these models are illustrated in Sec. IV, with particular emphasis laid on following approximately the evolution of the scalar field and the other dynamical parameters during the epoch immediately preceding the inflation.

II. PURE FLUIDS

A. The phase-plane equations

The analysis presented in this paper will mainly be restricted to the FLRW (Friedmann-Lemâitre-Robertson-Walker) universes with scale factor $S(t)$ and spatial curvature $K = k/S^2$, where the constant k can be normalized to $+1$, 0, or -1 (see, for example, Ref. [6]). Then in terms of the Hubble parameter $H \equiv \dot{S}/S$, the nontrivial relations are the Einstein equations

$$3H^2 = 8\pi G\rho - 3K, \tag{1}$$

$$3\dot{H} + 3H^2 + \tfrac{1}{2}8\pi G(\rho + 3p) = 0, \tag{2}$$

and the energy-conservation equation

$$\dot{\rho} + 3H(\rho + p) = 0, \tag{3}$$

where ρ is the energy density and p is the total pressure. An overdot appearing anywhere denotes a derivative with respect to the cosmic time t. These quantities are taken to be related by an equation of state of the form

$$p = (\gamma - 1)\rho. \tag{4}$$

Note that the relation (4) can be taken as the *definition* of the index γ. The physics of the model is then expressed in the specification of γ, indeed the equations are indeterminate until this is done. In general, γ can vary with time; it will be expressed here in terms of $S(t)$, as $\gamma = \gamma(S)$, because $S(t)$ determines the physical conditions prevailing at each time t. We shall make extensive use of the definition (4) throughout this paper. This specification then enables us to determine the (S, Ω) phase planes of Secs. II and III.

Defining the total density parameter Ω by

$$\Omega = 8\pi G\rho/3H^2 \tag{5}$$

leads, with Eq. (1), to the relation

$$K = (\Omega - 1)H^2 = (\Omega_0 - 1)H_0^2/y^2, \tag{6}$$

where $y \equiv S(t)/S_0$, and H_0, Ω_0, S_0 are the values at the present time $t = t_0$ of H, Ω, and S. This relation shows that $K > 0 (= 0, < 0)$ if $\Omega > 1 (= 1, < 1)$. Alternatively, one can solve for Ω as

$$\Omega = [1 - 3K/8\pi G\rho]^{-1}, \tag{7}$$

$$\Rightarrow \Omega = [1 + 3H_0^2(1 - \Omega_0)/8\pi G\rho y^2]^{-1}\ . \tag{8}$$

Although it provides an exact expression for Ω as a function of time, (8) does so only implicitly via the dependence on time of ρ and S, so in order for it to be useful, the complete solution of the field equations must be available. Of course, most of the questions which are asked within the context of standard cosmology are nontrivial precisely because ignorance of the exact solutions to the resulting equations is the normal state in which the cosmologist is found. It is the main purpose of this section, therefore, to replace the (largely useless) expression (8) by a different relation which will prove to be more useful in showing how the universe evolves. Such an equation can be derived as follows.

Defining the *deceleration parameter* q by

$$q \equiv -\left(\frac{\ddot{S}}{H^2 S}\right), \tag{9}$$

one finds from Eq. (2) that

$$q = -(1 - 3\gamma/2)\Omega, \tag{10}$$

showing that $\gamma = \frac{2}{3}$ is a critical value separating *decelerating* ($q > 0, \gamma > \frac{2}{3}$) from *accelerating* ($q < 0, \gamma < \frac{2}{3}$) periods in the universe.

Following [11] we will say that the universe is *inflationary* when $q < 0$, for this is the essential feature which can enable the horizon to grow to sizes larger than the observable region of our universe. This definition of what is actually meant by inflation is not the only one possible, but is the definition most commonly used in the study of inflationary models. It also has the further virtues of simplicity, generality, and locality, the question of whether a universe model is inflationary at a given time can be answered on the basis of quantities known only at that time. This situation should thus be contrasted with definitions of inflation which require knowledge of the past history of the cosmological model.

The time derivative of Eq. (5), together with Eqs. (1) and (2) shows that

$$d\Omega/dt = (2-3\gamma)H(1-\Omega)\Omega\,. \tag{11}$$

This shows that $d\Omega/dt = 0$ when (i) $H = 0$ [i.e., at a turning point: $\dot{S}(t_c) = 0$, and at all times in a static universe: $S(t)$ =const], (ii) $\Omega = 0$ (an empty universe: $\rho = 0$), (iii) $\gamma = \frac{2}{3}$ (the critical case with $q = 0$), and (iv) $\Omega = 1$ (the case of spatial flatness: $K = 0$).

Finally, the desired phase-plane equation for Ω is obtained by dividing $\dot{S}$ through Eq. (11):

$$d\Omega/dS = [2-3\gamma(S)](1-\Omega)\Omega/S. \tag{12}$$

This leads to the required phase-plane diagrams once $\gamma(S)$ is specified. Nonstatic solutions can be followed through turnaround points where $\dot{S} = 0$ because there $H \to 0, \Omega \to \infty$ like $1/H^2$, and $\dot{\Omega} \to \infty$ as well.

It follows immediately that both $\Omega = 0$ and $\Omega = 1$ are solutions of Eq. (12), no matter what form $\gamma(S)$ takes; on the other hand, if $\gamma(S) = \frac{2}{3}$ for all S, then $\Omega = \Omega_0$ =const is a solution for all values of Ω_0. Furthermore, combining (10) and (12) shows that $d\Omega/dS = -2q(1-\Omega)/S$, so that the signs of $d\Omega/dS$ and q are the same when $\Omega > 1$ and $d\Omega/dS = 0$ when $q = 0$.

Note that in Eq. (12), the scale factor S takes the role of a kind of conformal time variable. This can be seen even more clearly by writing Eq. (12) in terms of the logarithmic scale factor $s \equiv \ln S$:

$$d\Omega/ds = [2-3\gamma(s)](1-\Omega)\Omega. \tag{13}$$

It is worth noticing at this point that our analysis will be based on Eq. (12) which yields the (Ω, S) phase planes. Actually, the full set of dynamical equations would include the $\dot{K}$ equation and either of the $\dot{H}$ or $\dot{\rho}$ equations. The fact is, however, that these two latter equations are both implicitly included in the $\dot{\Omega}$ equation and no essential information comes from the other equation, since $\dot{\Omega}$ does not involve K or ρ explicitly.

B. Epochs of constant adiabatic ratio

Now consider a situation where $\gamma(S)$ is constant for a series of epochs each defined by initial and final values of $S: S_1 < S < S_2$. This corresponds to the universe being dominated by a simple one-component *adiabatic* matter field during that time. Then Eq. (12) holds for each such range of S, with γ constant; differentiating shows that

$$\frac{d^2\Omega}{dS^2} = -(2-3\gamma)\left[1-(2-3\gamma)(1-2\Omega)\right]\frac{(1-\Omega)\Omega}{S^2}\,. \tag{14}$$

Apart from the special cases $\gamma = \frac{2}{3}$, $\Omega = 0$, and $\Omega = 1$, this vanishes when

$$\Omega = \frac{1}{2} + \frac{1}{6(\gamma - 2/3)}. \tag{15}$$

One can obtain the explicit solution to (12) either directly or by integrating (2) to give

$$\rho = \rho_0 y^{-3\gamma}, \quad y \equiv S(t)/S_0, \tag{16}$$

and substituting in (8) to find

$$\Omega = \{1 + [(1-\Omega_0)/\Omega_0]y^{3\gamma-2}\}^{-1}\,. \tag{17}$$

The behavior is quite different, depending on whether $\gamma > \frac{2}{3}$ or $\gamma < \frac{2}{3}$. In the critical case ($\gamma = \frac{2}{3}$), as mentioned above, $\Omega = \Omega_0$ =const is a solution for all values of Ω_0, so the phase curves in the (S, Ω) plane are simply the horizontal lines. This conclusion is completely in accordance with the form of the differential equation (12) and the discussion following it.

C. The standard case: $\gamma > \frac{2}{3}$

In this case $d\Omega/dS > 0$ for $\Omega > 1$ but $d\Omega/dS < 0$ for $\Omega < 1$. Points where $d^2\Omega/dS^2$ changes sign occur for the values of Ω determined by (14); if $\frac{2}{3} < \gamma < 1$ they exist with $\Omega > 1$, but if $\gamma > 1$ they exist with $0 < \Omega < 1$. The shapes of the curves in the latter case are shown in Fig. 1; the basic features are the same when $\frac{2}{3} < \gamma \leq 1$. All the curves diverge from the fixed boundary point $\{S = 0, \Omega = 1\}$, which is a state resembling an Einstein–de Sitter universe. Those for $\Omega > 1$ increase monotonically, diverging to infinity at a finite value of S [found by setting the denominator to zero in Eq. (17)]; those for $\Omega < 1$ decrease monotonically toward zero. The line $\Omega = 1$ is the separatrix between these two types of behavior. It is useful to bring the infinities of both S and Ω to finite values by suitable transformations $s = f(S)$, $\omega = g(\Omega)$, for example, $s = \arctan(\log S), \omega = \arctan(\log \Omega)$. Then the value $\Omega = \infty$ becomes a boundary between the expanding and collapsing regions of the universe model, and the total phase-plane diagram is as in Fig. 2, the bottom half cor-

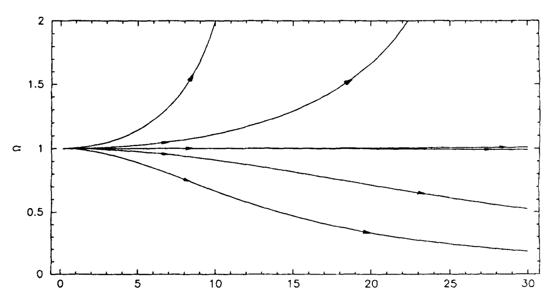

FIG. 1. Phase portraits for $\gamma > \frac{2}{3}$. Curves showing the evolution of Ω with S. The line $\Omega = 1$ is an unstable asymptote, the line $\Omega = 0$ is a stable asymptote for all curves lying below $\Omega = 1$, and the curves above $\Omega = 1$ all become vertical for finite S.

