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"Cosmic flashing" in four dimensions

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Quantum field theory allows violation of the weak energy condition in the form of locally negative energy densities and fluxes. If the laws of physics place no restrictions on the extent of energy condition breakdown, then dramatic violations of the second law of thermodynamics, causality, and cosmic censorship might become possible. In this paper, we explore the possibility that manipulation of negative energy fluxes could lead to the production of naked singularities. This might be accomplished by injecting a negative energy flux into an extreme ($Q=M$) charged black hole. However, quantum field theory requires that an initial negative energy flux due to quantum coherence effects must be followed by a more than compensating positive flux. Thus any singularity resulting from this process would be only "temporarily" naked. In an earlier publication, we dubbed the occurrence of a naked singularity with limited duration "cosmic flashing." There, in a two-dimensional analysis where the fluxes were produced by moving mirrors, we showed that quantum field theory imposed limits on the magnitude and duration of the negative energy flux in the form of an uncertainty-principle-type inequality. If $|\Delta M|$ is the magnitude of the change in the mass of the black hole due to the absorption of negative energy, and ΔT is the effective lifetime of the naked singularity thus produced, then we showed that $|\Delta M|\Delta T < 1$, in units where $c=\hbar=1$. The current paper analyzes the behavior of a minimally coupled, quantized, massless scalar field propagating in a four-dimensional extreme Reissner-Nordström black-hole background. In this case a similar inequality is shown to hold for a general negative energy flux, irrespective of how the flux is produced. A numerical analysis shows that the angular-momentum-dependent potential barrier around the black hole screens out the contributions to the flux from the higher l modes. We estimate the metric perturbations produced by the negative energy flux. In an order of magnitude estimate, we show that these are smaller than the metric fluctuations expected from quantum gravity. Therefore we conclude that quantum field theory prevents an unambiguous violation of cosmic censorship.

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I. INTRODUCTION

A remarkable feature of quantum field theory is that it allows states in which the local energy density can become negative [1]. This implies that at least local violations of the "weak energy condition" are permitted by quantum field theory. The weak energy condition (WEC) states that $T_{\mu\nu}U^{\mu}U^{\nu}\geq 0$, for all timelike or null vectors U^{μ}. This is the weakest of all the "standard" energy conditions [2] used in proving singularity theorems and it is obeyed by known forms of classical matter. Violations of the WEC in quantum field theory must be taken seriously, however, because of examples such as the Casimir effect and squeezed states, which have been experimentally confirmed [3]. Other examples of theoretical processes which involve negative energy densities and/or fluxes include radiation by moving mirrors [4] and black-hole evaporation [5]. On the other hand, if the laws of physics impose no limits on the degree of energy condition violation then the result could be possible gross violations of the second law of thermodynamics, causality, and cosmic censorship. The fact that the observed behavior of the physical world is consistent with at least the first two of these principles suggests that some such constraints exist.

One possible form of restrictions are "averaged energy conditions," i.e., energy conditions suitably averaged over null or timelike curves [6–8]. It can be shown [8], for example, that Penrose's singularity theorem (which uses only the WEC) will still hold if the WEC is replaced with the "averaged weak energy condition" (AWEC). (As used in Ref. [8], this condition effectively says that $\int_{\gamma}T_{\mu\nu}K^{\mu}K^{\nu}d\lambda\geq 0$, when averaged over a half-infinite null geodesic γ.) The extent to which quantum field theory enforces the AWEC is not yet definitely known, but it has been shown to hold in various special cases [9–11]. However, recent results indicate that it may not hold in an arbitrary curved four-dimensional spacetime [11]. The current resurgence of interest in averaged energy conditions stems not only from their role in singularity theorems, but also from the recent observation that violations of the AWEC are required to maintain traversable wormholes [12]. These wormholes have the disturbing (or intriguing, depending on one's point of view) property of being generically transformable into time machines [13,14].

Another type of possible restrictions on the behavior of negative energy are "quantum inequalities." These have taken the form of uncertainty-principle-type inequalities on the magnitude and duration of negative energy fluxes due to quantum coherence effects. (The integral of the energy flux in an inertial frame over all time is nonnegative, but this alone does not constrain the magnitude and duration of the negative-energy part of the flux.) It was shown some time ago [15] that negative energy fluxes seen by inertial observers in two-dimensional flat spacetimes appear to obey an inequality of the form

$$|F| < (\Delta T)^{-2} , \tag{1.1}$$

where $|F|$ is the average magnitude of the negative flux and ΔT is its duration. Let $|\Delta E| = |F|\Delta T$ be the amount of negative energy passing by a fixed spatial position in a time ΔT. Equation (1.1) implies

$$|\Delta E|\Delta T < 1 , \tag{1.2}$$

and hence $|\Delta T|$ is less than the quantum uncertainty in the energy, $(\Delta T)^{-1}$, on the time scale $|\Delta T|$. As originally formulated, the proof of this inequality was limited to only certain classes of quantum states in two-dimensional spacetimes. However, recently a more precise version of this type of inequality has been proven to hold for all quantum states of a free massless scalar field in both two- and four-dimensional flat spacetime [16]. The inequality may be expressed as an integral of the flux multiplied by a sampling function. This inequality will be discussed in more detail in Sec. II. This relation is of the kind required to prevent large-scale violations of the second law of thermodynamics.

In at least some circumstances quantum inequalities seem to be stronger constraints on the behavior of negative energy fluxes than averaged energy conditions. For example, consider an inertial observer in two-dimensional Minkowski spacetime at fixed position. At time $t=0$, a δ-function pulse of negative $(-)$ energy massless scalar radiation traveling along the $+x$ axis crosses the observer's world line, accompanied by a subsequent positive $(+)$ energy δ-function pulse. [This would seem to represent the most efficient separation of $(-)$ and $(+)$ energy.] If AWEC holds when applied to this observer's (timelike) geodesic [17], it would imply that the initial $(-)$ energy pulse must be followed at *some* time later by compensating $(+)$ energy. However, AWEC would seem to place no constraint on *when* the $(+)$ energy has to arrive. By contrast, it can be shown [16] that quantum inequalities imply that there exists a maximum time separation between the two pulses which is within the limits allowed by the uncertainty principle. This constraint requires the $(+)$ energy to arrive within a time $1/|\Delta E|$, where $|\Delta E|$ is the magnitude of the amount of $(-)$ energy transmitted.

A useful thought experiment to test whether $(-)$ energy fluxes could be manipulated to produce large-scale macroscopic effects is to consider a $(-)$ energy flux of massless scalar radiation injected into an extreme $(Q=M)$ charged black hole. An infinitesimal increase in the charge Q (or a decrease in the mass M) would destroy the hole and create a $Q>M$ naked singularity spacetime, in violation of the cosmic censorship hypothesis. One might think that such a situation could be realized by dropping or firing charged classical test particles into the hole to increase Q above the critical limit. Interestingly, this turns out not to be the case. It was shown some time ago [18,19] that in these scenarios either a dropped test particle will always be repelled by the (like) charge of the hole, or the increase in mass produced by an injected test particle is always enough to offset the increase in charge. Similar results were found for extreme $(a=M)$ rotating black holes. We can imagine repeating the above thought experiments, replacing the test particles with fluxes of $(-)$ energy. Here one is using the quantum properties of the matter fields to lower the mass of the hole, while keeping its charge fixed [20] (since the scalar radiation carries no charge). Classically *any* amount of $(-)$ energy injected into the hole would be sufficient to create a naked singularity. Figure 1 shows the spacetime of an eternal extreme $(Q=M)$ charged black hole. It was argued in Ref. [21] that this spacetime could be regarded as the limit of the more realistic $Q<M$ black-hole spacetimes formed by gravitational collapse, the limit in which the cosmic censorship hypothesis becomes most vulnerable. An inspection of Fig. 1 shows that an infinitesimal perturbation of the spacetime could allow a null ray to connect the singularity with future null infinity. In this case, it is not immediately obvious as to what mechanism might prevent a violation of cosmic censorship.

