Black-hole evaporation and the equivalence principle

Brian Punsly
4104 Emerald Street, No. 116, Torrance, California 90503
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This article investigates the underlying physics of Hawking radiation. It is proposed that global quantum field theory on the Schwarzschild background is such that its restriction to any point of spacetime is consistent with the field theory postulated by a freely falling observer at that point. The equivalence principle demands that the field theory defined by freely falling observers be the same as special relativisitic (flat-spacetime) field theory in a neighborhood of the observer. A minor technical point is that one needs to find a family of freely falling observers whose world lines form a space-filling congruence in order to synthesize the theory. Once this is accomplished, the global field theory as dictated by the equivalence principle predicts that a black hole experiences thermal evaporation in isolation. The main point of this paper is to attain a physical understanding of this phenomenon with particular emphasis on the renormalized stress-energy tensor. It is shown that this tensor is a measure of the change in the energy of the zero-point oscillations of the field theory which is formulated by inertial observers during free fall, as compared to a global standard. An external onlooker sees the zero-point energy in a freely falling coordinate patch decrease as it approaches the horizon. The freely falling coordinate patch was assigned a value of zero renormalized energy due to the oscillations of the field when it was released from rest near "infinity" in the distant past. This decrease in zero-point energy during free fall is shown to translate to a negative energy density of the field, near the horizon, in the components of the renormalized stress-energy tensor. The external onlooker interprets the zero-point energy lost during free fall as an outgoing stream of particle-antiparticle pairs.

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I. INTRODUCTION

The evaporation of black holes via Hawking radiation has attracted great interest in the physics literature since it was postulated in 1974 [1]. However, it is remarkable how many venerable theoretical particle physicists, astrophysicists, and relativists do not understand why black holes should radiate away their inertia, as evidenced by numerous personal communications that I have had "in confidence" over the years. This article is an attempt to reach a wider range of physicists by explaining the phenomenon rigorously in terms of well-known fundamental physical principles.

Furthermore, it may very well be that evaporating black holes are astrophysically unimportant. None have been observed to date. The only ones which might be detectable are micro black holes, possibly created early in the history of the Universe [2]. If these exist, they might have decayed long ago. More massive holes formed by self-gravitating cosmic gas and plasma are probably surrounded by some remnants of the original matter distribution (i.e., disks, clouds, or winds). The effects of the black-hole–matter interactions would swamp and neutralize the Hawking effect for these holes [3]. In light of this, it may be that the main motivation for studying black-hole evaporation is as a gendanken experiment to help understand the synthesis of general relativity and quantum field theory.

Based on the two considerations above, it is compelling to create physically dominated explanations of Hawking's discovery which are therefore more readily understandable by theorists who are not experts in the particulars of this problem. Thus, a physically motivated analysis is developed in this article that does not base the main conclusion on mathematically difficult or ambiguous steps such as difficult renormalization calculations or obscure methods of analytic continuation.

In this author's opinion, there are three main methods of arriving at Hawking's result that do not involve any major "hand-waving arguments" or assumptions at a crucial stage in their development (there might be more and I apologize to those who created them for not mentioning their analyses).

(1) Hawking's original past-directed classical ray-tracing argument couched in the language of scattering from catastrophically collapsing objects [4,5].

(2) Point-separated regularization of stress-energy tensors evaluated in an appropriate vacuum state [6–8].

(3) Analogies to the production of particles in a uniformly accelerating frame (Rindler space) which involves the analytic continuation of wave functions evaluated by noninertial observers into regions beyond their domain of causal contact. This can be extrapolated to static observers outside the horizon to give wave functions which have particle creation built into them [9]. However, the method of analytic continuation is not unique and the reason why Refs. [9,10] produce the "right" method is subtle and involved [10–12]. The bottom line is that the physics in this analysis results from one very slick math step in the complex plane.

Methods 1 and 3 demonstrate particle creation and the more sophisticated analysis in 2 gives the renormalized stress-energy tensor which is necessary for understanding the complete semiclassical problem, i.e., the back reaction on the metric. The following analysis is related to all three. The mathematical calculations look very similar in form to those in method 1, the physical explanation is related to method 3, and the final result gives a physical and relatively simple derivation of the asymptotic (near the horizon and at infinity) form of the renormalized stress-energy tensor in method 2.

The physics of the problem is contained in the renormalized stress-energy tensor. There are many ways to generate a finite tensor from a divergent tensor through regularization. This article gives a physically plausible description of why the regularized tensor in Refs. [8,13] is physically important besides the argument that the result is well behaved in freely falling frames near the horizon and self-consistently makes sense for computing the back reaction on the metric. A physical explanation is given for the fact that the energy density of the spacetime vacuum, near the horizon, is negative as viewed by observers at asymptotic infinity. Unfortunately, previous methods of computing the tensor involve the very complicated, tedious mathematics of point-separated bitensors [see (6.3)–(6.6) of Ref. [14]] [14,15]. The problem with such regularizations which extract a finite part from a formally divergent quantity is that they "always contain ambiguities which must be resolved by the application of additional criteria, such as physical reasonableness" [6]. For example, the final result of the calculation depends on the direction of the point separation which was used. Davies, Fulling, and Unruh [6] conclude that "It is hard to understand how a physical result can depend on such an arbitrary vector field. It appears that such terms, which evidently arise in any point-separation procedure, must be discarded." There are also the "usual" divergent covariant terms which are eliminated because we want a finite result. The procedure is vindicated since it reproduces standard field-theoretic results in the flat-spacetime limit and the final results in the black-hole case are plausible. The only complaint is that, after struggling through all of this algebra supplemented by the aforementioned mathematical slight of hand, what has one learned about the physics of particle creation in the gravitational field of a black hole?

The original past-directed ray-tracing argument proposed by Hawking was hard for many experts to accept in the early years because it is difficult to understand how such a classical analysis can be used to describe an intrinsically quantum-field-theoretic problem [16]. Treated as a mathematically well-posed scattering problem of a scalar field interacting with the gravitational potential of a catastrophically collapsing object, it is not clear what the relevant physics of particle creation is. Hawking states a relationship between $|R^{\mu\nu\lambda\delta}R_{\mu\nu\lambda\delta}|$ (where $R^{\mu\nu\lambda\delta}$ is the Riemann curvature tensor of spacetime) and an indeterminancy of particle number in the field as the fundamental physics of particle creation [4]. However, the Riemann tensor does not appear in his calculation, so it is not clear how the physical description is compatible with the mathematical exercise. It was the genius of Hawking that allowed him the confidence to know that his analysis was so sound that his conclusions must be correct. By contrast, in the following the math and physics are tied together at every juncture.

The main premise of this study is the equivalence principle. At every point of spacetime, it is conjectured that all inertial observers can accurately postulate the relativistic quantum field theory of flat spacetime on open sets with dimensions much less than the "radii of curvature" of spacetime (for modes with a local wavelength less than this same distance scale). Dual spaces to local Lorentz frames at every point of spacetime can be constructed outside the horizon which form the very useful momentum space representations of quantum field theory [17]. All of these local observers can formulate number representations of the field through particle creation and annihilation operators defined as a result of purely local consideration. Similarly, each local freely falling observer transports with him his own definition of the vacuum state. It is proposed in this article that the equivalence principle demands that a global vacuum state defined throughout the spacetime outside of the horizon is constructed by "integrating" the local vacua along a space-filling family of freely falling trajectories (i.e., the restriction of the global vacuum to a point on these trajectories is the vacuum state of the local freely falling observer).

When viewed from this perspective, the reason for particle creation is that the Fock space representations of quantum field theory on curved-spacetime backgrounds are not really generally covariant. The entire theory is in this formulation and it yields a consistent set of physics in any frame. But in defining a Fock space, one must single out the time direction in the observer's frame to define particles and antiparticles. This essentially breaks the invariance of the Fock space representation of the fields under general coordinate transformation and different observers can, in principle, detect different particle numbers in a given field state.

In this article, the aforementioned physics is demonstrated by comparing the formulation of quantum field theory in freely falling frames near the horizon and the same theory as posed by static observers at asymptotic infinity. First of all, one can foliate spacetime with a family of world lines of freely falling observers. The wave equation of the field in the local basis of the freely falling observers is then constructed and solved. These local wave functions are compared to the "global" wave functions computed from the wave equation of the static observers at asymptotic infinity. These "global" solutions separate into two categories: outgoing (propagating away from the horizon as viewed by a static observer) and ingoing quanta. The "global" outgoing solutions are not eigenstates of the local energy operator in the freely falling frames near the horizon. They can only be represented as a wave packet of local solutions which has negative energy components. This last statement results from the fact that, not only are the "global" outgoing waves blueshifted as viewed by freely falling ingoing observers near the horizon, but it is a differential blueshift which varies appreciably over a wavelength. The local decomposition of the "global" solutions involving negative energy local solutions necessarily requires (as one passes to the number representation of the field) particles to exist in the field state which represents the locally defined vacuum, when interpreted by static observers.

An analysis of spacetime "atomized" into the local neighborhoods of freely falling observers can be exploited to provide a quantitative physical explanation of the negative energy density of the vacuum near the horizon (the renormalized stress-energy tensor). The renormalized stress energy tensor of Refs. [8,18] is interpreted as a global quantity which compares the locally evaluated energies of the zero-point oscillations of the field at different points of spacetime. This is illustrated in the article by noting that (for eternal black holes) infalling inertial observers whose world lines represent a foliation of spacetime can be thought of as existing near the horizon as a result of falling freely from "infinity" (far away) after being released from rest at some time in the distant past. This tensor is indicative of the change in the local zero-point energy of the field theory formulated by these observers at different points along their world lines, when these energies are compared to a global standard. To an external static observer, the zero-point energy of the locally defined field appears to decrease in free fall.

To be more specific, when an observer is in free fall, he is constantly redefining his quantum field theory, wave functions, $\varphi_{\rm loc}$, and vacuum state as viewed by a static external observer. By contrast, the field theory as he defines it locally always looks the same, just like flat-spacetime field theory in a local neighborhood. There is a spectrum of momentum states which retains its uniform measure, the mass gap between positive and negative energy states remains the same and there are no particles in the local vacuum.

First, one can look at the field theory when the observer is released from rest at "infinity" in the distant past. The observer at $r\to\infty$ can construct a quantum field theory, wave functions, $\lim_{r\to\infty}\varphi_{\rm loc}$, and there is a natural definition of his vacuum state. This theory is the same as the one formulated by static observers at infinity. The wave functions which he measures are the "global" wave functions $\tilde{\varphi}$ (tildes are used to designate quantities in the static frames at infinity throughout the article) which were mentioned previously:

$$\lim_{r\to\infty}\varphi_{\rm loc}=\tilde{\varphi}\ .$$

These wave functions are characterized by a constant energy eigenvalue ω known as the redshifted energy as viewed from asymptotic infinity in the "membrane paradigm" [see Eq. (2.4) for a definition] [18]. Denote the wave function associated with ω as $\tilde{\varphi}(\omega)$ (to streamline the discussion, we drop the other quantum numbers of the field without loss of physical content in the Schwarzschild geometry).

One can extrapolate to very late times as the freely falling observer approaches near the event horizon. The local energy that he measures will be denoted by P^0. One can look at a local state which appears locally as well as globally outgoing (i.e., propagating away from the horizon). The local energy P^0 can be related to the globally redshifted energy ω for these modes using Schwarzschild coordinates (r is the coordinate of the observer)

$$P^0 \underset{r\to 2M}{\sim} \frac{4M}{r-2M}\omega\ , \tag{1.1}$$

where the "$\sim$" means asymptotically. At a given point of spacetime, one can label these outgoing states equally well by P^0 or ω as a consequence of (1.1). The local wave functions have a constant value of P^0 (not ω) in a neighborhood of the observer. Consider the local wave function with redshifted energy [via the inverse of (1.1)], ω_0, at the point of observation (only), near the horizon. One can designate such a wave function as

$$\lim_{r\to 2M}\varphi_{\rm loc}(\omega_0) \tag{1.2}$$

(where r is the coordinate of the observer). This wave function has a redshifted energy expectation value of $1/2\omega_0$ in the local vacuum.

The "global" wave functions can be interpreted by a freely falling observer near the horizon in the language of his local field theory. As noted earlier, $\tilde{\varphi}$ is not an eigenstate of the local energy operator near the horizon and has a wave-packet representation. To define the energy of $\tilde{\varphi}$ with respect to the local vacuum, one must compute a quantum-mechanical average $\langle\omega_{\rm pac}\rangle$ of the energy operator over the components of the wave packet. This wave form appears highly distorted from a plane wave to freely falling observers near $r=2M$ and it is found that the redshifted energy is larger as measured locally than it is when it is measured by a static observer at asymptotic infinity. This is because the global outgoing solution $\tilde{\varphi}(\omega_0)$ appears differentially blueshifted along a wavelength as perceived by a freely falling observer near the horizon. The effective Lorentz γ factor of the quanta in the language of special relativity is enhanced relative to its pointwise value by averaging over a local neighborhood. In particular, this average results in a locally measured redshifted energy in the wave packet

$$\langle\omega_{\rm pac}\rangle=\tfrac{1}{2}\omega_0\coth 4\pi M\omega_0>\tfrac{1}{2}\omega_0\ . \tag{1.3}$$

Result (1.3) is the essence of the negative energy density of the renormalized stress-energy tensor evaluated near the horizon. Consider an external observer fixed relative to the static background who assesses the situation. In the distant past, the external observer sees the freely falling frame released from rest at "infinity" and singles out a mode to monitor with redshifted energy ω_0, wave function $\tilde{\varphi}(\omega_0)$, and zero-point energy relative to the local freely falling vacuum of $\frac{1}{2}\omega_0$. At a much later time, the external observer sees that this freely falling frame is now very near to the horizon and again he singles out the mode with redshifted energy ω_0 evaluated at the origin of the freely falling tetrad via (1.1). This local mode is designated as $\lim_{r\to 2M}\varphi_{\rm loc}(\omega_0)$ and has a zero-point energy of $\frac{1}{2}\omega_0$ relative to the local freely falling vacuum near the horizon. This mode is the analog of $\tilde{\varphi}(\omega_0)=\lim_{r\to\infty}\varphi_{\rm loc}(\omega_0)$ when the tetrad was released from infinity in distant past.

Although, locally, the two modes appear similarly defined, globally they are quite different. The expectation value of the redshifted energy of the mode $\tilde{\varphi}(\omega_0)$ with respect to the freely falling vacuum is larger than the same for $\varphi_{\rm loc}(\omega_0)$ as $r \rightarrow 2M$ as evidenced by (1.3). Define the change in energy of the mode with redshifted energy ω_0, which results from the reformulation of the local field theory during free fall, as $\Delta\omega_0$:

$$\Delta\omega_0 = \langle \omega_{\rm pac} \rangle - \tfrac{1}{2}\omega_0 = \frac{\omega_0}{e^{8\pi M \omega_0} - 1} . \tag{1.4}$$

The global view of the external observer indicates that this energy must have been radiated away during free fall and it is shown in the text that half is lost in the particle channels and half in the antiparticle channels. Relation (1.4) holds for all globally outgoing modes. Consequently, an external onlooker concludes that the redshifted zero-point energy E_0 of the locally defined field theory is less near the horizon in the freely falling frame than it was in the distant past, "near infinity," by an amount ΔE_0:

$$\Delta E_0 = -\sum_{l,m} \int \frac{d\omega}{e^{8\pi M \omega} - 1} . \tag{1.5}$$

It is proposed that (1.5) represents the fundamental physics of the dynamical components of the renormalized stress-energy tensor near the horizon. All of these local patches of infalling vacuum can be integrated to reproduce the renormalized stress-energy tensor of spacetime near the horizon using (1.5).