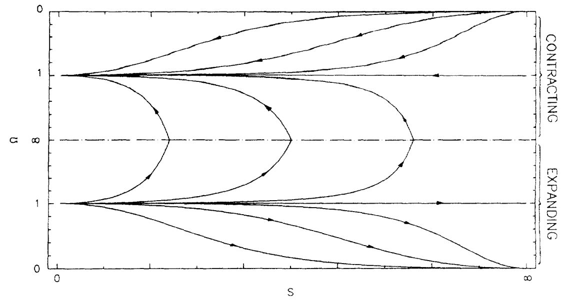

FIG. 2. Phase portraits for $\gamma > \frac{2}{3}$. The completion of Fig. 1 to include the points at infinity and the curves corresponding to contracting universes. The points at the top and bottom right are Milne universes, the dotted line corresponds to a set of states of instantaneous time symmetry, and the lines $\Omega = 1$ are a set of Einstein–de Sitter universes.

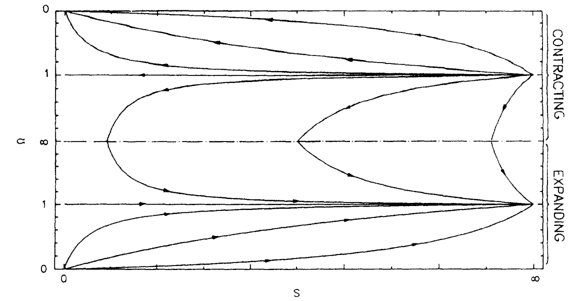

FIG. 4. Phase portraits for $\gamma < \frac{2}{3}$. The completion of Fig. 3 to include the points at infinity and curves corresponding to contracting universes. The points at the top and bottom left are Milne universes, the dotted line corresponds to a set of states of instantaneous time symmetry, and the lines $\Omega = 1$ are a set of Einstein–de Sitter universes.

responding to expanding universes and the top to collapsing universes with the models for $\Omega > 1$ making the transition from expansion to collapse. The time symmetry of the Friedmann equation (1) leads to a symmetry between the top and bottom halves of the diagram. The asymptotic states at the top and bottom right-hand corners are the collapsing and expanding Milne universes, respectively, while the singular source and sink points on the left-hand side are initial and final Einstein–de Sitter universe singularities. Particular cases of importance are pressureless matter ($\gamma = 1$) and pure radiation ($\gamma = \frac{4}{3}$).

D. The inflationary case: $0 \leq \gamma < \frac{2}{3}$

In this case $d\Omega/dS < 0$ for $\Omega > 1$ but $d\Omega/dS > 0$ for $\Omega < 1$. If $0 \leq \gamma \leq \frac{1}{3}$, points where $d^2\Omega/dS^2$ changes sign occur for $0 < \Omega \leq \frac{1}{4}$, but there are no such points if $\frac{1}{3} \leq \gamma < \frac{2}{3}$. The shapes of the curves in the former case are shown in Fig. 3; the basic features are the same in both cases (and also if $\gamma < 0$). All the curves converge toward the separatrix at $\Omega = 1$, those for $\Omega < 1$ diverging from the fixed boundary point $\{S = 0, \Omega = 0\}$ and monotonically increasing toward 1, while those for $\Omega > 1$

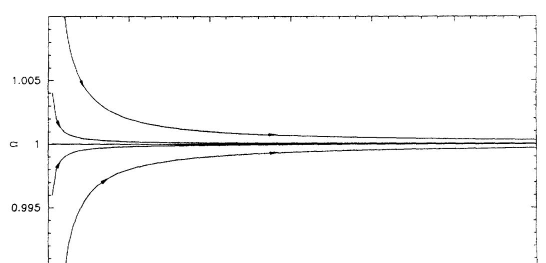

FIG. 3. Phase portraits for $\gamma < \frac{2}{3}$. Curves showing the evolution of Ω with S. The line $\Omega = 1$ is a stable asymptote, the line $\Omega = 0$ is an unstable asymptote for all curves below $\Omega = 1$, and the curves above $\Omega = 1$ approach $\Omega = 1$ asymptotically. The further the value of γ is from $\frac{2}{3}$, the sooner a given curve approaches $\Omega = 1$.

monotonically decrease toward 1. The curve $\Omega = 1$ is a separatrix between these behaviors. It is again useful to bring the infinities of both S and Ω to finite values by suitable transformations $s = f(S), \omega = g(\Omega)$, as in the previous section. The total phase-plane diagram is as in Fig. 3, the bottom half corresponding to expanding universes and the top to collapsing universes, with the models for $\Omega > 1$ this time making the transition from collapse to expansion. The asymptotic states at the right-hand edges are the collapsing and expanding de Sitter universes, respectively, while the singular source and sink points on the left-hand corners are "big-bang" models with $q > 0$ at the origin; the horizontal lines at $\Omega = 1$ correspond to models that start or end asymptotically at a state describing the Einstein–de Sitter universe. A particular case of importance is that of exponential expansion: $\gamma = 0$. In this case there is no longer an initial singularity, because an infinite proper time is required to reach the edge $S = 0$. Thus these steady-state universes expand forever without S ever being zero. Apart from the symmetry between the top and bottom halves of each of Figs. 2 and 4, another symmetry is apparent: the two figures look much like time reverses of each other. In fact, this is made explicit by the realization that Eq. (17) is invariant under the transformation $y \to 1/y, 2 - 3\gamma \to 3\gamma - 2$. Thus, in particular, the cases of radiation and a cosmological constant are exact inverses of each other in terms of the time measured by the universe's expansion.

E. A 4-epoch inflationary universe

It is not realistic to assume that γ is constant throughout the history of the universe, but one can obtain a reasonably realistic model if one assumes that the universe evolves through epochs during each of which γ is constant. Consider a universe which is radiation dominated ($\gamma = \frac{4}{3}$) for an initial epoch I, from the origin of the universe at $S = 0$ until the start of inflation at $S = S_i$, then undergoes exponential expansion (with $\gamma = 0$) during the inflationary epoch II, from S_i to S_f; undergoes radiation-

TABLE I. Equations of state for the 4-epoch inflationary model. The corresponding evolution curves are shown in Fig. 6.

Epoch	γ	Scale factor
I	$\frac{4}{3}$	$0 < S < S_i$
II	0	$S_i < S < S_f$
III	$\frac{4}{3}$	$S_f < S < S_c$
IV	1	$S_c < S$

dominated expansion after inflation for a third epoch III, from S_f to S_c; and finally enters a matter-dominated epoch IV, from S_c to the present (when $S = S_0$) and beyond. Thus one solves Eq. (12) for $\Omega(S)$ to obtain the 4-epoch solution in the form of Eq. (17) with the effective equation of state for each epoch as given in Table I. In effect, the universe moves on curves as those in Fig. 1 for epoch I, as those in Fig. 3 in epoch II, and then as those in Fig. 1 again in epochs III and IV. At each interface between epochs, the junction conditions of general relativity require that the scale factor S and its first derivative $\dot{S}$ be continuous [13]. From Eqs. (1) and (5), this means that Ω must be continuous there also, so that the joining requirement is simply that the curves representing the evolution of the universe in the (S, Ω) plane are continuous. However, the tangent vectors to the curves will not be continuous, since the equation of state of the matter is discontinuous there. The resulting phase plane is shown in Fig. 5. Its general form is not dependent on the details of the inflation chosen; as long as $q < 0$ in that epoch, the same general form will result. The basic feature is that Ω diverges away from 1 in the initial expansion of the universe: the inflationary epoch brings it back to very near unity,[1] but after inflation it again diverges away from 1, either to $+\infty$ (and recollapse) if $\Omega > 1$, or to zero if $\Omega < 1$. Qualitatively new behavior occurs: oscillating universe models are possible, because of the changes of the equation of state, that are never singular. To see what is happening, one should again bring the points at infinity on the (S, Ω) plane to finite values by transformations $s = f(S), \omega = g(\Omega)$ as before. The total phase-plane diagram can then be drawn as is shown in Fig. 6. As usual, the bottom half corresponds to expanding and the top half to collapsing universes. There are two new critical points at $\Omega = \infty$, with the corresponding separatrices as shown. Motion of the evolution curves at the new critical points is undefined because of the change in the equation of state that occurs there.[2]

[1]Notice that here, in contrast with many treatments, the effective energy density of the cosmological constant is *included* in Ω. This is done by representing Λ as a fluid with $p = -\rho$; $\rho = \Lambda$ ($\Longleftrightarrow \gamma = 0$). The contribution of Λ to Ω is then given by Eq. (5).

[2]Note that the left-hand critical point *can* be reached in a finite time from the origin of the universe.

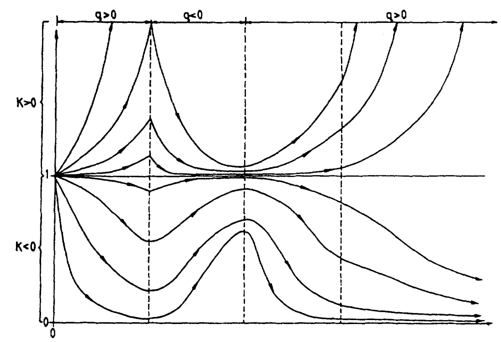

FIG. 5. Phase portraits for the 4-epoch inflationary models of Sec. II A. The evolution of Ω with S. Here q is the deceleration parameter. From left to right, the dotted vertical lines divide the regions of the initial radiation epoch, the inflationary epoch, the postreheating radiation epoch, and the postdecoupling matter-dominated epoch, respectively.