In an earlier paper [21], we studied negative energy fluxes produced by a moving mirror in two-dimensional black-hole spacetimes. There it was shown that, for physically reasonable mirror trajectories, any initial $(-)$ energy flux must be followed by a more than compensating $(+)$ energy flux. This implies that any naked singu-

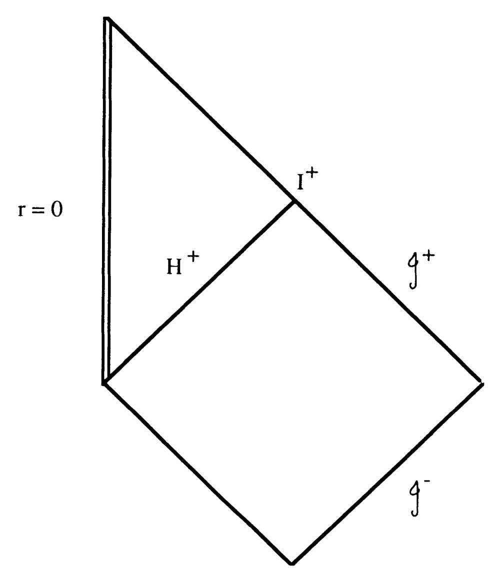

FIG. 1. The spacetime of an eternal, extreme $(M=Q)$ Reissner-Nordström black hole. Here H^+ is the future event horizon, the point I^+ represents future timelike infinity, and $r=0$ is the (timelike) singularity. The 45° line above I^+ is a Cauchy horizon; the spacetime may be extended beyond this line, although this is not shown in the figure.

larity created in this process would have a finite lifetime. We called the production of a naked singularity with limited duration "cosmic flashing." The limiting case, illustrated in Fig. 2, is obtained by using δ-function pulses of (−) and (+) energy which are separated by a time interval ΔT (as measured by an observer at infinity). The best chance of observing cosmic flashing is to make ΔT as large as possible. This can be done by a suitable choice of mirror trajectory. Let the magnitude of the change in the mass of the black hole due to the absorption of the (−) energy flux be $|\Delta M|$, and let ΔT denote the effective lifetime of the naked singularity. (It can be shown [21] that the latter is the same as the time interval between the pulses.) It was discovered that there exists a quantum inequality in this case also, similar to that in flat spacetime, and given by

$$|\Delta M|\Delta T<1 \ . \tag{1.3}$$

Surprisingly, our result did not depend on the initial distance of the mirror from the hole [22]. The change in the mass of the hole, due to the absorption of the (−) energy flux, is below the scale of the normal quantum fluctuations of the hole's mass expected from the uncertainty principle on the time scale ΔT. We therefore concluded that cosmic flashing would not be observable, since it would be "lost" in the background noise of quantum fluctuations.

The most serious drawback of our previous analysis is the two dimensionality of the model. The mirror is constrained to move in only one spatial dimension. It must therefore either fall into the black hole, in which case one

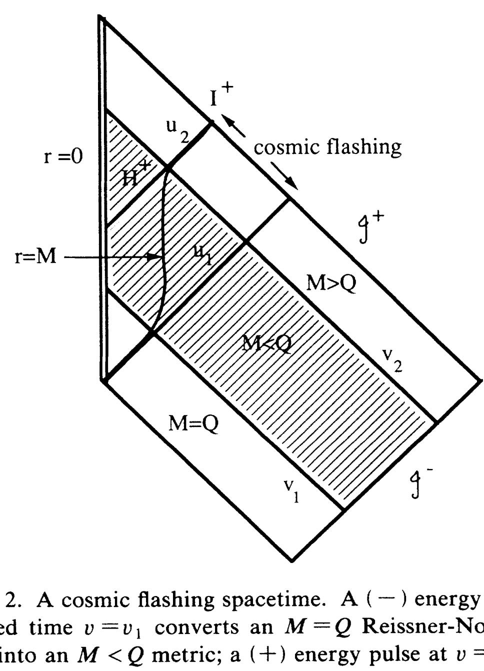

FIG. 2. A cosmic flashing spacetime. A (−) energy pulse at advanced time $v=v_1$ converts an $M=Q$ Reissner-Nordström metric into an $M<Q$ metric; a (+) energy pulse at $v=v_2$ converts it into an $M>Q$ metric. Between retarded time $u=u_1$ and $u=u_2$, outgoing null rays from the singularity reach future null infinity. The future horizon H^+ is the $u=u_2$ line. Here the pulses are depicted as originating from past null infinity.

has to take into account the (+) mass of the mirror, or it must stop before crossing the horizon. There is no potential barrier around the black hole in two dimensions, so one cannot model wave scattering off the hole. Lastly, there are no Einstein equations in two dimensions so that one cannot directly tie changes in mass to changes in spacetime geometry. It is conceivable that in four dimensions one could violate inequality (1.3) by superposing wave modes with different angular momentum quantum number l. Although each individual mode might obey the inequality, the combination need not. To overcome these difficulties, a more general analysis is required.

In the present paper, we analyze a minimally coupled massless scalar field propagating in a four-dimensional extreme Reissner-Nordström black-hole background. One can take the point of view that any allowed quantum state of the field should be physically realizable by *some* mechanism. Moving mirrors simply provide one such method for generating states that have associated (−) energy fluxes. Unfortunately, the radiation from a moving mirror in four dimensions is not known exactly even in flat spacetime, except in special simplified cases. Therefore, in this paper we will consider (−) energy fluxes in general, without regard to how they are produced. By generalizing the formalism of Ref. [16], we demonstrate the existence of a bound on the integrated (−) energy flux. Our numerical analysis shows that the angular-momentum-dependent potential barrier screens out the higher l modes, and leads to an inequality of the form of Eq. (1.3). This discussion is presented in Sec. II. In Sec. III we show, in an order of magnitude estimate, that the metric perturbations produced by (−) energy are smaller than the metric fluctuations expected from quantum gravity. This implies that the spacetime region in which cosmic flashing occurs is unavoidably "smeared" by quantum effects. Thus once again our conclusion is that quantum field theory prevents an unambiguous observation of a violation of cosmic censorship. We again stress that our conclusion is independent of how the (−) energy flux is produced. In passing, we also note that our current results, as well as our earlier two-dimensional mirror results, do not depend on a particular renormalization of the stress-energy tensor. This is because we work only with the "flux" components of the stress tensor which are finite without renormalization. Our metric convention is $(-,+,+,+)$ and, unless otherwise noted, we work in units where $G=c=\hbar=1$.

