Nucleation of relativistic first-order phase transitions

László P. Csernai
Theoretical Physics Institute, University of Minnesota, Minneapolis, Minnesota 55455
*and Physics Department, University of Bergen, N-5007 Bergen, Norway**

Joseph I. Kapusta
School of Physics and Astronomy, University of Minnesota, Minneapolis, Minnesota 55455
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We apply the general formalism of Langer to compute the nucleation rate for systems of relativistic particles with a zero or small baryon-number density and which undergo first-order phase transitions. In particular, we obtain an expression for the pre-exponential factor and it is proportional to the viscosity. The initial growth rate of a critical size bubble or droplet is limited by the ability of dissipative processes to transport latent heat away from the surface.

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I. INTRODUCTION

The dynamics of first-order phase transitions has fascinated scientists at least since the time of Maxwell and van der Waals. Much work on the classical theory of nucleation of gases and liquids was carried out in the early part of this century, culminating in the theory of Becker and Döring [1]. There were and still are many important applications, such as cloud and bubble chambers, freezing of liquids, and precipitation in the atmosphere. The modern theory of nucleation was pioneered by Langer [2]. Langer's theory is based in a more fundamental way on the microscopic interactions of atoms and molecules. It can also be applied close to a critical point where in fact most of the current interest in the condensed matter community has been. Finally, nucleation theory has been extended to relativistic quantum field theory by Callan and Coleman [3] for zero temperature and by Affleck [4] and Linde [5] for finite temperature. Applications here are to elementary particle phase transitions in the early Universe [6, 7] and even to the nucleation rate for black holes [8].

The goal of nucleation theory is to compute the probability that a bubble or droplet of the A phase appears in a system initially in the B phase near the critical temperature. Homogeneous nucleation theory applies when the system is pure; inhomogeneous nucleation theory applies when impurities cause the formation of bubbles or droplets. For the applications we have in mind, namely, the early Universe and very-high-energy nuclear collisions, it seems that homogeneous nucleation theory is appropriate. In the everyday world it is usually the opposite; dust or ions in the atmosphere are much more efficient in producing precipitation. Nucleation theory is applicable for first-order phase transitions when the matter is not dramatically supercooled or superheated. If substantial supercooling or superheating is present, or if the phase transition is second order, then the relevant dynamics is spinodal decomposition. In this paper we concern ourselves only with homogeneous nucleation theory. For an excellent overview of all these topics see Ref. [9].

Suppose that a system is cooled below its critical temperature. Then there exists a critical sized droplet (or bubble, depending on whether the energy density in the lower-temperature phase is greater or less than the higher-temperature phase). If a droplet that forms because of statistical fluctuations is too small, its surface free energy is relatively large and the cost in total free energy is positive. The droplet will evaporate. If the droplet is large, its surface free energy is unimportant, and the droplet will accrete molecules and grow. A droplet of critical size is metastable, it is balanced between evaporation and accretion. The classical theory of Becker and Döring [1], which is nicely reviewed by McDonald [10], says that the probability per unit time per unit volume to nucleate the dense liquid phase from a dilute gas is given by

$$I = a(i_*)\left(\frac{\epsilon''(i_*)}{2\pi T}\right)^{1/2}\bar{n}(1)e^{-\epsilon(i_*)/T}, \tag{1}$$

where $\epsilon(i_*)$ is the formation energy of a critical sized droplet consisting of i_* molecules, a prime denotes differentiation with respect to the number of molecules i, T is the temperature, $\bar{n}(1)$ is the density of single molecules and $a(i_*)$ is the accretion rate of single molecules on a critical droplet. Usually the accretion rate is taken to be proportional to the surface area of the critical droplet times the mean speed of molecules in the gas times a "sticking fraction" less than 1. The first term in Eq. (1) is a dynamical factor influencing the growth rate, the second term characterizes fluctuations about the critical droplet, and the product of the third and fourth terms gives the quasiequilibrium number density of crit-

*Permanent address.

ical sized droplets.

Langer's modern theory of nucleation yields the following formula for the rate:

$$I = \frac{\kappa}{2\pi}\Omega_0 e^{-\Delta F/T} \tag{2}$$

where ΔF is the change in the free energy of the system due to the formation of the critical droplet. Ω_0 is a statistical prefactor which measures the available phase space volume. κ is a dynamical prefactor which determines the exponential growth rate of critical droplets which are perturbed from their quasiequilibrium radius R_*.

$$\kappa = \frac{d}{dt}\ln[R(t) - R_*]. \tag{3}$$

The basic structure is the same as in the classical theory, but the prefactors are different. The dynamical prefactor has been calculated by Langer and Turski [11, 12] and by Kawasaki [13] for a liquid-gas phase transition near the critical point, where the gas is not dilute, to be

$$\kappa = \frac{2\lambda\sigma T}{\ell^2 n_\ell^2 R_*^3} \tag{4}$$

which involves the thermal conductivity λ, the surface free energy σ, the latent heat per molecule ℓ and the density of molecules in the liquid phase n_ℓ. The interesting physics in this expression is the appearance of the thermal conductivity. In order for the droplet to grow beyond the critical size latent heat must be conducted away from the surface into the gas. For a relativistic system of particles or quantum fields, which has no net conserved charge such as baryon number, the thermal conductivity vanishes. The reason is that there is no rest frame defined by the baryon density to refer to heat transport. Hence this formula obviously cannot be applied to such systems.

A relativistic quantum field theory approach has been worked out by Callan and Coleman [3] for nucleation from one vacuum to another and extended by Affleck [4] and Linde [5] to finite temperature. In the limit that thermal fluctuations dominate quantum fluctuations the rate is

$$I = \frac{\omega_-}{\pi}\left(\frac{S_3}{2\pi T}\right)^{3/2}\left(\frac{\det'[-\nabla^2 + V''(\phi,T)]}{\det[-\nabla^2 + V''(0,T)]}\right)^{-1/2} e^{-S_3/T}, \tag{5}$$

where S_3 is the three-dimensional action associated with the formation of a critical sized bubble or droplet. The ω_- is the frequency of the unstable mode. The ratio of determinants contains some interesting dynamics. The V'' is the second derivative of the potential with respect to the "order parameter" ϕ which is actually a field describing the shape of the critical bubble or droplet. This ratio of determinants is almost never evaluated because it must be done numerically. A notable example of such an evaluation is presented by McLerran *et al.* [14] for the sphaleron of electroweak theory. Usually dimensional analysis is invoked to approximate the preexponential factor by T^4 or by T_c^4, so that

$$I = T^4 e^{-\Delta F/T} \quad \text{or} \quad I = T_c^4 e^{-\Delta F/T}. \tag{6}$$

It is clear that the nonrelativistic expressions are inadequate when we deal with a relativistic quantum field theory at finite temperature. We have in mind the QCD and electroweak phase transitions when the baryon density is negligible. We want to do better than estimate the prefactor by dimensional analysis as quoted above. To evaluate the ratio of determinants is extremely cumbersome and must be done numerically. In addition it requires identification of a scalar field as an order parameter, which has not yet been accomplished in QCD. The classical expression is probably not accurate because both the hadronic matter and the quark-gluon plasma are expected to be rather dense and strongly interacting near the critical temperature, not dilute. Therefore we have set our goal in this paper to estimate the pre-exponential factor using the basic formalism of Langer as applied to a course-grained effective field theory with zero net baryon number.

In Sec. II we briefly review the essential features of Langer's nucleation formalism. In Sec. III we construct an effective, course-grained field theory to describe the state of matter which interpolates between the two equilibrium phases. This is done in the sense of a Landau expansion of the free energy away from equilibrium states. We determine the parameters in terms of physical quantities, such as the surface free energy and the surface thickness. In Sec. IV we calculate the prefactor using relativistic hydrodynamics. It turns out that κ has a form very similar to the nonrelativistic expression, but with the thermal conductivity replaced by the viscosity. In Sec. V we put everything together to obtain the nucleation rate as a function of temperature, and summarize our findings.