Models with $k = +1$ expand from an initial singularity, reach a maximum, and then recollapse to a final singularity circle around the separatrix, starting at the left-hand edge where $\Omega = 1$ and eventually ending on the left-hand edge again. Some models with $k = +1$ recollapse before inflation ever takes place: these are the ones lying to the left of the separatrices. Models with $k = -1$ have oscillating values of Ω, which starts at 1 and ends at 0. The oscillating nonsingular universes (with $k = +1$) will occur only if the inflationary mechanism is time symmetric, so that if an expanding universe inflates from S_i to S_f then a collapsing universe will "deflate" in the corresponding way for the same values of S. This may not be true for specific inflationary mechanisms [14]; then a single equation of state $\gamma = \gamma(S)$ cannot describe both the expansion and the collapse, so that none of the phase curves in Fig. 6 corresponds to such a case.

III. COMBINATIONS OF FLUIDS

We now consider the situation where the universe evolves through epochs when the energy-momentum content is the sum of simple fluids (each one characterized by constant γ). Equations (1)–(4) are therefore still valid, but now ρ and p are each the total energy resulting from

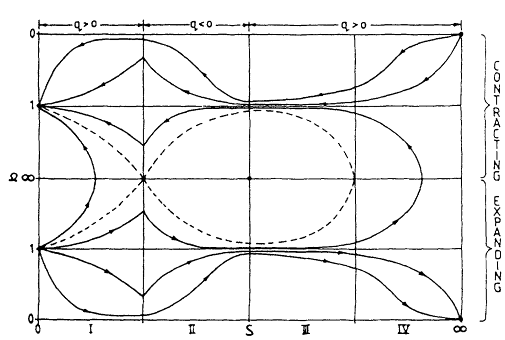

FIG. 6. Phase portraits for the 4-epoch inflationary models of Sec. II A. The completion of Fig. 5 to include the points at infinity and curves corresponding to contracting universes.

summing the contributions of the N components:

$$\rho = \sum_{i=1}^{N} \rho_i \,, \tag{18}$$

$$p = \sum_{i=1}^{N} p_i, \tag{19}$$

where the ith energy density ρ_i and pressure p_i are related by the usual equation of state

$$p_i = (\gamma_i - 1)\rho_i \,, \tag{20}$$

where the γ_i are, in the most general situation, not constants. Also, the contracted Bianchi identities yield the conservation equations

$$\sum_{i=1}^{N} \{\dot{\rho}_i + 3H(\rho_i + p_i)\} = 0 \,. \tag{21}$$

Now, comparing with the case of a single perfect fluid, we realize that the necessary assumption of one equation of state is no longer enough to render the system of field equations solvable for all the variables. In fact, if we want to trace the evolution of all the matter components, a number of assumptions is required. Actually, we would need N assumptions: one for each matter mode, motivated by the physics of the problem under consideration.

In this section we will deal with the simplest of these possible assumptions, namely, that the various fluid components are both noninteracting and adiabatic.

A. Many-component noninteracting fluids

Assuming the components are noninteracting, each one separately obeys the conservation equation (3):

$$\dot{\rho}_i + 3H(\rho_i + p_i) = 0 \,. \tag{22}$$

This should be at least a good approximation to the situation where there are a number of types of radiation and pressureless matter together with a cosmological constant for example, since these components will not exchange energy density at any noticeable rate. Summing (20) over all components and using the definition (1), the effective total index γ relating the total energy to the total pressure is given by

$$\gamma = \rho^{-1} \sum_i \gamma_i \rho_i \,. \tag{23}$$

The nonadiabatic character of the present configuration must be stressed here, given the time dependence of this γ index.

Remember that we normally require the sound speed in a fluid to be less than the speed of light, which constrains the γ_i to be less than 2. It is easy to see from the form of (23) that if these constraints are satisfied by the individual γ_i, then we also have $\gamma < 2$ in accordance with the causality requirement. Also, if the density parameter of the ith matter component is defined as

$$\Omega_i \equiv 8\pi G \rho_i / 3H^2 \,, \tag{24}$$

then summing these contributions using Eqs. (5) and (18) shows that

$$\Omega = \sum_i \Omega_i \,, \tag{25}$$

as is required for consistency.

The time dependence of the total density parameter Ω is still given by (11), where now γ is given by (23). Ω is expressed directly in terms of S using Eqs. (16), (17), and (25):

$$\Omega = \left[1 + (1 - \Omega_0) \Big/ \left(\sum_i \Omega_{i0} y^{2-3\gamma_i}\right)\right]^{-1} , \tag{26}$$

while from (16) and (23), $\gamma(S)$ is given by

$$\gamma = <\gamma_i> = \left(\sum_i \gamma_i \Omega_{i0} y^{-3\gamma_i}\right) \left(\sum_i \Omega_{i0} y^{-3\gamma_i}\right)^{-1} . \tag{27}$$

Despite the clarity of these last two equations, it is more transparent, from the point of view of trying to determine the evolutionary qualities of the universe, to derive evolution equations for the γ_i and the Ω_i. The derivation simply repeats the steps used in obtaining (11). One finds that the generalized form of that equation is now

$$\frac{d\Omega_i}{dt} = H\Omega_i \left[(2 - 3\gamma_i) - (2 - 3\gamma)\Omega\right], \tag{28}$$

while the relation corresponding to (10) is found to be

$$q = -\frac{1}{2}\left(2\Omega - 3\sum_i \gamma_i \Omega_i\right). \tag{29}$$

Equation (28) can also be written in other forms which illustrate different characteristics. One can factor out $(2 - 3\gamma)$, for example, to obtain

$$\frac{d\Omega_i}{dt} = (2 - 3\gamma)H\Omega_i \left[\left(\frac{2 - 3\gamma_i}{2 - 3\gamma}\right) - \Omega\right] \tag{30}$$

or one can rearrange the terms inside the square brackets in (28) and substitute from (23) to obtain

$$\frac{d\Omega_i}{dt} = H\Omega_i \left[2\left(1 - \sum_j \Omega_j\right) - 3\left(\gamma_i - \sum_j \gamma_j \Omega_j\right)\right]. \tag{31}$$

Similarly, it is easy to obtain an equation for the evolution of the mean adiabatic index γ by differentiating (23). The result is

$$\frac{d\gamma}{dt} = 3H\left[\gamma^2 - \sum_j \gamma_j^2 \left(\frac{\Omega_j}{\Omega}\right)\right]. \tag{32}$$

Finally, the previous two equations can be combined to construct an equation whose phase portraits show the evolutionary relation between the total density parameter and the total adiabatic index:

$$\frac{d\Omega}{d\gamma} = \frac{(2-3\gamma)(1-\Omega)\Omega^2}{3\left[\gamma^2\Omega - \sum_j \gamma_j^2\Omega_j\right]}. \tag{33}$$

Note that this equation is singular when there is only one matter component. This is not a problem with the formulation given here, but simply a reflection of the fact that there is no evolution of Ω with γ in this particular case.

We now proceed to apply some of the results of this section by presenting some models described in terms of multicomponent fluids.

B. Matter with three components

For our purposes, it is sufficient to restrict attention to the case of a 3-component fluid defined as follows: pressureless matter $p_1 = 0, \gamma_1 = 1$; radiation $p_2 = \rho_2/3, \gamma_2 = \frac{4}{3}$; cosmological constant $p_3 = -\rho_3, \gamma_3 = 0$. Using (16), Eq. (23) becomes

$$\gamma(S) = \frac{M_1/S^3 + 4M_2/3S^4}{M_1/S^3 + M_2/S^4 + M_3}\ , \tag{34}$$

where the M_i are constants representing the relative magnitudes of the three contributions to ρ. Equation (34) can also be written with the M_i replaced by the Ω_i. Equations (12) and (34) together determine the (S, Ω) phase-plane evolution of the universe with matter as described above. Explicitly, from (26),

$$\Omega = \frac{1}{1 + (1-\Omega_0)/(\Omega_1 y^{-1} + \Omega_2 y^{-2} + \Omega_3 y^2)}\ , \tag{35}$$

where the Ω_i are the values of the relevant parameters at the present time, and so are constants.

C. A 2-epoch inflationary universe

In contrast with the previous section, use of multicomponent fluids allows one to represent an inflationary universe by a model with only two epochs, the first one including an inflationary period (when $q < 0$).

In the *initial phase* from $S = 0$ until $S = S_f$, we include radiation and a cosmological constant, but no matter. The phase plane is then described by Eq. (12) where now, from Eq. (34)

$$\gamma = \frac{4/3}{1 + \alpha S^4}\ , \tag{36}$$

where α is a constant; γ decreases monotonically from $\frac{4}{3}$ at the beginning of the expansion to zero; it passes through the critical value $\frac{2}{3}$ when $S_{\text{crit}} = \alpha^{-1/4}$, the inflationary era ($q < 0$) occurring when S is larger than this value, that is, when $S_{\text{crit}} < S < S_i$. The phase plane shown in Fig. 7 is like the combination of epochs I and II in the previous section (see Fig. 6), except that the phase curves are now smooth everywhere in these eras. The explicit form for $\Omega(S)$ is given by Eq. (35) with $\Omega_1 = 0$. The

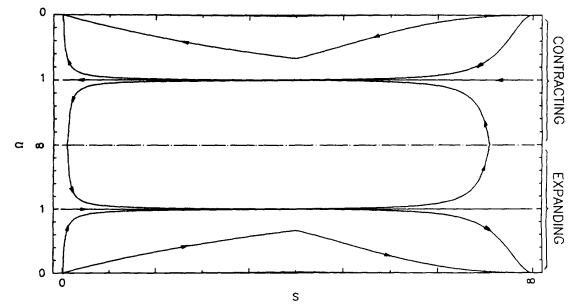

FIG. 7. Phase portraits for the 2-epoch inflationary models of Sec. III. Curves showing the evolution of Ω with S. The left half of the diagram represents the radiation and inflaton fluid together, while the right half represents the mixture of radiation and dust fluids.

diagram will be symmetric if the matter density is zero.