II. LIMITS ON COSMIC FLASHING

A. Back reaction in the Reissner-Nordström spacetime

We wish to consider a Reissner-Nordström black hole for which the metric is

$$ds^2=-C(r)dt^2+C^{-1}(r)dr^2+r^2(d\theta^2+\sin^2\theta\, d\varphi^2) \ , \tag{2.1}$$

where $C(r)=(1-2M/r+Q^2/r^2)$. Suppose that a quantized field propagates on this background for which the renormalized energy-momentum tensor is denoted by

$\langle T_{\mu\nu}\rangle$. In general there will be a flux of energy across the black hole's horizon, causing a change in mass given by

$$\dot{M}=F=\int \langle T_t{}^r\rangle r^2 d\Omega \ . \qquad (2.2)$$

Although this relation may seem obvious, it is instructive to derive it from the semiclassical Einstein equation

$$G_{\mu\nu}=8\pi(T^{\rm cl}_{\mu\nu}+\langle T_{\mu\nu}\rangle) \ . \qquad (2.3)$$

Here $T^{\rm cl}_{\mu\nu}$ is the energy-momentum tensor for classical matter. The energy as measured at infinity absorbed by the black hole is defined by

$$E=\int \langle T_{\mu\nu}\rangle \xi^\mu d\Sigma^\nu \ , \qquad (2.4)$$

where ξ^μ is the timelike Killing vector and $d\Sigma^\nu$ is the area element of the three-dimensional hypersurface defined by the outer horizon of the black hole. Here we are assuming that the fractional change in M is small over the time scale M, the light-travel time across the black hole. Therefore the metric is approximately Reissner-Nordström and hence has a timelike Killing vector. The timelike Killing vector has the explicit form $\xi^\mu=\delta^\mu{}_t$. The area element may be expressed as $d\Sigma^\nu=n^\nu d\Sigma$, where $n^\nu=\delta^\nu{}_r C^{1/2}$ is the unit normal in the radial direction, and $d\Sigma=C^{1/2}r^2 d\Omega\, dt$ is the scalar area element in a hypersurface of constant r. From these facts, and taking a time derivative of Eq. (2.4), we can express the energy flux into the black hole as

$$F=\dot{E}=\int \langle T_t{}^r\rangle r^2 d\Omega \ , \qquad (2.5)$$

which is the second half of Eq. (2.2). The Reissner-Nordström metric, Eq. (2.1), is a solution of Eq. (2.3) only if the right-hand side of the latter equation contains just a contribution from the classical electromagnetic field of the black hole, which we here take to be $T^{\rm cl}_{\mu\nu}$. If there is a flow of energy into or out of the black hole, the mass parameter M will become time dependent. If Q remains fixed, we may take the metric to be of the form of Eq. (2.1), but with $C=C(r,t)$ and M a function of time. The relevant component of the Einstein tensor is

$$G_t{}^r=\frac{2\dot{M}}{r^2} \ . \qquad (2.6)$$

The semiclassical Einstein equation tells us that

$$\langle T_t{}^r\rangle=\frac{1}{8\pi}G_t{}^r \ , \qquad (2.7)$$

as the (rt) component of $T^{\rm cl}_{\mu\nu}$ vanishes. If we insert into this relation Eq. (2.5), we discover that $F=\dot{M}$, which is the first half of Eq. (2.2).

B. Scalar field on the Reissner-Nordström background

We are interested in a massless scalar field, ϕ, propagating on the Reissner-Nordström background. The wave equation

$$\Box\phi=0 \qquad (2.8)$$

has solutions which we take to have the form

$$f_{\lambda\omega lm}=\eta_m e^{-i\delta_{\omega l}}\frac{U_{\lambda\omega l}(r)}{r\sqrt{4\pi\omega}}Y_{lm}(\theta,\varphi)e^{-i\omega t} \ . \qquad (2.9)$$

Here the $Y_{lm}(\theta,\varphi)$ are the usual spherical harmonics, $\eta_m=1$ if m is odd and $\eta_m=i$ if m is even, and $\delta_{\omega l}$ is a phase which will be defined below. The radial mode functions $U_{\lambda\omega l}(r)$ are solutions of the equation

$$\frac{d^2U}{dr^{*2}}+(\omega^2-V_{\rm eff})U=0 \ , \qquad (2.10)$$

where $V_{\rm eff}$ is the effective potential given by

$$V_{\rm eff}=C(r)\left[\frac{2(Mr-Q^2)}{r^4}+\frac{l(l+1)}{r^2}\right] \ . \qquad (2.11)$$

The independent variable r^* in Eq. (2.10) is the usual tortoise coordinate for which $r^*=\int C^{-1}(r)dr$. The case which is of particular interest to us is the extreme $Q=M$ black hole for which

$$V_{\rm eff}=\frac{(r-M)^2}{r^4}\left[l(l+1)+\frac{2M(r-M)}{r^2}\right] \ . \qquad (2.12)$$

The maximum of this potential occurs at $r=2M$ for all l and has the value

$$V_M=\frac{2l(l+1)+1}{32M^2} \ . \qquad (2.13)$$

The mode label λ takes on two values $(+)$ and $(-)$. The modes are defined so that the $(+)$ modes have no component which is outgoing from the past horizon, and the $(-)$ modes have no component which is incoming from past null infinity. The asymptotic forms of the radial mode functions near the horizons ($r^*\to-\infty$) and at large distances ($r^*\to\infty$) are given by

$$U_{+\omega l}\sim\begin{cases} e^{-i\omega r^*}+A_{\omega l}e^{i\omega r^*}, & r^*\to\infty ,\\ B_{\omega l}e^{-i\omega r^*}, & r^*\to-\infty ,\end{cases} \qquad (2.14)$$

and by

$$U_{-\omega l}\sim\begin{cases} B'_{\omega l}e^{i\omega r^*}, & r^*\to\infty ,\\ e^{i\omega r^*}+A'_{\omega l}e^{-i\omega r^*}, & r^*\to-\infty \ .\end{cases} \qquad (2.15)$$

Here $A_{\omega l}$ and $B_{\omega l}$ are the reflection and transmission coefficients, respectively, for the $(+)$ mode, and the primed quantities are the corresponding coefficients for the $(-)$ mode. The $(+)$ mode is a wave which is incoming from past null infinity, with a transmitted portion ingoing on the future horizon and a reflected portion which is outgoing to future null infinity. The $(-)$ mode is outcoming from the past horizon, with a transmitted portion outgoing to future null infinity and a reflected portion which is ingoing on the future horizon. We may see that the $(+)$ and $(-)$ modes are orthogonal to one another by regarding them as the limits of wave-packet modes. At early times, any $(+)$ mode will be localized at large distances ($r^* \to \infty$), whereas the ($-$) modes will be localized near the horizon, and hence the overlap between the two must vanish. The phase $\delta_{\omega l}$ which appears in the definition of the mode functions is the phase shift for the (+) mode transmitted wave:

$$B_{\omega l} = |B_{\omega l}| e^{i\delta_{\omega l}} \ . \tag{2.16}$$

We now wish to regard ϕ as a quantized field propagating on the Reissner-Nordström background. The mode functions $f_{\lambda\omega lm}$ have been normalized so as to have unit Klein-Gordon norm. Thus we may expand the quantized field ϕ in terms of creation and annihilation operators as

$$\phi = \sum_{\lambda\omega lm} (a_{\lambda\omega lm} f_{\lambda\omega lm} + a^\dagger_{\lambda\omega lm} f^*_{\lambda\omega lm}) \ . \tag{2.17}$$

Here $\sum_{\lambda\omega lm}$ denotes an integral on ω and a discrete sum on the remaining mode labels. The energy-momentum tensor for the scalar field is

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \tfrac{1}{2} g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha} \ . \tag{2.18}$$

The energy flux across the future horizon of the black hole is given by Eq. (2.2) with the expectation value taken in the quantum state of the field. We now wish to impose the restriction on this state that only (+) mode quanta be excited. This means that we require that $\langle a^\dagger_{-\omega' l' m'} a_{-\omega lm} \rangle = 0$ and $\langle a_{-\omega' l' m'} a_{-\omega lm} \rangle = 0$, and hence only $\lambda = +$ modes will contribute to F. Physically, this is the requirement that there be no particles emerging from the past horizon, and that the only particles that can cross the future horizon are those that have come from past null infinity. Near the future horizon, we may now write

$$\langle T_t{}^r \rangle = 2\,\mathrm{Re} \sum_{\substack{\omega lm \\ \omega' l' m'}} \omega\omega' (\langle a^\dagger_{+\omega' l' m'} a_{+\omega lm} \rangle f^*_{+\omega' l' m'} f_{\omega lm} - \langle a_{+\omega' l' m'} a_{+\omega lm} \rangle f_{+\omega' l' m'} f_{+\omega lm}) \ . \tag{2.19}$$