In order to isolate the physics of particle creation in the gravitational field of a black hole, we concentrate on the simplest case: a scalar field theory on the Schwarzschild background. The analysis is begun by describing the foliation of spacetime by hypersurfaces orthogonal to a space-filling family of timelike freely falling trajectories. A global coordinate system is constructed in which the "time" direction is the four-velocity of the freely falling observers. In Sec. III the scalar wave equation in the freely falling coordinate system is derived and then solved near the horizon. These are the wave functions defined by the freely falling observers. Section IV shows how the "global" solutions defined by static observers at infinity can be decomposed as an integral over frequency of the local solutions found in Sec. III. The inverse relation is derived as well and the corresponding transformation between the local and static frame creation and annihilation operators is also determined. Section V is a discussion of various stress-energy tensors evaluated in the local vacuum state and their physical meaning. Particular emphasis is given to what is called the renormalized stress-energy tensor of the local vacuum. Section VI synthesizes the results of Sec. V to construct the dynamic components of the renormalized stress-energy tensor of spacetime near the horizon and at asymptotic infinity. A physical description of the result is given. For completeness, a similar discussion for fermions is included in the Appendices.

II. THE FOLIATION OF SPACETIME BY FREELY FALLING FRAMES

In Ref. [19], it was demonstrated that the vacuum spacetime of a black hole can be faithfully represented by "piecing together" the local neighborhoods of freely falling observers. This exploitation of the equivalence principle is made mathematically rigorous in terms of completely integrable distributions and the reader is referred to Ref. [19] for details. The freely falling frames which are released from rest at "infinity" (denoted by $r \rightarrow \infty$ in Schwarzschild coordinates which are used throughout this article) in the distant past were used to accomplish this decomposition of spacetime. These were called the preferred freely falling (PFF) observers. The essence of the formalism of Ref. [19] is that the local neighborhoods of freely falling observers can be used to form a best linear approximation or local tangent space approximation to the manifold, then passing to the limit of arbitrarily high accuracy. This method reproduces the metric of the Kerr spacetime by considering special relativistic time dilation and Lorentz contraction effects restricted to these local frames.

The success of reproducing the general relativistic potential in terms of special relativity in local inertial frames suggests that it is insightful to formulate quantum field theory in the Schwarzschild spacetime as a synthesis of local special relativistic field theory in the freely falling frames. Of particular interest to this paper is that the Schwarzschild vacuum state of the field (ground state) is defined so that its restriction to an open neighborhood of a PFF observer is the local vacuum state of inertial observers in that subset of spacetime. This paper formulates the local quantum field theory at different points of spacetime and compares these theories in a global context. The following discussion develops the tools necessary to accomplish this.

A. Freely falling coordinates

In order to compare field theories at different points of spacetime, a global coordinate system is introduced in which one can formulate the local physics so that it takes the form of flat-space quantum field theory in a neighborhood of a PFF observer. Since spacetime is curved, this coordinate system cannot be orthonormal. For this reason it is useful to define the orthonormal tetrad $\hat{e}_\alpha$ carried by an observer that falls freely in the radial direction:

$$\begin{aligned} \begin{bmatrix} \hat{e}_u \\ \hat{e}_\rho \end{bmatrix} &= \frac{\omega_0}{m_0} \begin{bmatrix} \alpha^{-2} & V^r \\ \alpha^{-2} V^r & 1 \end{bmatrix} \begin{bmatrix} \partial/\partial t \\ \partial/\partial r \end{bmatrix} , \\ \hat{e}_\theta &= \frac{1}{r}\frac{\partial}{\partial\theta} , \\ \hat{e}_\phi &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} . \end{aligned} \tag{2.1}$$

Note that $\hat{e}_u$ is the four-velocity of the freely falling observers, α is the lapse function

$$\alpha \equiv \left[1 - \frac{2M}{r}\right]^{1/2} , \tag{2.2}$$

which measures the gravitational redshift between static observers at the coordinate r and the static observers at infinity [18]. The velocity of the observer in radial free fall as seen by a static observer at coordinate r, V^r, is [20]

$$V^r = \pm \left[1 - \alpha^2 \frac{m_0^2}{\omega_0^2} \right]^{1/2} . \tag{2.3}$$

The plus and minus signs are valid for globally outgoing (propagation away from the horizon) and globally ingoing trajectories, respectively. The quantity m_0 is the rest mass of the freely falling observer and ω_0 is its redshifted energy defined by

$$\omega = -\frac{\partial}{\partial t} \cdot P , \tag{2.4}$$

where P is the four-momentum of the trajectory [18].

The basis covectors are defined by

$$\begin{bmatrix} \hat{\omega}^u \\ \hat{\omega}^\rho \end{bmatrix} = \frac{\omega_0}{m_0} \begin{bmatrix} 1 & -V^r \alpha^{-2} \\ -V^r & \alpha^{-2} \end{bmatrix} \begin{bmatrix} dt \\ dr \end{bmatrix} . \tag{2.5}$$

The inverse transforms are

$$\begin{bmatrix} \dfrac{\partial}{\partial t} \\ \dfrac{\partial}{\partial r} \end{bmatrix} = \frac{\omega_0}{m_0} \begin{bmatrix} 1 & -V^r \\ -V^r \alpha^{-2} & \alpha^{-2} \end{bmatrix} \begin{bmatrix} \hat{e}_u \\ \hat{e}_\rho \end{bmatrix} , \tag{2.6}$$

$$\begin{bmatrix} dt \\ dr \end{bmatrix} = \frac{\omega_0}{m_0} \begin{bmatrix} \alpha^{-2} & \alpha^{-2} V^r \\ V^r & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega}^u \\ \hat{\omega}^\rho \end{bmatrix} . \tag{2.7}$$

The freely falling frames which are released from rest at "infinity" in the distant past, the PFF observers, are defined by $\omega_0 = m_0$ in (2.1), (2.3), and (2.5)–(2.7). This special case will be emphasized because the algebra is slightly simpler. It should be noted that the analysis in this article is equally valid for all radial freely falling observers with $\omega_0 \geq m_0$.

The four-velocity of the radial freely falling frames is a hypersurface orthogonal vector field since

$$d\hat{\omega}^u \wedge \hat{\omega}^u = 0 . \tag{2.8}$$

Thus, it should be possible to find a spacelike coordinate vector field in the hypersurface, $\hat{\omega}^u = \text{const}$, which is also orthogonal to $\hat{e}_\theta$ and $\hat{e}_\phi$. If one defines the vector field E_1,

$$E_1 = -\frac{m_0}{\omega_0} V^r \frac{\partial}{\partial X^\rho} , \tag{2.9}$$

then $[E_1, \hat{e}_0] = 0$ and one has the useful global orthogonal coordinate frame

$$E_0 = \hat{e}_0 \equiv \frac{\partial}{\partial X^0} , \tag{2.10a}$$

$$E_1 = -\frac{m_0}{\omega_0} V^r \hat{e}_\rho \equiv \frac{\partial}{\partial X^1} , \tag{2.10b}$$

$$E_\phi = \frac{\partial}{\partial \phi} , \tag{2.10c}$$

$$E_\theta = \frac{\partial}{\partial \theta} . \tag{2.10d}$$

Note that the transformations in (2.1) and (2.5)–(2.7) become undefined at the horizon as $\alpha \to 0$. However, the physics of interest occurs outside of the horizon; thus, there is no need to extend the coordinates in (2.10) across the horizon (which can be done).

One can define basis covectors

$$dX^0 = \hat{\omega}^u , \tag{2.11a}$$

$$dX^1 = -\frac{\omega_0}{m_0 V^r} \hat{\omega}^\rho , \tag{2.11b}$$

$$dX^\phi = d\phi , \tag{2.11c}$$

$$dX^\theta = d\theta . \tag{2.11d}$$

It is significant that, near the horizon, since $V^r \to -1$ for the PFF observers by (2.3), the coordinate basis is the same as the orthonormal tetrad carried by the PFF observers to $O(\alpha^2)$.

Using (2.5) and (2.11), one can define the useful relation

$$dX^0 - dX^1 = \frac{1}{V^r} dr . \tag{2.12a}$$

This suggests a simple form of the freely falling coordinates, X^0, X^1, θ, and ϕ obtained by integrating (2.11) since (2.12a) and (2.3) imply that

$$X^0 - X^1 = \frac{2}{3} \frac{r^{3/2}}{\sqrt{2M}} + \text{const} . \tag{2.12b}$$

Expanding (2.12b) in a Taylor series near the horizon yields the asymptotic relation

$$X^1 - X^0 \underset{r \to 2M}{\sim} [r - 2M + c][1 + O(\alpha^2)] , \tag{2.12c}$$

where c is a constant; equivalently,

$$\alpha^2(r) \underset{r \to 2M}{\sim} \frac{X^1 - X^0 - c}{2M} . \tag{2.12d}$$

This relation will be used frequently in the following discussions.

The metric in the PFF coordinate system is given by the line element

$$ds^2 = -dX^0 \otimes dX^0 + (V^r)^2 dX^1 \otimes dX^1 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \tag{2.13}$$

and the volume measure is

$$\sqrt{-g} = \sqrt{2M} r^{3/2} \sin\theta . \tag{2.14}$$

B. Local momentum

One can write the radial velocity of a particle or group velocity of a wave as measured in the static frames at a coordinate, r, for a general trajectory as in (2.3):

$$V \equiv \alpha^{-2} \frac{dr}{dt} = \frac{\tilde{p}^r}{\tilde{p}^t} \alpha^{-2} = \pm \left[1 - \alpha^2 \frac{m_e^2 + \mathcal{K}/r^2}{\omega^2} \right]^{1/2} , \tag{2.15}$$

where m_e is the rest mass of the quanta and $\mathcal{K}$ is a constant of geodesic motion representing the total angular momentum [20]. This relation allows one to use (2.5) to find the locally evaluated momentum in compact notation:

$$P^0=\frac{\omega_0}{m_0}\alpha^{-2}[1-VV^r]\omega\ , \tag{2.16a}$$

$$P^\rho=\frac{\omega_0}{m_0}\alpha^{-2}[V-V^r]\omega\ , \tag{2.16b}$$

$$P^1=-\left[\frac{\omega_0}{m_0}\right]^2\frac{\alpha^{-2}}{V^r}[V-V^r]\omega\ . \tag{2.16c}$$

First, consider globally outgoing waves and particles where $V>0$. By (2.16b), all of these trajectories appear locally outgoing, i.e., $P^\rho>0$. These are the modes with the interesting physics which is linked to particle creation in the gravitational field of the black hole. In a freely infalling frame near the horizon, all globally outgoing modes are locally ultrarelativistic, in particular,

$$P^0\simeq P^\rho\simeq\frac{2\omega_0}{m_0}\alpha^{-2}\omega\ . \tag{2.17}$$

Not only are these waves blueshifted in the freely falling frames, but for modes with ω equal to a constant, the gradient in the blueshift is unbounded as the freely falling observer approaches the horizon.

For globally ingoing waves one takes the minus sign in (2.15). In this case, by (2.16b), the waves still appear locally outgoing if $|V|<|V^r|$ and are locally ingoing otherwise. The globally ingoing waves have local momenta that do not scale with lapse function near the horizon by (2.3) and (2.15):

$$P^0\sim\alpha^0\ , \tag{2.18a}$$

$$P^\rho\sim\alpha^0\ . \tag{2.18b}$$

There is no differential blueshift in the asymptotic zone near the horizon for these waves.

III. THE LOCAL WAVE EQUATION

In this section, the scalar wave equation, which a freely falling observer would use to formulate a quantum field theory, is derived. This is naturally accomplished in the freely falling coordinate system of Sec. II since $\partial/\partial X^0$ is the four-velocity of these observers.

A. The scalar wave equation

Covariantly written, the wave equation is

$$\varphi_{;\alpha}^{;\alpha}+m_e^2\varphi=0\ , \tag{3.1}$$

where the semicolon signifies covariant differentiation. In a coordinate system this can be expressed as

$$\frac{1}{\sqrt{-g}}(\sqrt{-g}\,g^{\alpha\beta}\varphi_{,\beta})_{,\alpha}+m_e^2\varphi=0\ . \tag{3.2}$$

Using (2.13), this can be expanded in the freely falling coordinate system as

$$-\frac{\partial^2}{(\partial X^0)^2}\varphi+\frac{r}{2M}\frac{\partial^2}{(\partial X^1)^2}\varphi-\frac{3}{2}\frac{\sqrt{2M}}{r^{3/2}}\frac{\partial}{\partial X^0}\varphi+\frac{5}{2}\frac{1}{\sqrt{2Mr}}\frac{\partial}{\partial X^1}\varphi+m_e^2\varphi+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\varphi+\frac{1}{r^2}\cot\theta\frac{\partial}{\partial\theta}\varphi+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\varphi=0\ . \tag{3.3}$$

The coefficients are left in terms of the Schwarzschild coordinate r since the equation will only be looked at in the limit $r=2M[1+O(\alpha^2)]$. As is customarily done, we separate the angular dependence out of the solution in the form of spherical harmonics $Y_{lm}(\theta,\phi)$:

$$\varphi_{lmn}=\varphi_n(r)Y_{lm}(\theta,\phi)\ . \tag{3.4}$$

The quantum number n labels the local energy eigenvalue. Using (3.4), one gets an equation for the "radial function" φ_n:

$$-\frac{\partial^2}{(\partial X^0)^2}\varphi_n+\frac{r}{2M}\frac{\partial^2}{(\partial X^1)^2}\varphi_n-\frac{3\sqrt{2M}}{2r^{3/2}}\frac{\partial}{\partial X^0}\varphi_n+\frac{5}{2}\frac{1}{\sqrt{2Mr}}\frac{\partial}{\partial X^1}\varphi_n-\left[\frac{l(l+1)}{r^2}+m_e^2\right]\varphi_n=0\ . \tag{3.5}$$

B. The solution space

One can examine the solutions to (3.5) when the PFF observer is near the horizon. First of all, when $P^1\gg 1/4M$, the "radial" wave equation reduces to its flat space form as $r\to 2M$. Thus, we know that, in a local neighborhood of the observer, these solutions will appear to be plane waves

$$(\varphi_{\rm loc})_n\sim\exp[iP_n{}^\mu X_\mu],\quad P^1\gg\frac{1}{4M}\ , \tag{3.6}$$

where the local momentum P^μ is a constant in (3.6). To define the local neighborhood where (3.6) is valid, first signify the coordinates of the PFF observer as $(\bar{X}^0,\bar{X}^1)$. Then (3.6) is valid on any open set $\mathcal{V}$ defined by

$$\mathcal{V}=\{X^0,X^1,\theta,\phi\,||(X^0-\bar{X}^0)-(X^1-\bar{X}^1)|\ll 4M,0\le\theta\le\pi,\ 0\le\phi\le 2\pi\}\ . \tag{3.7a}$$

Sometimes it will be important to look at an open set in which (3.6) is valid and is outside of the horizon, $\mathcal{V}_+$. To generate, $\mathcal{V}_+$ (3.7a) is modified using (2.12c),

$$\mathcal{V}_+=\{X^0,X^1,\theta,\phi\,||(X^0-\bar{X}^0)-(X^1-\bar{X}^1)|\ll 4M,X^1-X^0>c,0\le\theta\le\pi,0\le\phi\le 2\pi\}\ . \tag{3.7b}$$

Now one can consider the case when $P^0 \sim 1/M$ and $P^1 \sim 1/M$. First, for globally outgoing solutions, this implies by (2.17) that

$$\omega \sim \alpha^2 M^{-1} . \tag{3.8}$$

Thus, as $\alpha \to 0$ these modes have an arbitrarily small effect on any global phenomenon. Also, in a realistic massive case $m_e \gg 1/M$, so the first derivative terms in (3.5) will always be negligible. (We are intentionally avoiding the case of black holes where the radius of curvature near the horizon is on the order of an electron Compton wavelength. Particle production in this case would not proceed via the Hawking effect for the most part. This would no longer be a discussion of the semiclassical Hawking effect but would require a knowledge of quantum gravity. No theory of quantum gravity is established at this time. It is implicit in the text that one is nowhere near this Planck limit. Some details of the modifications due to this effect are noted in Ref. [13].) One concludes that (3.6) will faithfully represent the globally outgoing solutions in the ω phase space when restricted to a neighborhood of a freely falling observer located at $r \gtrsim 2M$.

