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Evaporation of rotating black holes and the equivalence principle

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This article is an investigation of the physics which underlies Hawking radiation from rotating black holes. It is shown that a global scalar field theory which is compatible with the equivalence principle demands that a rotating black hole in isolation experience thermal evaporation. The main goal of this paper is to attain a physical understanding of the phenomenon with a particular emphasis on computing the renormalized stress-energy tensor in the asymptotic zones, near the event horizon and at stationary infinity. This tensor is shown to be a measure of the change in the zero-point oscillations of the local field theory which is formulated by inertial observers during free fall, as compared to a global standard. An external onlooker sees the zero-point energy in a freely falling coordinate patch decrease as it approaches the horizon. This translates to a negative energy density of the field, near the horizon, in the components of the renormalized stress-energy tensor. The external onlooker interprets the zero-point energy lost during free fall as an outgoing stream of particle-antiparticle pairs.

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I. INTRODUCTION

This article is a derivation of the asymptotic form of the dynamic components of the renormalized stress-energy tensor of a scalar field, near the event horizon of a rotating black hole and at asymptotic infinity, as a consequence of the equivalence principle. The analysis parallels the treatment of Ref. [1], which is a similar calculation for Schwarzschild black holes.

The Kerr geometry is far more complicated than the nonrotating case, requiring additional calculational machinery. More in-depth physical explanations appear in Ref. [1], and many steps are omitted here from calculations which are similar in the two cases. However, enough details are included so that this article can stand on its own.

There is not nearly as much published research concerning Hawking radiation from rotating black holes as there is from Schwarschild holes. The energy and angular momentum fluxes at asymptotic infinity have been inferred for scalar fields from partial calculations and plausibility arguments in Refs. [2,3]. In Ref. [4], Unruh's ansatz for particle creation is used by Iyer and Kumar to find the fluxes at asymptotic infinity. The ansatz imposes a boundary condition at the horizon which essentially puts the particle creation in the normalization amplitudes of the wave functions [5]. Some information on the renormalized stress-energy tensor is found in Ref. [6] through a plausibility argument based on analogies to accelerated observers in flat spacetime. In theory, point-separated bitensor regularization of the stress-energy tensor can be used as it has been for the Schwarzschild hole in Refs. [7–9]. Some of the preliminary steps can be found in Refs. [10,11]. In Ref. [12], a calculation exists for the renormalized stress-energy tensor of a massive scalar field in the Hartle-Hawking vacuum based on the renormalized effective action generated by a point-separated method [3]. This calculation is valid for Compton wavelengths that are much less than the radius of curvature of spacetime near the horizon. Thus, by (6.12) of this article there is very little radiation in this case and it is essentially a pure vacuum polarization phenomenon. There are also some heuristic results for the renormalized stress-energy tensor for electromagnetic fields in Ref. [13].

By contrast, one of the strengths of this article is that the results are derived from first principles—the equivalence principle. The mathematics of point-separated bitensors in Kerr geometry generates some algebra of astronomical proportions. In this formalism, the mathematics is much more tractible. As in Ref. [1], an in-depth expose of the underlying quantum physics which governs this problem is presented, as opposed to the pure calculational treatments based on point-separated bitensors. This analysis shows the utility of the method developed in Ref. [1], to study the Schwarzschild case, for understanding more complicated problems involving field theories in curved spacetimes. It should be noted that the following is valid for both massless and massive fields.

The main premise of this effort is that freely falling observers can formulate their version of quantum field theory so that it looks just like special relativistic field theory in their local neighborhood. As viewed globally, these locally formulated field theories differ from point to point of spacetime and, in particular, between those observers near the horizon and those near asymptotic infinity. This article compares the stress energy of the zero-point oscillations as measured by inertial observers near the horizon with the same as measured by stationary observers at asymptotic infinity. It is shown, when compared to a global standard, that the energy of the zero-point oscillations decreases during free fall and this is the essence of Hawking radiation. One advantage of this analysis is that, by considering the vacuum state of the freely falling observers near the horizon, one is forced to acknowledge that the "Unruh" vacuum approximates the only vacuum state of physical relevance for a black hole in isolation (as opposed to Hawking-Hartle- or

Boulware-type vacua).

The article is organized as follows. First, to be able to piece together the local inertial analyses in order to obtain a covariant global field theory, one needs a space-filling family of freely falling observers whose four-velocities are hypersurface orthogonal. Unfortunately, in the Kerr spacetime the tetrads carried by such observers can never be a coordinate frame. Thus, one is forced to calculate in an anholonomic basis throughout the article. This creates its biggest problem in Sec. III, where the wave equation for a scalar field in a freely falling frame is derived and solved, locally, near the horizon. In Sec. IV, the solutions of the wave equation as formulated by observers at stationary infinity, the so-called "global" solutions, are reviewed. The Fourier decomposition of the "global" solutions in terms of the "local" solutions defined in Sec. III is accomplished near the horizon. An inverse relation is found, as well as the Bogoliubov transformation relating the particle creation and annihilation operators in the two different formulations of field theory (local and global). In Sec. V, the renormalized stress-energy tensor of the local vacuum is found. Since this vacuum is tied to the motion of each of the freely falling observers, it is straightforward to find the renormalized stress-energy tensor of spacetime. This is deduced in Sec. VI by using the foliation of spacetime described in Sec. II to piece together the local results. Like the Schwarzschild case, it has a thermal component, but now there is a contribution in the superradiant modes as was found in Ref. [4]. The Hawking radiation is clearly shown to be the result of negative energy and angular momentum of the local vacuum (as viewed from stationary infinity), which is tied to each freely falling observer, flowing towards the hole along the congruence of observer's world lines.

II. THE FOLIATION OF SPACETIME

In this section a foliation of spacetime outside of the horizon by the world lines of a family of freely falling observers is described. This will provide the fundamental mathematical machinery necessary to elucidate the physics of black-hole evaporation.

A. Classical trajectories in the Kerr spacetime

To define an appropriate family of freely falling observers, one needs to classify the timelike geodesics outside of the horizon through Carter's equations [14].

The Kerr metric in Boyer-Lindquist coordinates is given in terms of the mass of the hole, M, and its angular momentum per unit mass, a:

$$ds^2=-\left[1-\frac{2Mr}{\rho^2}\right]dt^2+\rho^2 d\theta^2+\frac{\rho^2}{\Delta}dr^2 + \left[(r^2+a^2)+\frac{2Mra^2}{\rho^2}\sin^2\theta\right]\sin^2\theta\, d\phi^2 - \frac{4Mra}{\rho^2}\sin^2\theta\, d\phi\, dt \ , \tag{2.1}$$

where $\rho^2 \equiv r^2+a^2\cos^2\theta$ and $\Delta \equiv r^2-2Mr+a^2$. There are two event horizons which are given by the roots of the equation $\Delta=0$:

$$\Delta=(r-r_+)(r-r_-) \ , \tag{2.2a}$$

$$r_\pm = M\pm\sqrt{M^2-a^2} \ , \tag{2.2b}$$

where r_+ and r_- are the outer and inner event horizons, respectively.

The redshifted energy of a particle (the energy as seen as asymptotic infinity in the stationary frames), ω, is given in terms of the four-momentum in the stationary frames, $\tilde{P}_\mu$, by

$$-\tilde{P}_t \equiv \omega \ . \tag{2.3}$$

Tildes will be used to denote quantities evaluated in the stationary frames at asymptotic infinity throughout the remainder of the article. The component of angular momentum of a particle along the symmetry axis of the hole, $-m$, as seen in the stationary frames at asymptotic infinity, is defined by

$$\tilde{P}_\phi \equiv -m \ . \tag{2.4}$$

For geodesic motion, both ω, m, as well as the mass of the particle, m_e, are conserved. In Kerr geometry there is a fourth constant of motion K, Carter's fourth constant of motion, which can be given in positive-definite form:

$$K=\tilde{P}_\theta^2+\left[\omega a \sin\theta+\frac{m}{\sin\theta}\right]^2+m_e^2a^2\cos^2\theta \ , \tag{2.5}$$

where $\tilde{P}_\theta$ is the momentum conjugate to $\partial/\partial\theta$. A general nongeodesic trajectory can be parametrized by ω, m, and K, but they are no longer constants [15].

Carter's equations of geodesic motion are defined for trajectories in Boyer-Lindquist coordinates in terms of these parameters [16]:

$$\rho^2\tilde{P}^t=-a(\omega a\sin^2\theta+m)+\frac{r^2+a^2}{\Delta}P \ , \tag{2.6a}$$

$$\rho^2\tilde{P}^r=\pm\sqrt{R} \ , \tag{2.6b}$$

$$\rho^2\tilde{P}^\phi=-\left[\omega a+\frac{m}{\sin^2\theta}\right]+\frac{Pa}{\Delta} \ , \tag{2.6c}$$

where

$$P\equiv\omega(r^2+a^2)+ma \ , \tag{2.6d}$$

$$R\equiv P^2-\Delta(m_e^2r^2+K) \ . \tag{2.6e}$$

Taking a cue from Ref. [1], the most convenient choice for a foliation of spacetime is a set of freely falling observers defined by frames released from rest at asymptotic infinity with no angular momentum, in the distant past:

$$\omega_0=m_e \ , \tag{2.7a}$$

$$m_0=0 \ , \tag{2.7b}$$

$$K_0=\omega^2a^2 \ . \tag{2.7c}$$

Then Carter's equations (2.6) and (2.7) give the fourvelocity of this set of observers:

$$\tilde{u}_\mu = \frac{1}{\Delta \sin^2\theta}\left[\tilde{g}_{\phi\phi}\frac{\partial}{\partial t} - \tilde{g}_{\phi t}\frac{\partial}{\partial \phi}\right] - \frac{\sqrt{r^2+a^2}\sqrt{2Mr}}{\rho^2}\frac{\partial}{\partial r}\ , \tag{2.8}$$

where $\tilde{g}_{\mu\nu}$ is the metric in Boyer-Lindquist coordinates, (2.1).

B. The global frame field

One can define a global frame field which is carried by these freely falling observers that is denoted as the preferred freely falling (PFF) frames:

$$E_0 \equiv \tilde{u}_\mu = \frac{g_{\phi\phi}}{\Delta \sin^2\theta}\left[\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial \phi}\right] - \frac{\sqrt{r^2+a^2}\sqrt{2Mr}}{\rho^2}\frac{\partial}{\partial r}\ , \tag{2.9a}$$

$$E_1 = -\frac{(r^2+a^2)(2Mr)}{\Delta\rho^2}\left[\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial \phi}\right] + \frac{\sqrt{r^2+a^2}\sqrt{2Mr}}{\rho^2}\frac{\partial}{\partial r}\ , \tag{2.9b}$$

$$E_\phi = \frac{\partial}{\partial \phi}\ , \tag{2.9c}$$

$$E_\theta = \frac{\partial}{\partial \theta} + \mathcal{P}(r)a^2\sin 2\theta E_1\ , \tag{2.9d}$$

where

$$\Omega \equiv -\frac{\tilde{g}_{\phi t}}{\tilde{g}_{\phi\phi}} \tag{2.10}$$

and

$$\mathcal{P}(r) \equiv \int_{r_+}^{r}\frac{dr}{\sqrt{r^2+a^2}\sqrt{2Mr}}\ . \tag{2.11}$$

The basis covectors are

$$\omega^0 = dt + \frac{\sqrt{r^2+a^2}\sqrt{2Mr}}{\Delta}dr\ , \tag{2.12a}$$

$$\omega^1 = dt + \frac{[r^2+a^2]^{3/2}}{\Delta\sqrt{2Mr}}dr - d(a^2\mathcal{P}\sin^2\theta)\ , \tag{2.12b}$$

$$\omega^\phi = d\phi - \Omega\, dt\ , \tag{2.12c}$$

$$\omega^\theta = d\theta\ . \tag{2.12d}$$

Since $d\omega^0 \wedge \omega^0 = 0$, E_0 is a hypersurface orthogonal vector field and the world lines of the PFF observers foliate spacetime.