Here we have used the fact that $f_{+\omega lm, r} = C^{-1} f_{+\omega lm, r^*} \sim -i\omega C^{-1} f_{+\omega lm}$ near the future horizon. If we now insert this expression into Eq. (2.2), use the orthogonality relations for the spherical harmonics

$$\int Y_{lm} Y^*_{l'm'} d\Omega = \delta_{ll'}\delta_{mm'} \tag{2.20}$$

and

$$\int Y_{lm} Y_{l'm'} d\Omega = (-1)^m \delta_{ll'}\delta_{m,-m'} \ , \tag{2.21}$$

and the asymptotic form of the ingoing radial functions, Eq. (2.14), the result is

$$F = \frac{1}{2\pi} \mathrm{Re} \sum_{\omega\omega' lm} \sqrt{\omega\omega'} |B_{\omega l} B_{\omega' l}| (\langle a^\dagger_{\omega' lm} a_{\omega lm} \rangle e^{i(\omega'-\omega)v} + \langle a_{\omega lm} a_{\omega' l-m} \rangle e^{-i(\omega'+\omega)v}) \ . \tag{2.22}$$

Here $v = t + r^*$ is the advanced time coordinate, and the label + has been dropped from the $a_{\omega lm}$'s.

Strictly speaking, the above discussion (and the inequality on F to be derived below) applies only to the extreme, $Q = M$, Reissner-Nordström black hole. A nonextreme black hole will also have a Hawking radiation contribution to the flux. However, even in the nonextreme case, our analysis could be easily adapted to apply to that portion of the flux which is injected into the black hole from large distances.

C. A constraint on negative energy fluxes into the black hole

If the quantum state is such that F is instantaneously negative, we have a negative energy flux into the black hole, and according to Eq. (2.2), the black hole's mass will decrease. The integral of F over all v is necessarily positive for any nonvacuum state, i.e.,

$$\int_{-\infty}^{\infty} F(v)dv = \mathrm{Re} \sum_{\omega lm} \omega |B_{\omega l}|^2 \langle a^\dagger_{\omega lm} a_{\omega lm} \rangle > 0 \ , \tag{2.23}$$

since $a^\dagger_{\omega lm} a_{\omega lm}$ is the number operator for mode ωlm. However, this constraint alone is insufficient to prevent an observation of cosmic flashing. Our goal will be to establish an inequality which limits the magnitude and duration of a negative energy flux. The approach adopted here is similar to that used in Ref. [16] to prove inequalities which constrain negative energy fluxes in flat spacetime. Define an integrated energy flux $\hat{F}$ by

$$\hat{F} = \frac{v_0}{\pi} \int_{-\infty}^{\infty} \frac{F(v)dv}{v^2 + v_0^2} \ . \tag{2.24}$$

Here v_0 is an arbitrary time interval. The function $v_0/[\pi(v^2+v_0^2)]$ is a normalized sampling function with a characteristic width v_0. We can think of $\hat{F}v_0$ as being the average energy observed by a detector which measures the flux for a time v_0. The particular functional form for the sampling function which we have chosen is convenient, but one could presumably obtain similar results with other functions.

From Eqs. (2.22) and (2.24), we have that

$$\hat{F}=\frac{1}{2\pi}\mathrm{Re}\sum_{\omega\omega' lm}\sqrt{\omega\omega'}|B_{\omega l}B_{\omega' l}|[\langle a^{\dagger}_{\omega' lm}a_{\omega lm}\rangle e^{-|\omega-\omega'|v_0}+\langle a_{\omega lm}a_{\omega' l-m}\rangle e^{-(\omega+\omega')v_0}] . \tag{2.25}$$

The proof of the inequality on $\hat{F}$ follows closely the procedure used in Ref. [16] with minor modifications. The first step is to show that

$$\mathrm{Re}\sum_{\omega\omega' lm}\sqrt{\omega\omega'}|B_{\omega l}B_{\omega' l}|\langle a^{\dagger}_{\omega' lm}a_{\omega lm}\rangle e^{-|\omega-\omega'|v_0}\geq\sum_{\omega\omega' lm}\sqrt{\omega\omega'}|B_{\omega l}B_{\omega' l}|\langle a^{\dagger}_{\omega' lm}a_{\omega lm}\rangle e^{-(\omega+\omega')v_0} . \tag{2.26}$$

This inequality is very similar to that proven in Appendix B of Ref. [16]. Let the integration on ω be replaced by a discrete sum; take $\omega=\omega_n=n\Delta\omega$, $\omega'=\omega'_n=n'\Delta\omega$, where $n,n'=1,\ldots,\infty$, and let $\alpha=v_0\Delta\omega$. Furthermore, define the quantities

$$\tilde{B}_{nn'}=\sqrt{nn'}e^{-\alpha(n+n')}|B_{\omega l}B_{\omega' l}|\mathrm{Re}\langle a^{\dagger}_{\omega' lm}a_{\omega lm}\rangle \tag{2.27}$$

and

$$\tilde{A}_{nn'}=[e^{-\alpha[|n-n'|-(n+n')]}-1]\tilde{B}_{nn'} . \tag{2.28}$$

The quantities $\tilde{A}_{nn'}$ and $\tilde{B}_{nn'}$, are analogous to the quantities A_{mn} and B_{mn}, respectively, defined in Appendix B of Ref. [16]. In particular, for any two positive integers J and M,

$$\sum_{n,n'=J}^{J+M}\tilde{B}_{nn'}=\left|\sum_{n=J}^{J+M}\sqrt{n}\,|B_{\omega l}|e^{-\alpha n}a_{\omega lm}|\psi\rangle\right|\geq 0 . \tag{2.29}$$

Here $|\psi\rangle$ is the actual quantum state of the scalar field, and the above inequality follows by virtue of the left-hand side being the norm of another state vector. Thus the argument given in Appendix B shows that

$$\sum_{n,n'=1}^{\infty}\tilde{A}_{nn'}\geq 0 . \tag{2.30}$$

However, when further summed on l and m, this inequality is just Eq. (2.26). Thus we can now use Eq. (2.26) to rewrite Eq. (2.25) as

$$\hat{F}\geq\frac{1}{2\pi}\mathrm{Re}\sum_{\omega\omega' lm}\sqrt{\omega\omega'}|B_{\omega l}B_{\omega' l}|e^{-(\omega+\omega')v_0}[\langle a^{\dagger}_{\omega' lm}a_{\omega lm}\rangle+\langle a_{\omega lm}a_{\omega' l-m}\rangle] . \tag{2.31}$$

The establishment of a lower bound on $\hat{F}$ now follows closely the argument given in Appendix A of Ref. [16]. We take the quantum state to be an arbitrary state in the Fock representation:

$$|\psi\rangle=\sum_{\{n_i\}}c(\{n_i\})|\{n_i\}\rangle , \tag{2.32}$$

where $\sum_{\{n_i\}}$ is a sum over all possible sets of occupation numbers and the coefficients are normalized so that

$$\sum_{\{n_i\}}|c(\{n_i\})|^2=1 . \tag{2.33}$$

The right-hand side of Eq. (2.31) can be expressed as

$$S\equiv 2\,\mathrm{Re}\sum_{\omega\omega' lm}h_{\omega' l}h_{\omega l}(\langle a^{\dagger}_{\omega' lm}a_{\omega lm}\rangle+\langle a_{\omega' l-m}a_{\omega lm}\rangle) , \tag{2.34}$$

where

$$h_{\omega l}=\left[\frac{\omega}{4\pi}\right]^{1/2}|B_{\omega l}|e^{-\omega v_0} . \tag{2.35}$$