In a subsequent paper we develop a set of rate equations which govern the time evolution of the system as it converts from one phase to another using as input the nucleation rate. We propose that these are more accurate than the expression given by Guth and Weinberg [15] and widely used in the literature on phase transitions in the early Universe. We apply these phase transformation equations to two problems: the QCD phase transition in the early Universe and in high-energy heavy-ion collisions.

II. REVIEW OF LANGER'S FORMALISM

The general theory of nucleation developed in Ref. [2] starts with the introduction of a set of variables η_i, $i = 1, \ldots, N$, which describe N degrees of freedom of the system of interest. Oftentimes it is convenient to take these to be collective coordinates. In the system with which we are concerned, for example, the η_i will be the energy density $e(\mathbf{r})$ and flow momentum $\mathbf{M}(\mathbf{r})$ at positions $\mathbf{r}$ in the system. Thus a sum over the index i represents an integration over $\mathbf{r}$ and a sum over each of the density and flow fields.

We next introduce a distribution function $\rho(\{\eta\}, t)$, which is a probability density over configurations $\{\eta\}$ and is also a function of time t. We assume that $\rho(\{\eta\}, t)$ sat-

isfies a continuity equation of the form

$$\frac{\partial\rho}{\partial t} = \partial_t\rho = -\sum_{i=1}^{N}\frac{\partial J_i}{\partial\eta_i}, \tag{7}$$

where the probability current J_i is given by

$$J_i = -\sum_{j=1}^{N} M_{ij}\left(\frac{\partial F}{\partial\eta_j}\rho + T\frac{\partial\rho}{\partial\eta_j}\right) . \tag{8}$$

Here M_{ij} is a generalized mobility matrix and $F\{\eta\}$ is a coarse-grained free energy. Both of these latter quantities will be discussed in detail below in more specific connections. Note that Eqs. (7) and (8) can be derived via standard statistical techniques [2] by adding a suitable Langevin force to the Hamiltonian equations of motion

$$\partial_t\eta_i = -\sum_{j=1}^{N} A_{ij}\frac{\partial F}{\partial\eta_j} , \tag{9}$$

where A is an antisymmetric matrix with entries 0 or 1.

The choice of variables η_i should depend on the problem, but this general description is applicable to all systems that are characterized by the equation of motion of the form (9). If we have chosen a suitable set of coordinates we can describe the system in this η space. In this η space the equilibrium configurations, for which $\partial_t\rho = 0$, have the probability distribution of the form

$$\rho_{\rm eq}\{\eta\} \propto \exp[-F\{\eta\}/T] . \tag{10}$$

Such configurations are the initial metastable point in the η-space denoted by $\{\eta_0\}$, or the final state. In this space the phase transition starts from a metastable point $\{\eta_0\}$ and moves to the vicinity of a stable point: a point where F has its minimum. In this process the system most likely has to pass a saddle point $\{\bar\eta\}$. See Fig. 1. This configuration, $\{\bar\eta\}$, describes a situation close to $\{\eta_0\}$ except for the presence of one critical size droplet of the new phase. At the saddle point we can assume stationary flow $\partial_t\rho = 0$, and we can calculate the current across this saddle. The rate of probability flow across this saddle $\{\bar\eta\}$ determines the droplet formation rate in the system. This rate

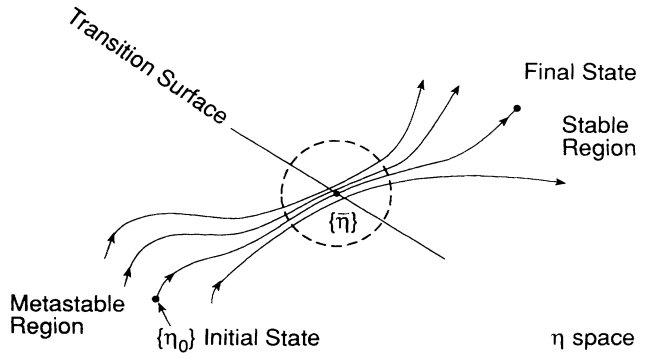

FIG. 1. Paths for nucleating a stable phase from a metastable phase in η space.

$$I = I_0\exp[-\Delta F/T] , \tag{11}$$

telling us the number of critical size droplets created in unit volume in unit time, was evaluated in Ref. [2]. The activation energy ΔF is given by

$$\Delta F = F\{\bar\eta\} - F\{\eta_0\} . \tag{12}$$

The prefactor I_0 is the product of two terms

$$I_0 = \frac{\kappa}{2\pi}\Omega_0 , \tag{13}$$

the dynamical prefactor κ (of dimension [c/fm]) and the statistical prefactor Ω_0 (of dimension [fm^{-3}]). In terms of the eigenvalues $\bar\lambda_\alpha$ and $\lambda_\alpha^{(0)}$ of the matrix

$$\frac{\partial^2 F\{\eta\}}{\partial\eta_i\partial\eta_j} ,$$

evaluated at points $\{\bar\eta\}$ and $\{\eta_0\}$ respectively, the statistical prefactor can be written as

$$\Omega_0 = \mathcal{V}\left(\frac{2\pi T}{|\bar\lambda_1|}\right)^{1/2}\prod_{\alpha=\alpha_0+2}^{N}\left(\frac{2\pi T}{\bar\lambda_\alpha}\right)^{1/2}\prod_{\alpha=1}^{N}\left(\frac{\lambda_\alpha^{(0)}}{2\pi T}\right)^{1/2} , \tag{14}$$

where $\mathcal{V}$ is the volume of the η-space available for the flux of probability flow [2]. We will evaluate Ω_0 in the next section.

Since $\{\eta_0\}$ is a minimum of F, all the $\lambda_\alpha^{(0)}$'s must be positive. Similarly, the $\bar\lambda_\alpha$ are eigenvalues of the above second-derivative matrix, in this case evaluated at the saddle point $\{\bar\eta\}$. Because $\{\bar\eta\}$ locates the highest point along the path of lowest energy leading away from $\{\eta_0\}$, there must be one and only one eigenvalue $\bar\lambda_\alpha$, which is negative. This is the eigenvalue denoted by $\bar\lambda_1$ in Eq. (14). If $F\{\eta\}$ has translational symmetry, then there must be at least three other eigenvalues $\bar\lambda_\alpha$ which are identically zero, corresponding to the three independent translations of the position of the symmetry-breaking fluctuation (bubble or droplet) described by $\{\bar\eta\}$. The product of $\bar\lambda$'s appearing in Eq. (14) starts with $\alpha = \alpha_0 + 2$, where α_0 is the total number of symmetries of F which are broken by $\{\bar\eta\}$. The integration over these α_0 degrees of freedom defines the factor $\mathcal{V}$ in Eq. (14). That is, $\mathcal{V}$ is the volume of the α_0-dimensional region of η-space spanned by the set of configurations $\{\bar\eta\}$ which leave $F\{\bar\eta\}$ invariant.

The dynamical factor κ is the exponential growth rate of the unstable mode of deformation $\{\bar\eta\}$. To compute κ, we linearize Eq. (9) about $\eta_i = \bar\eta_i$. That is, we write

$$\partial_t\nu_i = -\sum_{j,l=1}^{N} M_{ij}\frac{\partial F}{\partial\bar\eta_j\partial\bar\eta_l}\nu_l , \tag{15}$$

where $\nu_i = \eta_i - \bar\eta_i$. Then, setting $\nu \propto e^{\kappa t}$, we identify κ as the positive eigenvalue of the matrix

$$-\sum_{j=1}^{N} M_{ij}\frac{\partial F}{\partial\bar\eta_j\partial\bar\eta_l} . \tag{16}$$

In the nucleation problem the instability described by κ is the initial growth rate of a bubble or droplet which has just exceeded the critical size.

III. COURSE-GRAINED EFFECTIVE FIELD THEORY

The model of nucleation which we propose here will be defined by the choice of the statistical variables η_i and the corresponding coarse-grained free energy $F\{\eta\}$. The conventional formulation of classical many-body statistical mechanics in terms of particle positions and momenta is not very convenient for the present purpose. Nucleation is characterized by semimacroscopic fluctuations involving large numbers of particles. Therefore hydrodynamic-type collective variables are more appropriate to describe the formation of bubbles or droplets.