The metric in this frame is given by

$$g_{00} = -1\ , \tag{2.13a}$$

$$g_{11} = 1 - \frac{\Delta \sin^2\theta}{g_{\phi\phi}} \equiv (V^r)^2\ , \quad V^r < 0\ , \tag{2.13b}$$

$$g_{1\theta} = g_{\theta 1} = (V^r)^2\mathcal{P}a^2\sin 2\theta\ , \tag{2.13c}$$

$$g_{\theta\theta} = \rho^2 + [V^r\mathcal{P}a^2\sin 2\theta]^2\ , \tag{2.13d}$$

$$g_{\phi\phi} = \tilde{g}_{\phi\phi}\ . \tag{2.13e}$$

All of the other components of $g_{\mu\nu}$ are zero. The determinant of the metric, g, is

$$g = -(V^r)^2 g_{\phi\phi}[\rho^2 + \mathcal{P}^2\alpha^2 a^4\sin^2\theta]\ , \tag{2.14}$$

where α is the lapse function [6]

$$\alpha^2 = \frac{\Delta \sin^2\theta}{g_{\phi\phi}} = 1 - (V^r)^2\ . \tag{2.15}$$

Asymptotically (denoted by the symbol "$\sim$"),

$$g \underset{r \to r_+}{\sim} -[r^2+a^2]^2\sin^2\theta\ . \tag{2.16}$$

The inverse metric is

$$g^{00} = -1\ , \tag{2.17a}$$

$$g^{11} = \frac{1}{(V^r)^2}\frac{\rho^2 + (V^r)^2\mathcal{P}^2a^4\sin^2 2\theta}{\rho^2 + \mathcal{P}^2\alpha^2a^4\sin^2\theta} \underset{r \to r_+}{\sim} \frac{1}{(V^r)^2}\left[1 + \frac{(V^r)^2\mathcal{P}^2a^4\sin^2 2\theta}{\rho^2}\right]\ , \tag{2.17b}$$

$$g^{1\theta} = g^{\theta 1} = \frac{-(V^r)^2\mathcal{P}a^2\sin 2\theta}{\rho^2 + \mathcal{P}^2\alpha^2a^4\sin^2\theta}\ , \tag{2.17c}$$

$$g^{\theta\theta} = \frac{-(V^r)^2}{V^r[\rho^2 + \mathcal{P}^2\alpha^2a^4\sin^2\theta]} \underset{r \to r_+}{\sim} \frac{1}{\rho^2}\ , \tag{2.17d}$$

$$g^{\phi\phi} = \frac{1}{\tilde{g}_{\phi\phi}}\ , \tag{2.17e}$$

all other $g^{\mu\nu}$'s are zero.

C. Global coordinates

The PFF frame of (2.9) is not a coordinate frame. This is demonstrated with the commutator algebra which can be represented by the structure constants of the Lie algebra, $c^{\alpha}_{\mu\nu}$:

$$[E_\mu, E_\nu] = c^{\alpha}_{\mu\nu}E_\alpha\ , \tag{2.18a}$$

$$c^{\alpha}_{\mu\nu} = -c^{\alpha}_{\nu\mu}\ . \tag{2.18b}$$

One finds, from (2.9) and (2.18a),

$$c^{\phi}_{\theta 0} = \frac{2Mra^3\sin 2\theta}{\Delta\rho^4} + \frac{\mathcal{P}a^2\sin 2\theta\sqrt{r^2+a^2}\sqrt{2Mr}}{\rho^2}\frac{\partial}{\partial r}(\Omega)\ , \tag{2.19a}$$

$$c^{\phi}_{\theta 1} = -\frac{(r^2+a^2)(2Mr)}{\Delta}\frac{\partial}{\partial \theta}\left[\frac{\Omega}{\rho^2}\right] \underset{r \to r_+}{\sim} -\frac{2Mra^3}{\Delta\rho^4}\sin 2\theta\ , \tag{2.19b}$$

$$c^{\phi}_{10} = \frac{\sqrt{r^2+a^2}\sqrt{2Mr}}{\rho^2}\frac{\partial}{\partial r}(\Omega)\ , \tag{2.19c}$$

where the asymptotic form of Ω was used in (2.19b):

$$\Omega \underset{r \to r_+}{\sim} \left[\frac{a}{r^2+a^2}\right]\left[1 - \frac{\Delta\rho^2}{(r^2+a^2)^2}\right]\ . \tag{2.20}$$

Carter's equations in (2.6) can be used to show that there is no family of freely falling frames which can be a coordinate frame for all θ and ϕ even when restricted to a small neighborhood of the horizon. This is a significant difference from the Schwarzschild case that will add complications to the analysis in Kerr geometry. The root of the problem is a lack of symmetry in the θ direction (there is no longer spherical symmetry as in the nonrotating case) and nonvanishing contributions from $\partial/\partial\theta$ will occur in commutators between basis vector fields.

The frame in (2.9) is chosen to come as close to a coordinate basis as possible with $E_0=\tilde{u}_\mu$ [see Eq. (3.6)]. The dual covectors ω^0, ω^1, and ω^θ are all exact differentials, so their integrals (coordinate functions) are well defined. The problem is with ω^ϕ, since Ω is a function of r and θ. Thus, the following coordinates $(x^0,x^1,x^\phi,x^\theta)$ are introduced:

$$dx^0=\omega^0 \ , \tag{2.21a}$$

$$dx^1=\omega^1 \ , \tag{2.21b}$$

$$dx^\phi=d\phi-\Omega_H dt \ , \tag{2.21c}$$

$$dx^\theta=d\theta \ , \tag{2.21d}$$

where Ω_H is the angular velocity of the horizon as viewed from asymptotic infinity,

$$\Omega_H=\frac{a}{r_+^2+a^2} \ . \tag{2.21e}$$

These covectors are useful near the horizon since

$$\omega^\phi \underset{r\to r_+}{\sim} d\phi-\Omega_H dt+\left[\Omega_H+\frac{2r_+a}{(r_+-r_-)\rho^2}\right]d(x^0-x^1) +O(\alpha^2)dx^0+O(\alpha^2)d\theta \ . \tag{2.22}$$

In order to approximately integrate (2.22), a useful relation is derived from (2.21a) and (2.21b):

$$\alpha^2 \underset{r\to r_+}{\sim} 2\kappa[x^1-x^0-C] \ , \tag{2.23}$$

where

$$\kappa=\frac{r_+-r_-}{2(r_+^2+a^2)} \tag{2.24}$$

is the surface gravity of the hole and C is a constant. Thus, (2.23) and (2.22) imply that

$$\int\omega^\phi \underset{r\to r_+}{\sim} x^\phi+O(\alpha^2)=\phi-\Omega_H t+O(\alpha^2) \ . \tag{2.25}$$

D. Local momentum

There are two classes of nonspacelike trajectories which will be differentiated in this article. Those that are "globally outgoing," $\tilde{V}^r>0$, and those that are "globally ingoing," $\tilde{V}^r<0$, as viewed from asymptotic infinity:

$$\tilde{V}^r=\frac{\tilde{P}^r}{\tilde{P}^t} \ . \tag{2.26}$$

One can use (2.12) and (2.6) to express the momentum of a particle as viewed by a PFF observer near the horizon:

$$P^0 \underset{r\to r_+}{\sim} 2\alpha^{-2}[\omega+\Omega_H m] \ , \quad \tilde{V}^r>0 \ , \tag{2.27a}$$

$$P^0 \underset{r\to r_+}{\sim} \alpha^0 \ , \quad \tilde{V}^r<0 \ , \tag{2.27b}$$

$$P^1 \underset{r\to r_+}{\sim} 2\alpha^{-2}[\omega+\Omega_H m] \ , \quad \tilde{V}^r>0 \ , \tag{2.27c}$$

$$P^1 \underset{r\to r_+}{\sim} \alpha^0 \ , \quad \tilde{V}^r<0 \ . \tag{2.27d}$$

If one considers modes defined by observers at asymptotic infinity which are characterized by ω and m constant, then the globally outgoing modes have local momenta with huge gradients in blueshift, near the horizon, by (2.27a), and (2.27c). These are the modes which result in Hawking emission. The physics of this relationship between differential blueshifts and particle creation is discussed in detail in Ref. [1], particularly Sec. IV.

A phenomenon unique to the Kerr case occurs when $\omega+m\Omega_H<0$, $\omega>0$. These are well-defined globally outgoing modes, but by (2.27a) they have a local negative energy. Such modes are referred to as superradiant [17]. These global states "dive" into the negative energy continuum of the PFF observers (a Klein paradox) [18].

III. THE LOCAL WAVE EQUATION

In order to compare the field theory formulated by freely falling observers with the field theory defined by observers at asymptotic infinity, it is essential that the wave equation in the PFF basis be solved. Since (2.9) is an anholonomic basis, the scalar wave equation is far more complicated than it is in the Schwarzschild case. The connection coefficients $\Gamma_{\mu\alpha\beta}$ are tabulated in the Appendix.

A. The scalar wave equation

The wave equation in covariant form is

$$\varphi^{;\alpha}{}_{;\alpha}-m_e^2\varphi=0 \ , \tag{3.1}$$

where a semicolon represents covariant differentiation. This can be expanded out in terms of partial derivatives

$$g^{\alpha\beta}\varphi_{,\alpha,\beta}+g^{\alpha\beta}{}_{,\beta}\varphi_{,\alpha}+\Gamma^\nu{}_{\beta\nu}g^{\alpha\beta}\varphi_{,\alpha}-m_e^2\varphi=0 \ . \tag{3.2}$$

For this analysis, it is of interest to solve (3.2) only when the PFF observer is near the horizon. If $(\bar{x}^0,\bar{x}^1)$ are coordinates of the observer, then, by (2.23),

$$\bar{x}^1-\bar{x}^0 \underset{r\to r_+}{\sim} C \ . \tag{3.3}$$

One can define, according to special relativity and the equivalence principle, local energy eigenfunctions defined by

$$i\frac{\partial}{\partial x^0}\varphi\Big|_{(\bar{x}^0,\bar{x}^1)}=P^0\varphi\Big|_{(\bar{x}^0,\bar{x}^1)} \ , \tag{3.4}$$

where the identification

$$E_\alpha=\frac{\partial}{\partial x^\alpha} \tag{3.5}$$

has been made. One should note that

$$E_0 x^i = \delta^i{}_0 , \quad i=0,1,\theta , \tag{3.6a}$$

$$E_1 x^i = \delta^i{}_1 , \quad i=0,1,\theta , \tag{3.6b}$$

$$E_\theta x^i = \delta^i{}_\theta , \quad i=0,1,\theta , \tag{3.6c}$$

$$E_\phi x^i = \delta^i{}_\phi , \quad i=0,1,\theta,\phi . \tag{3.6d}$$

This signifies how close the frame in (2.9) is to being a coordinate frame.

Our main interest is with the globally outgoing modes which by (2.27a) are characterized by $P^0 \gg 1/r_+$. By (3.4), the wave equation in an order of magnitude scaling is

$$-E_0 E_0 \varphi + g^{11} E_1 E_1 \varphi + O\left[\frac{P^0}{r_+}\right] + O\left[\frac{1}{r_+^2}\right] = 0 . \tag{3.7a}$$

It follows from (3.4) and (2.27c) that there is no term to balance the $\partial^2/\partial x^{0^2}\varphi$ term in (3.7a) unless

$$\varphi_{\text{loc}} \underset{r\to r_+}{\sim} \exp[i(P_0 x^0 + P_1 x^1)] \simeq e^{iP_\mu x^\mu} . \tag{3.7b}$$

This is what one expects since the equivalence principle implies that the local solutions should look like plane waves near $(\bar{x}^0, \bar{x}^1)$. The form of the solution in (3.7b) is valid on any open set $\mathcal{V}$ defined by

$$\mathcal{V} = \{x^0, x^1, x^\theta, x^\phi \,|\, |(x^0 - \bar{x}^0)^2 - g_{11}(x^1 - \bar{x}^1)^2| \ll r_+^2\} . \tag{3.8a}$$

On this set $P^0 \simeq$ const. Normally, it will not be of interest to know the slow variation of P^0 on $\mathcal{V}$. However, its existence is acknowledged in (3.11c).