Let us denote the contributions to S in Eq. (2.34) which arise from the $m=0$ and the $m\neq 0$ terms as S_0 and S_n, respectively. Thus $S=S_0+S_n$. The quantity S_0 is exactly of the form to which the lemma proven in Appendix A of Ref. [16] may be applied; it shows that

$$S_0\geq-\sum_{\omega l}h^2_{\omega l} . \tag{2.36}$$

A lower bound on S_n is established using a slight modification of the argument of Ref. [16]. The explicit form of S_n is

$$S_n=\sum_{\substack{l\\ m\neq 0}}\sum_{\{n_{\omega lm}\}}\left\{\sum_{\omega} h^2_{\omega l}[n_{\omega lm}|c|^2+\sqrt{n_{\omega lm}n_{\omega l-m}}\,c^*(n_{\omega lm}-1,n_{\omega l-m}-1)c]\right.$$
$$+\sum_{\omega\neq\omega'} h_{\omega l}h_{\omega' l}[\sqrt{n_{\omega' lm}(n_{\omega lm}+1)}\,c^*(n_{\omega lm}+1,n_{\omega' lm}-1)c$$
$$\left.+\sqrt{n_{\omega lm}n_{\omega' l-m}}\,c^*(n_{\omega lm}-1,n_{\omega' l-m}-1)c]\right\}+\text{c.c.}, \quad (2.37)$$

where c.c. denotes complex conjugate and the notation convention of Ref. [16] has been adopted whereby the arguments of the coefficients c are written explicitly only if they have been raised or lowered from their original value $n_{\omega lm}$. We now add and subtract the term

$$\sum_{\{n_{\omega lm}\}}\sum_{\substack{\omega l\\ m\neq 0}} h^2_{\omega l}|c|^2=\sum_{\substack{\omega l\\ m\neq 0}} h^2_{\omega l} \quad (2.38)$$

and relabel the sums on the occupation numbers to write

$$S_n=\sum_{\substack{l\\ m\neq 0}}\sum_{\{n_{\omega lm}\}}\left[\sum_{\omega} h^2_{\omega l}\{n_{\omega lm}|c(n_{\omega lm}-1)|^2+(n_{\omega l-m}+1)|c(n_{\omega l-m}+1)|^2\right.$$
$$+\sqrt{n_{\omega lm}(n_{\omega l-m}+1)}[c^*(n_{\omega lm}-1)c(n_{\omega l-m}+1)+\text{c.c.}]\}$$
$$+\sum_{\omega\neq\omega'} h_{\omega' l}h_{\omega l}\{\sqrt{n_{\omega' lm}n_{\omega lm}}\,c^*(n_{\omega' lm}-1)c(n_{\omega lm}-1)$$
$$+\sqrt{(n_{\omega' l-m}+1)(n_{\omega l-m}+1)}\,c^*(n_{\omega' l-m}+1)c(n_{\omega l-m}+1)$$
$$\left.+[\sqrt{(n_{\omega' l-m}+1)n_{\omega lm}}\,c^*(n_{\omega lm}-1)c(n_{\omega' l-m}+1)+\text{c.c.}]\}\right]-\sum_{\substack{\omega l\\ m\neq 0}} h^2_{\omega l}\,. \quad (2.39)$$

We have also used the freedom to let $m\to -m$ inside the summation. Finally, we may write this as

$$S_n=\sum_{\substack{l\\ m\neq 0}}\sum_{\{n_{\omega lm}\}}\left|\sum_{\omega} h_{\omega l}[\sqrt{n_{\omega lm}}\,c(n_{\omega lm}-1)+\sqrt{(n_{\omega l-m}+1)}\,c(n_{\omega l-m}+1)]\right|^2-\sum_{\substack{\omega l\\ m\neq 0}} h^2_{\omega l}\,. \quad (2.40)$$

Thus,

$$S_n\geq -\sum_{\substack{\omega l\\ m\neq 0}} h^2_{\omega l} \quad (2.41)$$

and

$$S=S_0+S_n\geq -\sum_{\omega lm} h^2_{\omega l}=-\sum_{\omega l}(2l+1)h^2_{\omega l}\,. \quad (2.42)$$

[An alternative derivation of Eq. (2.42) will be given in the Appendix.] Because S is itself a lower bound on $\hat{F}$, we have, writing the sum on ω as an integral and using Eq. (2.35),

$$\hat{F}\geq\hat{F}_B\,, \quad (2.43)$$

where

$$\hat{F}_B=-\frac{1}{4\pi}\sum_{l=0}^{\infty}\int_0^{\infty} d\omega\,\omega(2l+1)|B_{\omega l}|^2 e^{-2\omega v_0}\,. \quad (2.44)$$

D. Numerical results: The square barrier approximation

Since the exact form of the field modes is not known analytically, we must evaluate $\hat{F}_B$ numerically. Accordingly, we can approximate Eq. (2.44) for $\hat{F}_B$ by letting the integral over ω go over into a summation, and defining the dimensionless variables

$$W\equiv\omega M \quad (2.45)$$

and

$$\bar{v}\equiv\frac{v_0}{M}. \quad (2.46)$$

We then obtain

$$\hat{F}_B\approx -\frac{1}{4\pi M^2}\sum_W\sum_l \Delta W W(2l+1)|B_{\omega l}|^2 e^{-2W\bar{v}}\,. \quad (2.47)$$

We will demonstrate the existence of a bound on $\hat{F}_B$ by approximating the actual potential barrier V_{eff}, with a square barrier potential V_S. This model barrier is "inscribed" in the actual barrier in the manner depicted in Fig. 3. The potential V_S is defined to be

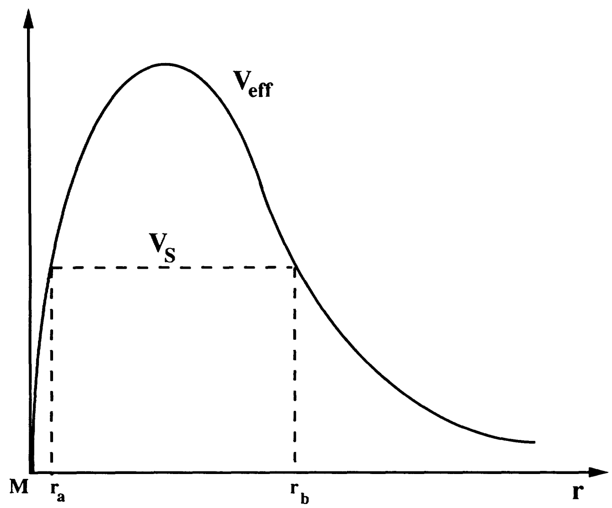

FIG. 3. The actual potential barrier V_{eff}, and the model square barrier potential V_S, here depicted as functions of $r(r^*)$. The height of V_S is half the height of the maximum of V_{eff}. The scalar waves are assumed to be incident from right to left.

$$V_S=\begin{cases} 0.5V_M=0.5\left[\dfrac{l(l+1)+0.5}{16M^2}\right] & \text{for } r_a^*<r^*<r_b^*, \\ 0 \text{ for } r^*>r_b^* \text{ and } r^*<r_a^*. \end{cases} \tag{2.48}$$

Here V_M is the maximum height of the actual potential, which depends on the angular momentum quantum number l. The square barrier is chosen to have height equal to half the maximum height of the actual barrier, and width equal to the width of the actual barrier at half the maximum height. The equation satisfied by the $U(r)$ modes with the model barrier is

$$\frac{d^2U}{dr^{*2}}+(\omega^2-V_S)U=0\ , \tag{2.49}$$

with $r(r^*)$. This has the same form as the Schrödinger equation for a particle moving in a one-dimensional square barrier potential.