Hydrodynamics can be derived from microscopic kinetic theory by a coarse-graining or cellular method [16, 17]. That is, one divides up the macroscopic system into semimacroscopic cells of given volume and assigns specific densities and flows to each of these cells. The free energy computed by performing a partition sum subject to the cellular constraints is the coarse-grained F that we are talking about. There is no problem, in principle, in summing over the cellular densities and flows to obtain the true equilibrium free energy. Moreover, as long as each cell comes to local thermal equilibrium rapidly compared to the times required for the hydrodynamic processes that one wants to consider, then one can use the coarse-grained F for computing nonequilibrium properties of the system.

The question which arises at this point is: What is a suitable size for the coarse-graining cells? In order for the hydrodynamic description to make sense, the cell volume must be much larger than the average volume per molecule. For our purposes, however, the cells cannot have linear dimensions appreciably larger than a correlation length. If the cells are chosen to be too large, phase separation will occur within single cells, and the interesting details of the condensation mechanism will be lost in the process of taking cellular averages. To put this another way, we expect F as a function of the average energy density e to be a nonconvex function with distinct minima corresponding to the two phases. But, if the cell size is large enough for well-defined phase separation to occur within a cell, then F must approach its convex envelope and cannot possibly have the above property. We conclude that the cell size can be neither much larger nor much smaller than a correlation length.

A. Relativistic hydrodynamics

The equations of motion of relativistic fluid dynamics, $\partial_\mu T^{\nu\mu} = 0$, can be given in terms of $E \equiv T^{00}$ and $M^i = T^{0i}$, that is, $E = (e+pv^2)\gamma^2$ and $\mathbf{M} = (e+p)\gamma^2\mathbf{v}$, where e is the energy density and p is the pressure [18]. The low speed limit of relativistic fluid dynamics ($\gamma^2 \approx 1$ and $pv^2 \ll e$, but p not assumed small compared to e) is given by

$$\partial_t e = -\nabla \cdot \mathbf{M} \tag{17}$$

and

$$\partial_t \mathbf{M} = -\nabla \cdot \left(\frac{1}{w}\mathbf{M} \otimes \mathbf{M}\right) - \nabla p \ . \tag{18}$$

Here $w = e + p$ is the enthalpy density, we assumed that the relativistic energy density is $E = (e+pv^2)\gamma^2 \approx e$, and that the relativistic momentum density is $\mathbf{M} = w\gamma^2\mathbf{v} \approx w\mathbf{v}$.

With the above mentioned restrictions in mind we will try to find a suitable form for the coarse-grained free energy F. This is not a trivial problem. We choose as our basic variables the local energy density and momentum density fields, $e(\mathbf{r},t)$ and $\mathbf{M}(\mathbf{r},t)$. The free energy F must consist of a kinetic energy F_K and an interaction term F_I. The kinetic term is simply

$$F_K(e,\mathbf{M}) = \frac{1}{2}\int d^3r\ w\mathbf{v}^{\,2} = \int d^3r\ \frac{1}{2w}\mathbf{M}^2 \ . \tag{19}$$

We shall assume that F_I is a functional of e only, and that it can be written in the form [19, 20]

$$F_I\{e(\mathbf{r})\} = \int d^3r\ \left[\frac{1}{2}K(\nabla e)^2 + f(e)\right] \ , \tag{20}$$

where $f(e)$ is the Helmholtz free energy density and $\frac{1}{2}K(\nabla e)^2$ is the usual gradient energy. K is a constant to be determined. Note that in this discussion we assume that the temperature T is constant.

Using the above F with the mobility matrix

$$\begin{aligned} M_{ij} &= \partial_j(M_i) + (M_i)\partial_j - \frac{M_j}{2w}(\partial_i w) \ , \\ M_{i0} &= -\partial_i e \ , \\ M_{00} &= 0 \ , \\ M_{0i} &= (\partial_i w) + w\partial_i \ , \end{aligned} \tag{21}$$

the equations of motion for e and $\mathbf{M}$, that is, Eqs. (17) and (18), are obtained as the low speed limit of relativistic fluid dynamics:

$$\begin{aligned} \partial_t e &= -(\nabla w)\cdot\frac{\delta F_K}{\delta \mathbf{M}(\mathbf{r})} - w\nabla\cdot\frac{\delta F_K}{\delta \mathbf{M}(\mathbf{r})} \\ &= -\nabla\cdot\mathbf{M}(\mathbf{r}) \end{aligned} \tag{22}$$

is the equation for energy conservation and

$$\begin{aligned} \partial_t \mathbf{M} &= -\left[\nabla\mathbf{M} + \mathbf{M}\,\nabla - \frac{\mathbf{M}}{2w}\nabla w\right]\cdot\frac{\delta F_K}{\delta \mathbf{M}(\mathbf{r})} + \frac{\delta F}{\delta e(\mathbf{r})}\nabla e \\ &= -\nabla\cdot\left(\frac{1}{w}\mathbf{M}\otimes\mathbf{M}\right) - K(\nabla^2 e)\nabla e + \frac{\partial f}{\partial e}\nabla e \end{aligned} \tag{23}$$

is the equation for momentum conservation, the Euler equation. From Eqs. (18) and (23) it is clear that we must identify the last term on the right-hand side with the gradient of the pressure; that is,

$$\frac{\partial f}{\partial e}\nabla e = \nabla f \longrightarrow -\nabla p \tag{24}$$

in the limit of a uniform system in equilibrium. Note that

when $e(\mathbf{r})$ is varying so slowly that the gradient energy can be neglected, Eq. (20) is consistent with

$$f(e) = e - Ts = -p \ . \tag{25}$$

At asymptotically high energy densities the equation of state of the QCD plasma is that of the Stefan-Boltzmann gas with a bag constant, $f(e) = -e/3 + B$. However, we have to bear in mind that near the phase transition, and particularly in the metastable region, the equation of state is not that of a Stefan-Boltzmann gas. Therefore we will keep $f(e)$ explicit throughout the equations.

B. Parametrization of the free energy

Imagine having two phases in equilibrium with each other at temperature T, and furthermore consider an interface separating them. This interface cannot be perfectly sharp. The interface must have a finite thickness on the order of a correlation length. In a local density picture the energy density e should vary smoothly from one phase to the other. Since first-order phase transitions have a latent heat, this means that we should know the free energy density $f(e)$ for the energy density ranging between one phase and the other. To be specific, in what follows we assume we are dealing with the QCD phase transition. Then the low-temperature, low-energy-density phase will be denoted by a subscript h (for hadron) and the high-temperature, high-energy-density phase will be denoted by a subscript q (for quark). These subscripts can be replaced by others if one considers the electroweak phase transition, for example, or some other phase transition. In addition to the need to know $f(e)$ for $e_h < e < e_q$ we would also encounter situations where we would need to know $f(e)$ for a range of values about e_h and e_q. Statistical fluctuations about local thermal equilibrium would require such knowledge, for example.