The restriction of the local wave function to the open set $\mathcal{V}_+$ is denoted by $\vec{u}_m(X^0,X^1,\theta,\phi)$ for the globally outgoing solutions

$$\begin{aligned} \vec{u}_m &= (\varphi_{\text{loc}})_m |_{\mathcal{V}_+} \\ &= \frac{1}{\sqrt{2\pi}\sqrt{2P_m^0}} \frac{1}{r} \exp[i[P^0]_m (X^0 - X^1)] , \end{aligned} \tag{3.9}$$

where the fact that $P^0 \simeq P^1$ for ultrarelativistic local quanta was used in the exponent of (3.9). [By (3.8) most of the ω phase space satisfies the ultrarelativistic criterion for globally outgoing modes.] The normalization used in (3.9) is a tricky issue and essentially (3.9) is written for later mathematical simplicity and these are treated as unnormalized states. Normalization is a global consideration and this analysis is local in nature. The reason that one needs to be cautious in choosing a normalization constant for φ_{loc} is that many treatments put the math of particle production into a multiplicative factor which is energy dependent, in front of the wave function (see Ref. [10] for an example and, in the case of accelerating observers, see Ref. [21]). There is also the question of what is the appropriate base manifold for the normalization. The Klein-Gordon inner product between states a and b is

$$\langle a,b \rangle = -i \int_{T=\text{const}} a^* \frac{\overleftrightarrow{\partial}}{\partial T} b \sqrt{-g^{(3)}} d^3x \left| \frac{\partial}{\partial T} \right|^{-1} g^{TT} , \tag{3.10}$$

where $\sqrt{-g^{(3)}}$ is the volume measure of the hypersurface orthogonal to $\partial/\partial T$ for some global time coordinate T. The problem with normalizing φ_{loc} is whether to perform the integration only over the space external to the horizon, $X^i - X^0 > c$ in (2.12c), or over all of X^1. Such issues involve analytic continuation, making the discussion too mathematical and physically obscure for our purposes. This issue will have to be side stepped again in the next section. The justification of the choice of normalization in (3.9) is that it might be guessed by the equivalence principle. A PFF observer approximates his situation by just choosing standard flat-space functions to order α^2.

To analyze the globally ingoing case (including when $P^1 \sim 1/M$), note that P^0 and P^1 do not experience the effects of a differential blueshift near the horizon as discussed in (2.18). Thus, there exists a solution to (3.5) near the horizon of the form

$$\begin{aligned} (\dot{\varphi}_{\text{loc}})_n \underset{r\to 2M}{\sim} & F_n(X^0,X^1) \\ & \times \exp\{ i[(P_0)_n X^0 + (P_1)_n X^1][1+O(\alpha^2)] \} , \end{aligned} \tag{3.11a}$$

$$P_0, P_1 \to \text{const} + O(\alpha^2) , \tag{3.11b}$$

where $F_n(X_0,X_1)$ is a slowly varying function (i.e., varying on distance scales on the order of M). Rearranging (3.11a) and using (2.3), (2.15), and (2.16), one gets

$$\begin{aligned} (\dot{\varphi}_{\text{loc}})_n \underset{r\to 2M}{\sim} & F_n(X^0,X^1) \\ & \times \exp\left\{ i \frac{P^0 - P_1}{2} (X^0 + X^1)[1+O(\alpha^2)] \right\} . \end{aligned} \tag{3.11c}$$

Applying (2.11)

$$\begin{aligned} (\dot{\varphi}_{\text{loc}})_n \underset{r\to 2M}{\sim} & F_n(r=2M) \\ & \times \exp\{ i\omega(P_n) v [1+O(\alpha^2)] \} , \end{aligned} \tag{3.11d}$$

where $\omega(P_n)$ is a constant defined via the inverse to (2.16) and v is the advanced coordinate

$$v \equiv t + r^* , \tag{3.12a}$$

$$r^* \equiv r + 2M \ln\alpha^2 . \tag{3.12b}$$

As in (3.9), we write the restriction to $\mathcal{V}_+$, $\dot{u}_m$, as

$$\dot{u}_m = (\dot{\varphi}_{\text{loc}})_m |_{\mathcal{V}_+} = \frac{1}{\sqrt{2\pi}\sqrt{2P_m^0}} \frac{1}{r} \exp[i\omega(P_m) v] . \tag{3.13}$$

Summarizing, the local solutions for $P^1 \gg 1/4M$ are characterized by P^0 and P^1, a constant in a neighborhood $\mathcal{V}$, about the PFF observers near the horizon. This is a manifestation of the approximate translational symmetry on $\mathcal{V}$ which is an axiom of special relativity [22]. Note that, since P^0 is a constant for globally outgoing solutions, (2.17) implies that ω is not constant in local time for the local solution (3.9) when viewed by a PFF observer near the horizon.

When $P^1 \sim 1/M$ or less, the waves vary negligibly in time and space as viewed by PFF observers near the hole on the dynamical time and distance scales of interest. For example, such an observer would pass inside of the horizon in a local time interval $\Delta X^0 \ll M$, and in this time interval he would detect only a small fraction of one oscillation of the wave. As an aside, it should be noted that the concept of a wave with a wavelength larger than the dimensions of the inertial coordinate patch can be well defined. It is commonly argued in the discussion of Hawking radiation that an inertial observer cannot detect modes with a wavelength larger than M because, due to curvature, the dimensions of an inertial coordinate patch is of order M and his detector is therefore too small [4]. However, electrical engineers detect 60-cps signals and their detectors are not 2000 miles long. Similarly, microwave engineers working at ultrahigh frequencies (UHF's) detect waves with scalar network analyzers which are much smaller than 4 ft.

Finally, the local solutions in the PFF frames can be thought of as having positive frequencies defined relative to the timelike vector field $\partial/\partial X^0$. This vector field is an ingoing principal null geodesic as seen from static infinity to order α^2 [20]. Thus, these solutions are similar to Unruh's solutions which have positive frequencies defined relative to the null generators of the past horizon (principal null congruences) [9]. The reader familiar with the particulars of Hawking radiation will immediately realize that particle creation with a blackbody spectrum must follow from these wave functions. What has been done in this section is to physically motivate Unruh's construct and remove the physically unfamiliar notion of defining positive and negative frequencies relative to a null and not a timelike vector.

IV. THE LOCAL FOURIER DECOMPOSITION OF THE STATIC FRAME WAVE FUNCTIONS

In this section, the wave functions as formulated by static observers at asymptotic infinity are decomposed as an integral of the local wave functions discussed in the last section. An inverse relation is derived as well as the Bogoliubov transformation between the two sets of particle creation and annihilation operators.

A. The global wave functions

The global wave functions $\tilde{\varphi}_{ml\omega}(r,\theta,\phi,t)$ defined by the static observers at infinity have the well-known asymptotic form near the horizon [4]:

$$\vec{\tilde{\varphi}}_{lm\omega}(r,\theta,\phi,t) \underset{r\to 2M}{\sim} \frac{e^{i\omega u}}{\sqrt{2\pi}\sqrt{2\omega r}} Y_{lm}(\theta,\phi)$$

$$\text{purely outgoing as } r\to 2M \ , \quad (4.1a)$$

$$\overleftarrow{\tilde{\varphi}}_{lm\omega}(r,\theta,\phi,t) \underset{r\to 2M}{\sim} \frac{e^{i\omega v}}{\sqrt{2\pi}\sqrt{2\omega r}} Y_{lm}(\theta,\phi)$$

$$\text{purely ingoing as } r\to 2M \ . \quad (4.1b)$$

The coordinate v is defined in (3.12) and u is the retarded coordinate

$$u = t - r^* \ . \quad (4.2)$$

Note the equivalence of (3.13) and (4.1b). These solutions are characterized by $\omega=$const. The following restrictions to the generic open set $\mathcal{V}_+$ defined in (3.7b) are

$$\vec{u}_{n'} Y_{lm}(\theta,\phi) \equiv \vec{\tilde{\varphi}}_{lm\omega_{n'}}|_{\mathcal{V}_+} \simeq \frac{e^{i\omega_{n'}u}}{\sqrt{2\pi}\sqrt{2\omega_{n'}r}} Y_{lm}(\theta,\phi) \ , \quad (4.3a)$$

$$\overleftarrow{u}_{n'} Y_{lm}(\theta,\phi) \equiv \overleftarrow{\tilde{\varphi}}_{lm\omega_{n'}}|_{\mathcal{V}_+} \simeq \frac{e^{i\omega_{n'}v}}{\sqrt{2\pi}\sqrt{2\omega_{n'}r}} Y_{lm}(\theta,\phi) \ . \quad (4.3b)$$

Since we are mainly interested in zero-point energies, the wave functions which were chosen in (4.3) are not the scattering states often implemented in the discussion of Hawking radiation. The scattering states incorporate the reflection and transmission of the waves in the curvature potential. The scattering states are defined by [5]

$$\chi(l,m,\omega/x) = \frac{1}{\sqrt{2\pi}\sqrt{2\omega r}} R_l(\omega/r) Y_{lm}(\theta,\phi) e^{-i\omega t} \ , \quad (4.4a)$$

$$\vec{R}_l(\omega/r) \to e^{i\omega r^*} + \vec{A}_l(\omega) e^{-i\omega r^*}, \quad r^* \to -\infty \ , \quad (4.4b)$$

$$\overleftarrow{R}_l(\omega/r) \to B_l(\omega) e^{-i\omega r^*}, \quad r^* \to \infty \ , \quad (4.4c)$$

where $\vec{A}_l(\omega)$ is the reflection coefficient in the curvature potential for a wave emanating from the asymptotic region near the horizon. Similarly, $B_l(\omega)$ is the transmission coefficient of a wave launched from asymptotic infinity toward the hole. They are related by the unitarity condition [5]

$$1 - |\vec{A}_l(\omega)|^2 = |B_l(\omega)|^2 \ . \quad (4.4d)$$

A very useful representation of (4.1) is the WKB-type approximation which was shown in Ref. [17] to be very accurate near the horizon:

$$\tilde{\varphi}_{lm\omega}(r,\theta,\phi,t) \underset{r\to 2M}{\sim} \frac{1}{\sqrt{2\pi}\sqrt{2\omega r}} \exp\left[i\int P^\mu dX_\mu\right] Y_{lm}(\theta,\phi)$$
$$\simeq \frac{1}{\sqrt{2\pi}\sqrt{2\omega r}} \exp\left[-i\int (P^0 dX^0 - P^1 dX^1)\right] Y_{lm}(\theta,\phi) \ . \quad (4.5)$$

The vector P^μ is the four-momentum of the wave which is chosen to be evaluated in the PFF coordinate system near the horizon in the second step in (4.5). When ω is a constant, (2.16) and (2.17) show that P^0 and P^1 are functions in the globally outgoing case evaluated near the horizon. Integrating (4.5) using (2.17), (2.16), (2.15), and (2.3) reproduces the exponent in (4.1) to order α^2. Relation (4.5) is invaluable for evaluating the local observables, in the PFF frames, of the "global" wave functions.

To compute the Fourier transform of (4.1), one needs the "global" wave functions expressed in local coordinates. To accomplish this, first use (4.2) and (2.7) to express the retarded coordinate as

$$du=(1-V^r)\alpha^{-2}[dX^0+V^r dX^1] \ . \tag{4.6a}$$

Near the horizon

$$du \underset{r\to 2M}{\sim} 2\alpha^{-2}[dX^0-dX^1] \ . \tag{4.6b}$$

Employing (2.12d) to eliminate the lapse function,

$$du \underset{r\to 2M}{\sim} \frac{-4M}{X^1-X^0-c} d[X^1-X^0-c] \tag{4.6c}$$

or

$$u \underset{r\to 2M}{\sim} -4M \ln\left[\frac{X^1-X^0-c}{4M}\right] \ . \tag{4.6d}$$

Thus, using (4.6d) in (4.1), near the horizon

$$\vec{\tilde{\varphi}}_{lm\omega}(X^0,X^1,\theta,\phi) \underset{r\to 2M}{\sim} \frac{1}{\sqrt{2\pi}\sqrt{2\omega}r}\exp\left\{-i4M\omega\ln\left[\frac{X^1-X^0-c}{4M}\right]\right\} \times Y_{lm}(\theta,\phi) \ . \tag{4.7}$$

It should be noted that the wave functions in (4.1) and (4.7) are undefined inside of the horizon because the coordinates u and v diverge at the horizon.

B. The local Fourier decomposition

The main mathematical step in this article is the expression of the "global" wave function $\tilde{\varphi}$ as a Fourier integral with respect to the local wave functions $\varphi_{\rm loc}$ near the horizon. The Fourier decomposition of the globally ingoing case is trivial by (3.13) and (4.1b):

$$\overleftarrow{\tilde{\varphi}}_{lm\omega}|_{\mathcal{V}_+}=\int \delta(\omega-\omega')\sqrt{P/\omega}(\overleftarrow{\varphi}_{\rm loc})_{lm\omega'(P)}|_{\mathcal{V}_+}d\omega'(P) \ . \tag{4.8}$$

The globally outgoing solutions are more interesting. The "global" solutions are characterized by ω=const. By (2.17), this means that the local momentum of the "global" wave varies greatly in a small neighborhood of a PFF observer near the horizon. Thus, unlike the globally ingoing case, the outgoing "global" wave functions can only be represented as a packet of local wave functions.

The interesting aspect of these wave packets are the negative energy components. This is a result of the fact that, near the horizon, the "global" states have a very large gradient in their locally measured energy. A way of quantifying the magnitude of the differential blueshift is with the ratio $\mathcal{R}$ defined by

$$\mathcal{R}\equiv\frac{|(\partial/\partial X^0)P^0|\lambda}{P^0}\simeq\frac{|(\partial/\partial X^1)P^0|\lambda}{P^0} \ , \tag{4.9}$$

where λ is the wavelength of the quanta. This quantity tells one the degree of the energy change in one wavelength of oscillation of the quanta. When $\mathcal{R}\sim 1$, one expects the negative energy modes to contribute significantly to the Fourier integral [23].

To find the wavelength of $\tilde{\varphi}_{m'}$ as measured by a PFF observer, one can use (4.5). (Since the interesting modes are the globally outgoing ones, the arrows are dropped in the remaining discussion to streamline notation. Also, since the angular dependence is the same in both frames, the Y_{lm}'s are dropped from the wave functions. The $\tilde{\varphi}_{m'}$ and φ_m are just the "radial" and "time" dependence of the "global" and local wave functions, respectively.) When the phase changes by 2π in the exponential, one wave oscillation is complete. Thus, one can find the local wavelength by solving for the half-wavelength ahead of the PFF observer $(\lambda/2)_+$ and for the half-wavelength behind the PFF observer $(\lambda/2)_-$:

$$(\lambda/2)_+\equiv\bar{X}^1-X^1_+>0 \ , \tag{4.10a}$$

$$(\lambda/2)_-\equiv X^1_- -\bar{X}^1>0 \ , \tag{4.10b}$$

$$\lambda\equiv(\lambda/2)_+ +(\lambda/2)_- = X^1_- - X^1_+ \ , \tag{4.10c}$$

and $(\bar{X}^0,\bar{X}^1)$ are the coordinates of the PFF observer. To solve for the half-wavelengths, one must evaluate

$$\pi=\int_{(\bar{X}^0,\bar{X}^1)}^{(X^0_+,X^1_+)} P_\mu dX^\mu \ , \tag{4.10d}$$

$$-\pi=\int_{(\bar{X}^0,\bar{X}^1)}^{(X^0_-,X^1_-)} P_\mu dX^\mu \ . \tag{4.10e}$$

Looking at (4.10d) and (4.10e) in a hypersurface orthogonal to $\partial/\partial X^0$ (i.e., $\bar{X}^0$ a constant), one finds that, for this observer, (2.17) and (2.12) imply that, near the horizon,

$$\lambda=\tfrac{1}{2}[\bar{X}^1-\bar{X}^0-c]\sinh\left[\frac{\pi}{4M\omega}\right] \ . \tag{4.11}$$

Then using (2.12d), (2.17), (4.9), and (4.11),

$$\mathcal{R}\equiv\tfrac{1}{2}\sinh\left[\frac{\pi}{4M\omega}\right] \ . \tag{4.12}$$

Thus, for $\omega\sim 1/M$ or less, one expects a large negative energy contribution to the wave packet which represents $\tilde{\varphi}_{m'}$ to a PFF observer near the horizon.