Our strategy is to use the model potential V_S to obtain a bound on $\hat{F}_B$ as follows. For modes with $\omega^2\geq V_S$, we will simply set the $B_{\omega l}$ equal to one. For modes with $\omega^2<V_S$, we will compute the $B_{\omega l}$ using Eqs. (2.48) and (2.49). We expect the transmission coefficients through the square barrier to be greater than those through the actual barrier for all modes [23]. Since this procedure *overestimates* the contributions to the flux, the proof of a bound on $\hat{F}_B$ using the model barrier potential implies an even more stringent bound on $\hat{F}_B$ for the actual potential, and thus on $\hat{F}$ as well. In this paper, our goal is only to establish the existence of a bound rather than the best possible bound. The requirement of the continuity of U and dU/dr^* at $r^*=r_a^*$ and $r^*=r_b^*$, and a standard although tedious calculation yields

$$B_{\omega l}=\frac{-4ik\kappa e^{-\kappa(r_b^*-r_a^*)}e^{-ik(r_b^*-r_a^*)}}{(\kappa-ik)^2-e^{-2\kappa(r_b^*-r_a^*)}(\kappa+ik)^2}\ , \tag{2.50}$$

where $\kappa=\sqrt{V_S-\omega^2}$, and $k=\omega$. In our formula for $\hat{F}_B$, we will need

$$|B_{\omega l}|^2=\frac{16k^2\kappa^2e^{-2\kappa(r_b^*-r_a^*)}}{(\kappa^2+k^2)^2(1+e^{-4\kappa(r_b^*-r_a^*)})-2e^{-2\kappa(r_b^*-r_a^*)}(\kappa^4-6k^2\kappa^2+k^4)}\ . \tag{2.51}$$

We will now split the right-hand side of Eq. (2.47) into two parts:

$$\hat{F}_B>\frac{-1}{4\pi M^2}\sum_W\sum_{l=0}^{l\leq l_c}\Delta WW(2l+1)e^{-2W\bar{v}} -\frac{1}{4\pi M^2}\sum_W\sum_{l>l_c}^{\infty}\Delta WW(2l+1)|B_{\omega l}|^2e^{-2W\bar{v}}\ . \tag{2.52}$$

We have set the $|B_{\omega l}|$'s equal to 1 in the first double summation, corresponding to modes with $\omega^2\geq V_S$. The $|B_{\omega l}|$'s in the second double summation, corresponding to modes with $\omega^2<V_S$, are evaluated numerically using Eq. (2.51). Here l_c, the critical value of l at which $\omega^2=V_S$, is given by

$$l_c=-0.5+0.5\sqrt{1+4(32W^2-0.5)}\ , \tag{2.53}$$

where we have used the fact that $l\geq 0$ for all l. The latter inequality also implies a minimum value for $W\equiv\omega M$ of

$$W_{min}=0.125\ . \tag{2.54}$$

A numerical computation of the right-hand side of Eq. (2.52) for $1\leq\bar{v}\leq 10$ yields the following bound on $\hat{F}$:

$$\hat{F}\geq\hat{F}_B>-\frac{C_0}{M^2}\left[\frac{M}{v_0}\right]^a\ , \tag{2.55}$$

where $a\approx 3.8$ and $C_0\approx 0.96$.

E. Quantum inequalities

In four dimensions, as in two dimensions, the best opportunity to violate cosmic censorship arises with the use of widely spaced δ-function pulses of (−) and (+) energy. Therefore, let

$$F(v)=|\Delta M|[-\delta(v)+\rho\delta(v-\Delta T)]\ , \tag{2.56}$$

where ρ is the fraction by which the (+) pulse overcompensates the (−) pulse. Note that this situation is depicted in Fig. 2 with $v_1=0$ and $v_2=\Delta T$. From Eqs. (2.24), (2.55), and (2.56), we find that

$$\hat{F}=\frac{|\Delta M|}{\pi\Delta T}\left[\frac{(\rho-1)\beta^2-1}{\beta(\beta^2+1)}\right]\gtrsim-\frac{C_0}{M^2}\left[\frac{M}{v_0}\right]^a , \quad (2.57)$$

where $\beta=v_0/\Delta T$. This inequality is nontrivial only when

$$\beta<\frac{1}{\sqrt{\rho-1}} . \quad (2.58)$$

In this case

$$|\Delta M|\Delta T\lesssim\pi C_0\left[\frac{M}{\Delta T}\right]^{a-2} G\left[\frac{v_0}{\Delta T}\right] , \quad (2.59)$$

where the function $G(\beta)$ is given by

$$G(\beta)=\frac{\beta^{1-a}(\beta^2+1)}{1-(\rho-1)\beta^2} . \quad (2.60)$$

Although the sampling time v_0 may be arbitrary, we wish to choose it so as to minimize the right-hand side of Eq. (2.59). The minimum of the function $G(\beta)$ occurs at

$$\beta=\beta_m=\left[\frac{\sqrt{(a+1)^2\rho^2+8(1-a)\rho}-(a+1)\rho+2(a-1)}{2(a-1)(\rho-1)}\right]^{1/2} . \quad (2.61)$$

For given values of a and C_0, the interpretation of the inequality, Eq. (2.59), is qualitatively different for different ranges of ρ. Recall that our numerical calculation gave us $a\approx3.8$ and $C_0\approx0.96$. With these values, we find three cases which may be characterized by whether $\rho-1\sim\frac{1}{10}$, $\rho\gg1$, or $\rho-1\ll\frac{1}{10}$. (Here $\frac{1}{10}$ is representative of any number that is an order of magnitude less than unity.) Each of these cases will be discussed below.

If $\rho-1\sim\frac{1}{10}$, the optimum bound in Eq. (2.59) is obtained when $\beta_m\sim1$, i.e., $v_0\sim\Delta T$, which leads to $G(\beta_m)\sim1$, and

$$|\Delta M|\Delta T\lesssim\left[\frac{M}{\Delta T}\right]^{a-2} . \quad (2.62)$$

In order to increase the chances of observing a violation of cosmic censorship, we would like to have ΔT as large as possible. Furthermore, if $\Delta T<M$, the effective lifetime of any naked singularity is of order M since the cosmic flash will decay on this time scale, as discussed in Ref. [21]. Hence the case of greatest interest is when $\Delta T\gtrsim M$. We then have an inequality on the magnitude of the change in mass similar to Eq. (1.3) in the two-dimensional case:

$$|\Delta M|\Delta T\lesssim1 . \quad (2.63)$$

In both cases, the change in the mass is of the order of the quantum energy fluctuations associated with the time scale ΔT. Recall that the inequality given in Eq. (2.63) was derived using the model square barrier V_S, where we *overestimated* the contributions of the various modes to the (−) energy flux. Therefore, the actual bound on $\hat{F}$ and the corresponding inequality on $|\Delta M|$ and ΔT will be even more stringent than Eq. (2.55) and Eq. (2.63).

When $\rho\gg1$, $G(\beta_m)$ grows as $\rho^{(a-1)/2}$, in which case the bound Eq. (2.59) is not especially strong. Equation (2.58) shows that β must be very small for large ρ. When ρ is large, there is a large overcompensation by the (+) energy pulse. Hence for large ρ, even sampling functions with small widths will pick up enough of the contribution from the (+) pulse to render $\hat{F}>0$ and the inequality Eq. (2.57) trivial. However, causality prohibits the magnitude of the subsequent (+) pulse from affecting the observability of the naked singularity. Therefore we would not expect large ρ pulses to be any more effective for violating cosmic censorship than those of small ρ.