For a range of temperatures about T_c, $f(e)$ should have minima located at $e_h(T)$ and $e_q(T)$. There should also be a barrier between these two minima located at some $e_0(T)$. We require that

$$\begin{aligned} f(e_h(T)) &= -p_h(T) \ , \\ f(e_q(T)) &= -p_q(T) \ . \end{aligned} \tag{26}$$

Therefore, at fixed T we shall parametrize $f(e)$ by a fourth-order polynomial in e. Because of the pinning of the two local minima shown above, $f(e)$ will have its global minimum at $e_q(T)$ when $T > T_c$, and will have its global minimum at $e_h(T)$ when $T < T_c$. At the critical temperature the two minima of $f(e)$ are equal. Our parametrization is

$$\begin{aligned} f(e) = f_0 + \frac{f_0''(e-e_0)^2}{2} - \frac{(e_h+e_q-2e_0)f_0''}{3(e_h-e_0)(e_q-e_0)}(e-e_0)^3 \\ + \frac{f_0''}{4(e_h-e_0)(e_q-e_0)}(e-e_0)^4, \end{aligned} \tag{27}$$

where $e_h(T)$, $e_q(T)$, $p_h(T)$, and $p_q(T)$ are specified functions of T, and f_0'' is the curvature of f at the top of the barrier located at e_0 ($f_0'' < 0$). Let us define $\Delta e \equiv e_q - e_h > 0$ and $\Delta p \equiv p_h - p_q$. In terms of these variables

$$e_0 = \frac{e_h+e_q}{2} + \frac{f_0''(\Delta e)^3}{12\Delta p} \pm \left[\left(\frac{f_0''(\Delta e)^3}{12\Delta p}\right)^2 + \frac{(\Delta e)^2}{4}\right]^{1/2} , \tag{28}$$

where the + (−) is chosen when $\Delta p > 0$ ($\Delta p < 0$) and

$$f_0 = -p_q + \frac{f_0''}{12}\frac{(e_q-e_0)^2(e_q-2e_h+e_0)}{e_h-e_0} \ . \tag{29}$$

The first derivative of f is

$$f'(e) = \frac{\partial f}{\partial e} = \frac{f_0''(e-e_0)(e-e_h)(e-e_q)}{(e_h-e_0)(e_q-e_0)} \ . \tag{30}$$

Thus, if the location of the two minima and their depths are given for fixed T, then only one free parameter f_0'' remains. In particular, this parameter determines the barrier height, position, and curvature at all energy densities:

$$\begin{aligned} f''(e) = \frac{f_0''}{(e_h-e_0)(e_q-e_0)} \ [&(e-e_0)(e-e_h) \\ &+(e-e_0)(e-e_q) \\ &+(e-e_h)(e-e_q)] \ . \end{aligned} \tag{31}$$

See Fig. 2 for illustrations of $f(e)$ when T is greater than, equal to, and less than T_c. Unless we can extract this free-energy function from the Lagrangian in a more fundamental way we shall be content to use this parametrization in the following analyses.

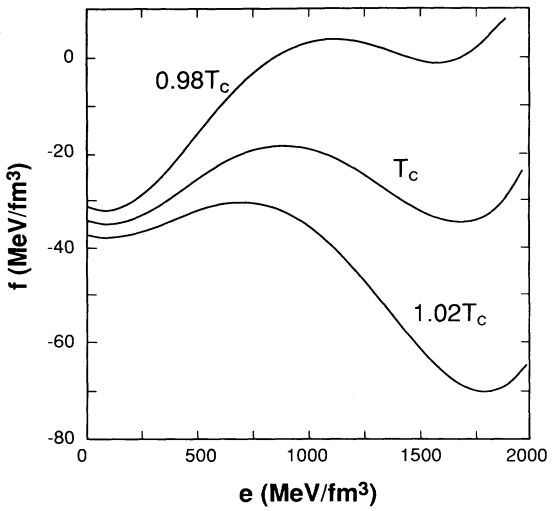

FIG. 2. The bulk free energy density at fixed temperatures of $T = 0.98\,T_c$, $T = T_c$ and $T = 1.02\,T_c$. The numerical scales are obtained from a particular model of the QCD phase transition as discussed in Sec. III.

C. Surface profile

For the remainder of this paper we restrict ourselves to the case of idealized bubbles or droplets. That is, we consider only the limit in which the nucleating fluctua-

tion described by $\{\bar{\eta}\}$ is, indeed, a well-defined sphere of the hadronic phase with radius R large compared to the interface thickness or the correlation length ξ (to be defined below). In principle we need not make this restriction in the present theory. As we shall see, however, this limit is the appropriate one in the cases of interest here. By going to this limit we can do all of our calculations analytically instead of having to resort to numerical methods.

The stationary point $\{\bar{\eta}\}$ is given in our model by $\mathbf{v}(\mathbf{r}) = 0$ and $e(\mathbf{r}) = \bar{e}(\mathbf{r})$, where $\bar{e}$ satisfies

$$\frac{\delta F_I}{\delta \bar{e}(r)} = -K\nabla^2\bar{e} + \frac{\partial f}{\partial \bar{e}} = 0 \ . \tag{32}$$

For a spherical bubble of hadronic matter surrounded by quark-gluon plasma at $T < T_c$ the energy density $\bar{e}$ depends only on the distance r from the center of the bubble. Deep inside the bubble the energy density should be e_h; far away from the bubble the energy density should be e_q. The energy density profile $\bar{e}(r)$ then describes a smooth transition from one phase to the other. As discussed above, we will assume that the surface is located at a distance R from the center that is much greater than the surface thickness. Analogous dropletlike solutions for nonrelativistic matter with conserved particle number were considered in Refs. [21, 22].

Using our parametrization of $f(e)$ the static profile equation becomes

$$-K\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right]\bar{e} + f_0''\frac{(\bar{e}-e_0)(\bar{e}-e_h)(\bar{e}-e_q)}{(e_h-e_0)(e_q-e_0)} = 0. \tag{33}$$

We introduce a correlation length defined at the top of the barrier by $\xi_0^2 \equiv -K/f_0''$. Then

$$\frac{d^2\bar{e}}{dr^2} + \frac{2}{r}\frac{d\bar{e}}{dr} + \frac{(\bar{e}-e_0)(\bar{e}-e_h)(\bar{e}-e_q)}{\xi_0^2(e_h-e_0)(e_q-e_0)} = 0. \tag{34}$$

Let us find the behavior of the solution in each of three regions: (i) the interior hadronic region $r < R - \xi_0$; (ii) the interface region around R; (iii) the exterior quark-gluon plasma region $r > R + \xi_0$.

(i) In the interior of the bubble one can assume that $\bar{e} = e_h + g_1(r)$ where $g_1(r)$ is a small deviation from the equilibrium hadronic energy density. g_1 should satisfy

$$\frac{d^2g_1}{dr^2} + \frac{2}{r}\frac{dg_1}{dr} - \xi_h^{-2}g_1 = 0 \ , \tag{35}$$

where

$$\xi_h^2 = \xi_0^2(e_q - e_0)/\Delta e$$

defines the correlation length in the hadronic phase. The solution of this equation is of the form

$$g_1 = r^{-1}[A_1\sinh(-r/\xi_h) + B_1\cosh(r/\xi_h)].$$

From the requirement that the solution be finite at the origin we get $B_1 = 0$. In order to match onto the interface region A_1 must be very small, proportional to e^{-R/ξ_h}. Then $\bar{e}(r) \approx e_h$ throughout most of the interior.

(ii) Near R we can write $\bar{e} = e_0 + g_2(r)$. Linearizing in g_2 leads to

$$\frac{d^2g_2}{dr^2} + \frac{2}{r}\frac{dg_2}{dr} + \xi_0^{-2}g_2 = 0. \tag{36}$$

The general solution is

$$g_2(r) = \frac{A_2}{r}\sin\left(\frac{r}{\xi_0}\right) + \frac{B_2}{r}\cos\left(\frac{r}{\xi_0}\right) \ . \tag{37}$$

We require that $g_2(R) = 0$, which is equivalent to defining the location of the surface by the equation $\bar{e}(R) = e_0$. Thus the solution for $\bar{e}$ near the center of the bubble's surface is

$$\bar{e} = e_0 + \frac{A_2}{r}\sin\left(\frac{r-R}{\xi_0}\right) \tag{38}$$

and $A_2 > 0$.

(iii) For the exterior the solution has the same functional form as in the interior except that only $B_3 = 0$ is required by the boundary condition. The exterior solution is therefore

$$\bar{e} = e_q - \frac{A_3}{r}e^{-r/\xi_q}, \tag{39}$$

where

$$\xi_q^2 = \xi_0^2(e_h - e_0)/\Delta e$$

defines the correlation length in the QCD plasma and $A_3 > 0$.