In the *subsequent phase* from $S = S_f$ onward, matter and radiation are included, and the cosmological constant omitted. The phase plane is again given by (12), this time with

$$\gamma(S) = \frac{4/3 + \beta S}{1 + \beta S}\ , \tag{37}$$

where β is a constant; γ decreases monotonically from $\frac{4}{3}$ to 1 as the universe expands. The phase plane is now like the combination of epochs III and IV in the previous section (again, see Fig. 6). The explicit form for $\Omega(S)$ is given now by (35) with $\Omega_3 = 0$.

The total phase plane is obtained by joining these two phases together at $S = S_f$; as in the previous section, the phase curves must be continuous at this join. The resulting diagram is again like that of Fig. 6, which represents the evolution of the entire universe from $S = 0$ to the present day and beyond. There is one significant difference in Fig. 6 now: the phase curves will be smooth at the junctions between epochs I and II, and between epochs III and IV, the only discontinuity in the tangent vectors being at the junction of epochs II and III, when $S = S_f$. The left-hand critical point at $\Omega = \infty$ will now be an ordinary saddle point, which represents the *Einstein static universe.* Thus the somewhat disturbing state of undecided motion which appeared in the model of the previous section no longer occurs. The stable critical point at $\Omega = \infty$ does not represent any universe model, for it lies on a surface of discontinuity of the equation of state. However, as in the previous case, one finds a family of oscillating universe models around this critical point. All the usual inflationary universe models, though, start at the initial singularity at the left-hand side of Fig. 6 where $\Omega = 1$, and then expand forever if $k = -1$ or 0, or, if $k = +1$, recollapse either before the inflationary era begins or after it ends. Finally, an unstable family with $k = +1$ tends asymptotically to the Einstein static universe in the infinite future.

D. The (q,σ) phase plane

A complementary approach to the study of the evolution of Ω when two noninteracting matter components are present was developed by Stabell and Refsdal [2] who produced phase-plane diagrams in the (q,σ) plane, where $\sigma = \Omega/2$. They only considered the case $p = 0$ but included an explicit cosmological constant that was not included in the total σ. Here we generalize their treatment to accommodate different equations of state, so as to allow comparison of their results with ours.

In the context of the inflationary universe, we wish to examine phase planes during a radiation plus Λ phase. With this in mind we will rewrite the equations of Ref. [2] in terms of the general perfect fluid equation of state. The reasoning used in deriving these forms is exactly analogous to that used in Ref. [2], but the results obtained are more general in that they apply to a fluid with an arbitrary equation of state. In plotting the final phase planes of this section it will be more transparent to use Ω as the density parameter instead of σ, but this change is only made at the end of the section. Some of the material contained in this section repeats earlier material, but this is in the interest of clarity and in order to aid direct comparison with the relations given in Ref. [2]. It also facilitates direct comparison with the derivations given in Ref. [2].

The Einstein equations with an explicit Λ term and ideal gas equation of state can be written

$$8\pi G\rho = \frac{3}{S^2}(k + \dot{S}^2) - \Lambda, \tag{38}$$

$$\frac{8\pi G}{3}(\gamma - 1)\rho = -2\frac{\ddot{S}}{S} - \frac{\dot{S}^2}{S} - \frac{k}{S^2} + \Lambda, \tag{39}$$

and, from (38)

$$\frac{3}{2}\left(H^2 + \frac{k}{S^2}\right) = \frac{1}{2}(8\pi G\rho + \Lambda). \tag{40}$$

Substituting Eq. (40) in Eq. (39) results in

$$-3\frac{\ddot{S}}{S} = \frac{3}{2}\left(H^2 + \frac{k}{S^2}\right) + \frac{3}{2}\left[8\pi G(\gamma - 1)\rho - \Lambda\right]. \tag{41}$$

Now multiplying Eq. (41) by $1/H^2$ and introducing q we get

$$3q = \frac{8\pi G\rho}{2H^2}(3\gamma - 2) - \frac{\Lambda}{H^2}, \tag{42}$$

and by introducing the new density parameter

$$\sigma \equiv \frac{4\pi G\rho}{3H^2} = \Omega/2, \tag{43}$$

we have

$$q = \sigma(3\gamma - 2) - \frac{\Lambda}{3H^2}. \tag{44}$$

We want to be able to express Λ and k in terms of σ_0, q_0, and y. From (44),

$$\Lambda = 3H^2[(3\gamma - 2)\sigma - q] = 3H_0^2[(3\gamma - 2)\sigma_0 - q_0]. \tag{45}$$

Multiplying (38) by $1/3H^2$ and substituting (45) for Λ,

$$k = H^2S^2[3\gamma\sigma - q - 1] = H_0^2S_0^2[3\gamma\sigma_0 - q_0 - 1]. \tag{46}$$

For a perfect fluid we have

$$\rho S^{3\gamma} = \rho_0 S_0^{3\gamma} = \text{const}, \tag{47}$$

so that inserting Eqs. (43), (46), and (47) into (38),

$$\dot{S}^2 = \dot{S}_0^2\left(2\sigma_0\frac{S_0^{3\gamma-2}}{S^{3\gamma-2}} + [(3\gamma - 2)\sigma_0 - q_0]\frac{S^2}{S_0^2} - (3\gamma\sigma_0 - q_0 - 1)\right). \tag{48}$$

For convenience in the discussion that follows we define

$$F(S) \equiv \dot{S}^2. \tag{49}$$

For $\gamma < \frac{2}{3}$ the universe contains two inflationary components (unless $\sigma_0 = q_0$) and the behavior is quite different from $\gamma > \frac{2}{3}$. The different cases are now considered separately, extending the descriptions given by Stabell and Refsdal in Ref. [2].

1. The case $\gamma > \frac{2}{3}$

a. $\Lambda > 0$ $[(3\gamma - 2)\sigma_0 - q_0 < 0]$. F is positive when S is less than some $S = S_c$ and negative when $S > S_c$. Since $\dot{S}^2$ must always be positive, S must always be less than S_c. These are thus oscillating universes (type O).

b. $\Lambda \geq 0$ and $k \leq 0$ $\{(3\gamma - 2)\sigma_0 - q_0 \geq 0$ and $3\gamma\sigma_0 - q_0 - 1 \leq 0\}$. By inspection, F is always positive. $\dot{S}$ can never change sign and, since $\dot{S}_0 > 0$, $\dot{S}$ is always greater than zero. These universes, denoted by M_1, are ever expanding and tend to the de Sitter universe.

c. $\Lambda \geq 0$ and $k > 0$ $\{(3\gamma - 2)\sigma_0 - q_0 \geq 0$ and $3\sigma_0 - q_0 - 1 > 0\}$. In this case we have to look more closely at Eq. (48) to understand the behavior. Differentiating F with respect to S,

$$\frac{dF}{dS} = \dot{S}_0^2\left(-2(3\gamma - 2)\sigma_0\frac{S_0^{3\gamma-2}}{S^{3\gamma-1}} + 2\left[(3\gamma - 2)\sigma_0 - q_0\right]\frac{S}{S_0^2}\right) \tag{50}$$

F has a minimum value, F_m, at $\frac{dF}{dS} = 0$, $S = S_m$:

$$F_m = 2\sigma_0\left(\frac{(3\gamma - 2)\sigma_0}{(3\gamma - 2)\sigma_0 - q_0}\right)^{\frac{2}{3\gamma}-1} + [(3\gamma - 2) - q_0]\left(\frac{(3\gamma - 2)\sigma_0}{(3\gamma - 2)\sigma_0 - q_0}\right)^{\frac{2}{3\gamma}} - [3\gamma\sigma_0 - q_0 - 1], \tag{51}$$

$$S_m = S_0 \left(\frac{(3\gamma - 2)\sigma_0}{(3\gamma - 2)\sigma_0 - q_0} \right)^{\frac{1}{3\gamma}} . \quad (52)$$

(a) If $F_m > 0$ then $\dot{S}$ is always positive and the universes are monotonically expanding (type M_1). Apart from the Einstein–de Sitter case ($q_0 = \sigma_0 = \frac{1}{2}$), the final state is the de Sitter universe.

(b) $F_m = 0$. This may occur in two regions.

1. $q_0 \leq -1$. These are universes of type A_2. They may be regarded as having asymptotically started from the static Einstein universe ($-q = \sigma = \infty$) at $t = -\infty$. They expand toward a final de Sitter state.

2. $\sigma_0 > \frac{1}{2}$ (obtained by solving $k = \Lambda = 0$). These universes (type A_2) start at $q = \sigma = \frac{1}{2}$ and expand toward the static Einstein case ($q = \sigma = \infty$).

(c) If $F_m < 0$ then F is negative for an interval around S_m. There are again two different cases:

1. $q_0 > 0$, $S_m > S_0$. The universe expands to some maximum S which is less than S_m and then contraction begins. This is a universe of type $O(5)$.

2. $q_0 < 0$. The universe contracted from an infinite time in the past until $S = S_{\min}(S_{\min} > S_m)$, when expansion began. These universes, which are type M_2, expand into the de Sitter universe.

Figure 10 shows the representation of the different models in the (q_0, σ_0) plane for the case $\gamma = 1$. The A_1 and A_2 curves are obtained by solving the equation $F_{\min} = 0$. The A_1 curve separates universes of type O from universes of type M_1. The A_2 curve separates universes of types M_1 and M_2.