For $(\rho-1)\ll\frac{1}{10}$, β grows as $(\rho-1)^{-1/2}$ and $G(\beta)$ decreases as $(\rho-1)^{(a-3)/2}$. This suggests a restriction on δ-function pulses which is even stronger than Eq. (2.62). If we create a (−)δ-function pulse with $\Delta M=-|\Delta M|$, and follow it with a (+)δ-function pulse with energy $\Delta M=\rho|\Delta M|$ a time ΔT later, then there is a minimum value of ρ for given $|\Delta M|$ and ΔT which increases as $|\Delta M|\Delta T$ increases. For $(\rho-1)\ll\frac{1}{10}$,

$$\rho\gtrsim1+K\left[\frac{\Delta T}{M}\right]^{2(a-2)/(a-3)}(|\Delta M|\Delta T)^{2/(a-3)} , \quad (2.64)$$

where K is a constant of order 1. In the case of δ-function plane wave pulses in four-dimensional flat spacetime, $a=4$, and it can be shown that an inequality similar to Eq. (2.64) holds:

$$\rho\gtrsim1+K\left[\frac{(\Delta T)^2}{A}\right]^2(|\Delta E|\Delta T)^2 , \quad (2.65)$$

where A is the area of the collecting surface, $|\Delta E|$ is the magnitude of the (−) energy pulse, and K is a (different) constant of order 1.

It is of interest to note that a similar restriction applies to δ-function pulses produced by moving mirrors in two-dimensional spacetimes. For example, consider a mirror initially at rest in flat spacetime, which instantaneously begins accelerating toward the observer with constant proper acceleration a, thereby emitting a (−)δ-function pulse of magnitude $|\Delta E|=a/(12\pi)$. A time ΔT later, its acceleration ceases and therefore it emits a (+)δ-function pulse of magnitude $\rho|\Delta E|$, where $\rho=\sqrt{1-V^2}/(1-V)^2$ and V is the velocity of the mirror [21]. In the nonrelativistic limit $V=a\Delta T=12\pi|\Delta E|\Delta T$ and $\rho=1+2V$, so the (+) pulse overcompensates by a factor of

$\rho=1+24\pi|\Delta E|\Delta T$. An examination of the spacetime diagrams for mirror trajectories involving a $(-)\delta$-function pulse followed by a $(+)\delta$-function pulse indicates that an analogous restriction must hold for the relativistic case as well.

We suggest that these are all examples of a more general principle which demands that an energy loan (i.e., negative energy) must always be repaid "with interest" depending on the magnitude and duration of the debt. This might be dubbed "the quantum interest conjecture."

III. INTERPRETATION

In this section, we wish to discuss the significance of the quantum inequality, Eq. (2.63), and to show that it implies that any violation of cosmic censorship produced by $(-)$ energy fluxes is not unambiguously observable. As discussed in Sec. II E, we want

$$\Delta T \gtrsim M \ . \tag{3.1}$$

If we use this condition and the quantum inequality, Eq. (2.63), we find

$$\frac{\Delta M}{M} \lesssim \frac{m_P^2}{M^2} \ , \tag{3.2}$$

where m_P is the Planck mass. Thus for a macroscopic black hole, the fractional change in the mass is extremely small. For example, in the case of a nondischarging black hole [20] with $M \geq 10^5 M_\odot$, we have

$$\frac{\Delta M}{M} \lesssim 10^{-86} \ . \tag{3.3}$$

Let the change in a typical component of the Riemann tensor due to the $(-)$ energy pulse be $\Delta R_{\mu\nu\rho\sigma}$, and the corresponding change in the metric be $\Delta g_{\mu\nu}$. Since ΔM is small, we have

$$\Delta R_{\mu\nu\rho\sigma} \propto \Delta M, \quad \Delta g_{\mu\nu} \propto \Delta M \ . \tag{3.4}$$

Figures 1 and 2 show that to obtain an observable violation of cosmic censorship we need to liberate null rays just inside the horizon. Hence, we are interested in evaluating these quantities near $r=M$. Equation (3.4) and dimensional considerations give

$$\Delta R_{\mu\nu\rho\sigma} \approx \frac{\Delta M}{M^3} \tag{3.5}$$

and

$$\Delta g_{\mu\nu} \approx \frac{\Delta M}{M} \ . \tag{3.6}$$

(These quantities are understood to be computed in a coordinate system which is well behaved near the horizon.) If we successively apply the quantum inequality, Eq. (2.63), and the condition Eq. (3.1), we find

$$\Delta R_{\mu\nu\rho\sigma} \lesssim \frac{1}{\Delta T M^3} \lesssim \frac{1}{M^4} \ , \tag{3.7}$$

and

$$\Delta g_{\mu\nu} \lesssim \frac{1}{\Delta T M} \lesssim \frac{m_P^2}{M^2} \ . \tag{3.8}$$

We now wish to compare the magnitude of the metric and curvature fluctuations expected from quantum gravity [24] with the perturbations in these same quantities due to the absorption of a $(-)$ energy flux. If the latter are smaller than the former, then the effects of the $(-)$ energy flux cannot be observed. The former may be estimated by the following argument. Let $\hat{h}_{\mu\nu}$ be the quantized operator of linear metric perturbations on the Reissner-Nordström background, and $\hat{R}_{\mu\nu\rho\sigma}$ be the corresponding Riemann tensor operator. The scale of the quantum gravity fluctuations of the metric and Riemann tensor are given by rms expectation values such as

$$\delta g_{QG} = [\langle 0|\hat{h}_{\mu\nu}\hat{h}^{\mu\nu}|0\rangle]^{1/2} \tag{3.9}$$

and

$$\delta R_{QG} = [\langle 0|\hat{R}_{\mu\nu\rho\sigma}\hat{R}^{\mu\nu\rho\sigma}|0\rangle]^{1/2} \ . \tag{3.10}$$

These quantities are formally infinite and require renormalization. However, we may estimate their magnitudes by a dimensional argument. The gravitational Lagrangian is of the form $L_G \propto (1/G)(\nabla\hat{h}_{\mu\nu})^2 \propto (\nabla\phi)^2$, where G is Newton's constant and ϕ has the usual dimensions for a quantum scalar field operator, i.e., $(\text{length})^{-1}$ in units where $\hbar=c=1$. Hence $\hat{h}_{\mu\nu} \propto \sqrt{G}\,\phi \propto \sqrt{G}\,(\text{length})^{-1}$. Thus δg_{QG} and δR_{QG} are proportional to $\sqrt{G}=l_P$, the Planck length. Because δg_{QG} is dimensionless and δR_{QG} has dimensions of $(\text{length})^{-2}$, near the black hole's horizon,

$$\delta g_{QG} \sim \frac{l_P}{M} \tag{3.11}$$

and

$$\delta R_{QG} \sim \frac{l_P}{M^3} \ . \tag{3.12}$$

A somewhat less rigorous though more physical argument is the following. We can estimate the typical scale of metric fluctuations using the expected energy density of gravitons on a curved background. For example, if we describe the back reaction of gravitons on a background vacuum solution of Einstein's equations, we would expand the Einstein equations to second order in the metric perturbation $h_{\mu\nu}$, and find the effective energy-momentum tensor of the gravitons to be $T_{\mu\nu}^{(G)}=(1/8\pi G)G_{\mu\nu}^{(2)}$. It is of the form

$$T_{\mu\nu}^{(G)} \sim \frac{(\nabla h)^2 + h(\nabla\nabla h)}{l_P^2} \ . \tag{3.13}$$