It will be of more interest to look at generic sets $\mathcal{V}_+$, which are restrictions of $\mathcal{V}$ to the region outside of the horizon:

$$\mathcal{V}_+ = \{x^0, x^1, x^\theta, x^\phi \,|\, |(x^0 - \bar{x}^0)^2 - g_{11}(x^1 - \bar{x}^1)^2| \ll r_+ , x^1 - x^0 > C\} . \tag{3.8b}$$

Since the coordinate x^ϕ does not appear in (3.2), the equation separates and relation (3.7b) can be modified:

$$\varphi_{\text{loc}} \underset{r\to r_+}{\sim} \exp[i(P_0 x^0 + P_1 x^1)] e^{imx^\phi} F(x^0, x^1, \theta) , \tag{3.9}$$

where $F(x^0, x^1, \theta)$ is a slowly varying function of x^0 and x^1 as well as θ near the horizon and m is an integer.

B. The angular functions

The angular dependence of the local wave functions is much more complicated than it is for their Schwarzschild counterparts which are simply spherical harmonics. For the present purpose, it need not be determined explicitly, but it will be shown that the angular dependence separates to a good approximation near the horizon and the angular functions which result from a complete orthonormal set.

To define the angular functions, an angular equation is generated near the horizon by substituting (3.9) into (3.2) and evaluating all of the terms at the horizon, i.e., $x^1 - x^0 = C$ or $r = r_+$. This leaves a second-order ordinary differential equation (ODE) with variable coefficients in the variable θ only. Since, physically, one is only interested in the dominant variation of φ_{loc} near the horizon, the exponential in (3.7b), the asymptotic expression (3.9), can be written approximately for the globally outgoing modes as

$$\vec{\varphi}_{\text{loc}} \underset{r\to r_+}{\sim} \exp[i(P_0 x^0 + P_1 x^1)] e^{imx^\phi} \exp[-i\Delta(\Gamma_{\omega m}(r,\theta)/2(r-M)^2)] \vec{F}(x^0, x^1, \theta)|_{x^1 - x^0 = C} , \tag{3.10a}$$

where $\Gamma_{\omega m}(r,\theta)$ is an arbitrary function to be chosen later to simplify the angular equation. It depends on the parameters m and $\omega(P^\mu)$, which is a function of the local momentum via the inverse to (2.9) or (2.27). The angular function $\vec{R}(\theta)$ is defined by (the arrows signify the globally outgoing condition)

$$\vec{R}(\theta) \equiv \vec{F}(x^0, x^1, \theta)|_{x^1 - x^0 = C} . \tag{3.10b}$$

To demonstrate the flavor of the calculation, some typical algebraic steps which are used to transform (3.2) into the angular equation for $\vec{R}(\theta)$ are given. Firstly,

$$\left[g^{00}\frac{\partial}{\partial x^0}\frac{\partial}{\partial x^0} + g^{11}\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^1}\right]\vec{\varphi} \underset{r\to r_+}{\sim} \left[P_0^2 - P_1^2 + \alpha^2 P_0^2 - \frac{2(\omega + m\Omega_H)}{\rho^4(r-M)}(r^2+a^2)a^2\sin^2\theta\,\Gamma + i\kappa P_0\right]\vec{\varphi} \tag{3.11a}$$

$$\underset{r\to r_+}{\sim} \left\{\frac{1}{\rho^2}(\omega + m\Omega_H)^2(\rho^2 - a^2\sin^2\theta) + m_e^2 r^2 + K + 2\rho^2 m\Omega_H(\omega + m\Omega_H) - \frac{2(\omega + m\Omega_H)(r^2+a^2)}{\rho^4(r-M)}a^2\sin^2\theta\,\Gamma + i\kappa P_0\right\}\vec{\varphi} . \tag{3.11b}$$

In deriving (3.11a), one needs to note the slow variation of P^0 and P^1 on $\mathcal{V}_+$,

$$-\frac{\partial}{\partial x^0}P_0+\frac{\partial}{\partial x^1}P_1=\kappa P_0\ , \tag{3.11c}$$

which follows from (2.9) and (2.27).

Then, after some more similar algebra and choosing

$$\Gamma_{\omega m}(\theta,r)=-\left[a^2\frac{\partial}{\partial r}P_0+P_1\mathcal{P}a^2\left[2\cos^2\theta+\frac{a^2}{2\rho^2}\sin^2 2\theta\right]+\frac{a^2\sin^2\theta}{2\rho^2}(\omega+m\Omega_H)(4r-2\kappa a^2\sin^2\theta)\right] \tag{3.11d}$$

in order to eliminate the imaginary terms in the equation, one gets, after multiplying through with ρ^2, the desired equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}\vec{R}_{km}(\omega;\theta)\right]+f(\omega,m,m_e;\theta)\vec{R}_{\kappa m}(\omega;\theta)+K\vec{R}_{km}(\omega;\theta)=0\ , \tag{3.12a}$$

where

$$f(\omega,m,m_e;\theta)=[\omega+m\Omega_H]^2[\rho^2-a^2\sin^2\theta]-\frac{4a^4\sin^2\theta(r^2+a^2)}{(r_+-r_-)^2\rho^6}-m_e^2a^2\cos^2\theta+2\rho^2m\Omega_H[\omega+m\Omega_H]+\frac{\rho^4m^2}{(r^2+a^2)^2\sin^2\theta}$$
$$+\frac{2(\omega+m\Omega_H)}{\rho^4(r-M)}(r^2+a^2)a^2\sin^2\theta\left\{P_1\mathcal{P}\left[2\cos^2\theta+\frac{a^2}{2\rho^2}\sin^2 2\theta\right]+\frac{a^2\sin^2\theta}{2\rho^4}[\omega+m\Omega_H][4r-2\kappa a^2\sin^2\theta]\right\}\ . \tag{3.12b}$$

The angular equation (3.12) is an eigenvalue equation where the eigenvalues K are restricted by the regularity of $R_{km}(\omega;\theta)$ at $\theta=\pi$ and $\theta=0$. The subscript k was added to the angular function to reflect this.

According to Sturm-Liouville theory, since $\sin\theta>0$ on the domain $0<\theta<\pi$, the functions R_{km} form a complete set which are chosen to be orthonormal for each pair $\omega(P^\mu)$ and m [19]:

$$\int\vec{R}_{km}(\omega(P^\mu);\theta)\vec{R}^{\,*}_{k'm}(\omega(P^\mu);\theta)\sin\theta\,d\theta=\delta_{kk'}\ . \tag{3.13a}$$

The appropriate angular measure used in (3.13a) follows from (2.16).

A similar analysis can be performed for the globally ingoing modes $\overleftarrow{\varphi}$. Another complete set of orthonormal functions is obtained:

$$\int\overleftarrow{R}_{km}(\omega(P^\mu);\theta)\overleftarrow{R}^{\,*}_{km}(\omega(P^\mu);\theta)\sin\theta\,d\theta=\delta_{kk'}\ . \tag{3.13b}$$

C. The solution space

The restriction of the local wave function to the generic open set $\mathcal{V}_+$ of (3.8b) is denoted by $\vec{\varphi}_{nmk}(x^0,x^1,\theta,x^\phi)$ for the globally outgoing modes:

$$(\vec{\varphi}_{\rm loc})_{nmk}|_{\mathcal{V}_+}=\vec{u}_ne^{imx^\phi}\vec{R}_{km}(\omega_n;\theta)$$
$$\equiv\vec{\varphi}_{nmk}(x^0,x^1,\theta,x^\phi)\ , \tag{3.14a}$$

$$\vec{u}_n=\frac{1}{\sqrt{2\pi}\sqrt{2P_n^0}\sqrt{r^2+a^2}}\exp[i(P^0)_n(x^0-x^1)]\ , \tag{3.14b}$$

where the fact that $P^0\simeq P^1$ for globally outgoing quanta near the horizon [see Eq. (2.27)] was used in the exponent of (3.14b).

Similarly, the local solutions which are globally ingoing near the horizon are of the form

$$(\overleftarrow{\varphi}_{\rm loc})_{nmk}\underset{r\to r_+}{\sim}\{\overleftarrow{F}_{nmk}(x^0,x^1,\theta)|_{x^1-x^0=C}e^{imx^\phi}\}$$
$$\times(\exp\{i[(P_0)_nx^0+(P_1)_nx^1]$$
$$\times[1+O(\alpha^2)]\})\ , \tag{3.15a}$$

$$P_0,P_1\underset{r\to r_+}{\sim}\text{const}+O(\alpha^2)\ , \tag{3.15b}$$

where $\overleftarrow{F}_{nmk}(x^0,x^1,\theta)$ is a slowly varying function of x^0 and x^1 near the horizon (i.e., varying on distance scales on the order of r_+). Rearranging (3.15a), using (2.21) and (2.27),

$$(\overleftarrow{\varphi}_{\rm loc})_{nmk}\underset{r\to r_+}{\sim}N_{nmk}e^{imx^\phi}\overleftarrow{R}_{km}(\omega_n;\theta)$$
$$\times\exp\left\{\frac{i(P^0-P_1)}{2}(x^0+x^1)[1+O(\alpha^2)]\right\}\ , \tag{3.16}$$

where N_{nmk} is a normalization constant. Using (2.21) and (2.27) again, (3.16) is

$$(\overleftarrow{\varphi}_{\rm loc})_{nmk}|_{\mathcal{V}_+}\simeq\overleftarrow{u}_ne^{imx^\phi}\overleftarrow{R}_{km}(\omega_n;\theta)\equiv\overleftarrow{\varphi}_{nmk}\ , \tag{3.17a}$$

$$\overleftarrow{u}_n=\frac{1}{\sqrt{2\pi}\sqrt{2(P^0)_n}}\frac{1}{\sqrt{r^2+a^2}}$$
$$\times\exp\{i[\omega(P^\mu)_n+m\Omega_H]v\}\ , \tag{3.17b}$$

where v is the advanced coordinate:

$$v = t + r_* \ , \tag{3.18a}$$

$$r_* = \int \frac{r^2+a^2}{\Delta} dr \ . \tag{3.18b}$$

The advantage of the normalization constant implemented in (3.14b) and (3.17b) will be made apparent in the next section. For more details on the validity and the mathematical rigor of the approximations used, as well as more physical insight, see Sec. III of Ref. [1].

IV. THE LOCAL FOURIER DECOMPOSITION OF THE STATIONARY FRAME WAVE FUNCTIONS

In this section, the wave functions, as formulated by stationary observers at asymptotic infinity, are decomposed as generalized sums of the local wave functions discussed is the last section. An inverse relation is derived, as well as the Bogoliubov transformation between the two sets of particle creation and annihilation operators.