2. The case $\gamma = \frac{2}{3}$

In this case (48) becomes

$$F = \dot{S}_0^2 \left(q_0 + 1 - q_0 \frac{S^2}{S_0^2} \right) , \quad (53)$$

and its derivative is

$$\frac{dF}{dS} = -2q_0 \frac{S}{S_0^2} . \quad (54)$$

The only maxima or minima of F are at $S = 0$ (maximum if $q_0 > 0$ and minimum if $q_0 < 0$. There are three cases to consider:

1. $q_0 < -1$. F is positive for all S greater than

$$S_m = S_0 \sqrt{\frac{q_0 + 1}{q_0}} . \quad (55)$$

These are M_2 models and behave as previously described.

2. $-1 \leq q_0 \leq 0$. F can never be negative. The models (type M_1) monotonically expand into a de Sitter state.

3. $q_0 > 0$. F is positive for all S less than

$$S_m = S_0 \sqrt{\frac{q_0 + 1}{q_0}} . \quad (56)$$

These are therefore oscillating models of type O.

3. The case $\gamma < \frac{2}{3}$

The behavior is significantly different from $\gamma > \frac{2}{3}$. There are four cases to consider.

1. $\Lambda > 0$, $k < 0$. The universes are of the type M_1 described earlier.

2. $\Lambda > 0$, $k > 0$. The universes are of type M_2 and expand into the de Sitter universe.

3. $\Lambda < 0$, $k < 0$. The universes are oscillating (type O). They expand up to some maximum y when contraction begins.

4. $\Lambda < 0$, $k > 0$. This is a new type of universe not previously encountered in this section. F is positive for some range $y_{\min} < y < y_{\max}$. The universe oscillates between these bounds without y ever becoming singular. These universes will be denoted type N.

E. Relations for arbitrary γ

We now derive equations for the phase planes in the (q, σ) plane and the (σ, y) plane. Equations (40) and (47) combine to give

$$\sigma = \frac{8\pi G \rho S_0^{3\gamma}}{3 S^{3\gamma - 2} \dot{S}^2} , \quad (57)$$

which simplifies to

$$\sigma = \frac{2\sigma_0}{2\sigma_0 + \{(3\gamma - 2)\sigma_0 - q_0\} y^{3\gamma} - [3\gamma\sigma_0 - q_0 - 1] y^{3\gamma - 2}} . \quad (58)$$

Now we use (45) in (44) to obtain

$$q = (3\gamma - 2)\sigma - \frac{\dot{S}_0^2}{S_0^2} \frac{S^2}{\dot{S}^2} [(3\gamma - 2)\sigma_0 - q_0] . \quad (59)$$

Again we substitute for $\dot{S}^2$ and simplify the result into

$$q = \frac{(3\gamma-2)\sigma + [q_0 - (3\gamma-2)\sigma_0]\, y^{3\gamma}}{2\sigma_0 + [(3\gamma-2)\sigma_0 - q_0]\, y^{3\gamma} - [3\gamma\sigma_0 - q_0 - 1]\, y^{3\gamma-2}}. \tag{60}$$

Equations (58) and (59) are the phase-plane equations and may be used to produce diagrams in the (q,σ) plane or the (q,y) plane as preferred. If it is desired to include the contribution of Λ in Ω then we can add a term $[(3\gamma-2)\sigma_0 - q_0]\, y^{3\gamma}$ in (58), giving

$$\sigma = \frac{\sigma_0 - [q_0 - (3\gamma-2)\sigma_0]\, y^{3\gamma}}{2\sigma_0 + [(3\gamma-2)\sigma_0 - q_0]\, y^{3\gamma} - [3\gamma\sigma_0 - q_0 - 1]\, y^{3\gamma-2}}, \tag{61}$$

so that Eq. (60) then becomes simply

$$q = (3\gamma_{\rm eff} - 2)\sigma. \tag{62}$$

F. The (q,Ω) phase plane

Figures 8–11 show the phase planes produced, which illustrate the descriptions given above. Figure 10 ($\gamma = 1$) is similar to that given in [2], but with Ω replacing σ as the density parameter. The lines $k = 0$ and $\Lambda = 0$ are seen to act as boundaries between the different classes of models. All the models which asymptotically approach the de Sitter universe may be regarded in some sense as "late inflationary." In Sec. III D it was stated for the one-component case that all universes with $\Lambda > 0$ approached a final state of the de Sitter universe. This is clearly not true for the two-component model, when universes with a positive cosmological constant may also be of type O. Figure 11 shows the phase plane for Λ and radiation. It is this case that we expect to be most relevant to the inflationary scenario as in the new inflationary models it is quite plausible that radiation density was significant at the start of inflation. There is still no universal agreement on the question of whether or not a radiation-dominated preinflationary epoch will affect the course of inflation [12]. It can be seen that Figs. 10 and 11 are qualitatively very similar, with the only noticeable difference lying in the slopes of the lines.

Although Figs. 8–11 are restricted to two dimensions, they give a hint as to how the phase structure of the three-dimensional (q,Ω,γ) system might look. To see this, one should imagine these diagrams to be stacked vertically in order of increasing γ.

Figure 9 shows the model for cosmological constant and a $\gamma = \frac{2}{3}$ component. It can be seen that all the paths are now straight lines. The models cannot cross the $k = 0$ or $\Lambda = 0$ lines, this accounts for the unusual shape of the phase plane in the region $q > 0$. If the paths of models in this region are extrapolated they converge at $q = -1$, $\sigma = 0$ but in fact the models are doomed to oscillate forever with $q > 0$. Figure 8 shows the two-inflationary-component phase plane. The curvature of the lines has changed sign from the case $\gamma > \frac{2}{3}$. Universes which used to be of type M_1 are now of type N, it is impossible for them to cross $k = 0$ or $\Lambda = 0$ so they can never reach the de Sitter universe. This is of interest because with two inflationary components, we might naively expect all model universes to tend to de Sitter universes.

IV. SCALAR FIELD FLUIDS

The problem of constructing the phase portraits of the inflationary models is considered in this section. As in [1], inhomogeneities will be neglected in this treatment, however, see [12] for further discussion. These phase portraits are harder to construct in the case now under consideration, where the scalar field is treated completely dynamically, than the corresponding phase planes of the previous paper. This is mainly because there, it was assumed that the history of the Universe could be modeled by specifying the effective equation of state of the dom-

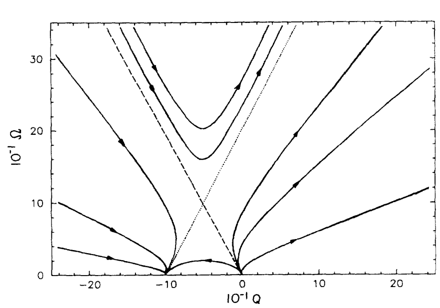

FIG. 8. The (q,Ω) phase plane for nonzero Λ and $\gamma = \frac{1}{3}$. The dashed line represents the $\Lambda = 0$ and the dotted line the $k = 0$ case, respectively.

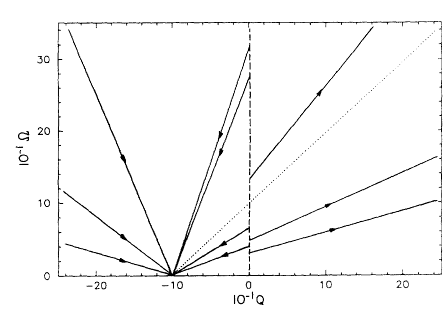

FIG. 9. (q,Ω) phase plane for nonzero Λ and $\gamma = \frac{2}{3}$. The dashed line represents the $\Lambda = 0$ and the dotted line the $k = 0$ case, respectively.

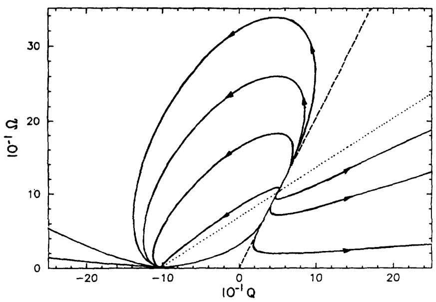

FIG. 10. (q, Ω) phase plane for nonzero Λ and $\gamma = 1$. The dashed line represents the $\Lambda = 0$ and the dotted line the $k = 0$ case, respectively.

inant component during eras labeled by the initial and final values of the scale factor. In addition, as mentioned in the Introduction, the explicit inclusion of the inflaton field transforms the plane autonomous system studied in [1] into a dynamical system of minimum dimension 3, in general nonautonomous, so that the phase curves can be vastly more complicated than in the plane case. Nevertheless, a number of conclusions can be drawn from an examination of the equations for the phase portraits provided certain approximations can be taken to hold throughout the periods examined. Some of the specific cases dealt with below are those where the spatial curvature is negligible, and where the effective adiabatic index is taken to be constant. Either or both of these cases can provide a good approximation at different epochs, although it is unlikely in the general case that they will describe the behavior of the dynamical system for a large fraction of the time during which inflation occurs. Nevertheless, in view of the complexity of the general case compared to these special cases, their treatment provides a useful amount of insight into the type of behavior possible for the evolution of the Universe.