At the critical temperature $f(e_h) = f(e_q)$. Then the free energy, Eq. (27), becomes symmetric, $e_0 = (e_h + e_q)/2$ and $\xi_q^2 = \xi_h^2 = \xi_0^2/2$. In this case the interfacial profile has a nice analytical solution in the planar ($R \to \infty$) limit:

$$\bar{e}(x) = \frac{1}{2}\left[e_h + e_q + \Delta e\tanh\left(\frac{x}{2\xi_q}\right)\right] \ . \tag{40}$$

Here the surface is located at $x = 0$ with hadronic matter on the left and QCD plasma on the right.

Suppose that a hadronic bubble has formed in the QCD plasma at $T < T_c$ because of statistical fluctuations. The change in free energy of the system is

$$\Delta F = \frac{4\pi}{3}(f_h - f_q)R^3 + 4\pi R^2\sigma, \tag{41}$$

where σ is the surface free energy. In our case, for baryon free matter,

$$\Delta F = \frac{4\pi}{3}[p_q(T) - p_h(T)]R^3 + 4\pi R^2\sigma. \tag{42}$$

The hadronic droplet is stationary if $\partial_R\Delta F = 0$, which leads to Laplace's formula

$$p_h(T) - p_q(T) = \frac{2\sigma}{R(T)} \ . \tag{43}$$

Thus the activation energy, in our approximation, is

$$\Delta F = \frac{4}{3}\pi\sigma R^2 \ . \tag{44}$$

The surface free energy can be calculated from our parametrization of F_I. For a planar interface or for a sphere whose radius is much greater than its surface thickness, the formula is [19]

$$\sigma = K \int_{-\infty}^{\infty} dx \left(\frac{d\bar{e}}{dx}\right)^2 \ . \tag{45}$$

Inserting the solution for the planar interface at T_c from Eq. (40), this integral takes the form

$$\sigma = K \left(\frac{\Delta e}{2}\right)^2 \frac{1}{2\xi_q} \int_{-\infty}^{\infty} dz \frac{1}{\cosh^4(z)} = \frac{K(\Delta e)^2}{6\xi_q} \ . \tag{46}$$

We can estimate the thickness of the planar interface, $2\xi_0$, as being of the order of the thermal de Broglie wavelength for massless particles at the critical temperature

$$2\xi_0 \approx \frac{2\pi}{3T_c} \ . \tag{47}$$

This yields

$$\xi_0 \approx 1 \text{ fm} \ , \ \xi_q \approx \xi_h \approx 0.7 \text{ fm} \ , \tag{48}$$

if $T_c \approx 200$ MeV. According to lattice gauge theory simulations with no dynamical quarks the surface free energy is about [23, 24]

$$\sigma \approx 50 \text{ MeV fm}^{-2} \ . \tag{49}$$

The correlation length and the surface free energy determine the parameters $-f_0''$ and K in the course-grained free energy. In principle these parameters are temperature dependent. Their temperature dependence is, however, unknown to us. When numerical calculations are made later in this paper we shall use these values independent of T. To get an idea how bad this assumption is, we can calculate the radius of a critical bubble at zero temperature. From Eq. (43), and assuming the pressure of the perturbative phase of QCD is smaller than the true one by the bag constant $B \approx (200 \text{ MeV})^4$, we get the radius to be $R \approx 0.5$ fm. This is a typical hadronic radius, so perhaps the temperature dependence is in reality not very strong.

IV. THE PREFACTOR

In this section we evaluate the exponential prefactor in the nucleation rate. This prefactor is a product of two terms: the statistical prefactor and the dynamical prefactor. The statistical prefactor Ω_0 is a measure of both the available phase space as the system goes over the saddle and of statistical fluctuations at the saddle relative to the equilibrium states. The dynamical prefactor κ is the exponential growth rate of the bubble or droplet sitting on the saddle. The latter is the more difficult to calculate. We shall evaluate it using techniques exactly analogous to those employed by Turski and Langer [11, 12].

A. Statistical prefactor

The general expression for the statistical prefactor was given in Eq. (14). To evaluate it, we first consider the eigenvalues of the matrix of second derivatives of F, the λ_α. The $\lambda_\alpha^{(0)}$ are eigenvalues of the operator

$$\left.\frac{\delta^2 F_I}{\delta e(\mathbf{r})\delta e(\mathbf{r}\,')}\right|_{e=e_q} = \left(-K\nabla^2 + \frac{\partial^2 f}{\partial e_q^2}\right)\delta(\mathbf{r}-\mathbf{r}\,') \ . \tag{50}$$

Here by $\partial^2 f/\partial e_q^2$ we mean the second derivative of f with respect to e at fixed temperature evaluated in the equilibrium quark-gluon phase. This is a measure of fluctuations in the system and cannot be determined from knowledge of the equation of state alone. Since the right side of Eq. (50) depends on $\mathbf{r}$ only through ∇^2, its eigenfunctions are plane waves, with wave vectors $\mathbf{q}$ and eigenvalues

$$\lambda_{\mathbf{q}}^{(0)} = K\mathbf{q}^{\,2} + \frac{\partial^2 f}{\partial e_q^2} . \tag{51}$$

There is also a set of eigenvalues, formally to be included among the $\lambda_\alpha^{(0)}$, which come from the kinetic term F_K. In Ref. [12] it was concluded that these eigenvalues are spurious; that is, they do not describe physically relevant fluctuations and hence do not appear in the final formula for any observable quantity. This is true in our case as well.

At the saddle point, $e(\mathbf{r}) = \bar{e}(r)$, the operator

$$\left.\frac{\delta^2 F_I}{\delta e(\mathbf{r})\delta e(\mathbf{r}\,')}\right|_{e=\bar{e}(r)} = \left(-K\nabla^2 + \frac{\partial^2 f}{\partial \bar{e}^2}\right)\delta(\mathbf{r}-\mathbf{r}\,') \tag{52}$$

is no longer translationally invariant because of the r dependence of $\bar{e}$. As has been discussed previously in Ref. [21], the resulting spherically symmetric Schrödinger-like eigenvalue equation has an s-wave ground state with a radial eigenfunction proportional to $d\bar{e}/dr$ and a negative eigenvalue

$$\bar{\lambda}_1 \approx -2K/R^2 \ . \tag{53}$$

This eigenstate is associated with the instability of the critical bubble against uniform expansions or contractions. The next states are the three p waves, with eigenvalues $\bar{\lambda} = 0$, which occur because of the broken translational symmetry. Then there are higher-order partial waves with positive $\bar{\lambda}$ corresponding to volume-conserving deformations of the shape of the droplet. Finally, there is a continuum of nonlocalized eigenfunctions of (52) starting at $\bar{\lambda} = \partial^2 f/\partial e_q^2$. These eigenfunctions are similar to the states associated with the $\lambda^{(0)}$ in that they describe fluctuations in the bulk plasma, but here these fluctuations are perturbed by the presence of the bubble. As before, the eigenvalues associated with the kinetic part of F are spurious and can be disregarded.

We can recognize the products over α in Eq. (14) as representing fluctuation corrections to the mean-field excess free energy of the bubble. If we were to evaluate ΔF using measured values of the surface energy and thermodynamic potential, it would be inconsistent to include fluctuation corrections to ΔF in the prefactor Ω_0. Strictly speaking, the nucleation formula used here requires that ΔF first be evaluated at the stationary point obtained from Eq. (32), and then be corrected by the fluctuation terms in Ω_0. But this procedure would imply that the radius of the critical droplet be determined by the σ given in Eq. (45), which is not necessarily the same as the experimental surface free energy because of the fluctuation corrections. What we shall do, instead, is delete the explicit fluctuation terms in Ω_0, and interpret σ everywhere as the true surface energy; we shall make a similar assumption concerning other thermodynamic quantities that appear. This procedure possibly can be justified by going beyond the Gaussian approximations for η-space integrations which were used in deriving Eq. (14); that is, by constructing a renormalized perturbation expansion in the neighborhood of $\{\bar{\eta}\}$. If this program can be carried out, we might also be able to compute systematically curvature corrections to the surface energy. These corrections will be omitted here, and we shall focus our attention on other ingredients of the nucleation formula, particularly the dynamical prefactor. See also Ref. [12] for additional comments on this aspect of the analysis.