A. The global wave functions

The global wave functions $\tilde{\varphi}_{n'm'l'}(r,\theta,\phi,t)$ defined by the stationary observers at asymptotic infinity have a well-known asymptotic form near the horizon [20]:

$$\vec{\tilde{\varphi}}_{nml}(r,\theta,\phi,t) \underset{r\to r_+}{\sim} \frac{e^{i(\omega_n + m\Omega_H)u}}{\sqrt{2\pi}\sqrt{\omega_n m\Omega_H}\sqrt{r^2+a^2}} S_{lm}(\omega_n;\theta) e^{im(\phi-\Omega_H t)} \quad \text{purely outgoing as } r \to r_+ \ , \tag{4.1a}$$

$$\overleftarrow{\tilde{\varphi}}_{nml}(r,\theta,\phi,t) \underset{r\to r_+}{\sim} \frac{e^{i(\omega_n + m\Omega_H)v}}{\sqrt{2\pi}\sqrt{\omega_n + m\Omega_H}\sqrt{r^2+a^2}} S_{lm}(\omega_n;\theta) e^{im(\phi-\Omega_H t)} \quad \text{purely ingoing as } r \to r_+ \ . \tag{4.1b}$$

The coordinate u is the retarded coordinate

$$u = t - r_* \ . \tag{4.2}$$

The functions $S_{lm}(\omega_n;\theta)$ are spheroidal harmonics which form a complete set for each value of ω_n:

$$\int S_{lm}(\omega_n;\theta) S_{l'm}(\omega_n;\theta) \sin\theta \, d\theta = \delta_{ll'} \ . \tag{4.3}$$

The quantum number l is intimately related to Carter's constant K. In fact, the angular equation defined by these observers can be written as an eigenvalue equation in K as was done in (3.12) [20]. A complete set of eigenfunctions exists for this equation as well, but they are not well-studied functions like spheroidal harmonics.

These solutions are characterized by ω and m equal to a constant. The following restrictions to a generic open set $\mathcal{V}_+$ are implied:

$$(\vec{\tilde{\varphi}})_{nml}|_{\mathcal{V}_+} = \vec{\tilde{u}}_n e^{im(\phi-\Omega_H t)} S_{lm}(\omega_n;\theta) \equiv \vec{\tilde{\varphi}}_{nml} \ , \tag{4.4a}$$

$$\vec{\tilde{u}}_n = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\omega_n + m\Omega_H}} \frac{1}{\sqrt{r^2+a^2}} e^{i(\omega_n + m\Omega_H)u} \ , \tag{4.4b}$$

$$(\overleftarrow{\tilde{\varphi}})_{nml}|_{\mathcal{V}_+} = \overleftarrow{\tilde{u}}_n e^{im(\phi-\Omega_H t)} S_{lm}(\omega_n;\theta) \equiv \overleftarrow{\tilde{\varphi}}_{nml} \ , \tag{4.4c}$$

$$\overleftarrow{\tilde{u}} = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\omega_n + m\Omega_H}} \frac{1}{\sqrt{r^2+a^2}} e^{i(\omega_n + m\Omega_H)v} \ . \tag{4.4d}$$

Using (2.12), (2.21), and (2.23), a very useful expression for the global outgoing modes can be derived in terms of local coordinates:

$$\vec{\tilde{\varphi}}_{nml} \underset{r\to r_+}{\sim} \frac{e^{imx^\phi} S_{lm(\omega_n;\theta)}}{\sqrt{2\pi}\sqrt{\omega_n + m\Omega_H}\sqrt{r^2+a^2}} \times \exp\left\{ -\frac{i(\omega_n + m\Omega_H)}{\kappa} \ln\left[\frac{x^1 - x^0 - C}{D} \right] \right\} , \tag{4.5a}$$

where D is a constant. Another useful representation, near the horizon, is a WKB-type solutions [21];

$$\vec{\tilde{\varphi}}_{nml} \underset{r\to r_+}{\sim} \frac{e^{imx^\phi} S_{lm(\omega_n;\theta)}}{\sqrt{2\pi}\sqrt{\omega_n + m\Omega_H}\sqrt{r^2+a^2}} \times \exp\left[-i \int (P^0 dx^0 - P^1 dx^1) \right] , \tag{4.5b}$$

where P^0 and P^1 are functions when ω is a constant [see (2.27), for example].

B. Transformation between sets of angular functions

There will be no need to find the explicit transformation between the two sets of angular functions $S_{l'm'}$ and R_{km}. However, the unitarity properties of the transformation will be exploited and these are explored in this section. First, the transformation is defined using the completeness proved in Sec. III B:

$$S_{l'm}(\omega_{n'};\theta) = \sum_k \vec{\theta}_{l'k}(m,\omega_{n'},P_n) \vec{R}_{km}(\omega(P_n);\theta) \ , \tag{4.6a}$$

$$S_{l'm}(\omega_{n'};\theta) = \sum_k \overleftarrow{\theta}_{l'k}(m,\omega_{n'},P_n) \overleftarrow{R}_{km}(\omega(P_n);\theta) \ . \tag{4.6b}$$

Substituting (3.13b) and (4.6) into (4.3) yields

$$\sum_k \vec{\theta}_{lk}(m,\omega_{n'},P_n)\vec{\theta}^{\,*}_{l'k}(m,\omega_{n'},P_n)=\delta_{ll'}\ , \tag{4.7a}$$

$$\sum_k \overleftarrow{\theta}_{lk}(m,\omega_{n'},P_n)\overleftarrow{\theta}^{\,*}_{l'k}(m,\omega_{n'},P_n)=\delta_{ll'}\ . \tag{4.7b}$$

By unitarity,

$$\vec{R}_{km}(\omega(P_n);\theta)=\sum_{l'}\vec{\theta}^{\,*}_{kl'}(m,\omega_{n'},P_n)S_{l'm}(\omega_{n'};\theta)\ , \tag{4.8a}$$

$$\overleftarrow{R}_{km}(\omega(P_n);\theta)=\sum_{l'}\overleftarrow{\theta}^{\,*}_{kl'}(m,\omega_{n'},P_n)S_{l'm}(\omega_{n'};\theta)\ . \tag{4.8b}$$

C. The local Fourier decomposition

The main mathematical step used in deriving the renormalized stress-energy tensor is the expression of the "global" wave functions of (4.4) as a Fourier integral with respect to the locally defined wave functions, near the horizon. The Fourier decomposition of the globally ingoing case is trivial by (4.6b), (4.1b), and (3.17):

$$\overleftarrow{\varphi}_{n'm'l'}(r,\theta,\phi,t)|_{\mathcal{V}_+}=\sum_k\sum_m\int_0^{\infty}d\omega(P_n)\delta(\omega(P_n)-\omega')\left[\frac{(P^0)_n}{\omega_{n'}+m'\Omega_H}\right]^{1/2}(\overleftarrow{\varphi}_{\rm loc})_{nmk}\overleftarrow{\theta}_{l'm'km}(m,\omega_{n'},P_n)\ , \tag{4.9a}$$

where

$$\theta_{l'm'km}\equiv\theta_{l'k}\delta_{mm'}\ . \tag{4.9b}$$

The globally outgoing solutions are physically more interesting. The global solutions are characterized by $\omega+m\Omega_H$ equal to a sum of constants. By (2.27), this means that the local momentum of the "global" wave varies greatly and has many oscillations in a small neighborhood of a PFF observer, near the horizon. Thus, unlike the globally ingoing case, the outgoing "global" wave functions can only be represented by a packet of local wave functions. For a discussion of the relationship of this large differential blueshift to particle creation, see Sec. IV B of Ref. [1].

Dropping the arrows to streamline the notation, one expects an expansion of the general form

$$\tilde{u}_{n'}|_{\mathcal{V}_+}=\int_0^{\infty}(A_{nn'}u_n+B_{nn'}u_n^{\dagger})|_{\mathcal{V}_+}dP_n^0\ . \tag{4.10}$$

Using the same Fourier techniques employed in Ref. [1], one finds

$$A_{nn'}=-\frac{ie^{(\pi/2\kappa)[\omega_{n'}+m\Omega_H]}}{2\pi\sqrt{P_n(\omega_{n'}+m\Omega_H)}}D^{-i[\omega_{n'}+m\Omega_H]/\kappa}e^{iP_nC}\times P_n^{-i[\omega_{n'}+m\Omega_H]/\kappa}\Gamma\left[1+i\frac{\omega_{n'}+m\Omega_H}{\kappa}\right]\ , \tag{4.11a}$$

$$B_{nn'}=\frac{ie^{-(\pi/2\kappa)[\omega_{n'}+m\Omega_H]}}{2\pi\sqrt{P_n(\omega_{n'}+m\Omega_H)}}D^{-i[\omega_{n'}+m\Omega_H]/\kappa}e^{-iP_nC}\times P_n^{-i[\omega_{n'}+m\Omega_H]/\kappa}\Gamma\left[1+\frac{i(\omega_{n'}+m\Omega_H)}{\kappa}\right]\ . \tag{4.11b}$$

One can obtain an expression analogous to (4.10) for the complete wave function $\vec{\varphi}_{n'm'l'}$ from (4.11) and (4.6a):

$$\vec{\varphi}_{n'm'l'}|_{\mathcal{V}_+}=\sum_k\sum_m\int_0^{\infty}dP_n^0[A_{nn'mm'kl'}\vec{\varphi}_{nml}+B_{nn'mm'kl'}\vec{\varphi}^{\,\dagger}_{nml}]|_{\mathcal{V}_+} \tag{4.12a}$$

where

$$A_{nn'mm'kl'}=A_{nn'}\vec{\theta}_{l'm'km}(\omega_{n'},P_n)\ , \tag{4.12b}$$

$$B_{nn'mm'kl'}=B_{nn'}\vec{\theta}^{\,*}_{l'm'km}(\omega_{n'},P_n)\ . \tag{4.12c}$$

D. Inverting the Fourier decomposition

The expressions (4.10)–(4.12) contain certain information on the inverse transformation to (4.12) for φ_{nml} as well as the Bogoliubov transformation relating creation and annihilation operators. In order to obtain a rigorously derived result, one must introduce some mathematical abstraction as in Sec. IV C of Ref. [1].

First, an extended manifold M is introduced:

$$M=\{x^0,x^1,\theta,x^\phi|-\infty<x^0<\infty,-\infty<x^1<\infty,-\infty<x^\phi<\infty\ ,\ \ 0\le\theta\le\pi\}\ . \tag{4.13a}$$

The region outside of the horizon M_+, is given by

$$M_+=\{x^0,x^1,\theta,x^\phi|x^1>x^0+C,0\le\theta\le\pi,-\infty<x^\phi<\infty\}\ . \tag{4.13b}$$

The ambiguities of analytic continuation arguments (such as the future and past histories of the PFF observers and the hole) are avoided by saying that the manifold M is merely a mathematical construct of convenience. There may or may not be any physical significance to the space $M-\overline{M}_+$. To see the ambiguity of analytic continuation arguments, compare the results of Refs. [5,22] to Refs. [23,24] in the Schwarzschild case. The metric on M is given by the asymptotic forms in (2.17) and the volume measure is given by (2.16) everywhere. Physically only the subset of M, $\mathcal{V}_+$, is of interest.

Consider the function ϕ_{nmk} defined on M

$$\phi_{nmk}=\frac{\vec{R}_{km}(\omega_n;\theta)}{\sqrt{2\pi}\sqrt{2P_n^0}\sqrt{r^2+a^2}}\times\exp\{i[(P^0)_n(x^0-x^1)]\}e^{imx^\phi}\ . \tag{4.14}$$

The inner product associated with the Klein-Gordon operator on the hypersurface orthogonal to $\partial/\partial x^0$ is

$$\langle a,b\rangle=-i\int_{x^0=\text{const}}a^*\frac{\overleftrightarrow{\partial}}{\partial x^0}b\sqrt{g^{(3)}}dx^1dx^\phi dx^\theta \tag{4.15}$$

from (2.16),

$$\sqrt{g^{(3)}}=(r^2+a^2)\sin\theta\ . \tag{4.16}$$

Thus, the functions in (4.14) are normalized on these hypersurfaces:

$$\langle\phi_{nmk},\phi_{jih}\rangle=\delta(P_n^0-P_j^0)\delta_{mi}\delta_{kh}\ , \tag{4.17a}$$

$$\langle\phi_{nmk}^\dagger\phi_{jih}^\dagger\rangle=-\delta(P_n^0-P_j^0)\delta_{mi}\delta_{kh}\ . \tag{4.17b}$$

ϕ_{nmk} is not a solution of the free particle equation on M, (3.2). It is merely a mathematical construct. By (4.14) and (3.14) for any $\varepsilon>0$, there exists an open set $\mathcal{V}_+$ such that

$$|\phi_{nmk}-\vec{\varphi}_{nmk}|_{\mathcal{V}_+}<\varepsilon\ . \tag{4.18}$$

So one can say that ϕ_{nmk} is a good approximation to $\vec{\varphi}_{nmk}$ on $\mathcal{V}_+$ or φ_{nmk} is the restriction of ϕ_{nmk} to $\mathcal{V}_+$.