Therefore, we are looking for an expansion of $\tilde{\varphi}_{m'}$ near the horizon, $\tilde{u}_{m'}$, of the form

$$\tilde{u}_{m'}\equiv\tilde{\varphi}_{m'}|_{\mathcal{V}_+}=\int_0^\infty (A_{mm'}u_m+B_{mm'}u_m^\dagger)dP_m \ . \tag{4.13}$$

The following abbreviations are implicit:

$$\tilde{\varphi}_{m'}\equiv\tilde{\varphi}(\omega_{m'})$$

and

$$u_m\equiv u(P_m)\simeq\varphi_{\rm loc}(P_m)|_{\mathcal{V}_+}\equiv\varphi_m|_{\mathcal{V}_+} \ ,$$

$$P_m\equiv P_m^0 \ .$$

One could compute the coefficients in (4.13) by using (4.1), (3.9), and the inner product in (3.10). However, as mentioned in the discussion of normalization following (3.10), we do not want to introduce an ambiguity associated with choosing the "right" manifold to integrate over. Instead, the form of u_n in (3.9) suggests using Fourier integral techniques. To get the desired quantities, one needs the inversion theorem, but to be rigorous this requires integrating over spacetime variables which range from $-\infty$ to ∞ [24]. To motivate this idea, one can write (4.7) as

$$\tilde{\varphi}_{m'} = \frac{1}{\sqrt{2\pi}\sqrt{2\omega_{m'}}}\frac{1}{r}R_{m'}(X^0,X^1) , \tag{4.14a}$$

$$R_{m'}(X^0,X^1) \equiv \exp\left\{ i4M\omega_{m'}\ln\left[\frac{X^1-X^0-c}{4M}\right]\right\} . \tag{4.14b}$$

The function $R_{m'}$ is not defined for $X^1-X^0<c$ and for the sake of physical clarity it is desirable to not get involved in a discussion of how to extend the function to values of X^1 and X^0 satisfying $X^1-X^0-c<0$. So, to be able to integrate from $-\infty$ to ∞, one can change variables

$$z \equiv [X^1-X^0-c]-A . \tag{4.15}$$

Then the Fourier transform of $R_{m'}$, $F_{m'}(P)$, is

$$F_{m'}(P) = \lim_{A\to\infty}\int_{-A}^{\infty} R_{m'}(z)e^{-iPz}\frac{dz}{\sqrt{2\pi}} \tag{4.16a}$$

and the inversion theorem then gives the quantities of interest in (4.13):

$$R_{m'} = \int_{-\infty}^{\infty} F_{m'}(P)e^{iPz}\frac{dz}{\sqrt{2\pi}} . \tag{4.16b}$$

Combining (4.16b), (3.9), and (4.14), one has

$$\tilde{\varphi}_{m'}|_{\mathcal{V}_+} = \frac{1}{\sqrt{2\pi}\sqrt{2\omega_{m'}}r}\int_0^{\infty} F_{m'}(P)e^{-iP(c+A)}\sqrt{2p}\,\sqrt{2\pi}ru_m\frac{dP_m}{\sqrt{2\pi}} + \frac{1}{\sqrt{2\pi}\sqrt{2\omega_{m'}}r} \times\int_0^{\infty} F_{m'}(-P)e^{iP(c+A)}\sqrt{2p}\,\sqrt{2\pi}ru_m^{\dagger}\frac{dP_m}{\sqrt{2\pi}} . \tag{4.17}$$

Computing the Fourier transform of (4.14b) in (4.16a) and substituting into (4.17), the coefficients in (4.13) are found to be

$$A_{mm'} = \frac{-ie^{2\pi M\omega_{m'}}}{2\pi\sqrt{P_m\omega_{m'}}}4M^{-i4M\omega_{m'}} \times e^{iP_m c}P_m^{-i4M\omega_{m'}}\Gamma(1+i4M\omega_{m'}) , \tag{4.18a}$$

$$B_{mm'} = \frac{ie^{-2\pi M\omega_{m'}}}{2\pi\sqrt{P_m\omega_{m'}}}4M^{-i4M\omega_{m'}} \times e^{-iP_m c}P_m^{-i4M\omega_{m'}}\Gamma(1+i4M\omega_{m'}) . \tag{4.18b}$$

Direct substitution of (4.18) in (4.13) regains $\tilde{\varphi}_{m'}$ upon integration. This validates the use of the inversion theorem in (4.16).

C. Inverting the Fourier decomposition

The expressions (4.13) and (4.18) contain information on the inverse transformation for φ_m as well as the Bogoliubov transformation relating creation and annihilation operators. In order to obtain a rigorously derived result, one must introduce some mathematical abstraction. Quantum-field-theoretic expressions are most naturally manipulated if the wave functions are normalized. However, as discussed before in relation to (3.10), it is ambiguous to define a normalization for this problem.

The "global" wave functions with the normalizations in (4.3) are normalized on the manifold outside of the horizon on a $t=$const hypersurface [5]. De Witt argues this result by looking at very broad wave packets in the asymptotic regions where they originate in the very distant past. He then claims that if they are substituted into (3.10) and the integral is approximated in the limit of infinitely broad packets, one gets the appropriate normalization [5]

$$\langle \vec{\tilde{\varphi}}_{lm\omega_n}, \vec{\tilde{\varphi}}_{l'm'\omega_k}\rangle = \langle \overleftarrow{\tilde{\varphi}}_{lm\omega_n}, \overleftarrow{\tilde{\varphi}}_{l'm'\omega_k}\rangle = \delta_{ll'}\delta_{mm'}\delta(\omega_n-\omega_k) , \tag{4.19a}$$

$$\langle \overleftarrow{\tilde{\varphi}}_{lm\omega_n}, \vec{\tilde{\varphi}}_{l'm'\omega_k}\rangle = 0 . \tag{4.19b}$$

Implicit in this normalization argument is that any global outgoing wave observed by a PFF observer at $(\bar{X}^0,\bar{X}^1)$ can be thought of as originating at very large negative values of X^0 in the region where $X^1-X^0 \gtrsim c$.

One still does not have any idea whether the $\varphi_{\rm loc}$ are normalized. This problem can be circumvented by looking at a larger manifold M:

$$M = \{X^0,X^1,\theta,\phi \,|\, -\infty<X^0<\infty, -\infty<X^1<\infty, 0\le\theta\le\pi, 0\le\phi\le 2\pi\} . \tag{4.20a}$$

The region outside the horizon M_+ is given by

$$M_+ = \{X^0,X^1,\theta,\phi \,|\, X^1>X^0+c, 0\le\theta\le\pi, 0\le\phi\le 2\pi\} . \tag{4.20b}$$

The ambiguities of analytic continuation arguments (such as the future and past histories of the PFF observers and the hole) are avoided by saying that the manifold M is merely a mathematical construct of convenience. There may or may not be a physical significance to the space $M-\bar{M}_+$.

Consider the normalizable function ϕ_m defined on M [see (2.14) for the origin of the normalization factor]:

$$\phi_m(X^0,X^1,\theta,\phi)=Y_{lm}(\theta,\phi)\frac{e^{-iP_m(X^0-X^1)}}{(2M)^{1/4}r^{3/4}\sqrt{2\pi}\sqrt{2P_m}} . \quad (4.21a)$$

By (3.10) on an X^0=const hypersurface in M,

$$\langle \phi_m, \phi_n \rangle = \delta(P_m - P_n) , \quad (4.21b)$$

$$\langle \phi_m^\dagger, \phi_n^\dagger \rangle = -\delta(P_m - P_n) . \quad (4.21c)$$

ϕ_m is not a solution of the free-particle wave equation, (3.3), it is merely a mathematical construct. By (3.9), (3.7b), and (4.21a) for any $\varepsilon>0$ there exists an open set $\mathcal{V}_+$ such that

$$\left|\phi_m - \left[\frac{r}{2M}\right]^{1/4} u_m\right| < \varepsilon . \quad (4.22a)$$

So, one can say that ϕ_m is a good approximation to u_m and φ_m on the set $\mathcal{V}_+$ or u_m is the restriction of ϕ_m to $\mathcal{V}_+$:

$$\varphi_m \underset{r\to 2M}{\sim} \left[\frac{2M}{r}\right]^{1/4} \phi_m . \quad (4.22b)$$

Recall that $\tilde{u}_{m'}$ is valid only on $\mathcal{V}_+$. We extend $\tilde{u}_{m'}$ to all of M as $\tilde{\phi}_{m'}$ by using (4.13) and (4.18),

$$\tilde{\phi}_{m'} \equiv \int_0^\infty (A_{mm'}\phi_m + B_{mm'}\phi_m^\dagger) dP_m , \quad (4.23a)$$

and one has

$$\left[\frac{2M}{r}\right]^{1/4} \tilde{\phi}_{m'}|_{\mathcal{V}_+} \simeq \tilde{\varphi}_{m'}|_{\mathcal{V}_+} = \tilde{u}_{m'} . \quad (4.23b)$$

The extended function $\tilde{\phi}_{m'}$ may or may not be physical, but for this analysis it does not matter. $\tilde{\phi}_{m'}$ is merely a mathematical construct of convenience. In this new language (4.13) becomes

$$\left[\frac{2M}{r}\right]^{1/4} \tilde{\phi}_{m'}|_{\mathcal{V}_+} = \left\{ \int_0^\infty \left[A_{mm'} \left[\frac{2M}{r}\right]^{1/4} \phi_m + B_{mm'} \left[\frac{2M}{r}\right]^{1/4} \phi_m^\dagger \right] dP_m \right\} \bigg|_{\mathcal{V}_+} . \quad (4.23c)$$

If one uses (4.18), (4.21), and (4.23a) in (3.10), then, on any X^0=const hypersurface in M,

$$\langle \tilde{\phi}_{m'}, \tilde{\phi}_{n'} \rangle|_M = \delta(\omega_{m'} - \omega_{n'}) , \quad (4.24a)$$

$$\langle \tilde{\phi}_{m'}^\dagger, \tilde{\phi}_{n'}^\dagger \rangle|_M = -\delta(\omega_{m'} - \omega_{n'}) . \quad (4.24b)$$

The δ-function representation

$$\delta(x) = \frac{1}{2\pi}\int_0^\infty P^{-1\pm ix} dP \quad (4.25)$$

is useful in deriving (4.24).

Since the ϕ_m's have a δ-function normalization on spacelike hypersurfaces in M, the expansion (4.23a) can be written as

$$\tilde{\phi}_{m'} = \int_0^\infty dP_m [\phi_m \langle \phi_m, \tilde{\phi}_{m'} \rangle - \phi_m^\dagger \langle \phi_m^\dagger, \tilde{\phi}_{m'} \rangle] , \quad (4.26)$$

where inner products are taken on X^0=const hypersurfaces. Using (4.23a), one can make the identifications

$$\langle \phi_m, \tilde{\phi}_{m'} \rangle = A_{mm'} , \quad (4.27a)$$

$$\langle \phi_m^\dagger, \tilde{\phi}_{m'} \rangle = -B_{mm'} . \quad (4.27b)$$

One can use the symmetry of the inner product in (3.10) to establish

$$\langle A^*, B^* \rangle = -\langle \overline{A, B} \rangle = -\langle B, A \rangle . \quad (4.28)$$

From (4.27) and (4.28), one finds

$$\langle \tilde{\phi}_{m'}^\dagger, \phi_m \rangle = -\langle \phi_m^\dagger, \tilde{\phi}_{m'} \rangle = B_{mm'} , \quad (4.29a)$$

$$\langle \tilde{\phi}_{m'}^\dagger, \phi_m^\dagger \rangle = -\langle \phi_m, \tilde{\phi}_{m'} \rangle = -A_{mm'} . \quad (4.29b)$$

One can use (4.24) to derive an inverse relation to (4.23a). Since the $\tilde{\phi}_{m'}$'s have a δ-function normalization on spacelike hypersurfaces in M, there exists an expansion analogous to (4.26):

$$\phi_m = \int_0^\infty d\omega_{m'} [\tilde{\phi}_{m'} \langle \tilde{\phi}_{m'}, \phi_m \rangle - \tilde{\phi}_{m'}^\dagger \langle \tilde{\phi}_{m'}^\dagger, \phi_m \rangle] . \quad (4.30a)$$

Then using (4.29a) this can be put in the useful form

$$\phi_m = \int_0^\infty d\omega_{m'} [A_{mm'}^* \tilde{\phi}_{m'} - B_{mm'} \tilde{\phi}_{m'}^\dagger] . \quad (4.30b)$$

Finally, one can get the inverse to (4.13) by restricting the expression (4.30b) which is valid on all of M to the subset $\mathcal{V}_+$:

$$u_m = \int_0^\infty d\omega_{m'} [A_{mm'}^* \tilde{\varphi}_{m'} - B_{mm'} \tilde{\varphi}_{m'}^\dagger]|_{\mathcal{V}_+} . \quad (4.31)$$

This is the desired result. There is no further need for the abstract extended manifold M. The remainder of the discussion is restricted to M_+.