Note, now, that there are $\alpha_0+1=4$ more terms in the product over the $\lambda_\beta^{(0)}$ than in that over the $\bar{\lambda}_\alpha$ in Eq. (14). This means that the logarithm of the combined products is not precisely a free-energy difference. To see what is happening here, it is useful to think in terms of a one-to-one pairing between the $\lambda_\beta^{(0)}$ and the $\bar{\lambda}_\alpha$. At the top of the spectra (large positive $\lambda^{(0)}$ and $\bar{\lambda}$) both kinds of eigenvalues correspond to short-wavelength fluctuations which extend throughout the volume of the system V. We can pair these eigenvalues so that their contributions cancel each other to the extent that the droplet volume is negligible compared to the total volume of the system. At the bottom of the continuum a finite set of $\bar{\lambda}$, which correspond to localized deformations of the bubble, fall appreciably below their associated $\lambda^{(0)}$. Thus, by pairing the λ's as described, the correction to ΔF remains of order R^3 in the limit $V\to\infty$, as it must. This procedure leaves four unpaired $\lambda^{(0)}$'s at the bottom of the spectrum which are not accounted for by the revised ΔF. Specifically, we have

$$\lim_{V\to\infty}\prod_{\beta=1}^{4}\left(\frac{\lambda_\beta^{(0)}}{2\pi T}\right)^{1/2}=\left(\frac{1}{2\pi T}\,\frac{\partial^2 f}{\partial e_q^2}\right)^2 \tag{54}$$

remaining as the sole explicit contribution from the complicated products over the α Eq. (14).

Having written down the value for $\bar{\lambda}_1$ in Eq. (53), we need only evaluate the factor $\mathcal{V}$ to complete the calculation of Ω_0. The formula for $\mathcal{V}$ has been given in Refs. [21, 2]. It is

$$\mathcal{V}=V\left[\frac{1}{3}\int dr(\nabla\bar{e})^2\right]^{3/2}=V\left[\frac{4\pi R^2\sigma}{3\,K}\right]^{3/2}. \tag{55}$$

Here we have used (45) for σ and that $d\bar{e}/dr$ is appreciable only in a narrow region near $r=R$ for the bubble.

The resulting expression for Ω_0 is

$$\Omega_0=V\left(\frac{4\pi R^2\sigma}{3\,K}\right)^{3/2}\left(\frac{\pi TR^2}{K}\right)^{1/2}\left(\frac{1}{2\pi T}\,\frac{\partial^2 f}{\partial e_q^2}\right)^2. \tag{56}$$

Identifying the correlation length ξ_q in the quark-gluon phase as in Sec. III by

$$\frac{1}{K}\,\frac{\partial^2 f}{\partial e_q^2}=\frac{1}{\xi_q^2}, \tag{57}$$

we can write (56) in the form

$$\Omega_0=\frac{2}{3\sqrt{3}}\left(\frac{\sigma}{T}\right)^{3/2}\left(\frac{R}{\xi_q}\right)^4 V. \tag{58}$$

If one considers the nucleation rate to be per unit volume then the volume V should be divided out of the above expression. Usually in this paper we do mean the rate per unit volume and so Ω_0 will not include the factor V in subsequent discussion.

B. Dynamical prefactor

The dynamical prefactor κ should be obtained as the positive eigenvalue of the matrix given in Eq. (16). Using the mobility matrix, Eqs. (21), and the fact that the bubble solution is spherically symmetric and satisfies Eq. (32), one finds that $\kappa=0$. This means that the bubble does not grow. The reason was discussed by Langer and Turski [12]. In order for a bubble (or droplet) to grow, latent heat must be transported away from the surface region. For the nonrelativistic systems they were considering, they discovered that heat conduction was necessary to allow for growth. This eventually led to Eq. (4), which says that κ is proportional to the thermal conductivity λ. It is clear that to get our bubble to grow we must include the effects of dissipation on the dynamics.

In relativistic dissipative fluid dynamics one adds new terms to the stress-energy-momentum tensor and to the baryon current as follows [18, 25]:

$$T^{\mu\nu}=-pg^{\mu\nu}+(p+e)u^\mu u^\nu+\Delta T^{\mu\nu}, \tag{59}$$

$$N^\mu=nu^\mu+\Delta N^\mu. \tag{60}$$

In the absence of dissipation $\Delta T^{\mu\nu}$ and ΔN^μ are both zero, and u^μ is the flow four-velocity of the matter. When dissipation is present one has a choice of defining u^μ to be the velocity of baryon flow or the velocity of energy flow. These are known as the Eckart and Landau-Lifshitz approaches, respectively. In the Eckart approach ΔN^μ is zero by definition. Then $\Delta T^{\mu\nu}$ is a linear combination of a shear tensor, a projection tensor on the hyperplane normal to u^μ, and a heat flow vector, with coefficients be-

ing the shear viscosity η, the bulk viscosity ζ, and heat conductivity λ, respectively. In the Landau-Lifshitz approach $\Delta T^{\mu\nu}$ does not have the term involving the thermal conductivity; rather, ΔN^{μ} is nonzero and is in fact proportional to λ. These two approaches are completely equivalent and describe the same physics. However, the systems we are dealing with in this paper have no net baryon number, no baryon chemical potential, and so the Eckart approach is indeterminate. We must define u^{μ} to be the velocity of energy flow. The thermal conductivity simply is not defined because there is no net baryon density to define a frame of reference with respect to which energy can be conducted.

The contribution to the stress-energy-momentum tensor due to dissipation is

$$\Delta T_{\mu\nu} = -\eta(\partial_{\mu}u_{\nu} + \partial_{\nu}u_{\mu} - u_{\mu}u^{\iota}\partial_{\iota}u_{\nu} - u_{\nu}u^{\iota}\partial_{\iota}u_{\mu}) - (\zeta - \tfrac{2}{3}\eta)(\partial_{\iota}u^{\iota})(g_{\mu\nu} - u_{\mu}u_{\nu}) \ . \tag{61}$$

The coefficients of shear and bulk viscosity must be positive from the requirement that the entropy should not decrease. In fact, the divergence of the entropy current is

$$\partial_{\mu}(su^{\mu}) = -\frac{1}{T}\Delta T^{\mu}_{\nu}\partial_{\mu}u^{\nu} \ , \tag{62}$$

from which one can calculate the total entropy change of the system. This equation is a direct consequence of the equations of motion, which are just the usual ones of conservation of energy and momentum, $\partial_{\mu}T^{\mu\nu} = 0$. In general in our applications the flow three-velocity is small compared to the speed of light. The relativistic treatment is required mainly by the fact that the pressure is comparable to the energy density, and by the absence of a net baryon number to define the motion of the matter.

We now want to determine the equations of motion of dissipative fluid dynamics for small deviations about the stationary configuration $e(\mathbf{r},t) = \bar{e}(r), \mathbf{v}(\mathbf{r},t) = 0$. To that end we write $e = \bar{e}(r) + \nu(\mathbf{r},t)$ and $\mathbf{v} = \mathbf{v}(\mathbf{r},t)$ and linearize the full equations of motion, including the gradient term F_K, in terms of ν and $\mathbf{v}$:

$$\partial_t\nu = -\nabla\cdot\mathbf{M} = -\nabla\cdot(\bar{w}\mathbf{v}) \ , \tag{63}$$

$$\partial_t(\bar{w}\mathbf{v}) = \nabla\bar{e}\left[-K\nabla^2\nu + f''\nu\right] + \nabla\left[(\zeta + 4\eta/3)\nabla\cdot\mathbf{v}\right] \ . \tag{64}$$

Here and after when we write f, f', or f'' we intend that they be evaluated at the stationary configuration, so that they are complicated functions of r.