Recall that the expression for $\vec{\varphi}_{n'm'l'}$ in (4.4a) is valid only near the horizon and on $\mathcal{V}_+$. One extends $\vec{\varphi}_{n'm'l'}$ to all of as M as $\tilde{\phi}_{n'm'l'}$ by using (4.11), (4.12), and (4.14):

$$\tilde{\phi}_{n'm'l'}\equiv\sum_k\sum_m\int_0^\infty dP_n^0[A_{nn'mm'kl'}\phi_{nmk}+B_{nn'mm'kl'}\phi_{nmk}^\dagger]\ . \tag{4.19}$$

Direct substitution of (4.19) into (4.15) gives the normalization condition on $x^0=$const hypersurfaces in M:

$$\langle\tilde{\phi}_{n'm'l'},\tilde{\phi}_{j'i'h'}\rangle=\delta(\omega_{n'}-\omega_{j'})\delta_{m'i'}\delta_{l'h'}\ , \tag{4.20a}$$

$$\langle\tilde{\phi}^\dagger_{n'm'l'},\tilde{\phi}^\dagger_{j'i'h'}\rangle=-\delta(\omega_{n'}-\omega_{j'})\delta_{m'i'}\delta_{l'h'}\ . \tag{4.20b}$$

Since the ϕ_{nmk}'s have a δ-function normalization on spacelike hypersurfaces in M, the expansion (4.19) can be written as

$$\tilde{\phi}_{n'm'l'}=\sum_k\sum_m\int_0^\infty dP_n^0[\phi_{nmk}\langle\phi_{nmk},\tilde{\phi}_{n'm'l'}\rangle-\phi_{nmk}^\dagger\langle\phi_{nmk}^\dagger,\tilde{\phi}_{n'm'l'}\rangle]\ , \tag{4.21}$$

where the inner products are taken on $x^0=$const hypersurfaces. Using (4.19), one can make the identifications

$$\langle\phi_{nmk},\tilde{\phi}_{n'm'l'}\rangle=A_{nn'mm'kl'}\ , \tag{4.22a}$$

$$\langle\phi_{nmk}^\dagger,\tilde{\phi}_{n'm'l'}\rangle=-B_{nn'mm'kl'}\ . \tag{4.22b}$$

From the symmetry of the inner product in (4.15) and (4.22), one finds

$$\langle\tilde{\phi}^\dagger_{n'm'l'},\phi_{nmk}\rangle=-\langle\phi_{nmk}^\dagger,\tilde{\phi}_{n'm'l'}\rangle=B_{nn'mm'kl'}\ , \tag{4.23a}$$

$$\langle\tilde{\phi}^\dagger_{n'm'l'},\phi_{nmk}\rangle=-\langle\phi_{nmk}^\dagger,\tilde{\phi}_{n'm'l'}\rangle=-A_{nn'mm'kl'}\ . \tag{4.23b}$$

One can use (4.20) and (4.23) to find an inverse relation to (4.19). Since the $\tilde{\phi}$'s have a δ-function normalization on spacelike hypersurfaces in M, there exists an expansion analogous to (4.21):

$$\phi_{nmk}=\sum_{m'}\sum_{l'}\int_0^\infty d\omega_{n'}[\tilde{\phi}_{n'm'l'}\langle\tilde{\phi}_{n'm'l'},\phi_{nmk}\rangle-\tilde{\phi}^\dagger_{n'm'l'}\langle\tilde{\phi}^\dagger_{n'm'l'},\phi_{nmk}\rangle]\ . \tag{4.24a}$$

Then, using (4.23a), this can be expressed as

$$\phi_{nmk}=\sum_{m'}\sum_{l'}\int_0^\infty d\omega_{n'}[A^*_{nn'mm'kl'}\tilde{\phi}_{n'm'l'}-B_{nn'mm'kl'}\tilde{\phi}^\dagger_{n'm'l'}]\ . \tag{4.24b}$$

Finally, one can get the inverse to (4.12) by restricting the expression (4.24b) to the subset $\mathcal{V}_+$:

$$\vec{\varphi}_{nml}|_{\mathcal{V}_+}=\sum_{l'}\sum_{m'}\int_0^\infty d\omega_{n'}[A^*_{nn'mm'kl'},\vec{\varphi}_{n'm'l'}-B_{nn'mm'kl'}\vec{\varphi}^\dagger_{n'm'l'}]|_{\mathcal{V}_+}\ . \tag{4.25}$$

This is the desired result and there is no further need to use the abstract manifold M in the remainder of the text.

E. The Bogoliubov transformation

The local and global representations of the field are compared on $\mathcal{V}_+$ in order to determine the Bogoliubov transformation which relates particle creation and annihilation operators between the two formulations of the field theory. The stationary observers at asymptotic infinity expand the field $\tilde{\Phi}$ in the "global" modes:

$$\tilde{\Phi}=\sum_{l'}\sum_{m'}\int_{-\infty}^\infty d\tilde{P}^r_{n'}\sqrt{-g}\,\{[\vec{\varphi}_{n'm'l'}\vec{a}_{n'm'l'}+(\vec{\varphi}_{n'm'l'})^\dagger(\vec{a}^{\,\dagger}_{n'm'l'})^\dagger]\Theta(\tilde{V}^r_{n'})+[\overleftarrow{\varphi}_{n'm'l'}\overleftarrow{a}_{n'm'l'}+(\overleftarrow{\varphi}_{n'm'l'})^\dagger(\overleftarrow{a}_{n'm'l'})^\dagger]\Theta(-\tilde{V}^r_{n'})\}\ , \tag{4.26}$$

where the "radial" momentum $\tilde{P}^{\,r}_{n'}$ is determined in (2.6) through the quantum numbers l, m, and $\omega_{n'}$. Step functions were introduced to segregate the globally outgoing modes from the globally ingoing modes for later convenience. At this point a difference from the Schwarzschild case arises, the sign of $\tilde{V}^{\,r}$ is not necessarily the same as $\tilde{P}^{\,r}$. For the superradiant modes discussed in relation to (2.27), $\omega+m\Omega_H<0$, and $\tilde{P}^{\,r}$ is of the opposite sign to $\tilde{V}^{\,r}$. The volume measure $\sqrt{-\tilde{g}}$ is

$$\sqrt{-\tilde{g}}=\rho^2\sin\theta \ . \tag{4.27}$$

Near the horizon, by (2.6),

$$\tilde{P}^{\,r}_{n'} \underset{r\to r_+}{\sim} \operatorname{sgn}(\tilde{V}^{\,r})[\omega_{n'}+m\Omega_H]\frac{r^2+a^2}{\rho^2} \ . \tag{4.28a}$$

Thus, when integrating inside of a sum over quantum numbers m for each term in the sum, one can make the following useful substitution in the integrand of the asymptotic expressions:

$$\sqrt{-\tilde{g}}\,d\tilde{P}^{\,r}_{n'}\to(r^2+a^2)\sin\theta\,d\omega_{n'} \ . \tag{4.28b}$$

The operators $(\vec{a}_{n'm'l'})^\dagger$ and $(\overleftarrow{a}_{n'm'l'})^\dagger$ create modes from the vacuum defined by the stationary observers at asymptotic infinity, $|0_\infty\rangle$, that are outgoing with quantum numbers l', m', and $\omega_{n'}$ and ingoing with quantum numbers l', m', and $\omega_{n'}$, respectively. Similarly, $\vec{a}_{n'm'l'}$ and $\overleftarrow{a}_{n'm'l'}$ annihilate the stationary vacuum:

$$\vec{a}_{n'm'l'}|O_\infty\rangle=0 \ , \tag{4.29a}$$

$$\overleftarrow{a}_{n'm'l'}|O_\infty\rangle=0 \ . \tag{4.29b}$$

Analogously, the PFF observers describe the same field in terms of local modes:

$$\Phi_{\rm loc}=\sum_k\sum_m\int_{-\infty}^{\infty}\sqrt{-g}\,dP^1_n\{[\vec{\phi}_{nmk}\vec{a}_{nmk}+\vec{\varphi}^{\,\dagger}_{nmk}\vec{a}^{\,\dagger}_{nmk}]\Theta[-(P_0+P_1)(P^1-P^0)]$$
$$+[\overleftarrow{\varphi}_{nmk}\overleftarrow{a}_{nmk}+\overleftarrow{\varphi}^{\,\dagger}_{nmk}\overleftarrow{a}^{\,\dagger}_{nmk}]\Theta[(P_0+P_1)(P^2-P^0)]\} \ , \tag{4.30}$$

where (3.9) and (2.12) were used to produce step functions which segregate the solutions into globally outgoing and globally ingoing subsets in terms of locally evaluated momenta. The operators $\vec{a}^{\,\dagger}_{nmk}$ and $\overleftarrow{a}^{\,\dagger}_{nmk}$ creates particles out of the local vacuum of the PFF observers, $|O_{\rm loc}\rangle$, with quantum number K, m, and P^0_n which are globally outgoing and ingoing, respectively. Also, one has

$$\vec{a}_{nmk}|O_{\rm loc}\rangle=0 \ , \tag{4.31a}$$

$$\overleftarrow{a}_{nmk}|O_{\rm loc}\rangle=0 \ . \tag{4.31b}$$

Both representations of the field must agree on $\mathcal{V}_+$:

$$\Phi_{\rm loc}|_{\mathcal{V}_+}=\tilde{\Phi}|_{\mathcal{V}_+} \ . \tag{4.32}$$

By (3.17), (4.1), and (4.8),

$$\overleftarrow{a}_{n'm'l'}=\sum_{km}\int_0^\infty dP^0_n\left\{\left[\frac{P^0_n}{\omega_{n'}+m'\Omega_H}\right]^{1/2}\delta(\omega(P^\mu_n)-\omega_{n'})\times\overleftarrow{\theta}^{\,*}_{l'm'km}(P_n,\omega_{n'})\overleftarrow{a}_{nmk}\right\} , \tag{4.33}$$

where $\omega(P^\mu_n)$ is a constant by (3.15b) on $\mathcal{V}_+$ defined via the inverse to (2.9):

$$\omega(P^\mu_n)\underset{r\to r_+}{\sim}-(P_0+P_1)-\Omega_H m \ . \tag{4.34}$$

If one replaces φ_{nmk} by the approximation on $\mathcal{V}_+$, (4.25), and inserts this expression into (4.30) and solves (4.32), it will be found that

$$\vec{a}_{n'm'l'}=\sum_m\sum_k\int_0^\infty dP^0_n[A^*_{nn'mm'kl'}\vec{a}_{nmk}-B^*_{nn'mm'kl'}\vec{a}^{\,\dagger}_{nmk}] \ , \tag{4.35a}$$

$$(\vec{a}_{n'm'l'})^\dagger=\sum_m\sum_k\int_0^\infty dP^0_n[-B_{nn'mm'kl'}\vec{a}_{nmk}+A_{nn'mm'kl'}\vec{a}^{\,\dagger}_{nmk}] \ . \tag{4.35b}$$

Similarly, if one substitutes the expansion for $\tilde{\varphi}_{n'm'l'}$ of (4.12) into (4.26) and collects terms in (4.32),

$$\vec{a}_{nmk}=\sum_{m'}\sum_{l'}\int_0^\infty d\omega_{n'}[A_{nn'mm'kl'}\vec{a}_{n'm'l'}+B^*_{nn'mm'kl'}(\vec{a}_{n'm'l'})^\dagger] \ , \tag{4.36a}$$

$$\vec{a}^{\,\dagger}_{nmk}=\sum_{m'}\sum_{l'}\int_0^\infty d\omega_{n'}[B_{nn'mm'kl'}\vec{a}_{n'm'l'}+A^*_{nn'mm'kl'}(\vec{a}_{n'm'l'})^\dagger] \ . \tag{4.36b}$$

V. THE STRESS-ENERGY TENSOR OF THE FREELY FALLING VACUUM

In this section, the dynamic components of the stress-energy tensor of the freely falling vacuum state are evaluated, near the horizon, by different observers. One is then led naturally to the concept of the renormalized stress-energy tensor of the vacuum state transported by a PFF observer..