D. The Bogoliubov transformation

The local and "global" representations of the field are compared on $\mathcal{V}_+$ in order to determine the Bogoliubov transformation relating particle creation and annihilation operators. The static observers at asymptotic infinity expand this field $\tilde{\Phi}$ in the "global" modes:

$$\tilde{\Phi} = \sum_{lm} d\tilde{P}_{n'}^r \{ [\vec{\tilde{\varphi}}_{lmn'} \vec{\tilde{a}}_{lmn'} + (\vec{\tilde{\varphi}}_{lmn'})^\dagger (\vec{\tilde{a}}_{lmn'})^\dagger] \Theta(\tilde{P}_{n'}^r) + [\overleftarrow{\tilde{\varphi}}_{lmn'} \overleftarrow{\tilde{a}}_{lmn'} + (\overleftarrow{\tilde{\varphi}}_{lmn'})^\dagger (\overleftarrow{\tilde{a}}_{lmn'})^\dagger] \Theta(-\tilde{P}_{n'}^r) \} , \quad (4.32)$$

where the radial momentum $\tilde{P}_{n'}^r$ is determined by the quantum numbers l, m, and $\omega_{n'}$. Step functions were introduced to segregate the globally outgoing from globally ingoing modes for later convenience. Near the horizon, by (2.6) and (2.7),

$$|\tilde{P}_{n'}^r| \underset{r\to 2M}{\sim} \omega_{n'} \quad (4.33)$$

and $d\omega_{n'}$ can be substituted into (4.32) as the differential in the integrand for asymptotic expressions. The operators $(\vec{a}_{lmn'})^\dagger$ and $(\overleftarrow{a}_{lmn'})^\dagger$ create modes from the vacuum defined by the static observers at infinity, $|0_\infty\rangle$, that are outgoing with quantum numbers l, m, and $\omega_{n'}$ and ingoing with quantum numbers l, m, and $\omega_{n'}$, respectively. Similarly, $\vec{a}_{lmn'}$ and $\overleftarrow{a}_{lmn'}$ annihilate the static vacuum:

$$\vec{a}_{lmn'}|O_\infty\rangle=0\ , \tag{4.34a}$$

$$\overleftarrow{a}_{lmn'}|O_\infty\rangle=0\ . \tag{4.34b}$$

Analogously, the PFF observers describe the same field in terms of local modes

$$\Phi_{\rm loc}=\sum_{l,m}\int_{-\infty}^{\infty}dP_n^1\{[\vec{\varphi}_{lmn}\vec{a}_{lmn}+\vec{\varphi}_{lmn}^\dagger\vec{a}_{lmn}^\dagger]\Theta(P^1-P^0)+[\overleftarrow{\varphi}_{lmn}\overleftarrow{a}_{lmn}+\overleftarrow{\varphi}_{lmn}^\dagger\overleftarrow{a}_{lmn}^\dagger]\times\Theta(P^0-P^1)\}\ , \tag{4.35}$$

where (2.7) was used to produce the step functions in the expression required to segregate the solutions into the globally outgoing and globally ingoing subsets. The operators $\vec{a}_{lmn}^\dagger$ and $\overleftarrow{a}_{lmn}^\dagger$ create particles out of the local vacuum of the PFF observers, $|O_{\rm loc}\rangle$, with quantum numbers l, m, and P_n^0 which are globally outgoing and ingoing, respectively. Also,

$$\vec{a}_{lmn}|O_{\rm loc}\rangle=0\ , \tag{4.36a}$$

$$\overleftarrow{a}_{lmn}|O_{\rm loc}\rangle=0\ . \tag{4.36b}$$

Now, both representations of the field must agree on $\mathcal{V}_+$:

$$\Phi_{\rm loc}|_{\mathcal{V}_+}=\tilde{\Phi}|_{\mathcal{V}_+}\ . \tag{4.37}$$

By (4.1), (3.13), and (4.37) for the globally ingoing modes,

$$\overleftarrow{a}_{lmn'}=\int_0^\infty dP_n\sqrt{P_n/\omega_{n'}}\delta[\omega(P_n)-\omega_{n'}]\overleftarrow{a}_{lmn}\ , \tag{4.38}$$

where $\omega(P_n)$ is a constant defined via the inverse to (2.16). If one replaces φ_m by the approximation u_m in (4.35) and expands u_m as in (4.31), then substitution into (4.37) yields

$$\tilde{a}_{m'}=\int_0^\infty[A_{mm'}^*a_m-B_{mm'}^*a_m^\dagger]dP_m\ , \tag{4.39a}$$

$$\tilde{a}_{m'}^\dagger=\int_0^\infty[-B_{mm'}a_m+A_{mm'}a_m^\dagger]dP_m\ , \tag{4.39b}$$

where the arrows and l and m quantum numbers are dropped in the discussion of the outgoing modes as before. Similarly, if one substitutes the expansion for $\tilde{\varphi}_{m'}$, (4.13), into (4.32) and collects terms in (4.37),

$$a_m=\int_0^\infty[A_{mm'}\tilde{a}_{m'}+B_{mm'}^*\tilde{a}_{m'}^\dagger]d\omega_{m'}\ , \tag{4.39c}$$

$$a_m^\dagger=\int_0^\infty[B_{mm'}\tilde{a}_{m'}+A_{mm'}^*\tilde{a}_{m'}^\dagger]d\omega_{m'}\ . \tag{4.39d}$$

V. THE STRESS-ENERGY TENSOR OF THE FREELY FALLING VACUUM

In this section, the dynamic components of the stress-energy tensor of the freely falling vacuum state is evaluated, near the horizon, by different observers. One is then led naturally to the concept of the renormalized stress-energy tensor of the vacuum state transported by a PFF observer. In this discussion, being local in nature, there is no reference to scattering in the curvature potential. Thus, the physics of particle creation is easier to see and the "global" wave functions of (4.1) are appropriate.

A. The vacuum stress-energy tensor: A local evaluation

The globally interesting coordinates for analyzing the stress-energy tensor are the Schwarzschild coordinates. In particular, the components T_{tt}, T_{tr}, and T_{rr} are the dynamical components of Hawking radiation. On the other hand, computation in the local vacuum states is most naturally accomplished in a freely falling frame. Thus, we need the transformations

$$\langle O_{\rm loc}|T_{tt}|O_{\rm loc}\rangle=\langle O_{\rm loc}|T_{00}|O_{\rm loc}\rangle-2V^r\langle O_{\rm loc}|T_{0\rho}|O_{\rm loc}\rangle+(V^r)^2\langle O_{\rm loc}|T_{\rho\rho}|O_{\rm loc}\rangle\ , \tag{5.1a}$$

$$\langle O_{\rm loc}|T_{rt}|O_{\rm loc}\rangle=-\alpha^{-2}V^r\langle O_{\rm loc}|T_{00}|O_{\rm loc}\rangle+\alpha^{-2}[1+(V^r)^2]\langle O_{\rm loc}|T_{0\rho}|O_{\rm loc}\rangle-\alpha^{-2}V^r\langle O_{\rm loc}|T_{\rho\rho}|O_{\rm loc}\rangle\ , \tag{5.1b}$$

$$\langle O_{\rm loc}|T_{rr}|O_{\rm loc}\rangle=\alpha^{-4}(V^r)^2\langle O_{\rm loc}|T_{00}|O_{\rm loc}\rangle-2\alpha^{-4}V^r\langle O_{\rm loc}|T_{\rho 0}|O_{\rm loc}\rangle+\alpha^{-4}\langle O_{\rm loc}|T_{\rho\rho}|O_{\rm loc}\rangle\ . \tag{5.1c}$$

Near the horizon, one has

$$\langle O_{\rm loc}|T_{tt}|O_{\rm loc}\rangle\simeq\alpha^2\langle O_{\rm loc}|T_{tr}|O_{\rm loc}\rangle\simeq\alpha^4\langle O_{\rm loc}|T_{rr}|O_{\rm loc}\rangle\ . \tag{5.2}$$

The stress-energy tensor of a scalar field, Φ, in curved spacetime is

$$T_{ab}=\Phi_{;a}\Phi_{;b}-\tfrac{1}{2}g_{ab}[g^{cd}\Phi_{;c}\Phi_{;d}+m_e^2\Phi^2]\ . \tag{5.3}$$

The stress-energy tensor evaluated by a PFF observer is computed using local wave functions, denoted by

$$(T_{\mu\nu})_{\rm loc}\equiv T_{\mu\nu}(\varphi_m,\varphi_m)\ . \tag{5.4}$$

To find $\langle O_{\rm loc}|T_{tt}(\varphi_m,\varphi_m)|O_{\rm loc}\rangle$ one can substitute the expansion $\Phi_{\rm loc}$ of (4.35) into (5.3) and (5.1a):

$$\langle O_{\rm loc}|T_{tt}(\varphi_n,\varphi_n)|O_{\rm loc}\rangle|_{\mathcal{V}_+}=\frac{1}{16\pi^2r^2}\sum_{lm}\int_{-\infty}^{\infty}[(P_n^0)^2+2V^r(P_n^0P_n^\rho)+(V^r)^2(P_n^\rho)^2]\frac{dP_n^\rho}{P_n^0}\ . \tag{5.5}$$

There is a generally covariant measure in the momentum space in analogy with special relativity as a consequence of (2.1) and (2.5):

$$\frac{dP^{\rho}}{P^{0}}=\frac{d\tilde{P}^{r}}{\omega} . \tag{5.6}$$

From (2.6), one has

$$\omega=P^{0}+V^{r}P^{\rho} . \tag{5.7}$$

Combining (5.7), (5.6), and (5.5) yields

$$\langle O_{\rm loc}|T_{tt}(\varphi_m,\varphi_m)|O_{\rm loc}\rangle|_{\mathcal{V}_+}=\frac{1}{8\pi^2r^2}\sum_{lm}\int_{-\infty}^{\infty}\frac{\omega}{2}d\tilde{P}^{r} . \tag{5.8}$$

It is not a coincidence that one has the same result for the static observers at infinity in their vacuum state:

$$\langle O_{\infty}|T_{tt}(\tilde{\varphi}_{m'},\tilde{\varphi}_{m'})|O_{\infty}\rangle=\frac{1}{8\pi^2r^2}\sum_{lm}\int_{-\infty}^{\infty}\frac{\omega}{2}d\tilde{P}^{r} . \tag{5.9}$$

In (5.9), the notation $T_{\mu\nu}(\tilde{\varphi}_{m'},\tilde{\varphi}_{m'})$ signifies the stress-energy tensor evaluated by static observers at asymptotic infinity as a result of using the "global" wave functions. The computation is done by inserting the expression (4.32) for $\tilde{\Phi}$ into (5.3) and (5.1a).

One should note that, by (5.2), the values of

$$\langle O_{\rm loc}|T_{tr}(\varphi_m,\varphi_m)|O_{\rm loc}\rangle|_{\mathcal{V}_+}$$

and

$$\langle O_{\rm loc}|T_{rr}(\varphi_m,\varphi_m)|O_{\rm loc}\rangle|_{\mathcal{V}_+}$$

have been determined by (5.8) as well.

B. The stress-energy tensor of the local vacuum evaluated in the static frames

The stress-energy tensor of the local vacuum carried by a PFF observer as measured in the static frames at asymptotic infinity, $\langle O_{\rm loc}|T_{\mu\nu}(\tilde{\varphi}_{m'},\tilde{\varphi}_{m'})|O_{\rm loc}\rangle$, is calculated near the horizon. This computation requires using the field representation $\tilde{\Phi}$ of (4.32) in (5.3). However, this must be modified so that the creation and annihilation operators can act on the states belonging to the local number representation of the field. Thus, the Bogoliubov transformation, (4.39a) and (4.39b), must be used to reexpress these operators for the globally outgoing states. The ingoing states are trivial by (4.38). To utilize (5.1), the calculation is performed in the local basis using the simplifying form of $\tilde{\varphi}$ in (4.5):

$$\langle O_{\rm loc}|T_{00}(\tilde{\varphi}_{m'},\varphi_{m'})|O_{\rm loc}\rangle \underset{r\to 2M}{\sim} \frac{1}{16\pi^2r^2}\sum_{lm}\left[\int_{\tilde{P}_r<0}[P_0(\omega_{m'})]^2\frac{d\tilde{P}^{r}_{m'}}{\omega_{m'}}+\int_{\tilde{P}_r>0}\coth(4\pi M\omega_{m'})[P_0(\omega_{m'})]^2\frac{d\tilde{P}^{r}_{m'}}{\omega_{m'}}\right] , \tag{5.10a}$$

$$\langle O_{\rm loc}|T_{0\rho}(\tilde{\varphi}_{m'},\tilde{\varphi}_{m'})|O_{\rm loc}\rangle \underset{r\to 2M}{\sim} \frac{1}{16\pi^2r^2}\sum_{lm}\left[\int_{\tilde{P}_r<0}[P_0(\omega_{m'})P_\rho(\omega_{m'})]\frac{d\tilde{P}^{r}_{m'}}{\omega_{m'}}+\int_{\tilde{P}_r>0}\coth(4\pi M\omega_{m'})[P_0(\omega_{m'})P_\rho(\omega_{m'})]\frac{d\tilde{P}_{m'}}{\omega_{m'}}\right] , \tag{5.10b}$$

$$\langle O_{\rm loc}|T_{\rho\rho}(\tilde{\varphi}_{m'},\tilde{\varphi}_{m'})|O_{\rm loc}\rangle \underset{r\to 2M}{\sim} \frac{1}{16\pi^2r^2}\sum_{lm}\left[\int_{\tilde{P}_r<0}[P_\rho(\omega_{m'})]^2\frac{d\tilde{P}^{r}_{m'}}{\omega_{m'}}+\int_{\tilde{P}_r>0}\coth(4\pi M\omega_{m'})[P_\rho(\omega_{m'})]^2\frac{d\tilde{P}^{r}_{m'}}{\omega_{m'}}\right] . \tag{5.10c}$$

The notation $P_\mu(\omega_{m'})$ means to evaluate the local momentum P_μ from $\tilde{P}^{r}_{m'}$ and $\omega_{m'}$ using (2.1) and $\omega_{m'}$=const which typifies one of the wave functions of the static observers. Combining (5.1a) and (5.10) yields

$$\langle O_{\rm loc}|T_{tt}(\tilde{\varphi}_{m'},\tilde{\varphi}_{m'})|O_{\rm loc}\rangle|_{\mathcal{V}_+}=\frac{1}{16\pi^2r^2}\sum_{lm}\left[\int_{\tilde{P}_r<0}\omega\, d\tilde{P}^{r}+\int_{\tilde{P}_r<0}\omega\coth(4\pi M\omega)d\tilde{P}^{r}\right] . \tag{5.11}$$

Divergent results such as the one in (5.11) are ambiguous in quantum field theory. One way to regularize this quantity is to compute the normal-ordered operator with respect to the static observers at asymptotic infinity (i.e., normal order the $\tilde{a}$ and $\tilde{a}^{\dagger}$):

$$\langle O_{\rm loc}|:T_{\mu\nu}(\tilde{\varphi}_{m'},\tilde{\varphi}_{m'}):O_{\rm loc}\rangle=\langle O_{\rm loc}|T_{\mu\nu}(\tilde{\varphi}_{m'},\tilde{\varphi}_{m'})|O_{\rm loc}\rangle-\langle O_{\infty}|T_{\mu\nu}(\tilde{\varphi}_{m'},\tilde{\varphi}_{m'})|O_{\infty}\rangle . \tag{5.12a}$$

By (5.9) and (5.11) the normal-ordered expectation value is

$$\langle O_{\rm loc}|:T_{tt}(\tilde{\varphi}_{m'},\tilde{\varphi}_{m'}):|O_{\rm loc}\rangle|_{\mathcal{V}_+}=\sum_{lm}\frac{2}{16\pi^2r^2}\int_{\tilde{P}_r>0}\frac{\omega}{e^{8\pi M\omega}-1}d\tilde{P}^{r}\simeq\frac{1}{8\pi^2r^2}\sum_{lm}\int_{m_e}^{\infty}\frac{\omega}{e^{8\pi M\omega}-1}d\omega , \tag{5.12b}$$

where the last approximate equality follows from (2.6) and (2.7) evaluated as $\alpha \to 0$, as discussed in (4.33).