To determine κ we look for radial perturbations of the form

$$\nu(\mathbf{r},t) = \nu(r)e^{\kappa t} \ , \tag{65}$$

$$\mathbf{v}(\mathbf{r},t) = v(r)\hat{\mathbf{r}}e^{\kappa t} \ . \tag{66}$$

These radial deviations are governed by the equations of motion

$$\kappa\nu(r) = -\frac{1}{r^2}\frac{d}{dr}\left[r^2\bar{w}v(r)\right] \ , \tag{67}$$

and

$$\kappa\bar{w}v(r) = -\frac{d\bar{e}}{dr}\left[-K\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right) + f''\right]\nu(r) + \frac{d}{dr}\left[(\zeta + 4\eta/3)\frac{1}{r^2}\frac{d}{dr}[r^2v(r)]\right] \ . \tag{68}$$

Eliminating $\nu(r)$ using the first equation we obtain a linear, third-order differential equation for the velocity profile:

$$\kappa^2\bar{w}v(r) = -\frac{d\bar{e}}{dr}\left[K\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right) - f''\right] \times \left[\frac{1}{r^2}\frac{d}{dr}[r^2\bar{w}v(r)]\right] + \frac{d}{dr}\left[\kappa(\zeta + 4\eta/3)\frac{1}{r^2}\frac{d}{dr}[r^2v(r)]\right] \ . \tag{69}$$

Self-consistent solutions of this equation, together with the boundary conditions, should provide us with the allowed values of κ. Unfortunately, this is not a trivial equation to solve. Therefore we will first analyze the behavior of the solution in three regions: the interior of the bubble, the exterior of the bubble, and the surface region. However, we first note a constraint which follows from Eq. (67) together with the conditions that $v(r)$ vanish at the origin and at infinity, namely

$$\int_0^{\infty} dr\, 4\pi r^2\nu(r) = 0 \ . \tag{70}$$

In the interior region, from the origin to within a few correlation lengths of the surface, recall that $\bar{e} \approx$const. Then the first term on the right side of Eq. (69) vanishes and the equation for $v(r)$ reduces to

$$r^2v'' + 2rv' - (a_h^2r^2 + 2)v = 0 \ , \tag{71}$$

where $a_h^2 = \kappa w_h/(\zeta + 4\eta/3)$. The general solution of this *Forsyth-Jacobssthal* differential equation is

$$v(r) = A\left(\frac{a_h}{r} - \frac{1}{r^2}\right)e^{a_hr} + B\left(\frac{a_h}{r} + \frac{1}{r^2}\right)e^{-a_hr} \ , \tag{72}$$

where A and B are constants. We must require that v and v' vanish at $r = 0$. Consequently both A and B are zero, so that the velocity vanishes in the interior of the bubble. (This is true to the extent that $\bar{e}$=const in this region.)

In the exterior region, far outside the surface, the energy and enthalpy densities approach their equilibrium values in the bulk quark-gluon phase, $\bar{e} \to e_q$ and $\bar{w} \to w_q$. Then the first term on the right side of Eq. (69) can again be neglected as a first approximation. The solution with the proper large-r behavior is

$$v(r) = C\left(\frac{a_q}{r} + \frac{1}{r^2}\right)e^{-a_qr} \ , \tag{73}$$

where C is a constant and $a_q^2 = \kappa w_q/(\zeta + 4\eta/3)$.

In the region of the surface, $r \approx R$, the stationary configuration $\bar{e}(r)$ is varying rapidly, and $d\bar{e}/dr$ is nonzero.

Therefore, unlike the deep interior or exterior of the bubble, the first term on the right side of Eq. (69) cannot be dropped. In fact, as we shall see, κ is proportional to the viscosity which we assume to be very small. Then the other two terms in Eq. (69) are of second order in the viscosity, and we shall ignore them. Thus, to good approximation, in the surface region $\nu(r)$ satisfies

$$\left[-K\nabla^2 + f''\right]\nu(r) = 0 \; . \tag{74}$$

Given that $\bar{e}(r)$ satisfies Eq. (32), and that $\nu(r)$ must go to zero at the origin and at infinity, the solution to the above equation is

$$\nu(r) \sim \frac{d\bar{e}}{dr} \; . \tag{75}$$

Together with (67) this implies that in the surface region

$$v(r) = \frac{D}{r^2 \bar{w}(r)} \int_0^r dr' r'^2 \frac{d\bar{e}}{dr'} \; , \tag{76}$$

where D is a constant. For distances r exceeding the bubble radius R by more than a few correlation lengths, but less than $2R$, Eq. (76) can be integrated to give

$$v(r) \approx \frac{D\,\Delta e}{w_q} \frac{R^2}{r^2} \; . \tag{77}$$

Remember, as always, we are assuming weak to moderate supercooling so that $R \gg \xi$.

Let us summarize what we have learned of the general behavior of $v(r)$. In the interior, up to a few correlation lengths of the surface, v is essentially zero. It rises rapidly and reaches a maximum somewhere in the surface. A few correlation lengths outside the surface $v(r)$ begins to fall like r^{-2}, according to Eq. (77). Eventually $v(r)$ falls exponentially according to Eq. (73). If (77) and (73) are to join together smoothly at, say, 10 correlation lengths outside the surface, it must be that $a_q \ll R^{-1}$. Now one could entertain the possibility of smoothly joining together (73) and (77) to get an approximation to the exact solution $v(r)$ of Eq. (69). There are three unknowns: the amplitudes C and D, and the decay constant a_q (or equivalently κ, which is what we are after in the end). The ratio C/D can be determined from the condition (70). The magnitudes of C and D are irrelevant. This still does not suffice to determine κ. Therefore we turn to a different way of analyzing the growth of the bubble. The above analysis is not wasted, though, as we shall see.

It is necessary to distinguish between the actual radius of the bubble, R, and the radius of the bubble in the stationary or metastable configuration, R_*, determined by Laplace's formula (43). If the stationary bubble is perturbed only slightly then the energy density profile will only change by a minute amount. Transport of heat away from the surface will be a very slow process because of the assumed smallness of the viscosity. As the bubble slowly begins to expand, the energy density profile will not change much, only the profile needs to be moved out a small distance dR in a time dt. The energy flux density (energy per unit area per unit time) which must be transported outwards is $\Delta w\, dR/dt$. Here we do not distinguish between the difference of energy densities and the difference in enthalpy densities of the two bulk phases because the pressure difference is small compared to the energy density differences; we shall refer to them interchangeably as the latent heat. This energy flux must be balanced by that due to dissipation, which is $-(\zeta + 4\eta/3) v\, dv/dr$ [18]. We evaluate the flow velocity just outside the surface of the bubble. According to Eq. (77) the derivative is $dv/dr \approx -2v/R$. Therefore energy balance gives us the relation

$$\Delta w \frac{dR}{dt} = 2(\zeta + 4\eta/3)\frac{v^2}{R} \; . \tag{78}$$

The outward momentum flux density (momentum per unit area per unit time) is $\Delta w\, v^2$. (This neglects a small contribution from viscous terms which can be considered a higher-order effect.) The momentum flux density must be equated to the force per unit area which comes from the Laplace formula

$$\Delta w\, v^2 = 2\sigma\left(\frac{1}{R_*} - \frac{1}{R}\right) . \tag{79}$$

Again, the velocity is to be evaluated just outside the surface.

The Eqs. (78) and (79) are not identical to the Rankine-Hugoniot-Taub relations [26, 27] for discontinuities across a sharp surface because the surface under consideration is curved and because the velocity is not constant on the outside.

Using both energy and momentum conservation we can eliminate the velocity and solve for dR/dt:

$$\frac{dR}{dt} = \frac{4(\zeta + 4\eta/3)\sigma(R - R_*)}{(\Delta w)^2 R^2 R_*} \; . \tag{80}$$

This is a differential equation for $R(t)$ from which we can read off the value of κ. It is

$$\kappa = \frac{4\sigma(\zeta + 4\eta/3)}{(\Delta w)^2 R_*^3} \; . \tag{81}$$

This may be considered the principal result of our paper.