A. The vacuum stress-energy tensor: A local evaluation

The globally interesting coordinates for analyzing the stress-energy tensor are the Boyer-Lindquist coordinates. In particular, the dynamic components of Hawking radiation are $\tilde{T}_{tt}$, $\tilde{T}_{\phi t}$, $\tilde{T}_{rt}$, $\tilde{T}_{\phi r}$, $\tilde{T}_{\phi\phi}$, and $\tilde{T}_{rr}$. On the other hand, computations in the local vacuum states are most naturally accomplished in the PFF frames. Thus, the following transformations are useful:

$$\langle O_{\rm loc}|\tilde{T}_{tt}|O_{\rm loc}\rangle=\langle O_{\rm loc}|T_{00}|O_{\rm loc}\rangle-2\Omega\langle O_{\rm loc}|T_{\phi 0}|O_{\rm loc}\rangle+\Omega^2\langle O_{\rm loc}|T_{\phi\phi}|O_{\rm loc}\rangle$$
$$+2\langle O_{\rm loc}|T_{10}|O_{\rm loc}\rangle-2\Omega\langle O_{\rm loc}|T_{\phi 1}|O_{\rm loc}\rangle+\langle O_{\rm loc}|T_{11}|O_{\rm loc}\rangle\ , \tag{5.1a}$$

$$\langle O_{\rm loc}|\tilde{T}_{\phi t}|O_{\rm loc}\rangle=\langle O_{\rm loc}|T_{\phi 0}|O_{\rm loc}\rangle-\Omega\langle O_{\rm loc}|T_{\phi\phi}|O_{\rm loc}\rangle+\langle O_{\rm loc}|T_{\phi 1}|O_{\rm loc}\rangle\ , \tag{5.1b}$$

$$\langle O_{\rm loc}|\tilde{T}_{\rm tr}|O_{\rm loc}\rangle\underset{r\to r_+}{\sim}\frac{r^2+a^2}{\Delta}\{[\langle O_{\rm loc}|T_{00}|0_{\rm loc}\rangle+2\langle O_{\rm loc}|T_{10}|O_{\rm loc}\rangle+\langle O_{\rm loc}|T_{11}|O_{\rm loc}\rangle]$$
$$-\Omega[\langle O_{\rm loc}|T_{\phi 0}|O_{\rm loc}\rangle+\langle O_{\rm loc}|T_{\phi 1}|O_{\rm loc}\rangle]\}\ , \tag{5.1c}$$

$$\langle O_{\rm loc}|\tilde{T}_{\phi r}|O_{\rm loc}\rangle\underset{r\to r_+}{\sim}\frac{r^2+a^2}{\Delta}[\langle O_{\rm loc}|T_{\phi 0}|O_{\rm loc}\rangle+\langle O_{\rm loc}|T_{\phi 1}|O_{\rm loc}\rangle]\ , \tag{5.1d}$$

$$\langle O_{\rm loc}|\tilde{T}_{\phi\phi}|O_{\rm loc}\rangle=\langle O_{\rm loc}|T_{\phi\phi}|O_{\rm loc}\rangle\ , \tag{5.1e}$$

$$\langle O_{\rm loc}|\tilde{T}_{rr}|O_{\rm loc}\rangle\underset{r\to r_+}{\sim}\frac{(r^2+a^2)^2}{\Delta^2}[\langle O_{\rm loc}|T_{00}|O_{\rm loc}\rangle+2\langle O_{\rm loc}|T_{01}|O_{\rm loc}\rangle+\langle O_{\rm loc}|T_{11}|O_{\rm loc}\rangle]\ . \tag{5.1f}$$

The stress-energy tensor of a scalar field Φ in curved spacetime is

$$T_{ab}=\Phi_{;\alpha}\Phi_{;\beta}-\tfrac{1}{2}g_{ab}[g^{cd}\Phi_{;c}\Phi_{;d}+m_e^2\Phi^2]\ . \tag{5.2}$$

The stress-energy tensor evaluated by a PFF observer is computed using local wave functions and is denoted by

$$(T_{\mu\nu})_{\rm loc}\equiv T_{\mu\nu}(\varphi,\varphi)\ . \tag{5.3}$$

To find $\langle O_{\rm loc}|\tilde{T}_{tt}(\varphi,\varphi)|O_{\rm loc}\rangle$, one can substitute the expansion for $\Phi_{\rm loc}$, (4.30), and for R_{km} in (4.8) into (5.2) and (5.1a).

$$\langle O_{\rm loc}|\tilde{T}_{tt}|O_{\rm loc}\rangle|_{\mathcal{V}_+}=\frac{1}{8\pi^2(r^2+a^2)}\sum_{km}\sum_{l'm'}\sum_{l''m''}\int_{-\infty}^{\infty}\frac{dP_n^1}{P_n^0}\{\theta^*_{kl'mm'}(P_n,\omega_{n'})\theta_{kl''mm''}(P_n,\omega_{n'})S_{l'm'}(\omega_{n'};\theta)S_{l''m''}(\omega_{n'};\theta)$$
$$\times[(P_0)^2+(P_1)^2+2P_0P_1+\Omega^2m^2-2(P_0+P_1)\Omega m]\}\ . \tag{5.4}$$

From (2.6), (2.12), and (4.28a),

$$\frac{dP^1}{P^0}\underset{r\to r_+}{\sim}\frac{\rho^2}{r^2+a^2}\frac{d\tilde{P}^r}{\omega+m\Omega_H}=\pm\frac{d(\omega+m\Omega_H)}{\omega+m\Omega_H}\ . \tag{5.5}$$

The plus (minus) sign corresponds to the globally outgoing (ingoing) case. Applying (5.5), (5.1b), and (4.7a) to (5.4) and collecting terms yields

$$\langle O_{\rm loc}|\tilde{T}_t{}^t(\varphi,\varphi)|O_{\rm loc}\rangle_{\mathcal{V}_+}=-\frac{\alpha^{-2}}{4\pi^2(r^2+a^2)}\sum_{lm}\int_{-\infty}^{\infty}[S_{lm}(\omega_n;\theta)]^2\frac{\omega}{2}d\tilde{P}^r\ . \tag{5.6}$$

To interpret (5.6), note that the total redshifted energy E_∞ in a volume element V (defined at a particular value of global t) is given by [6]

$$E_\infty=-\int_V\tilde{T}_t{}^t\sqrt{-\tilde{g}}\,dr\,d\theta\,d\phi\ . \tag{5.7}$$

Look at the volume element dV transported by a PFF observer:

$$dV=\sqrt{g^{(3)}}dx^1dx^\theta dx^\phi\ . \tag{5.8a}$$

Asymptotically, by (2.12a), a global observer would see a $t=$const slice of the volume element:

$$dV_{t={\rm const}}\underset{r\to r_+}{\sim}\alpha^{-2}\rho^2\sin\theta\,dr\,d\theta\,d\phi\ . \tag{5.8b}$$

Thus, (5.6) and (5.8b) imply that the amount of redshifted energy of the zero-point oscillations of the local field in a freely falling volume element at any value of global time t is

$$dE_\infty\underset{r\to r_+}{\sim}\frac{dV_{t={\rm const}}}{4\pi^2(r^2+a^2)}\sum_{lm}\int_{-\infty}^{\infty}[S_{lm}(\omega_n;\theta)]^2\frac{\omega}{2}d\tilde{P}^r\ . \tag{5.9}$$

It is not a coincidence that one has the same result for the stationary observers at infinity in their vacuum state:

$$dE_\infty\underset{r\to\infty}{\sim}-dV_{t={\rm const}}\langle O_\infty|\tilde{T}_t{}^t(\tilde{\varphi},\tilde{\varphi})|O_\infty\rangle$$
$$=\frac{dV_{t={\rm const}}}{4\pi^2(r^2+a^2)}\sum_{lm}\int_{-\infty}^{\infty}[S_{lm}(\omega_n;\theta)]^2\frac{\omega}{2}d\tilde{P}^r\ . \tag{5.10}$$

In (5.10), the notation $T_{\mu\nu}(\tilde{\varphi},\tilde{\varphi})$ signifies that the stress-energy tensor evaluated by the stationary observers at asymptotic infinity is obtained by using the "global" wave functions. The computation is accomplished by inserting the expression (4.26) for $\tilde{\Phi}$ into (5.1a) and (5.2). The quality of (5.9) and (5.10) is a consequence of the equivalence principle. The physical relationship of this result to Hawking radiation is explained in Sec. V of Ref. [1].

B. The stress-energy tensor of the local vacuum evaluated in the stationary frames

The stress-energy tensor of the local vacuum transported by the PFF observers as measured in the stationary frames at asymptotic infinity, $\langle O_{\rm loc}|T_{\mu\nu}(\tilde{\varphi},\tilde{\varphi})|O_{\rm loc}\rangle$, is calculated near the horizon. This computation requires using the field representation $\tilde{\Phi}$ of (4.26) in (5.2). However, this must be modified so that the creation and annihilation operators can act on states belonging to the local number representation of the field. Thus, the Bogoliubov transformation, (4.35a), and (4.35b), must be used to reexpress these operators for the globally outgoing states. The globally ingoing states are trivial by (4.33). To utilize (5.1), the calculation is performed in the local basis. The algebraic manipulations are facilitated by using the general form of $\tilde{\varphi}$ given in (4.5b):

$$\langle O_{\rm loc}|T_{00}(\tilde{\varphi},\tilde{\varphi})|O_{\rm loc}\rangle \underset{r\to r_+}{\sim} \frac{1}{8\pi^2[r^2+a^2]}\sum_l \sum_m [S_{lm}(\omega;\theta)]^2 \left\{ \int_{\tilde{P}^r<0} [P_0(\omega,m,k)]^2 \frac{d\tilde{P}^r}{\omega+m\Omega_H} + \int_{\tilde{P}^r>0} \coth\left[\frac{\pi}{\kappa}(\omega+m\Omega_H)\right][P_0(\omega,m,k)]^2 \frac{d\tilde{P}^r}{\omega+m\Omega_H} \right\}, \tag{5.11a}$$

$$\langle O_{\rm loc}|T_{0\phi}(\tilde{\varphi},\tilde{\varphi})|O_{\rm loc}\rangle \underset{r\to r_+}{\sim} \frac{1}{8\pi^2[r^2+a^2]}\sum_l \sum_m [S_{lm}(\omega;\theta)]^2 \left\{ \int_{\tilde{P}^r<0} [P_0 m] \frac{d\tilde{P}^r}{\omega+m\Omega_H} + \int_{\tilde{P}^r>0} \coth\left[\frac{\pi}{\kappa}(\omega+m\Omega_H)\right][P_0 m] \frac{d\tilde{P}^r}{\omega+m\Omega_H} \right\}, \tag{5.11b}$$

$$\langle O_{\rm loc}|T_{01}(\tilde{\varphi},\tilde{\varphi})|O_{\rm loc}\rangle \underset{r\to r_+}{\sim} \frac{1}{8\pi^2[r^2+a^2]}\sum_l \sum_m [S_{lm}(\omega;\theta)]^2 \left\{ \int_{\tilde{P}^r<0} [P_0 P_1] \frac{d\tilde{P}^r}{\omega+m\Omega_H} + \int_{\tilde{P}^r>0} \coth\left[\frac{\pi}{\kappa}(\omega+m\Omega_H)\right][P_0 P_1] \frac{d\tilde{P}^r}{\omega+m\Omega_H} \right\}, \tag{5.11c}$$