This result relates to Hawking's original calculation [4]. The number operator in the static frames $\tilde{N}(\omega_{m'})$ is

$$\tilde{N}(\omega_{m'}) \equiv \tilde{a}^{\dagger}_{m'} \tilde{a}_{m'} \ . \tag{5.13}$$

Computing the expectation value of $\tilde{N}$ in the freely falling vacuum state, using (4.39a) and (4.39b), yields

$$\langle O_{\rm loc} | \tilde{N}(\omega_{m'}) | O_{\rm loc} \rangle = \sum_{lm} \int_0^\infty dP_n |B_{nm'}|^2 \ . \tag{5.14a}$$

Thus, as seen from static infinity, there exists particles in the local vacuum. This is exactly the same result which Hawking found [4]. It can be shown that the energy density of these particles is the origin of the stress energy of the normal-ordered operator (5.12b). To understand this, Hawking's remark that the result is divergent is noted. Using (4.18) this is seen explicitly by expanding (5.14a) as

$$\langle O_{\rm loc} | \tilde{N}(\omega_{m'}) | O_{\rm loc} \rangle = \sum_{lm} \frac{1}{e^{8\pi M \omega} - 1} \left\{ \lim_{\omega_{m'} \to \omega_{n'}} \left[\int_0^\infty \frac{dP_n}{P_n} P_n^{-i4M(\omega_{m'} - \omega_{n'})} \right] \right\} \tag{5.14b}$$

or, using (4.25),

$$\langle O_{\rm loc} | \tilde{N}(\omega_{m'}) | O_{\rm loc} \rangle = \frac{1}{4\pi M} \sum_{lm} \frac{2\pi\delta(0)}{e^{8\pi M\omega} - 1} \ . \tag{5.14c}$$

The singular function $\delta(0)$ commonly results in quantum mechanics from the fact that infinite plane waves have an infinite norm [the δ-function normalization in (4.19)]. When the δ function of zero argument occurs, it can be reinterpreted as corresponding to a time rate of change of a quantity with respect to a time coordinate τ [25]:

$$\frac{dN}{d\tau} = \sum_{lm} \frac{1}{4\pi M} \frac{1}{e^{8\pi M\omega} - 1} \ . \tag{5.14d}$$

There is an ambiguity in this case in that one does not know how the relevant time coordinate τ is defined. The infinite value of $\tilde{N}$ in (5.14b) results from the value of $\langle \tilde{\varphi}_{m'}, \tilde{\varphi}_{m'} \rangle$ discussed in (4.19). At any point $r \gtrsim 2M$ there exists an infinite amount of support for $\tilde{\varphi}_{m'}$ between r and $2M$ [this is the asymptotic zone of wave origin referred to in Ref. [5] in relation to (4.19).] To an observer at asymptotic infinity, there always exists a PFF frame at r [however, it is the frame carried by a different freely falling observer for each value of time coordinate t]. Thus, as he looks at the local vacuum at the fixed point r, he will see the outgoing mode $\tilde{\varphi}_{m'}$ passing through the local coordinate patch about r for all times t due to the infinite support of $\tilde{\varphi}_{m'}$. He concludes that there are an infinite total number of particles in the local vacuum at r.

Hawking interpreted the fact that an infinite number of particles exist in the vacuum analogously to (5.14d). Particles are created continuously for an infinite amount of time [4]. He resolves the issue by constructing wave packets with a finite norm. A different tactic will be implemented here. An operator will be defined that separates out the effect of the norm of $\tilde{\varphi}_{m'}$ from $\tilde{N}$.

Define $\mathcal{N}(\omega_{m'})$ as

$$\mathcal{N}(\omega_{m'}) = \tfrac{1}{2} \lim_{\omega_{m'} \to \omega_{n'}} [\tilde{a}^{\dagger}_{n'} \tilde{a}_{m'} + \tilde{a}^{\dagger}_{m'} \tilde{a}_{n'}] \ . \tag{5.15a}$$

The two operators will have the same expectation values for any state vector A:

$$\langle A | \tilde{N}(\omega_{m'}) | A \rangle = \langle A | \mathcal{N}(\omega_{m'}) | A \rangle \ . \tag{5.15b}$$

Thus, using (4.39a), (4.39b), (4.19), and (5.15a),

$$\begin{aligned} \langle O_{\rm loc} | \tilde{N}(\omega_{m'}) | O_{\rm loc} \rangle &= \lim_{\omega_{m'} \to \omega_{n'}} \sum_{lm} \left[\frac{1}{e^{8\pi M\omega_{m'}} - 1} \right] \times \delta(\omega_{m'} - \omega_{n'}) \\ &= \lim_{\omega_{m'} \to \omega_{n'}} \sum_{lm} \left[\frac{\langle \tilde{\varphi}_{m'}, \tilde{\varphi}_{n'} \rangle}{e^{8\pi M\omega_{n'}} - 1} \right] \\ &= \sum_{lm} \frac{\|\tilde{\varphi}_{m'}\|^2}{e^{8\pi M\omega_{m'}} - 1} \ . \end{aligned} \tag{5.16}$$

Thus, (5.16) allows (5.12b) to be rewritten as

$$\langle O_{\rm loc} | {:}T_{tt}(\tilde{\varphi}_{m'}, \tilde{\varphi}_{m'}){:} | O_{\rm loc} \rangle |_{\mathcal{V}_+} = \frac{1}{8\pi^2 r^2} \int_{m_e}^\infty \omega_{m'} \frac{\langle O_{\rm loc} | \tilde{N}(\omega_{m'}) | O_{\rm loc} \rangle}{\|\tilde{\varphi}_{m'}\|^2} d\omega_{m'} \ . \tag{5.17a}$$

One can interpret (5.12a) as the energy density of the Hawking pairs in the local vacuum as seen by observers at static infinity at any instant of time [i.e., one can construct sharply peaked wave packets about each $\omega_{m'}$ with unit norm and substitute into (5.17a)]. The wave-packet interpretation of (5.17a) is equivalent to using the discrete representation of phase space defined in (5.26d). In that notation, one picks a spread in the energy which is small and centered on each discrete value of ω_k. It follows from (5.16), (5.26d), and (5.12b) that

$$\langle O_{\rm loc} | {:}T_{tt}(\tilde{\varphi}_{m'}, \tilde{\varphi}_{m'}){:} | O_{\rm loc} \rangle |_{\mathcal{V}_+} = \frac{1}{8\pi^2 r^2} \sum_{lm} \sum_{k} \langle O_{\rm loc} | \tilde{N}_{lm}(\omega_k) | O_{\rm loc} \rangle \omega_k \ . \tag{5.17b}$$

C. The renormalized stress-energy tensor

The renormalized stress-energy tensor of the freely falling vacuum is given by

$$\langle O_{\rm loc} | T_{\mu\nu} | O_{\rm loc} \rangle_{\rm ren} \equiv \langle O_{\rm loc} | T_{\mu\nu}(\varphi_{m'} \varphi_m) | O_{\rm loc} \rangle - \langle O_{\rm loc} | T_{\mu\nu}(\tilde{\varphi}_{m'}, \tilde{\varphi}_{m'}) | O_{\rm loc} \rangle \ . \tag{5.18}$$

When these local vacuum expectation values are pieced together in the global spacetime in the next section, one will get the dynamical components of the renormalized stress-energy tensor of spacetime discussed in Refs. [8, 18]. From (5.8), (5.11), and (5.18), one finds

$$\langle O_{\rm loc}|T_{tt}|O_{\rm loc}\rangle_{\rm ren}|_{\mathcal{V}_+} = -\frac{1}{8\pi^2 r^2}\sum_{lm}\int_{m_e}^{\infty}\frac{\omega\,d\omega}{e^{8\pi M\omega}-1} \ . \tag{5.19}$$

As it stands, the relation is infinite since there is an equal contribution in each channel (l,m). However, when (5.19) is utilized in piecing together spacetime to form $\langle T_{\mu\nu}\rangle_{\rm ren}$ of the global spacetime, the effects of reflection from the curvature potential get integrated into this expression. This will yield a finite result. If the scattering states of (4.4) were used in the analysis instead of those in (4.1), then these effects would already be built into a finite expression analogous to (5.19).

It is important to physically understand why (5.18) is the fundamental quantity representing the stress-energy tensor, globally, as opposed to another regularized tensor such as (5.12). First of all, it singles out the local vacuum as the only relevant vacuum state, as expected by the equivalence principle. Relation (5.18) can be thought of as being computed from local fields with a globally defined zero-point energy subtracted off (the second term). As such, it is a global measure which compares the vacuum energy density of the local field description at different points of spacetime.

One can be more specific about the physical nature of the tensor in (5.18). First, let us understand why the quantity in (5.19) is negative in terms of a qualitative description of (5.18) via the equivalence principle. Consider an observer who falls freely from $r \gg 2M$ to very close to the horizon, $\alpha \ll 1$. As the observer falls, he formulates a quantum scalar field theory. At each point of his world line $(\bar{X}^0,\bar{X}^1)$, he defines his local wave function $\varphi_{\rm loc}$ as

$$\varphi_{\rm loc}(\bar{X}^0,\bar{X}^1;X^1,X^0,P_n)Y_{lm} \equiv \varphi_{lmn} \ , \tag{5.20}$$

where P_n is the locally measured energy and (X^1,X^0) are the coordinates at which the wave function is evaluated. Near the horizon on the open set $\mathcal{V}_+$, these modes are characterized by

$$\dot{\varphi}_{\rm loc}(\bar{X}^0,\bar{X}^1;X^1,X^0,P_n)|_{\mathcal{V}_+} = \dot{u}_n \ , \tag{5.21a}$$

$$P_n^\mu = \text{const on } \mathcal{V}_+ \ , \tag{5.21b}$$

$$|P_n| \geq m_e c^2 \ . \tag{5.21c}$$

In general, one can construct an open neighborhood $\mathcal{V}(\bar{X}^0,\bar{X}^1)$ about the observer at any point in free fall where

$$\varphi_{\rm loc}(\bar{X}^0,\bar{X}^1;X^1,X^0,P_n^\mu)|_{\mathcal{V}} = \frac{1}{\sqrt{2\pi}\sqrt{2P_n^0}}\frac{1}{r}\exp[\pm i(P_n^0X^0-P_n^1X^1)] \ , \tag{5.21d}$$

$$P_n^\mu|_{\mathcal{V}} = \text{const} \ , \tag{5.21e}$$

$$|P_n^0| \geq m_e c^2 \ . \tag{5.21f}$$

This is not always easy to see from (3.3) since the coordinate X^1 is not well behaved as $r \to \infty$. These wave functions appear to be just like flat-space wave functions on $\mathcal{V}$ as seen by this freely falling observer. This observer also defines particle creation and annihilation operators and a local vacuum state which obey the usual relationship

$$\dot{a}_{lmn}(\bar{X}^0,\bar{X}^1)|O_{\rm loc}(\bar{X}^0,\bar{X}^1)\rangle = 0 \ , \tag{5.22a}$$

$$\dot{a}_{lmn}(\bar{X}^0,\bar{X}^1)|O_{\rm loc}(\bar{X}^0,\bar{X}^1)\rangle = 0 \ . \tag{5.22b}$$

All of the time that a PFF observer is in free fall, his local physics looks exactly the same in his local neighborhood as defined via (5.20)–(5.22). But this is not the case as viewed by an external observer.

The tensor in (5.18) is actually a measurement of the zero-point stress energy of the local fields defined in (5.20)–(5.22) at different points of spacetime, compared to a global standard. Now (5.8) and (5.9) tell us that the redshifted energy density of the zero-point oscillations of the field relative to the local vacuum is constant during free fall. Globally, this is not the whole story since $|0_{\rm loc}\rangle$ changes during free fall. First, it can be established that, in a global sense, the redshifted energy of the local zero-point oscillations decreases during free fall. When the local observer computes his zero-point energy, he has, for all $\bar{X}^0$ during free fall, the same energy spectrum as given by (5.21f) and the same uniform measure in the momentum space which is dual to his local Lorentz frame. By (2.17), the redshifted energy in a local mode, φ_n, decreases for each term in the zero-point energy corresponding to a globally outgoing mode near the horizon according to

$$\omega_n(P_n) = \tfrac{1}{2}\alpha^2 P_n \ . \tag{5.23}$$

Thus, redshifted energy is lost during free fall from the local zero-point oscillations as viewed globally. Clearly, (5.8) and (5.9) imply that the local vacuum state is losing energy during free fall in order to be consistent with (5.23). This can be thought of as the source of Hawking radiation.

Taking the limit at $r \to \infty$ of (5.18), one has

$$\lim_{r\to\infty}\langle O_{\rm loc}|T_{\mu\nu}|O_{\rm loc}\rangle_{\rm ren} = 0 \ . \tag{5.24}$$

Thus, the standard for the global comparison of the locally evaluated zero-point energy is the zero-point energy computed using the wave functions of a static observer at asymptotic infinity. Equations (5.23) and (5.18) show why

$$\lim_{r\to r_+}\langle O_{\rm loc}|T_{tt}|O_{\rm loc}\rangle_{\rm ren} < 0 \ .$$

However, as evidenced by (5.8) and (5.9), summing the energies in (5.23) is not the complete story since it does not contain any information as to how the local vacuum state evolves during free fall. Another reason why (5.23) is not in itself the essence of (5.19) is because the wave functions of the observers at asymptotic infinity look like wave packets to PFF observers near the horizon. Thus, there is no one-to-one identification of local and global wave functions. To proceed beyond this qualitative argument, one needs to average over this wave packet in order to reveal the physics in (5.19) more precisely. This motivates an analysis of the vacuum expectation value of the Hamiltonian. The discussion of (5.23) is made to show that the underlying physics is gravitational redshifting.

Consider a local mode with a redshifted energy ω_n derived from the local momentum of a PFF observer at $(\bar{X}^0,\bar{X}^1)$. At the point $(\bar{X}^0,\bar{X}^1)$, $\omega_n \equiv \omega_0$ [see (2.16)]:

$$\omega_n[P_n^\mu(\bar{X}^0,\bar{X}^1)]=\omega_0 \ . \tag{5.25}$$

One can compute the expectation value of the energy of this mode with respect to the local vacuum by introducing the Hamiltonian operator defined by a PFF observer, H:

$$H=\tfrac{1}{2}\sum_{lm}\int_{-\infty}^{\infty}(a_{lmn}a_{lmn}^\dagger+a_{lmn}^\dagger a_{lmn})P_n^0 dP_n^\rho \ . \tag{5.26a}$$

The eigenvalues of this operator are locally measured energies. To get the redshifted zero-point energy E_0 of the local modes with respect to the local vacuum, one must introduce the local momentum operator $\mathcal{P}^\rho$ in the local number representation

$$\mathcal{P}^\rho=\tfrac{1}{2}\sum_{lm}\int_{-\infty}^{\infty}(a_{lmn}a_{lmn}^\dagger+a_{lmn}^\dagger a_{lmn})P_n^\rho dP_n^\rho \ , \tag{5.26b}$$

and utilizing (5.7) yields

$$E_0=\langle O_{\rm loc}|H|O_{\rm loc}\rangle+V^r\langle O_{\rm loc}|\mathcal{P}^\rho|O_{\rm loc}\rangle \ . \tag{5.26c}$$

To evaluate E_0, one needs to deal with normalization issues which manifest themselves as $\langle O_{\rm loc}|a_m a_m^\dagger|O_{\rm loc}\rangle=\delta(0)$. Relying on the equivalence principle, this issue is resolved as in flat spacetime (see Ref. [22]). The phase space is discretized into cells (one dimensional), ΔP_n. The following modifications occur [22]:

$$\int dP_n^\rho \to \sum_n \Delta P_n, \quad \delta(P_n^\rho-P_m^\rho)\to\frac{\delta_{mn}}{\Delta P_n} \ , \qquad a_n \to \Delta P_n a_n \ . \tag{5.26d}$$

Making these substitutions into (5.26c) yields

$$E_0=\tfrac{1}{2}\sum_{lm}\sum_k \omega_k(P_k^\mu)\equiv\sum_{lm}\sum_k E_k \ . \tag{5.26e}$$

Thus, the zero-point redshifted energy in the mode defined in (5.25), E_n, is, as expected,

$$E_n=\tfrac{1}{2}\omega_0 \ . \tag{5.26f}$$

Now contrast this situation with the "global" mode with redshifted energy $\omega_{m'}=\omega_0$. The local observer can compute the expectation value of the redshifted energy of this mode in the local vacuum state $\langle\omega_{m'}\rangle$ by means of the Hamiltonian operator defined by the observers at static infinity, $\tilde{H}$:

$$\tilde{H}(\tilde{\varphi}_{n'})=\tfrac{1}{2}\sum_{lm}\int_{-\infty}^{\infty}\omega_{n'}[\tilde{a}_{lmn'}\tilde{a}_{lmn'}^\dagger+\tilde{a}_{lmn'}^\dagger\tilde{a}_{lmn'}]d\tilde{P}_{n'}^r \ . \tag{5.27}$$