Now consider a region of characteristic size l. The metric perturbation $h_{\mu\nu}$ varies on a scale l, so $\nabla h \sim \delta g_{\mu\nu}/l$, where $h_{\mu\nu} \sim \delta g_{\mu\nu}$ is the typical scale of metric fluctuations. Therefore, $(\nabla h)^2 \sim h(\nabla\nabla h) \sim (\delta g_{\mu\nu})^2/l^2$, and hence

$$T_{\mu\nu}^{(G)} \sim \frac{(\delta g_{\mu\nu})^2}{l_P^2 l^2} \ . \tag{3.14}$$

The typical zero point energy in gravitons should be $T_{00}^{(G)}$. The dominant contribution to this zero point energy should come from modes whose wavelengths are of order l, i.e., $\omega \sim 1/l$. Therefore,

$$T_{00}^{(G)} \sim \frac{\omega}{l^3} \sim \frac{1}{l^4} \ . \tag{3.15}$$

A comparison of Eqs. (3.14) and (3.15) yields

$$\delta g_{\mu\nu} \sim \frac{l_P}{l} \ . \tag{3.16}$$

If we are interested in metric fluctuations on a given curved background, then l is the typical radius of curvature, e.g., M in the black-hole case. In our case $l \sim M$, which again gives Eq. (3.11). These estimates agree with those given by previous authors using different arguments [24].

From Eqs. (3.7), (3.8), (3.11), and (3.12), we see that

$$\Delta R_{\mu\nu\rho\sigma} \ll \delta R_{QG} \tag{3.17}$$

and

$$\Delta g_{\mu\nu} \ll \delta g_{QG} \ . \tag{3.18}$$

This shows that the metric perturbations due to the effects of ($-$) energy are below the scale of the normal metric fluctuations expected from quantum gravity.

In our earlier paper, where the ($-$) energy pulses were produced by a moving mirror in a two-dimensional spacetime, we argued that a necessary condition for the cosmic censorship violation to be observable is $|\Delta M|\Delta T \gtrsim 1$. There we found that this condition was not satisfied, leading us to conclude that the resulting violation of cosmic censorship was unobservable. Our argument was based on a common interpretation of the energy-time uncertainty principle which assumes that to measure the energy of any system to within an accuracy δE, the time ΔT required is $\Delta T \gtrsim 1/\delta E$. The validity of this interpretation has been criticized by Aharonov and Bohm [25], and also by Sorkin [26]. In the present paper, we arrived at a similar conclusion as in our earlier paper without using the energy-time uncertainty principle. In particular, note that the energy-time uncertainty principle was *not* used in the derivation of Eq. (2.63) [nor was it used to derive the similar inequality Eq. (1.3) in our earlier paper].

We conclude that quantum field theory prevents at least the *unambiguous* observation of any naked singularity produced in this process. The spacetime region in which cosmic flashing occurs (i.e., the region between the lines u_1 and u_2 in Fig. 2) will be "blurred" by quantum effects. An observer at future null infinity would not be able to say that a given null ray in this region, when "traced backward," definitely originated at the singularity. This conclusion seems to be supported by recent work of Kuo and Ford [27] which indicates that in flat spacetime, states of quantum fields involving ($-$) energy fluxes are accompanied by large fluctuations in the stress-energy tensor. When the fluctuations in $\langle T_{\mu\nu} \rangle$ are large, the semiclassical approximation $G_{\mu\nu} = 8\pi \langle T_{\mu\nu} \rangle$ is not expected to hold. Their results and ours suggest that at least in certain circumstances involving ($-$) energy fluxes, the predictions of the semiclassical theory of gravity (e.g., the production of a naked singularity from an extreme black hole) may be suspect. In our case, the effects of ($-$) energy predicted on the basis of semiclassical theory are below the scale of quantum gravity effects, a scale at which we expect the semiclassical picture do break down.

The existence of quantum inequality restrictions shows that quantum field theory does impose *some* constraints on the manipulation of negative energies. It is important to determine the generality of these constraints. The work of Wald and Yurtsever [11] hints that there may be connections between quantum inequalities and averaged energy conditions. It would also be desirable to find an analytic form for the numerical quantum inequality which we have established. Such a formulation might suggest a more general inequality which would hold on a wider class of spacetime backgrounds than those examined thus far. It would seem especially important to examine black-hole evaporation. The integrated negative energy flux through the horizon due to the normal evaporation of the hole, as measured by a static observer outside the horizon, would not seem at first sight to obey a quantum inequality-type restriction. To what special features of black-hole radiation can this be attributed? The negative energy fluxes that we have considered are those which can be manipulated (e.g., shot into the black hole from infinity). Black-hole radiation cannot be manipulated, except in a rather limited sense by changing the temperature of the hole through the injection of mass or charge, for example. The difference between these two kinds of situations involving negative energy fluxes is not immediately clear and warrants further study. It would also be interesting to examine effective ($-$) energy fluxes due to the motion of an observer through a static ($-$) energy background. These questions are currently under investigation.

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APPENDIX

In this appendix we will present an alternative derivation of Eq. (2.42) which was suggested to us by Flanagan. Define the operator $\hat{O}_{lm}$ by

$$\hat{O}_{lm} = \int_0^\infty d\omega\, h_{\omega l}(a_{\omega lm} + a^\dagger_{\omega l-m}) \ . \tag{A1}$$

It satisfies the property that $\hat{O}^\dagger_{lm} = \hat{O}_{l-m}$. Consequently, the operator $\hat{O}_{lm}\hat{O}_{l-m}$ is a Hermitian operator and has real expectation values. It is also a positive operator in that

$$\langle \hat{O}_{lm}\hat{O}_{l-m}\rangle \geq 0 \ . \tag{A2}$$

This follows from the fact that the left-hand side is just the norm of the state vector $|\Psi\rangle = \hat{O}_{l-m}|\psi\rangle$, where $|\psi\rangle$ is the quantum state in which we are taking the expectation value. We may write this expectation value explicitly as

$$\langle \hat{O}_{lm}\hat{O}_{l-m}\rangle = \int_0^\infty d\omega\, d\omega' h_{\omega l} h_{\omega' l}[\langle a^\dagger_{\omega' lm} a_{\omega lm}\rangle + \langle a^\dagger_{\omega l-m} a_{\omega' l-m}\rangle + \langle a_{\omega lm} a_{\omega' l-m}\rangle + \langle a^\dagger_{\omega l-m} a^\dagger_{\omega' lm}\rangle] + \int_0^\infty d\omega\, h^2_{\omega l} \ . \tag{A3}$$

Finally, we may use the fact that

$$\sum_m \langle a_{\omega lm} a_{\omega' l-m}\rangle^* = \sum_m \langle a^\dagger_{\omega l-m} a^\dagger_{\omega' lm}\rangle , \tag{A4}$$

and that $\int_0^\infty d\omega\, d\omega' h_{\omega l} h_{\omega' l}\langle a^\dagger_{\omega' lm} a_{\omega lm}\rangle$ is real to write

$$\sum_{lm}\langle \hat{O}_{lm}\hat{O}_{l-m}\rangle = \sum_{\omega lm} h^2_{\omega l} + S \ . \tag{A5}$$

Here S is defined in Eq. (2.34). It follows from Eqs. (A2) and (A5) that

$$S \geq -\sum_{\omega lm} h^2_{\omega l} \ , \tag{A6}$$

which is Eq. (2.42).

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[23] It is possible for resonance effects to cause the transmission coefficients for particular modes to be larger for a higher and wider barrier than for a lower and narrower barrier. However, this will only happen for above-barrier modes, that is, those for which $\omega^2 > V_S$. However, since we have set the transmission coefficients equal to one for these modes, the resonance effects are irrelevant.

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