Note that Eq. (81) justifies *a posteriori* our assertion that $a_q \ll R^{-1}$. Note also the similarity of Eq. (81) to the result of Langer and Turski, Eq. (4). In the relativistic case the linear combination of viscosity coefficients replaces the thermal conductivity times the temperature, and there is an extra factor of 2 because here the operator d/dr acts on the function $1/r^2$ while for them it acts on the function $1/r$. Actually, it may be worthwhile to extend the nonrelativistic treatment to include viscosity as well as heat conduction. To our knowledge this has never been done in the literature. Our result suggests that viscosity is as important as heat conduction in allowing the initial growth of a bubble or droplet. It may even be possible to find real materials in which heat conduction is anomalously small, so that the growth rate would be dominated by viscous forces.

For an ultrarelativistic gas the coefficients of viscosity can be estimated as

$$\eta = \frac{4}{15} a T^4 \tau \quad \text{and} \quad \zeta = 4 a T^4 \tau \left[\frac{1}{3} - \left(\frac{\partial p}{\partial e} \right) \right]^2 , \tag{82}$$

where τ is the collision time, and a is the Stefan-Boltzmann constant, defined such that the energy density is $e = aT^4$ [25]. Since the square of the sound velocity $(\partial p/\partial e)$ is close to $\frac{1}{3}$ for ultrarelativistic gases, the bulk viscosity ζ is usually much smaller than the shear viscosity η.

Recently [28] the shear viscosity of quark-gluon plasma was estimated in leading order of QCD. In this case the viscosity is an additive sum of the quark and gluon viscosities, $\eta = \eta_q + \eta_g$. Both components are given in terms of viscous relaxation times, τ_q and τ_g, by Eq. (82). For a zero baryon chemical potential

$$\tau_g^{-1} = 4.11\, T \left(1 + \frac{N_f}{6} \right) \alpha_s^2 \, | \ln \alpha_s | , \tag{83}$$

where $\alpha_s = g^2/4\pi$ is the QCD fine structure constant. The two terms 1 and $N_f/6$ come from the contribution of gluon-gluon and gluon-quark scatterings. The viscous relaxation rate for quarks is

$$\tau_q^{-1} = 0.39\, \tau_g^{-1} . \tag{84}$$

Around the critical temperature of 200 MeV, α_s has been estimated to be about 0.23 [29]. The relaxation times at $T = 200$ MeV thus are on the order of 1 fm/c. For QCD with 2 flavors the viscosity is

$$\eta = \frac{1.12 T^3}{\alpha_s^2 \ln(1/\alpha_s)} . \tag{85}$$

The possibilities and difficulties of evaluating transport coefficients in lattice QCD were pointed out by Karsch and Wyld [30].

V. DISCUSSION

Inserting Eqs. (58) and (81) into Eq. (2) gives us the nucleation rate

$$I = \frac{4}{\pi} \left(\frac{\sigma}{3T} \right)^{3/2} \frac{\sigma(\zeta_q + 4\eta_q/3) R_*}{\xi_q^4 (\Delta w)^2} e^{-\Delta F/T} , \tag{86}$$

where $\Delta F = 4\pi\sigma R_*^2/3$ and R_* is given by the Laplace formula (43). This is the probability per unit volume per unit time to nucleate a hadronic bubble out of the QCD plasma. If one considers nucleating a QCD plasma droplet in a hadronic gas instead, one just needs to evaluate the correlation length and the viscosities in the hadronic phase rather than the quark-gluon phase. At the critical temperature, $R_* \to \infty$, and the rate vanishes because of the exponential. The system must supercool at least a minute amount in order that the rate attain a finite value. Note that at the critical temperature the preexponential factor is linearly divergent in R_* which is qualitatively unlike the simple dimensionless estimates of Eqs. (6).

In determining the nucleation rate for hadronic bubbles in a supercooled quark-gluon plasma we ignore the relatively small variation of certain quantities with temperature. We take

$$\sigma = 50 \text{ MeV/fm}^2 , \tag{87}$$

$$\alpha_s = 0.23 , \tag{88}$$

$$\xi_q = 0.7 \text{ fm} , \tag{89}$$

$$B^{1/4} = 235 \text{ MeV} . \tag{90}$$

Since we are assuming massless pions, and massless quarks and gluons, with the deconfinement dynamics modeled by the bag constant, we have

$$p_h = \frac{3\pi^2}{90} T^4 , \tag{91}$$

$$p_q = \frac{37\pi^2}{90} T^4 - B , \tag{92}$$

$$\Delta w = \frac{68\pi^2}{45} T^4 . \tag{93}$$

Finally, the stationary bubble radius is given by the Laplace formula to be

$$R_*(T) = \frac{2\sigma}{p_h(T) - p_q(T)} . \tag{94}$$

In Fig. 3 we plot the radius of the critical size bubble as a function of temperature. As the critical temperature is approached the radius R_* diverges. However, it quickly reduces to typical hadronic sizes as the temperature falls. When T/T_c=99.5%, R_*=12.7 fm; when T/T_c=97%, R_*=2.2 fm. When the temperature has fallen to 90% of its critical value, R_* is about 0.73 fm; this is comparable to the correlation length ξ_q and to the surface thickness of the bubble. At this point one could say that many of our approximations have broken down. This is probably the limit where the notion of homogeneous nucleation theory can be applied. For lower temperatures we suspect that spinodal decomposition becomes the primary means of driving the phase transition.

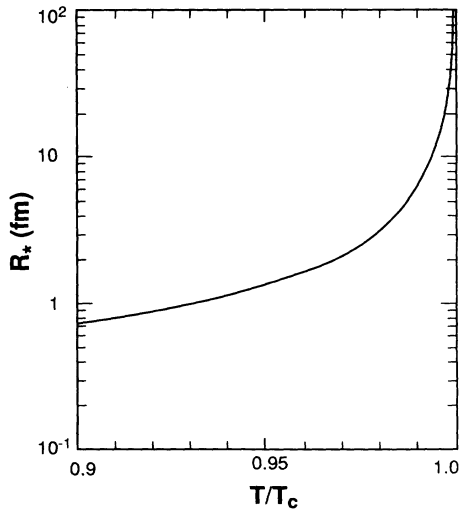

FIG. 3. The radius of a critical size hadronic bubble in a supercooled quark-gluon plasma. It diverges at the critical temperature.

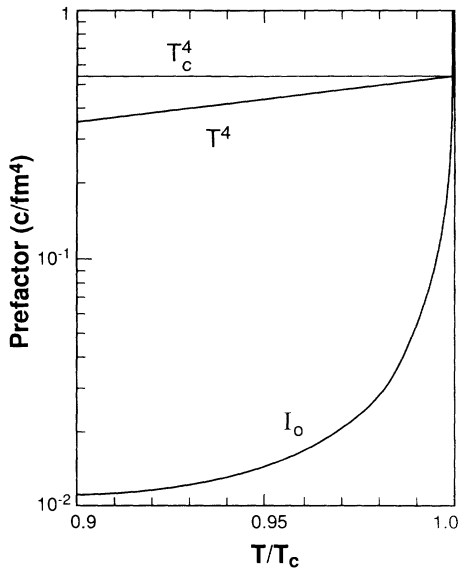

FIG. 4. The exponential prefactor I_0 from Eq. (86) using the numerical estimates given in the text is plotted as a function of temperature and compared to the simple dimensional estimates T^4 and T_c^4.

Whether the transition is completed before the temperature has fallen this much depends upon the expansion time scale of the system. Undoubtedly the transition would have been completed well before T/T_c=90% in the early Universe because of the relatively slow expansion, about 10^{-6} sec. For ultrarelativistic nuclear collisions the answer is so far unclear.

In Fig. 4 we show the preexponential factor calculated in this paper as a function of temperature. For comparison we also plot the simple dimensionless estimates T_c^4 and T^4. It is clear that neither of the latter two are good approximations of the former. The preexponential factor diverges as the critical temperature is approached because it is proportional to R_*. It is very temperature dependent, especially near T_c. However, it is important to remember that the exponential drives the full rate to zero at T_c.

This completes our calculation and analysis of the thermal nucleation rate for systems with zero or negligibly small baryon number. In a subsequent paper we shall use it in a set of rate equations for the time evolution of the QCD phase transition in the early Universe and in ultrarelativistic nuclear collisions.

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