As this stands, $\tilde{H}$ suffers from the same problems as the number operator in (5.13), arising from the infinite plane-wave normalization. This is resolved as in (5.15) by introducing a new operator:

$$\tilde{\mathcal{H}}\equiv\lim_{\omega_{m'}\to\omega_{n'}}\left\{\tfrac{1}{4}\sum_{lm}\int_{-\infty}^{\infty}\omega_{m'}[\tilde{a}_{lmn'}\tilde{a}_{lmn'}^\dagger+\tilde{a}_{lmn'}\tilde{a}_{lmm'}^\dagger+\tilde{a}_{lmm'}^\dagger\tilde{a}_{lmn'}+\tilde{a}_{lmn'}^\dagger a_{lmm'}]d\tilde{P}_{m'}^r\right\} \ . \tag{5.28a}$$

For any state vector A,

$$\langle A|\tilde{H}|A\rangle=\langle A|\tilde{\mathcal{H}}|A\rangle \ . \tag{5.28b}$$

Then, using (4.39a), (4.39b), (5.28a), and (5.28b),

$$\langle O_{\rm loc}|\tilde{H}(\varphi_{m'})|O_{\rm loc}\rangle \underset{r\to 2M}{\sim} \lim_{\omega_{m'}\to\omega_{n'}}\left\{\frac{1}{2}\sum_{lm}\int_{\tilde{P}_r>0}\coth(4\pi M\omega_{m'})\omega_{m'}\delta(\omega_{m'}-\omega_{n'})d\tilde{P}^r+\tfrac{1}{2}\sum_{lm}\int_{\tilde{P}_r<0}\omega_{m'}\delta(\omega_{m'}-\omega_{n'})d\tilde{P}_{m'}^r\right\} \ . \tag{5.29a}$$

Implementing discrete notation as in (5.26d) in the phase space of the static observers at infinity, (5.29a) becomes

$$\langle O_{\rm loc}|\tilde{H}(\varphi_{m'})|O_{\rm loc}\rangle\equiv\sum_{lm}\sum_{m'}\langle\omega_{m'}\rangle\underset{r\to 2M}{\sim}\tfrac{1}{2}\sum_{lm}\sum_{m'}\omega_{m'}[\coth(4\pi M\omega_{m'})\Theta(\tilde{P}_{m'}^r)+\Theta(-\tilde{P}_{m'}^r)] \ . \tag{5.29b}$$

A mode $\tilde{\varphi}_{m'}$ contributes an amount $\langle\omega_{m'}\rangle$ to the expectation value of the redshifted energy of the "global" solutions in the freely falling vacuum near the horizon. When $\omega_{m'}=\omega_0$, for the interesting globally outgoing solutions,

$$\langle\omega_{m'}\rangle=\tfrac{1}{2}\omega_0\coth(4\pi M\omega_0) \ . \tag{5.29c}$$

This energy is larger than the redshifted zero-point energy, (5.16f), of the local solution of (5.25). This difference in energies is the vacuum expectation of the normal-ordered operator $\tilde{H}$, $\langle O_{\rm loc}|{:}\tilde{H}{:}|O_{\rm loc}\rangle$, where normal ordering is with respect to the PFF observer. Denoting the normal-ordered energies as $\langle{:}\omega_{m'}{:}\rangle$, (5.26f) and (5.29c), implies

$$\langle{:}\omega_{m'}{:}\rangle=\frac{\omega_0}{e^{8\pi M\omega_0}-1}>0 \ . \tag{5.29d}$$

The result (5.29d) is a consequence of the differential blueshifting of the "global" solution in a neighborhood of the local observer at $(\bar{X}^0,\bar{X}^1)$. Energy-momentum operators are differential operators and therefore connect nearby points of spacetime. The results (5.29c) and (5.29d) represent an average value that takes into account the increase in locally evaluated energy and momentum due to blueshifting at points nearby to $(\bar{X}^0,\bar{X}^1)$.

From (5.29) and (5.26f), one can see that $\langle O_{\rm loc}|T_{\mu\nu}|O_{\rm loc}\rangle_{\rm ren}$, defined by (5.18), is a global measure of the redshifted energy momentum of the local zero-point oscillations of the field in the PFF frames. If one considers the observers near the horizon to have fallen from $r \gg 2M$ in the very distant past, then one can make the following observation. During free fall, by (5.23) or by comparing (5.26f) to (5.29c) [which is essentially (5.19)], the total redshifted energy of their zero-point oscillations decreases as viewed by an external, static observer. In particular,

$$\lim_{(\bar{X}^0,\bar{X}^1)\to(X^0,X^1)} \{\langle O_{\rm loc}|T_{tt}[\varphi_{\rm loc}(\bar{X}^0,\bar{X}^1;X^0,X^1),\varphi_{\rm loc}(\bar{X}^0,\bar{X}^1;X^0,X^1)]|O_{\rm loc}\rangle - \lim_{r(\bar{X}^0_\infty,\bar{X}^1_\infty)\to\infty} \langle O_{\rm loc}|T_{tt}[\varphi_{\rm loc}(\bar{X}^0_\infty,\bar{X}^1_\infty;X^0,X^1),\varphi_{\rm loc}(\bar{X}^0_\infty,\bar{X}^1_\infty;X^0,X^1)]|O_{\rm loc}\rangle\} < 0 . \quad (5.30)$$

Asymptotically, near the horizon, (5.30) is illustrated explicitly by (5.19). This statement was repeated in different forms throughout this section because this result is very significant since it was demonstrated in Ref. [19] that the Schwarzschild spacetime can be faithfully represented as the local neighborhoods of PFF observers pieced together in a trivial way.

VI. THE PHYSICS OF EVAPORATION

It was established in Sec. V that the redshifted energy of the zero-point oscillations in a freely falling frame decreases during free fall as viewed by an external, static onlooker. Consequently, the renormalized stress energy of the freely falling vacuum decreases during free fall from zero at $r\to\infty$ to negative values near the horizon. In this section it is illustrated how this effect is the essence of Hawking radiation.

The freely falling vacuum is "tied" to the set of freely falling observers. Since the energy density of the vacuum, near the horizon, is negative in a global context and it is flowing towards the hole with the PFF observers [see (5.1)], this is equivalent to an outgoing flux of positive energy (Hawking radiation). To understand the global phenomenon, it is necessary to piece together the local vacuum in a manner which must take into account the curvature potential.

A. Pair creation

It was determined that the energy density of the local vacuum decrease during free fall. One can show that the energy is radiated away in globally outgoing pairs. Consider (4.13), the positive frequency (defined locally) part of $\tilde{\varphi}_{m'}, \tilde{\varphi}^{(+)}_{m'}$, is

$$\tilde{\varphi}^{(+)}_{m'}|_{\mathcal{V}_+} = \int_0^\infty A_{mm'} u_m \, dP_m \, . \quad (6.1a)$$

Similarly, the negative frequency part $\tilde{\varphi}^{(-)}_{m'}$ is

$$\tilde{\varphi}^{(-)}_{m'}|_{\mathcal{V}_+} = \int_0^\infty B_{mm'} u^\dagger_m \, dP_m \, . \quad (6.1b)$$

Direct integration yields

$$\tilde{\varphi}^{(+)}_{m'}|_{\mathcal{V}_+} = \frac{e^{4\pi M\omega_{m'}}}{e^{4\pi M\omega_{m'}} - e^{-4\pi M\omega_{m'}}} \tilde{\varphi}_{m'} \Big|_{\mathcal{V}_+} , \quad (6.2a)$$

$$\tilde{\varphi}^{(-)}_{m'}|_{\mathcal{V}_+} = \frac{-e^{-4\pi M\omega_{m'}}}{e^{4\pi M\omega_{m'}} - e^{-4\pi M\omega_{m'}}} \tilde{\varphi}_{m'} \Big|_{\mathcal{V}_+} . \quad (6.2b)$$

Thus, one finds

$$\lim_{r\to 2M} \langle \tilde{\varphi}_{m'}, \tilde{\varphi}^{(+)}_{m'} \rangle = \frac{e^{4\pi M\omega_{m'}}}{e^{4\pi M\omega_{m'}} - e^{-4\pi M\omega_{m'}}} \langle \tilde{\varphi}_{m'}, \tilde{\varphi}_{m'} \rangle , \quad (6.3a)$$

$$\lim_{r\to 2M} \langle \tilde{\varphi}_{m'}, \tilde{\varphi}^{(-)}_{m'} \rangle = \frac{-e^{-4\pi M\omega_{m'}}}{e^{4\pi M\omega_{m'}} - e^{-4\pi M\omega_{m'}}} \langle \tilde{\varphi}_{m'}, \tilde{\varphi}_{m'} \rangle . \quad (6.3b)$$

Again, one is faced with the problem of interpreting the infinite plane-wave representation and its norm. As before, this is accomplished by making a norm-invariant statement. For each particle in the mode $\tilde{\varphi}_{m'}$, there is a conditional probability $P(\tilde{\varphi}^{(+)}_{m'}/\tilde{\varphi}_{m'})$ of the particle existing as a local positive frequency quanta:

$$\lim_{r\to 2M} P(\tilde{\varphi}^{(+)}_{m'}/\tilde{\varphi}_{m'}) = \frac{e^{4\pi M\omega_{m'}}}{e^{4\pi M\omega_{m'}} - e^{-4\pi M\omega_{m'}}} . \quad (6.4a)$$

There is also a conditional probability $P(\tilde{\varphi}^{(-)}_{m'}/\tilde{\varphi}_{m'})$ of the particle existing as a local negative energy state:

$$\lim_{r\to 2M} P(\tilde{\varphi}^{(-)}_{m'}/\tilde{\varphi}_{m'}) = \frac{e^{-4\pi M\omega_{m'}}}{e^{4\pi M\omega_{m'}} - e^{-4\pi M\omega_{m'}}} . \quad (6.4b)$$

The positive sign in the probability (6.4b) arises by reinterpreting the negative energy solutions in terms of antiparticles in the customary way [23].

Clearly when the PFF frame was released from infinity in the distant past, it was true that

$$\lim_{r\to\infty} \langle \tilde{\varphi}_{m'}, \tilde{\varphi}^{(+)}_{m'} \rangle = \langle \tilde{\varphi}_{m'}, \tilde{\varphi}_{m'} \rangle , \quad (6.5a)$$

$$\lim_{r\to\infty} \langle \tilde{\varphi}_{m'}, \tilde{\varphi}^{(-)}_{m'} \rangle = 0 . \quad (6.5b)$$

The solution appeared to have all of its support in the local positive frequency modes:

$$\lim_{r\to\infty} P(\tilde{\varphi}^{(+)}_{m'}/\tilde{\varphi}_{m'}) = 1 , \quad (6.6a)$$

$$\lim_{r\to\infty} P(\tilde{\varphi}^{(-)}_{m'}/\tilde{\varphi}_{m'}) = 0 . \quad (6.6b)$$

Combining (6.4) and (6.6), one has

$$\lim_{r\to 2M} P(\tilde{\varphi}_{m'}^{(+)}/\varphi_{m'}) - \lim_{r\to\infty} P(\tilde{\varphi}_{m'}^{(+)}/\varphi_{m'}) = \lim_{r\to 2M} P(\tilde{\varphi}_{m'}^{(-)}/\tilde{\varphi}_{m'}) - \lim_{r\to\infty} P(\tilde{\varphi}_{m'}^{(-)}/\tilde{\varphi}_{m'}) = \frac{1}{e^{8\pi M\omega_{m'}}-1} \ . \tag{6.7}$$

This means that the probability that a particle is created (as viewed by a static observer), as the local vacuum falls inward from infinity, is equally likely in the particle channels and antiparticle channels.

A question of interpretation arises at this juncture. Do these expressions imply that there are outgoing pairs or is there a negative energy particle flux, $w<0$, propagating toward the singularity which is balanced by an outgoing flux of Hawking radiation to infinity [29]? To answer this, one can examine the mathematical steps leading to (5.19). Firstly, since the negative energy of the vacuum in (5.19) is tied to the infalling motion of the PFF frames, one expects an outgoing redshifted energy flux. Proceeding as in the derivation of (5.19), Eqs. (5.1b), (5.10), and (5.18) imply

$$\langle O_{\text{loc}}|T_t{}^r|O_{\text{loc}}\rangle_{\text{ren}}|_{\mathcal{V}_+} = -\frac{1}{8\pi^2 r^2}\sum_{lm}\int_{m_e}^{\infty}\frac{\omega\, d\omega}{e^{8\pi M\omega}-1} \ . \tag{6.8a}$$

This outward flow of energy near the horizon is partially reflected and partially transmitted through the curvature potential as discussed in (4.4). Using the conservation of redshifted energy flux S^r [18],

$$S^r = -\int T_t{}^r\sqrt{-\tilde{g}}\, d\theta\, d\phi \ ,$$

one can find the energy flux at infinity, (6.16). Duly motivated, a spectral analysis of (6.8a) in terms of contributions from positive and negative local frequency channels is initiated.

If one inspects the steps used to generate (6.8a), the nonvanishing part of (5.18) comes from only the second (globally outgoing) terms in (5.10a)–(5.10c). The decomposition of the energy flux in (5.10) is analogous to that in (6.4). To see this, note the role of the Bogoliubov transformation, (4.39b), in deriving the second term in (5.10). The amplitude for a global particle creation operator to be a creation operator for local positive energy modes is given by $A_{mm'}$. Thus, one finds in deriving (5.10) that a portion of the energy flux

$$-\frac{1}{16\pi^2 r^2}\sum_{lm}\int_{m_e}^{\infty}\left[\frac{e^{4\pi M\omega_{m'}}}{e^{4\pi M\omega_{m'}}-e^{-4\pi M\omega_{m'}}}\right]\omega_{m'}d\omega_{m'} \tag{6.8b}$$

is in the form of locally positive energy modes. Similarly, the expression in square brackets is $P(\tilde{\varphi}_{m'}^{(+)}/\varphi_{m'})$ in (6.4a). Also, (4.39b) shows that a global particle creation operator has an amplitude $-B_{mm'}$ of annihilating a local positive energy state. However, anticipating an interest in non-Hermitian fields, this can be reinterpreted as is done for the Dirac field by calling a_m the creation operator for a negative energy state. In a complex field theory, such as the spin-$\frac{1}{2}$ field discussed in Appendix A, a_m would be replaced by d_m in expression (4.39b) [explicitly, see the complex conjugate of (A22a)], where d_m is the annihilation operator for antiparticles. In formulating Dirac field theory, d_m is initially interpreted as a creation operator for negative energy states and then is reinterpreted in terms of antiparticles [23]. Similar results hold for complex scalar fields. Thus, to elucidate the physics of complex fields, there is an amplitude $-B_{mm'}$ that a global particle creation operator is a creation operator for local negative energy states. The consistency of this interpretation is demonstrated by the amount of energy flux in (5.10), which is carried in the local negative energy channels,

$$-\frac{1}{16\pi^2 r^2}\sum_{lm}\int_{m_e}^{\infty}\left[\frac{e^{-4\pi M\omega_{m'}}}{e^{4\pi M\omega_{m'}}-e^{-4\pi M\omega_{m'}}}\right]\omega_{m'}d\omega_{m'} \ , \tag{6.8c}$$

and this quantity in square brackets is also $P(\tilde{\varphi}_{m'}^{(-)}/\tilde{\varphi}_{m'})$ in (6.4b).