$$\langle O_{\rm loc}|T_{1\phi}(\tilde{\varphi},\tilde{\varphi})|O_{\rm loc}\rangle \underset{r\to r_+}{\sim} \frac{1}{8\pi^2[r^2+a^2]}\sum_l \sum_m [S_{lm}(\omega;\theta)]^2 \left\{ \int_{\tilde{P}^r<0} [P_1 m] \frac{d\tilde{P}^r}{\omega+m\Omega_H} + \int_{\tilde{P}^r>0} \coth\left[\frac{\pi}{\kappa}(\omega+m\Omega_H)\right][P_1 m] \frac{d\tilde{P}^r}{\omega+m\Omega_H} \right\}, \tag{5.11d}$$

$$\langle O_{\rm loc}|T_{1\phi\phi}(\tilde{\varphi},\tilde{\varphi})|O_{\rm loc}\rangle \underset{r\to r_+}{\sim} \frac{1}{8\pi^2[r^2+a^2]}\sum_l \sum_m [S_{lm}(\omega;\theta]^2 \left\{ \int_{\tilde{P}^r<0} m^2 \frac{d\tilde{P}^r}{\omega+m\Omega_H} + \int_{\tilde{P}^r>0} \coth\left[\frac{\pi}{\kappa}(\omega+m\Omega_H)\right] m^2 \frac{d\tilde{P}^r}{\omega+m\Omega_H} \right\}, \tag{5.11e}$$

$$\langle O_{\rm loc}|T_{11}(\tilde{\varphi},\tilde{\varphi})|O_{\rm loc}\rangle \underset{r\to r_+}{\sim} \frac{1}{8\pi^2[r^2+a^2]}\sum_l \sum_m [S_{lm}(\omega;\theta)]^2 \left\{ \int_{\tilde{P}^r<0} [P_1]^2 \frac{d\tilde{P}^r}{\omega+m\Omega_H} + \int_{\tilde{P}^r>0} \coth\left[\frac{\pi}{\kappa}(\omega+m\Omega_H)\right][P_1]^2 \frac{d\tilde{P}^r}{\omega+m\Omega_H} \right\}. \tag{5.11f}$$

The local momentum P_μ, in the integrands of (5.11) is derived from the momentum which is measured by the observers at stationary infinity [as indicated in the notation of (5.11a)], using (2.9) and remembering that ω is a constant for global modes. Also notice that the modal contributions are segregated as to whether they are global ingoing or outgoing, not by the sign of $\tilde{P}^r$.

Combining (5.1a) and (5.11),

$$\langle O_{\rm loc}|\tilde{T}_t{}^t(\tilde{\varphi},\tilde{\varphi})|O_{\rm loc}\rangle|_{\mathcal{V}_+}=\frac{\alpha^{-2}}{8\pi^2[r^2+a^2]}\sum_l\sum_m[S_{lm}(\omega;\theta)]^2\left\{\int_{\tilde{V}^r<0}\omega\,d\tilde{P}^r+\int_{\tilde{V}^r>0}\omega\coth\left[\frac{\pi}{\kappa}(\omega+m\Omega_H)\right]d\tilde{P}^r\right\}. \tag{5.12}$$

Similar expressions can be found for the other components of $\langle O_{\rm loc}|\tilde{T}_{\mu\nu}(\tilde{\varphi},\tilde{\varphi})|O_{\rm loc}\rangle$.

C. The renormalized stress-energy tensor

The renormalized stress-energy tensor of the freely falling vacuum was shown in Ref. [1] to be

$$\langle O_{\rm loc}|T_{\mu\nu}|O_{\rm loc}\rangle_{\rm ren}\equiv\langle O_{\rm loc}|T_{\mu\nu}(\varphi,\varphi)|O_{\rm loc}\rangle-\langle O_{\rm loc}|T_{\mu\nu}(\tilde{\varphi},\tilde{\varphi})|O_{\rm loc}\rangle\ . \tag{5.13}$$

The physical significance of this quantity is discussed in Secs. V and VI of Ref. [1].

Using (5.5), (5.6), (5.12), (4.28), and (5.13),

$$\langle O_{\rm loc}|\tilde{T}_t{}^t|O_{\rm loc}\rangle_{\rm ren}\underset{r\to r_+}{\sim}\frac{\alpha^{-2}}{4\pi^2[r^2+a^2]}\sum_{lm}\int_{m_e}^{\infty}[S_{lm}(\omega;\theta)]^2\frac{\omega\,d\omega}{\exp[(2\pi/\kappa)(\omega+m\Omega_H)]-1}\ . \tag{5.14}$$

In combining the terms to arrive at (5.14), one must keep close track of the appropriate limits of integration due to the existence of superradiant modes. It is interesting that the superradiant modes are not segregated in (5.14), yet they are in all of the intermediate stages. For local modes $p^0>m_e$, which, from (2.6) and (2.12), implies that $\omega+m\Omega_H>0$ for all modes. Therefore, the allowed limits on integrals over $\tilde{P}^r$ for quantities derived from local fields, such as (5.4), become, with the aid of (5.5),

$$\sum_m\int_{\tilde{V}^r>0}d\tilde{P}^r\underset{r\to r_+}{\sim}\sum_m\int_0^{\infty}\frac{r^2+a^2}{\rho^2}d(\omega+m\Omega_H)\ , \tag{5.15a}$$

$$\sum_m\int_{\tilde{V}^r<0}d\tilde{P}^r\underset{r\to r_+}{\sim}\sum_m\int_{-\infty}^{0}-\frac{r^2+a^2}{\rho^2}d(\omega+m\Omega_H)\ , \tag{5.15b}$$

For "global" modes $\omega>m_e$, but $\omega+m\Omega_H$ can be negative. These are superradiant modes which have $\tilde{P}^t<0$ and $P^0<0$. The allowed limits of integration for quantities derived from "global" fields, such as (5.11), for globally outgoing modes are

$$\frac{\rho^2}{r^2+a^2}\sum_m\int_{\tilde{V}^r>0}d\tilde{P}^r\underset{r\to r_+}{\sim}\sum_{m_e+\Omega_H m<0}\int_{m_e+\Omega_H m}^{\infty}d(\omega+m\Omega_H)+\sum_{m_e+\Omega_H m>0}\int_0^{\infty}d(\omega+m\Omega_H)\ , \tag{5.16a}$$

where the contribution

$$\sum_{m_e+\Omega_H m<0}\int_{m_e+\Omega_H m}^{0}d(\omega+m(\Omega_H) \tag{5.16b}$$

corresponds to the superradiant globally outgoing modes. For the globally ingoing modes, quantities involving integrands which are functions of global fields have the following allowed values of $\tilde{P}^r$ in the integration:

$$\frac{\rho^2}{r^2+a^2}\sum_m\int_{\tilde{V}^r<0}d\tilde{P}^r\underset{r\to r_+}{\sim}\sum_{m_e+\Omega_H m<0}\int_{-\infty}^{-m_e-m\Omega_H}-d(\omega+m\Omega_H)+\sum_{m_e+\Omega_H m>0}\int_{-\infty}^{0}-d(\omega+m\Omega_H)\ , \tag{5.16c}$$

where the term

$$\sum_{m_e+\Omega_H m<0}\int_0^{-m_e-m\Omega_H}-d(\omega+m\Omega_H) \tag{5.16d}$$

is the contribution from the globally ingoing superradiant modes.

Returning to the result (5.14), by (5.7) the redshifted energy of the freely falling vacuum is negative as viewed from stationary infinity. Since this vacuum is tied to the PFF observers as viewed from infinity, the motion of the PFF vacuum gives rise to an energy flux. From (4.28), (5.5), (5.1), (5.11), (5.13), (5.15), and (5.16) one finds, in analogy with (5.14),

$$\langle O_{\rm loc}|\tilde{T}_t{}^r|O_{\rm loc}\rangle_{\rm ren}\underset{r\to r_+}{\sim}\frac{-1}{4\pi^2\rho^2}\sum_{lm}\int_{m_e}^{\infty}[S_{lm}(\omega;\theta)]^2\frac{\omega\,d\omega}{\exp[(2\pi/\kappa)(\omega+m\Omega_H)]-1}\ . \tag{5.17}$$

Similar expressions hold for the angular momentum density and flux. However, the physically relevant quantity is the renormalized stress-energy tensor of spacetime. Thus, these expressions will be introduced in the more important discussion of the next section.

VI. THE RENORMALIZED STRESS-ENERGY TENSOR OF SPACETIME

The energy flux associated with the infall of the freely falling vacuum, (5.17), was interpreted spectrally as outgoing radiation of particle-antiparticle pairs in Sec. VI A of Ref. [1]. Using this insight, the previous calculation of $\langle O_{\rm loc}|T_{\mu\nu}|O_{\rm loc}\rangle_{\rm ren}$ of the freely falling vacuum can be synthesized with the foliation of spacetime by PFF frames to find the renormalized stress-energy tensor of spacetime, $\langle T_{\mu\nu}\rangle_{\rm ren}$.

These pairs are generated during free fall, but the predominant effect occurs as $\alpha\to 0$, near the horizon. The pairs have an amplitude to reflect from and be transmitted through the curvature potential of spacetime. The reflection coefficient is designated as $A(l,m,\omega)$ and the transmission coefficient is $B(l,m,\omega)$. One has the relationship [17]

$$1-|A(l,m,\omega)|^2=\frac{\omega+m\Omega_H}{\omega}|B(l,m,\omega)|^2 \ . \tag{6.1a}$$

The superradiance condition is

$$|A(l,m,\omega)|^2>1 \ , \quad \omega(\omega+m\Omega_H)>0 \ . \tag{6.1b}$$

Thus, near the horizon, there are two components of the radial momentum flux: one results from the bulk motion of the negative energy density of the freely falling vacuum, (5.17), the other is the ingoing flux of the reflected pairs. Denote the stress-energy tensor of the radiated stream of Hawking radiation which has scattered off of the curvature potential by $\langle T_{\mu\nu}\rangle_{\rm rad}$. The renormalized stress energy of spacetime can be defined in the asymptotic zones

$$\langle T_{\mu\nu}\rangle_{\rm ren}=\langle O_{\rm loc}|T_{\mu\nu}|O_{\rm loc}\rangle_{\rm ren}+\langle T_{\mu\nu}\rangle_{\rm rad} \ . \tag{6.2}$$

A. The asymptotic zone near the horizon

To analyze the propagation of the pairs, it is convenient to introduce the conserved, integrated radial component of the redshifted energy flux, S^r: [6]

$$S^r=-\int \tilde{T}_t{}^r\sqrt{-g}\,d\theta\,d\phi \ . \tag{6.3}$$

By conservation of energy and the physical origin of the pairs, (5.17), (6.1), and (6.3) imply that the energy flux of the reflected pairs, $\langle \tilde{T}_t{}^r\rangle_{\rm ref}$, satisfies

$$\langle \tilde{T}_t{}^r\rangle_{\rm ref}\underset{r\to r_+}{\sim}\frac{1}{4\pi^2\rho^2}\times\sum_l\sum_m\int_{m_e}^{\infty}[S_{lm}(\omega;\theta)]^2|A(l,m,\omega)|^2\times\frac{\omega\,d\omega}{\exp[(2\pi/\kappa)(\omega+m\Omega_H)]-1} \ , \tag{6.4}$$

By (6.2), the renormalized stress-energy tensor of spacetime near the horizon is

$$\lim_{r\to r_+}\langle T_{\mu\nu}\rangle_{\rm ren}=\langle O_{\rm loc}|T_{\mu\nu}|O_{\rm loc}\rangle+\langle T_{\mu\nu}\rangle_{\rm ref} \ . \tag{6.5}$$