A. The basic equations

Here we shall need to begin with the field equations for an isotropic and spatially homogeneous cosmological model whose matter content consists of a fluid, which we shall be most interested in taking as radiation, and a scalar field, henceforth the *inflaton*, which is also spatially homogeneous:

$$3H^2 + 3K = 8\pi G[\tfrac{1}{2}\dot{\phi}^2 + V(\phi) + \rho] , \quad (63)$$

$$3\dot{H} + 3H^2 = 8\pi G[V(\phi) - \dot{\phi}^2 - \tfrac{1}{2}(\rho - 3p)] , \quad (64)$$

$$\ddot{\phi} + 3H\dot{\phi} + \partial V/\partial\phi = 0 , \quad (65)$$

$$\dot{\rho} + 3H(\rho + p) = 0 . \quad (66)$$

Equation (63) is the Friedmann equation, (64) is the Raychaudhuri equation, and (65) is the inflaton field equation, which is equivalent to the conservation equations derived from the Bianchi identities for the scalar field stress tensor (a detailed discussion is given in [16]) and (66) is the usual equation for the conservation of energy of a fluid. Here an overdot denotes differentiation with respect to cosmic proper time, H is the Hubble parameter, defined in terms of the cosmic scale factor S as $H \equiv \dot{S}/S$, ϕ is the inflaton field and $V(\phi)$ is its corresponding potential, $3K$ is the purely spatial part of the scalar curvature, $K = k/S^2$ with k a constant, ρ is the fluid energy density and p its pressure, and the units are chosen so that $\hbar = c = 1$, and the gravitational constant $G = 1/m_P^2$, where m_P is the Planck mass.

It is crucially important to the developments of this section that the system (63)–(66) actually contains only *three* independent equations, rather than the four which are apparently present. This is easily seen by differentiating (63) and substituting Eqs. (64)–(66) to get (64). Of course, this is merely a consequence of the fact that the Bianchi identities are not independent dynamical equations, but just express the integrability conditions for the equations of motion. Practically, the result of all this is that we can choose to examine the system constructed from any three of Eqs. (63)–(66).

Another important remark concerns Eq. (65) which governs the behavior of the scalar field. We mentioned earlier, at the end of the previous section, that a combination of interacting fluids could provide the conditions needed to set the appropriate inflationary stage. Now, Eq. (65) can be written as

$$\frac{d}{dt}\left(\frac{\dot{\phi}^2}{2}\right) + 6H\frac{\dot{\phi}^2}{2} = -\dot{V}(\phi) , \quad (67)$$

which allows us to identify this equation with the corresponding equation for the combination of two interactive fluids. The first fluid is just the kinetic energy of the scalar field and is characterized by the $\gamma = 2$ value which is expected for a free field (see below). The second fluid is the potential energy and is characterized by $\gamma = 0$, but contrary to what was considered earlier cannot be straightforwardly associated with a cosmological constant precisely because of its interaction with the kinetic energy. (If it were a constant then the scenario would be identical to that with a combination of noninteracting $\gamma = 2$ and $\gamma = 0$ fluids.)

Before proceeding further, it will be useful to pause in order to notice a few useful relations obeyed by the

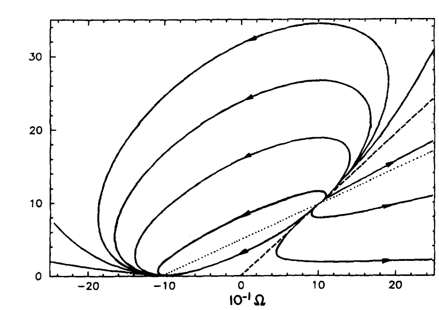

FIG. 11. (q, Ω) phase plane for nonzero Λ and $\gamma = \frac{4}{3}$. The dashed line represents the $\Lambda = 0$ and the dotted line the $k = 0$ case, respectively.

quantities appearing in Eqs. (63)–(66). The first is that, since $K = k/S^2$,

$$\dot{K} = -2HK \ . \tag{68}$$

Also, it will be necessary to use the fact that the scalar field can be treated as a fluid whose energy density and pressure are

$$\rho_\phi = \tfrac{1}{2}\dot{\phi}^2 + V(\phi) \ , \tag{69}$$

$$p_\phi = \tfrac{1}{2}\dot{\phi}^2 - V(\phi) \ , \tag{70}$$

so that the total energy density and pressure are

$$\rho_T = \tfrac{1}{2}\dot{\phi}^2 + V(\phi) + \rho \ , \tag{71}$$

$$p_T = \tfrac{1}{2}\dot{\phi}^2 - V(\phi) + p \ . \tag{72}$$

It is a simple matter to see that ρ_T obeys the same conservation law, Eq. (66), as ρ. The total density parameter obeys the differential equation given in an earlier section,

$$\dot{\Omega} = (2 - 3\gamma_T)H\Omega(1 - \Omega) \ , \tag{73}$$

where γ_T is the usual index, defined by the equation

$$p_T = (\gamma_T - 1)\rho_T \ , \tag{74}$$

exactly as in an earlier section. This is entirely consistent with Eqs. (69) and (70), and also with the definition

$$\gamma_\phi \equiv \frac{\rho_\phi + p_\phi}{\rho_\phi} \tag{75}$$

$$= \frac{\dot{\phi}^2}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \ . \tag{76}$$

By use of the field equations, this can also be written as

$$\gamma_\phi = 2\left(\frac{3H^2 + 3K - 8\pi G V(\phi) - 8\pi G\rho}{3H^2 + 3K - 8\pi G\rho}\right) \ . \tag{77}$$

The above equations will be useful in analyzing the motion of the dynamical system. Before dealing with the general case, we show that in a class of restricted cases there is still a useful phase plane construction, although this is now different from that of Ref. [1].

We shall now ignore the matter component, for the sake of avoiding any unnecessary difficulty in the subsequent analysis. Using the definition of the γ index (76), we can express either of the components of the energy density of the scalar field in terms of the other and of γ_ϕ. Then, substitution into the original field equations (63)–(65), yields a system of equations which clearly requires the specification of the γ behavior in order for it to be solvable. In the general case, γ depends both on ϕ and on $\dot{\phi}$ whose behavior we ignore. This means that an *ad hoc* assumption is required.

Two simple cases can, however, be distinguished for reasons which will become immediately clear. These are the γ =const case and the situation where the potential is constant: $V = V_0$ =const. The first of these cases corresponds to demanding adiabatic dynamics from the model and is clearly of no interest. The second case, on the contrary, though representing a simplification, given that we make the explicit dependence of γ on ϕ disappear, preserves the nonadiabatic character of the model. Futhermore, as we will discuss in what follows, not only does this case match what is expected to be the convenient shape of the inflationary potential, but also provides the "most unfavorable " one, in the sense that this potential provides the smallest maximum damping of the inflaton's roll.

B. Scalar field models with a flat potential

1. Solutions for a flat potential

Let us suppose now that the potential is flat, $V(\phi) = V_0$, at least over a sufficiently wide region of the domain of variation of ϕ . Then from Eq. (65) we obtain

$$\ddot{\phi} + 3H\dot{\phi} = 0 \quad , \tag{78}$$

which implies that

$$\dot{\phi} = \dot{\phi}_0 S^{-3}(t) \quad , \tag{79}$$

where $\dot{\phi}_0$ is a constant of integration giving the value of $\dot{\phi}$ when $S = 1$. Substitution into (63) yields

$$H^2 + \frac{k}{S^2} = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}_0^2 S^{-6} + V_0\right) \quad , \tag{80}$$

which leads to the differential equation

$$\frac{1}{S}\frac{dS}{dt} = \left[\frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}_0^2 S^{-6} + V_0\right) - \frac{k}{S^2}\right]^{1/2} \quad . \tag{81}$$

Equation (80) is the Friedmann equation for a universe with *stiff matter* (that is, the equation of state $p = \rho$ [26,21]) plus a cosmological constant (corresponding to the flat potential V_0). It is interesting to note that although there has been much discussion of the stiff matter equation of state in the literature, no full proposal has been given so far for its physical realization. We now see that it represents the pure kinetic effect of a scalar field (with a flat potential); it is exactly realizable as a classical scalar field with potential V_0 with value $V_0 = 0$ (which is of course not at all the same thing as no scalar field at all). Finally the value of $\phi(t)$ is given by integrating (79), or we can obtain $\phi(S)$ from

$$\phi = \int_{t_0}^{t} \dot{\phi}\, dt + \phi_0 = \dot{\phi}_0 \int_{t_0}^{t} \frac{dS}{S^4 H(S)} + \phi_0 \quad ,$$

with $H(S)$ given by (80).

The exact integration of Eq. (81), although possible in general, involves either elliptic or quasielliptic integrals [22] or a parametric solution [4]. Some particular cases however, admit a representation of the solutions in terms of time t and nontranscendental functions.

(1) For the $\dot{\phi}_0 = 0$ (cosmological constant only) case, one obtains the de Sitter solution in its different forms:

$$k = -1 \Longrightarrow S(t) = \sqrt{\frac{3}{8\pi G}}\sinh(t - t_0), \tag{82}$$

$$k = 0 \Longrightarrow S(t) = \sqrt{\frac{3}{8\pi G}} \exp{(t - t_0)}, \quad (83)$$

$$k = +1 \Longrightarrow S(t) = \sqrt{\frac{3}{8\pi G}} \cosh{(t - t_0)}. \quad (84)$$

We will not discuss these standard solutions any further; thus from now on we assume $\dot{\phi}_0 \neq 0$.

(2) For the $k = 0$ case (flat model), one derives

$$S(t) = \left\{ \frac{\dot{\phi}_0^2}{2V_0} \right\}^{1/6} \left[\sinh\left(\sqrt{24\pi G}(t - t_0)\right)\right]^{1/3} \quad , \quad (85)$$

when $V_0 \neq 0$, and

$$S(t) = \left\{ \frac{3}{8\pi G} \right\}^{1/6} (t - t_0)^{1/3} \quad (86)$$

for the "stiff matter" case $V_0 = 0$.

We may note that all of (82)–(86) have the same asymptotic form at large time. From (79), in all cases the kinetic term will die away rapidly as the universe expands (whether $\dot{\phi}_0$ is positive or negative). This is generally true. See, for example, Refs. [23,24].