Inserting (5.17) and (6.4) into (6.5) yields

$$\langle \tilde{T}_t{}^r\rangle_{\rm ren}\underset{r\to r_+}{\sim}-\frac{1}{4\pi^2\rho^2}\sum_l\sum_m\int_{m_e}^{\infty}[S_{lm}(\omega;\theta)]^2[1-|A(l,m,\omega)|^2]\frac{\omega\,d\omega}{\exp[(2\pi/\kappa)(\omega+m\Omega_H)]-1} \ . \tag{6.6}$$

The reflected pairs form a highly relativistic stream by (2.27). So, by evaluating (5.1) and (5.2) near the horizon, one finds that

$$\lim_{r\to r_+}\langle \tilde{T}_t{}^t\rangle_{\rm ref}=-\frac{\alpha^{-2}\rho^2}{(r^2+a^2)}\lim_{r\to r_+}\langle \tilde{T}_t{}^r\rangle_{\rm ref} \ . \tag{6.7}$$

Thus, (6.4), (6.7), and (5.14) inserted into (6.5) produces the desired result

$$\langle \tilde{T}_t{}^t\rangle_{\rm ren}\underset{r\to r_+}{\sim}\frac{\alpha^{-2}}{4\pi^2[r^2+a^2]}\sum_l\sum_m\int_{m_e}^{\infty}[S_{lm}(\omega;\theta)]^2[1-|A(l,m,\omega)|^2]\frac{\omega\,d\omega}{\exp[(2\pi/\kappa)(\omega+m\Omega_H)]-1} \ . \tag{6.8a}$$

Similarly, one can compute from (5.1), (5.11), (5.13), and (6.5),

$$\langle \tilde{T}_\phi{}^r\rangle_{ren}\underset{r\to r_+}{\sim}-\frac{1}{4\pi^2\rho^2}\sum_l\sum_m\int_{m_e}^{\infty}[S_{lm}(\omega;\theta)]^2[1-|A(l,m,\omega)|^2]\frac{m\,d\omega}{\exp[(2\pi/\kappa)(\omega+m\Omega_H)]-1} \ , \tag{6.8b}$$

$$\langle \tilde{T}_\phi{}^t\rangle_{\rm ren}\underset{r\to r_+}{\sim}\frac{\alpha^{-2}}{4\pi^2[r^2+a^2]}\sum_l\sum_m\int_{m_e}^{\infty}[S_{lm}(\omega;\theta)]^2[1-|A(l,m,\omega|^2]\frac{m\,d\omega}{\exp[(2\pi/\kappa)(\omega+m\Omega_H)]-1} \ , \tag{6.8c}$$

$$\langle \tilde{T}_r{}^r\rangle_{\rm ren}\underset{r\to r_+}{\sim}-\frac{r^2+a^2}{4\pi^2\rho^2\Delta}\sum_l\sum_m\int_{m_e}^{\infty}[S_{lm}(\omega;\theta)]^2[1-|A(l,m,\omega|^2]\frac{[\omega+\Omega_H m]d\omega}{\exp[(2\pi/\kappa)(\omega+m\Omega_H)]-1} \ , \tag{6.8d}$$

$$\langle \tilde{T}_\phi{}^\phi \rangle_{\mathrm{ren}} \underset{r\to r_+}{\sim} -\frac{\Omega_H \alpha^{-2}}{4\pi^2(r^2+a^2)} \sum_l \sum_m \int_{m_e}^{\infty} [S_{lm}(\omega;\theta)]^2 [1-|A(l,m,\omega)|^2] \frac{m\, d\omega}{\exp[(2\pi/\kappa)(\omega+m\Omega_H)]-1} , \tag{6.8e}$$

$$\langle \tilde{T}_r{}^t \rangle_{\mathrm{ren}} \underset{r\to r_+}{\sim} \frac{\alpha^{-2}}{4\pi^2 \Delta} \sum_l \sum_m \int_{m_e}^{\infty} [S_{lm}(\omega;\theta)]^2 [1-|A(l,m,\omega)|^2] \frac{[\omega+\Omega_H m]\, d\Omega}{\exp[(2\pi/\kappa)(\omega+m\Omega_H)]-1} , \tag{6.8f}$$

$$\langle \tilde{T}_r{}^\phi \rangle_{\mathrm{ren}} \underset{r\to r_+}{\sim} -\frac{\alpha^{-2}\Omega_H}{4\pi^2 \Delta} \sum_l \sum_m \int_{m_e}^{\infty} [S_{lm}(\omega;\theta)]^2 [1-|A(l,m,\omega)|^2] \frac{[\omega+\Omega_H m]\, d\omega}{\exp[(2\pi/\kappa)(\omega+m\Omega_H)]-1} , \tag{6.8g}$$

$$\langle \tilde{T}_t{}^\phi \rangle_{\mathrm{ren}} \sim_{r\to r_+} \frac{\Omega_H \alpha^{-2}}{4\pi^2(r^2+a^2)} \sum_l \sum_m \int_{m_e}^{\infty} [S_{lm}(\omega;\theta)]^2 [1-|A(l,m,\omega)|^2] \frac{\omega\, d\omega}{\exp[(2\pi/\kappa)(\omega+m\Omega_H)]-1} . \tag{6.8h}$$

B. The asymptotic form near stationary infinity

To compute $\langle T_\mu^\nu \rangle_{\mathrm{ren}}$ near stationary infinity from (6.5), first note that (5.13) implies

$$\langle O_{\mathrm{loc}} | T_{\mu\nu} | O_{\mathrm{loc}} \rangle_{\mathrm{ren}} \underset{r\to\infty}{\sim} 0 \tag{6.9}$$

and, therefore, by (6.2),

$$\langle T_{\mu\nu} \rangle_{\mathrm{ren}} \underset{r\to\infty}{\sim} \langle T_{\mu\nu} \rangle_{\mathrm{trans}} , \tag{6.10}$$

where $\langle T_{\mu\nu} \rangle_{\mathrm{trans}}$ is the stress-energy tensor of the radiated pairs which have been transmitted through the curvature potential. To find $\langle T_\mu^\nu \rangle_{\mathrm{trans}}$, one can use (6.6), (6.8), and energy conservation through $r=$const, $t=$const surfaces in each asymptotic zone. In analogy to (6.13), one must also introduce the conserved, integrated radial redshifted angular momentum flux L^r:

$$L^r = \int \tilde{T}_\phi{}^r \sqrt{-\tilde{g}}\, d\theta\, d\phi . \tag{6.11}$$

Combining these facts, a straightforward calculation of the stress-energy tensor of a stream of radiated particles yields [16]

$$\langle \tilde{T}_\mu{}^\nu \rangle_{\mathrm{ren}} \underset{r\to\infty}{\sim} \frac{\mathcal{L}}{4\pi^2 r^2} \circ \begin{bmatrix} -\dfrac{\omega}{\tilde{V}^r} & -\omega & \dfrac{m}{\tilde{V}^r r^2 \sin^2\theta} & 0 \\ \omega & -\omega \tilde{V}^r & \dfrac{-m}{r^2\sin^2\theta} & 0 \\ -\dfrac{m}{\tilde{V}^r} & -m & \dfrac{m^2}{\omega \tilde{V}^r r^2 \sin^2\theta} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} , \tag{6.12a}$$

where $\mathcal{L}$ is the spectral luminosity

$$\mathcal{L} \equiv \sum_l \sum_m \int_{m_e}^{\infty} d\omega \frac{[S_{lm}(\omega;\theta)]^2 [1-|A(l,m,\omega)|^2]}{e^{2\pi/\kappa(\omega+m\Omega_H)}-1} \tag{6.12b}$$

and the composition symbol "∘" in (6.12a) means that the matrix is to be considered inside of the generalized sum in (6.12b).

From (2.6),

$$\tilde{P}^r \underset{r\to\infty}{\sim} \left[\omega^2 - m_e^2 + \frac{2Mm_e^2}{r} \right]^{1/2} + O\left[\frac{1}{r^2} \right] , \tag{6.13a}$$

$$\tilde{P}^t \underset{r\to\infty}{\sim} \omega \left[1 + \frac{2M}{r} \right] + O\left[\frac{1}{r^2} \right] . \tag{6.13b}$$

Thus, for massless quanta (these modes carry off most of the black-hole energy, except possibly for microblack holes), $\tilde{V}^r \to 1$ in (6.12). This constraint on $\tilde{V}^r$ is a consequence of angular momentum conservation. Since m and K are constant in each modal contribution to (6.12) all of the way from the horizon to asymptotic infinity, and the effective lever arm increases without bound, the norm of the linear momenta in the plane orthogonal to $\hat{e}_r$ must go to zero as $1/r$ in the radiated stream.

APPENDIX: THE CONNECTION IN THE PFF BASIS

Since the PFF frame in (2.9) is anholonomic and is not orthonormal, there is no expected symmetry in the connection coefficients $\Gamma_{\mu\beta\gamma}$. To find the connection, the general equation is implemented [16]:

$$\Gamma_{\mu\beta\gamma} = \tfrac{1}{2}(g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu} + c_{\mu\beta\gamma} + c_{\mu\gamma\beta} - c_{\beta\gamma\mu}) . \tag{A1}$$

The metric is found in (2.13) and the structure constants in (2.19). The nonzero connection coefficients are listed:

$$-\Gamma_{011} = \Gamma_{101} = \Gamma_{110} = \tfrac{1}{2}(g_{11})_{,0} , \tag{A2}$$

$$\Gamma_{01\phi} = \Gamma_{0\phi 1} = -\Gamma_{1\phi 0} = \Gamma_{\phi 10} = -\Gamma_{10\phi} = \tfrac{1}{2}c_{01\phi} , \tag{A3}$$

$$-\Gamma_{0\phi\phi} = \Gamma_{\phi\phi 0} = \Gamma_{\phi 0\phi} = \tfrac{1}{2}(g_{\phi\phi})_{,0} , \tag{A4}$$

$$-\Gamma_{0\theta 1} = 2\Gamma_{\theta 01} = (g_{\theta 1})_{,0} , \tag{A5}$$

$$\Gamma_{0\theta\phi} = -\Gamma_{\phi 0\theta} = -\Gamma_{\theta 0\phi} = -\Gamma_{\theta\phi 0} = \Gamma_{\phi\theta 0} = \tfrac{1}{2}c_{0\theta\varphi} , \tag{A6}$$

$$-\Gamma_{0\theta\theta} = \Gamma_{\theta 0\theta} = \Gamma_{\theta\theta 0} = \tfrac{1}{2}(g_{\theta\theta})_{,0} , \tag{A7}$$

$$\Gamma_{111} = \tfrac{1}{2}(g_{11})_{,1} , \tag{A8}$$

$$\Gamma_{11\theta} = \Gamma_{1\theta 1} = \tfrac{1}{2}(g_{11})_{,\theta} , \tag{A9}$$

$$\Gamma_{1\theta\phi} = -\Gamma_{\phi 1\theta} = \Gamma_{\phi\theta 1} = -\Gamma_{\theta 1\phi} = \tfrac{1}{2}c_{1\theta\phi} , \tag{A10}$$

$$\Gamma_{1\theta\theta}=(g_{1\theta})_{,\theta}-\tfrac{1}{2}(g_{\theta\theta})_{,1}\ , \tag{A11}$$

$$\Gamma_{\phi 1\phi}=\Gamma_{\phi\phi 1}=\tfrac{1}{2}(g_{\phi\phi})_{,1}\ , \tag{A12}$$

$$\Gamma_{\phi\phi 0}=\Gamma_{\phi\theta\phi}=-\Gamma_{\theta\phi\phi}=\tfrac{1}{2}(g_{\phi\phi})_{,\theta}\ , \tag{A13}$$

$$\Gamma_{\theta 11}=(g_{\theta 1})_{,1}\ , \tag{A14}$$

$$\Gamma_{\theta 1\theta}=\Gamma_{\theta\theta 1}=\tfrac{1}{2}(g_{\theta\theta})_{,1}\ , \tag{A15}$$

$$\Gamma_{\theta\theta\theta}=\tfrac{1}{2}(g_{\theta\theta})_{,\theta}\ . \tag{A16}$$

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