2. The effective equation of state

Although the set of solutions derived provides a detailed knowledge of the behavior of the dynamical variables involved, namely, $S(t), H(t), T(\phi), \ldots$, a clear physical picture of the features of these solutions can be better drawn by discussing the evolution of the adiabatic ratio $\gamma(S(t))$.

Using Eq. (79) we may (using the assumption $\dot{\phi}_0 \neq 0$) rewrite (76) as

$$\gamma = \frac{2}{1 + 2\frac{V_0}{\dot{\phi}_0^2} S(t)^6} \quad . \quad (87)$$

It is immediately apparent that γ will undergo a decrease from its initial value as the universe expands, provided $V_0 > 0$. Indeed,

$$\frac{d\gamma}{dt} = \frac{d\gamma}{dS}\frac{dS}{dt} = -24 \frac{V_0}{\dot{\phi}_0^2} \frac{S^5}{\left(1 + 2\frac{V_0}{\dot{\phi}_0^2} S^6\right)^2} \frac{dS}{dt}, \quad (88)$$

which is negative when $V_0 \dot{S} > 0$. If, when the field starts along the flat region of the potential, the value of $S(t)$ is

$$S(t) \ll \frac{1}{2} \frac{\dot{\phi}_0^2}{V_0} \quad , \quad (89)$$

the effective value of γ will be

$$\gamma \simeq 2 \quad , \quad (90)$$

that is, as mentioned above, the inflaton field will be a massless field behaving as stiff matter. Subsequent decrease brings γ to lower values, reaching radiation-type behavior ($\gamma = \frac{4}{3}$) when

$$S(t) = \left(\frac{1}{4} \frac{\dot{\phi}_0^2}{V_0} \right)^{1/6} \quad , \quad (91)$$

and dust-type behavior ($\gamma = 1$) when

$$S(t) = \left(\frac{\dot{\phi}_0^2}{2V_0} \right)^{1/6} \quad , \quad (92)$$

and entering the inflationary domain of values when it crosses the critical value $\gamma = \frac{2}{3}$, for

$$S(t) \geq \left(\frac{\dot{\phi}_0^2}{V_0} \right)^{1/6} \quad . \quad (93)$$

Then it asymptotically approaches zero, namely, the de Sitter regime. This happens for

$$S(t) \ll \left(\frac{\dot{\phi}_0^2}{V_0} \right)^{1/6} \quad . \quad (94)$$

We thus realize that no matter which initial conditions apply to the inflaton field when it enters the flat region of the potential, if this region is wide enough for S to reach the value (93), exponential inflation inevitably occurs, and occurs increasingly rapidly.

Until now, no assumptions have been made about the ratio $\dot{\phi}_0^2/V_0$ other than it is positive. It is known that violation of the strong energy condition results when

$$\frac{\dot{\phi}^2}{V} < 1 \quad , \quad (95)$$

which for the case under consideration becomes

$$\frac{\dot{\phi}_0^2}{V_0} S^{-6}(t) < 1 \quad , \quad (96)$$

which is clearly equivalent to (93). Thus, if we demand ϕ_0 be sufficiently small compared to V_0, we may guarantee that inflation occurs as early as we like. However such a restriction needs physical justification; if on the other hand we allow all initial conditions for $\dot{\phi}$ for a given flat potential section V_0, in many cases inflation will not start until late [maybe too late, if ϕ runs off the flat part of the potential before (96) becomes true].

If $V_0 \leq 0$, life is quite different. When $V_0 = 0$, γ stays at the value 2 and no inflation takes place; we have the stiff matter solutions. When $V_0 < 0$, as the universe expands, γ increases from 2 upward, and no inflation takes place; recollapse follows soon. This case is presumably unphysical.

3. Exit from inflation

As a flat potential drives the scalar field toward inflation, independently of its initial state of motion (which probably corresponds to stiff matter), so clearly some change in the shape of the potential must be responsible for moving the universe away from inflation. Supposing V is no longer taken to be flat, we reconsider Eq. (76) and investigate the sign of $d\gamma/dt$. We have

$$\frac{d\gamma}{dt} = -\frac{1}{2}\gamma(t)\frac{d}{dt}\left(\frac{V}{\dot{\phi}^2}\right) \quad , \tag{97}$$

showing that the sign of $\dot{\gamma}$ depends on the evolution of the ratio $V/\dot{\phi}^2$. Evaluating $\frac{d}{dt}(V/\dot{\phi}^2)$, we find that (1) γ increases if

$$\frac{dV}{dt} < -3HV\gamma \quad , \tag{98}$$

and (2) γ decreases if

$$\frac{dV}{dt} > -3HV\gamma \quad , \tag{99}$$

where both H and γ will be approximately constant in the exponential inflationary era (in fact γ will be very nearly zero). Thus V should decrease sharply from its flat value $V = V_0$ if it is to trigger an increase in γ sufficient to drive it away from the $\gamma < \frac{2}{3}$ region, as happens with the "new inflationary" potential.

4. (Ω, S) phase planes

Let us now investigate the effect of a flat potential on the evolution of the density parameter Ω, extending to this particular case the study developed by two of us in a previous work [1]. It is important to notice that there are no further assumptions necessary about the behavior of the scalar field during the evolution described by these phase planes. In particular, there is no "slow roll" assumption made in constructing the diagrams shown. Introducing the density parameter Ω,

$$\Omega \equiv \frac{8\pi G\rho}{3H^2} \quad , \tag{100}$$

the field equations show that

$$\dot{\Omega} = H\Omega(1-\Omega)(2-3\gamma), \tag{101}$$

$$\dot{S} = HS. \tag{102}$$

We now assume $V_0 > 0$, $\dot{\phi} \neq 0$, for the cases $\dot{\phi} = 0$ and $V_0 = 0$ are covered in detail in [1] (being, respectively, the cases $\gamma = 0$ and $\gamma = 2$), and $V_0 < 0$ is nonphysical. Dividing these two equations, and using (87) then yields

$$\frac{d\Omega}{dS} = 2\left(1 - 3\frac{1}{1 + 2\frac{V_0}{\dot{\phi}_0^2}S^6}\right)(1-\Omega)\frac{\Omega}{S} \tag{103}$$

which is equivalent to the autonomous dynamical system resulting from transforming to a conformal time τ such that $d/dt = Hd/d\tau$ in (101) and (102). Equation (103) is easily integrated, and we obtain the well-known form

$$\frac{\Omega}{|1-\Omega|} = C\left\{S^2 + \frac{\dot{\phi}_0^2}{2V_0}S^{-4}\right\}. \tag{104}$$

where C is a constant of integration.

Using this result, and assuming the potential to be flat during the whole evolution of the Universe, so that the effect of this class of potentials is made explicit, we are able to draw the phase diagrams (Ω, S) (cf. [1]). However, some of the essential features of the qualitative behavior attached to these diagrams are immediately accessible. Considering just the expanding universes (since the contracting phase amounts to a time reversal of these), we see, from the previous result, that the phase trajectories are asymptotic, for $S \to 0$, and for $S \to +\infty$ to the value $\Omega = 1$. From Eq. (39) we see that for $S = (\dot{\phi}_0/V_0)$ (which corresponds to $\gamma = \frac{2}{3}$, as we have seen before) the trajectories have an extremum corresponding to their maximum deviation from the flat $\Omega = 1$ model, since at the left of this point the derivative $d\Omega/dS$ is positive (negative), for a $\Omega > 1$ $(\Omega < 1)$ universe, and the converse happens on the right. Therefore, one expects the phase trajectories to emerge from the $(\Omega = 1, S = 0)$ singular point (which corresponds to an Einstein–de Sitter universe for stiff matter) at the left of the phase plane, move away from the $\Omega = 1$ flat model, reach a maximum deviation, and approach the flat model again, which is reached asymptotically at the singular point $(\Omega = 1, S = +\infty)$ (this time, the de Sitter universe).

Actually, this is exactly what happens, with two important exceptions. These are, firstly, the $\Omega > 1$ trajectories which reach $\Omega = +\infty$ before entering the inflationary region, starting at the maximum deviation point, and which correspond to the closed models which thus recollapse before inflation (note that the $\Omega = +\infty$ value of the density parameter occurs when $H = 0$, which is also the situation for which our conformal time coordinate breaks down). Secondly, the exceptional Einstein static universe that is the fixed point of the phase plane, and the four models that asymptotically approach it, that form separatrices in the phase plane.

The behavior of these inflaton models under the influence of a flat potential is represented in Figs. 12 and 13, where the (Ω, S) phase diagrams are drawn (representing the curves on which both $\dot{\phi}_0$ is constant and C is constant, these being integration constants of the field equations). We have drawn them in two ways (1) with

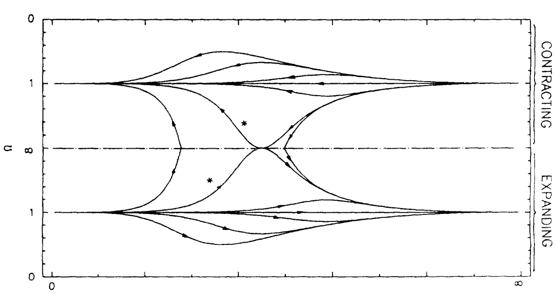

FIG. 12. Compactified phase diagram showing the evolution of Ω with S for a scalar field fluid with a flat potential. The lower half of the plane corresponds to expanding universes, the upper half to contracting universes. The integration constant C is held fixed and the ratio $\dot{\phi}^2/(2V_0)$ varied. Higher values of this ratio correspond to curves closer to $\Omega = 1$. The critical trajectories marked with an asterisk separate the universes which recollapse (bounce) from the ever-expanding (contracting) universes. These trajectories are characterized by $C(\dot{\phi}^2/2V_0)^{1/3} = \frac{2}{3}$.

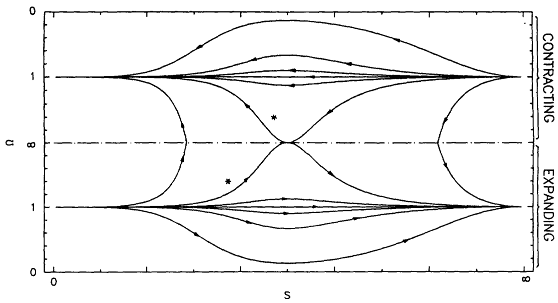

FIG. 13. Compactified phase diagram showing the evolution of Ω with S for a scalar fluid with a flat potential. The lower half of the plane corresponds to an expanding universe, the upper half to a contracting universe. The ratio $\dot{\phi}^2/(2V_0)$ is held fixed and the integration constant C is varied. Higher values of this correspond to curves closer to $\Omega = 1$. The critical trajectories marked with an asterisk separate the universes which recollapse from the ever-expanding (or contracting) universes. These trajectories are characterized by $C(\dot{\phi}^2/2V_0)^{1/3} = \frac{2}{3}$.

the same value of $\dot{\phi}_0$ on all curves (C varying between the curves), and (2) with C taking the same value on all curves (and $\dot{\phi}_0$ varying between them). As was done in [1] we have brought the infinities of both S and Ω to finite values by suitable transformations, such that the global dynamical behavior of the models might be fully apparent. The main differences between the two phase diagrams concern the relative position of the points of maximum deviation from the $\Omega = 1$ model. For the fixed $\dot{\phi}_0$ case, all the trajectories which exhibit inflationary behavior reach the critical $\gamma = \frac{2}{3}$ value for the same S, namely, for $S = (\dot{\phi}_0/V_0)$. For the second case above, that critical value is reached for values of S which increase with $\dot{\phi}_0$. It was implicit up to this point, that we were keeping the value of V_0 fixed in both figures. There is no loss of generality in doing so, since what really matters is the ratio $\dot{\phi}_0/V_0$ and therefore, a phase diagram with V_0 varying but nonzero would look like the second of the cases described above (the case $V_0 = 0$ being of different character, see [1]).

C. Scalar fluid with a steady-slope potential

Let us now consider a steady-slope potential in order to try and improve on the results of the preceding subsection. This is suggested by our discussion that a nonvanishing time derivative of the potential induces variations of the effective γ index. One requires that, at a certain stage and during a long enough period, the potential increases so that the kinetic energy of the inflaton field is efficiently reduced. The simplest potential which enables this is one exhibiting a steady slope over a sufficiently wide region of its domain [25]. Using the scale factor S as the time variable, the scalar field equation becomes

$$\frac{d}{dS}\left(\frac{\dot{\phi}^2}{2}S^6\right) = -\frac{dV}{dS}S^6 \,. \tag{105}$$

Making the steady-slope ansatz $V = \omega_0\phi + V_0$, where V_0 and ω_0 are constants, yields

$$\frac{d}{dS}\left(\frac{\dot{\phi}^2}{2}S^6\right) = -\omega_0\frac{d\phi}{dS}S^6 \,. \tag{106}$$

Now, we make the assumption that $\frac{d\phi}{dS} > 0$ and that, in particular,

$$\frac{d\phi}{dS} = S^{Q-1} \tag{107}$$

where Q is a constant. The equations can be integrated to yield

$$\frac{\dot{\phi}^2}{2} = \frac{\psi_0}{S^6} - \frac{\omega_0}{Q}S^Q, \tag{108}$$

$$V = V_0 + \frac{\omega_0}{Q}S^Q \,. \tag{109}$$

Here ψ_0 is a constant of integration giving the initial value of $\dot{\phi}^2$.

It becomes clear that, dependent on appropriate signs and values of ω_0 and K, the kinetic energy is now brought to a minimum in a finite time. Actually, in the cases for which it vanishes, the steady-slope approximation breaks down at that particular point, since the kinetic energy cannot afterward become negative.

The (Ω, S) phase planes for this shape of potential are given by the equation

$$\frac{d\Omega}{dS} = \left(2 - 3\frac{\psi_0 - \frac{\omega_0}{Q}S^{Q+6}}{V_0S^6 + \psi_0}\right)(1-\Omega)\frac{\Omega}{S}. \tag{110}$$

Different diagrams result according to the relative values of ω_0 and Q. Using the Friedmann equation it is possible to derive the $k = 0$ solution. We have

$$\frac{\dot{S}}{S} = \pm\sqrt{\frac{8\pi G}{3}\left(\frac{\psi_0}{S^6} + V_0\right) - \frac{k}{S^2}} \,. \tag{111}$$

This, taking Q to vanish, leads to the identity

$$t - t_0 = \pm\int\frac{S^2\,dS}{\sqrt{\frac{8\pi G}{3}\left(\frac{\psi_0}{S^6} + V_0\right)}} \tag{112}$$

which yields

$$S^3 = \sqrt{\frac{\psi_0}{V_0}}\sinh\left\{\sqrt{24\pi G V_0}(t - t_0)\right\} \,. \tag{113}$$

In the case that the inflaton potential has a constant slope, then, we can conclude that there exist a variety of solutions, in particular some like that shown in (113) which are plausible inflationary solutions.

V. CONCLUSIONS

The phase plane for a standard FLRW model with matter, radiation, and a cosmological constant is like that shown in Fig. 7; for a standard FLRW model with just matter and radiation ($\Lambda = 0$) it is like Fig. 1; and for a complete inflationary model ($\Lambda > 0$ for $S < S_f$, but $\Lambda = 0$ for $S > S_f$) it is like a combination of these. (Imagine Fig. 6 with the evolution curves smooth at all boundaries, as in fact Fig. 5 has been drawn.) These curves help one to visualize the probabilities of different values of Ω arising from specification of the microphysics, i.e., the function $\gamma(S)$. However, as noted above, the physics of the universe may not be time symmetric between expansion and collapse; if not, then separate forms of $\gamma(S)$ are appropriate for the expansion and collapse phases, so that corresponding adjustments must be made to the above-mentioned diagrams. In particular, $\gamma(S)$ will no longer be symmetric between the top and bottom halves of Figs. 2, 4, 6, and 7. The oscillating (nonsingular) models occurring in the inflationary cases are possible because a reversible inflation mechanism has been assumed in the form of $\gamma = \gamma(S)$. This will not be true for all inflationary mechanisms: some will be *time irreversible*, and these will not allow such cyclic behavior [14, 15]. If the inflationary mechanism is not time reversible, but γ can still be represented in the form $\gamma = \gamma(S, \Omega)$ separately in the expanding and collapsing phases, then similar diagrams to those presented here will be possible in that case too.

Note also that there may be no era before the inflationary epoch when the Universe is dominated by radiation, so the history of the Universe could plausibly begin on the left of the epoch labeled II in Table I and Fig. 5. For example, the Universe may initially be dominated by curvature, inhomogeneities, or part of the stress tensor of a scalar field [16]. These possibilities are examined in detail in Ref. [12] in the context of chaotic inflation [17].

Whatever happens at early times, it is clear that at any given value of S, one can choose any value for Ω and find an evolution curve for an inflationary universe that will lead to that value of Ω at that stage of evolution of the universe. That is, inflation does not, without some further input, imply that $\Omega = 1$ at the present: in fact, by itself it does not solve the "flatness problem," as it does not automatically require Ω to be within 2 orders of magnitude of unity.

Conversely, the fact that Ω is now measured to lie in the range $(10^{-1}, 10)$ cannot be taken as providing conclusive evidence that our Universe evolved through an inflationary epoch sometime in the past. Such an input will probably consist of a restriction on the allowed range of values of the spatial curvature. Most work on inflation implicitly assumes that such constraints hold, as indeed seems reasonable (see [12] for a discussion). It should be emphasized, however, that any constraint on the initial value of the spatial curvature applied near the classical singularity must, of necessity, be derived by taking quantum effects into account, since the classical dynamical system (1)–(3) describes the behavior of systems with *all* possible initial spatial curvature values. The dimensional arguments used in deriving quantal constraints are strongly physically motivated, but even so, the actual problem of defining a useful measure over the space of initial data remains without a total solution [18].

It should also be remarked that the alternative phase curves drawn in the (q, Ω) plane, as in the pioneering work of [2], are simply obtained by changing the independent variable from S to q, the relation being given by Eq. (10). Although the phase planes show very clearly the nature of the evolution of these universes, they do not, of course, show the relation between the chosen variables and cosmic proper time. In order to see at what point during an inflationary period the cosmological problems are solved still requires detailed calculations of the sort performed in Ref. [19].

In this paper we have also shown that the models constructed as having a potential of constant slope form a bridge between the qualitative investigations of [1] and the physical models normally used to describe inflation. The simplest of these are achieved when the scalar field has a completely level and flat potential, which provides an easily soluble model, while approximating the dynamics of a typical inflationary model for a relatively long period. Even more interesting from the point of view of trying to find exact solutions which model the behavior desired of inflationary models are those described at the end of the previous section in which the potential has a constant slope. Clearly a considerable amount of work remains to be done with these models.

Overall, then, the figures presented here demonstrate that the qualitative conclusions of [1] remain valid even when the effective equation of state is determined by the behavior of a scalar field rolling on a simple potential. Furthermore, the present treatment comprises a considerable improvement over the earlier work. This is because the transition from the velocity-dominated noninflating phase (when $\dot{\phi}^2 \gg V[\phi]$) to the slow-rolling inflationary phase (with $\dot{\phi}^2 \ll V[\phi]$) is not set arbitrarily, as it was modeled in [1], but is given as part of the dynamics specified by the initial conditions of the models.

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