The progression from (5.10) to (6.8a) is analogous to that from (6.4) to (6.7). The renormalized (physical) energy flux has two contributions near the horizon. Firstly, in the positive local energy channels there is a flux

$$-\frac{1}{16\pi^2 r^2}\sum_{lm}\int_{m_e}^{\infty}\left[\frac{e^{4\pi M\omega_{m'}}}{e^{4\pi M\omega_{m'}}-e^{-4\pi M\omega_{m'}}}-1\right]\omega_{m'}d\omega_{m'} \ . \tag{6.8d}$$

The second term in square brackets is the renormalization term which subtracts out the momentum flux due to the oscillations of the local field in the globally outgoing modes. Similarly, an equal portion of the flux in (6.8a) is transmitted through the local negative energy channels, (6.8c). Applying these results to (6.15) implies that equal amounts of energy flux reach infinity through the local positive and negative energy channels defined by PFF observers near the horizon. This seems most naturally interpreted as pair creation globally since P^0 and w have the same sign outside of the horizon by (2.16a).

Contrast this with the concept of negative energy ingoing modes pairing off with outgoing positive energy flux near the horizon. By (2.16a), outside of the horizon, the negative energy local components would be of negative energy globally as well, $w<0$. Even in Boulware's collapsing shell model, the ingoing solution has $w<0$ and, since the shell is outside of the horizon until the final stages of evaporation, $P^0<0$ in the region between the horizon and the shell [29]. The splitting of energy between a $w<0$ ingoing wave and a $w>0$ outgoing wave makes the interaction appear local in nature. (One should note the motivation of Boulware to put this scenario in perspective. He believed it was shown in Refs. [11, 28] that an eternal black hole does not radiate. Thus, Ref. [29] was an attempt to show that the energy flux develops as a result of the collapse, originating near the horizon. However, it is apparent from Ref. [8] and this article that the collapse need not be considered in order to understand black-hole radiance.)

Another attempt to elucidate the physics using negative energy waves was presented by Hawking [4]. He proposed that positive-negative energy virtual pairs can be created from the vacuum. The negative energy partners tunnel through the horizon where their locally evaluated energy is positive and they can exist as real quanta. The positive energy partners are radiated to infinity. This scenario is not equivalent to the treatment of the negative energy states just presented. If there is a local process of virtual pair creation near the horizon, then the equivalence principle demands that a consistent picture emerge in the PFF frames. It was shown in this analysis that local negative energy modes contribute to the Hawking flux, so far so good. However, P^0 is ultrarelativistic for the relevant outgoing modes. Thus, how do local vacuum fluctuations produce local modes of such high energy consistently with the uncertainty principle?

In the present pair creation interpretation, the interaction is a global phenomenon without an unambiguous local interpretation. In fact, it is necessary to acknowledge the free-fall history of the PFF frames which transport the local vacua. In essence, this analysis is a set of axioms for defining the vacuum of a field theory on a curved spacetime background via the equivalence principle. The result on a Schwarzschild background is that the vacuum state is dynamic by nature. There does not seem to be a local explanation such as virtual pairs separating near the horizon within the framework of this calculation.

It should be noted that a separate issue from the focus of this article is how the energy is transferred from inside the horizon to outside (the entire analysis was chosen to be outside the horizon for reasons which were stressed in the Introduction). As mentioned previously, this topic is looked at by Boulware and Hawking in Refs. [29,4], respectively. The analysis here does not preclude any of the possible means of energy flow inside of the horizon, whether it be carried by $w<0$ particles or by the gravitational field spontaneously decaying into Hawking particles (which seems plausible from the insights of York in Ref. [30]). It is quite likely that there is no unique interpretation of the physical situation inside of the horizon. If this energy flux is carried by $w<0$ particles inside of the horizon, due to their large local negative energy outside of the horizon (these are not antiparticles, they do not propagate outside of the horizon) they can only exist very close to the horizon and for a very short time. Thus, it seems clear that these are not the same as the $P^0<0$ modes which were discussed in terms of the spectrum of Hawking radiation since they are not restricted to the horizon.

In summary, the result (6.7) verifies Hawking's conjecture that energy is spontaneously created from the gravitational field in the form of particle-antiparticle pairs. Strictly speaking, this discussion has been developed with Hermitian fields for simplicity. Thus, the particles are their own antiparticles. However, by separating the positive and negative frequencies, the physics of a complex scalar field has been elucidated. Pair creation is more applicable for this theory and it is easily verified that (6.7) is the essence of Hawking radiation for complex fields. This is also true for fermions as discussed in the Appendices.

B. The renormalized stress-energy tensor of spacetime

The previous description of $\langle O_{\rm loc}|T_{\mu\nu}|O_{\rm loc}\rangle_{\rm ren}$ of the freely falling vacuum can be synthesized with the foliation of spacetime by PFF frames to find the renormalized stress-energy tensor of spacetime, $\langle T_{\mu\nu}\rangle_{\rm ren}$. The infall of the negative energy of the local vacuum is equivalent to an outflow of particle-antiparticle pairs as discussed in Sec. VI A. The question of whether this is an inflow of negative energy due to vacuum polarization as opposed to radiated particles is not well defined.

These pairs are generated during free fall, but predominantly the effect occurs as $\alpha\to 0$, near the horizon. The pairs have an amplitude to reflect from and be transmitted through the curvature potential of spacetime. As in (4.4), the reflection coefficient is designated as $A_l(\omega)$ (it is independent of the azimuthal quantum number). Thus, near the horizon there are two components of the radial momentum flux: one results from the bulk motion of the negative energy density of the freely falling vacuum; the other is the ingoing flux of the reflected pairs. Based on the discussion of the origin of the pairs in (6.7) and the explicit form of (5.19), energy conservation requires that the momentum flux of the reflected pairs, $\langle T_{\mu\nu}\rangle_{\rm ref}$, satisfies

$$\langle T_{\rm tr}\rangle_{\rm ref}\underset{r\to 2M}{\sim}\sum_l\int_0^\infty \frac{2l+1}{8\pi^2 r^2}|A_l(\omega)|^2\frac{\omega\, d\omega}{e^{8\pi M\omega}-1}\alpha^{-2}\ . \tag{6.9}$$

The renormalized stress-energy tensor of spacetime near the horizon is

$$\lim_{r\to 2M}\langle T_{\mu\nu}\rangle_{\rm ren}=\langle O_{\rm loc}|T_{\mu\nu}|O_{\rm loc}\rangle_{\rm ren}+\langle T_{\mu\nu}\rangle_{\rm ref}\ . \tag{6.10}$$

In general,

$$\langle T_{\mu\nu}\rangle_{\rm ren}=\langle O_{\rm loc}|T_{\mu\nu}|O_{\rm loc}\rangle_{\rm ren}+\langle T_{\mu\nu}\rangle_{\rm rad}\ , \tag{6.11}$$

where $\langle T_{\mu\nu}\rangle_{\rm rad}$ is the stress-energy tensor of the radiated stream of Hawking pairs. The decomposition in (6.11) is only obvious in the two asymptotic zones.

Using (4.4d), (6.9), and (6.10),

$$\langle T_{tr}\rangle_{\rm ren}\underset{r\to 2M}{\sim}-\frac{\alpha^2}{8\pi^2 r^2}\sum_l(2l+1)\int_0^\infty\frac{|B_l(\omega)|^2\omega\, d\omega}{e^{8\pi M\omega}-1}\ . \tag{6.12}$$

The reflected pairs form a highly relativistic stream so the energy density in the static basis satisfies

$$\lim_{r\to 2M}\langle T_{tt}\rangle_{\rm ref}=\lim_{r\to 2M}\langle T_{\rm tr}\rangle_{\rm ref}\alpha^2\ . \tag{6.13}$$

Thus, in analogy with (6.12),

$$\langle T_{tt}\rangle_{\rm ren}\underset{r\to 2M}{\sim}\frac{-1}{8\pi^2 r^2}\sum_l(2l+1)\int_0^\infty\frac{|B_l(\omega)|^2\omega\, d\omega}{e^{8\pi M\omega}-1}\ . \tag{6.14}$$

The asymptotic form of $\langle T_{rr}\rangle_{\rm ren}$ follows (5.2). Equations (6.12) and (6.14) reproduce the results of Ref. [8] which were based on a regularized stress-energy tensor derived by using point-separation techniques.

One should note that, if the scattering states of (4.4) were used as the set of basis functions defined by the static observers at asymptotic infinity, then one would have derived (6.12) and (6.14) instead of (5.19) in the discussion of $\langle O_{\rm loc}|T_{\mu\nu}|O_{\rm loc}\rangle_{\rm ren}$.

For the sake of completeness, this same technique can be used to compute $\lim_{r\to\infty}\langle T_{\mu\nu}\rangle_{\rm ren}$. By (5.24), the redshifted energy density of the PFF vacuum vanishes as $r\to\infty$ (as does the infall velocity, V^r). This simplifies (6.11) to be

$$\langle T_{\mu\nu}\rangle_{\rm ren} \underset{r\to\infty}{\sim} \langle T_{\mu\nu}\rangle_{\rm trans} , \tag{6.15}$$

where $\langle T_{\mu\nu}\rangle_{\rm trans}$ is the stress-energy tensor of the radiated pairs which have been transmitted through the curvature potential. Using (5.19), (4.4d), and the conservation of energy,

$$\langle T_\mu{}^\nu\rangle_{\rm ren} \underset{r\to\infty}{\sim} \frac{1}{8\pi^2 r^2}\sum_l (2l+1) \times\int_0^\infty \frac{|B_l(\omega)|^2\omega\, d\omega}{e^{8\pi M\omega}-1} \times\begin{bmatrix} -1 & -1 & 0 & 0\\ 1 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} . \tag{6.16}$$

APPENDIX A: SPIN-$\frac{1}{2}$ EVAPORATION

For the sake of completeness, the spin-$\frac{1}{2}$ field in the Schwarzschild geometry is studied. This appendix is a sketch of the steps paralleled in the text for the scalar field.

1. Local representation of Dirac algebra

The Dirac matrices satisfy

$$\tfrac{1}{2}\{\gamma^\mu,\gamma^\nu\} = -g^{\mu\nu} . \tag{A1}$$

A convenient representation for local computation is therefore

$$\gamma^0 = \begin{bmatrix} I & 0\\ 0 & -I\end{bmatrix} , \tag{A2a}$$

$$\gamma^1 = -\frac{1}{V^r}\begin{bmatrix} 0 & \sigma^z\\ -\sigma^z & 0\end{bmatrix} = \sqrt{r/2M}\begin{bmatrix} 0 & \sigma^z\\ -\sigma^z & 0\end{bmatrix} , \tag{A2b}$$

$$\gamma^\phi = \frac{1}{r}\begin{bmatrix} 0 & \sigma^1\\ -\sigma^1 & 0\end{bmatrix} , \tag{A2c}$$

$$\gamma^\theta = \frac{1}{r\sin\theta}\begin{bmatrix} 0 & \sigma^2\\ -\sigma^2 & 0\end{bmatrix} . \tag{A2d}$$

2. Helicity representation of the field

To streamline the discussion, the analysis is restricted to the globally outgoing modes (as these contain the interesting physics). These modes appear ultrarelativistic as viewed by PFF observers near the horizon. The spin operator on a spacelike hypersurface is (ε_{ijk} is the completely antisymmetric tensor)

$$S_i = \frac{i}{4}\sqrt{g^{(3)}}\varepsilon_{ijk}[\gamma^j,\gamma^k] . \tag{A3}$$

The helicity operator is

$$\Lambda_s = \frac{\hbar}{2}\vec{S}\cdot\frac{\vec{P}}{|\vec{P}|} . \tag{A4a}$$

Near the horizon $P^i \simeq P^1$ and $\tilde{P}^i \simeq \tilde{P}^r$; thus, in both frames the helicity operator is the same as $r\to 2M$,

$$\Lambda_s \simeq \frac{\hbar}{2}\begin{bmatrix} \sigma_z & 0\\ 0 & \sigma_z\end{bmatrix} . \tag{A4b}$$

The eigenvectors of the helicity operator are

$$\begin{bmatrix} u_1\\ 0\end{bmatrix},\quad \begin{bmatrix} u_{-1}\\ 0\end{bmatrix},\quad \begin{bmatrix} 0\\ u_1\end{bmatrix},\quad \begin{bmatrix} 0\\ u_{-1}\end{bmatrix} , \tag{A5a}$$

where

$$u_1 = \begin{bmatrix} 1\\ 0\end{bmatrix},\quad u_{-1} = \begin{bmatrix} 0\\ 1\end{bmatrix} . \tag{A5b}$$

The subscript ± 1 refers to the helicity. In analogy to flat space, a local solution of the Dirac equation is (see Appendix B)

$$\psi^{(\lambda)}_{\sigma|_{\mathcal{V}_+}} = N_\lambda \frac{\mathcal{Y}_{lm}(\theta,\phi)}{\sqrt{2\pi}r}\mu_{\sigma\lambda}e^{-\lambda P_\alpha X^\alpha} , \tag{A6a}$$

σ is the helicity and λ is the sign of the local energy ($\alpha=0,1$). The function $\mathcal{Y}_{lm}$ is the appropriate spin-weighted spherical harmonic, chosen to have the same normalization as the spherical harmonics used in the main body of the work [17]. The normalization is given by its flat-spacetime value [23]

$$N_\lambda = \left[\frac{m_e c^2 + \lambda P^0}{2\lambda P^0}\right]^{1/2} \simeq \frac{1}{\sqrt{2}} . \tag{A6b}$$

The helicity states are boosted into the form

$$u_{\sigma\lambda} = \begin{bmatrix} \mu_\sigma\\ \dfrac{c\sigma_z P^1}{mc^2+\lambda|P^0|}\end{bmatrix},\quad \tilde{u}_{\sigma\lambda} \underset{r\to 2M}{\sim} \begin{bmatrix} u_\sigma\\ \dfrac{c\sigma_z\tilde{P}_r}{\lambda|\tilde{P}^t|}u_\sigma\end{bmatrix} , \tag{A6c}$$

where the u_σ are defined in (A5b). Thus,

$$u_{++} = \begin{bmatrix} 1\\ 0\\ 1\\ 0\end{bmatrix} \equiv \mu_+(P^1>0) \equiv \nu_+(P^1<0) , \tag{A6d}$$

$$u_{-+} = \begin{bmatrix} 0\\ 1\\ 0\\ -1\end{bmatrix} \equiv \mu_-(P^1>0) \equiv \nu_-(P^1<0) , \tag{A6e}$$

$$u_{+-} = \begin{bmatrix} 1\\ 0\\ -1\\ 0\end{bmatrix} \equiv \nu_+(P^1>0) \equiv \mu_+(P^1<0) , \tag{A6f}$$

$$u_{--}=\begin{bmatrix}0\\1\\0\\1\end{bmatrix}\equiv\nu_-(P^1>0)\equiv\mu_-(P^1<0)\ . \tag{A6g}$$

Define the positive energy wave functions as

$$(\psi_{lmn}^{(+)})_a|_{\mathcal{V}_+}=\frac{1}{\sqrt{2}\sqrt{2\pi}r}\mathcal{Y}_{lm}(\theta,\phi)\mu_a \times\exp[-iP^0(X^0-X^1)]\ , \tag{A7a}$$

and, similarly, the negative energy wave function is

$$(\psi_{lmn}^{(-)})_a|_{\mathcal{V}_+}=\frac{1}{\sqrt{2}\sqrt{2\pi}r}\mathcal{Y}_{lm}(\theta,\phi)\nu_a\exp[iP^0(X^0-X^1)]\ . \tag{A7b}$$

The field is defined by

$$\Psi=\sum_{(a)}\sum_{lm}\int_{-\infty}^{\infty}dP_n^1[(\psi_{lmn}^{(+)})_a(b_{lmn})_a +(\psi_{lmn}^{(-)})_a(d_{lmn}^\dagger)_a]\ , \tag{A8}$$

where the globally ingoing states can be trivially incorporated into the notation.

3. Global representation of the field

In analogy with (A6a), the "global" solutions satisfy [17]

$$\tilde{\psi}_\sigma^{(\lambda)}\underset{r\to 2M}{\sim}\tilde{N}_\lambda\frac{\mathcal{Y}_{lm}(\theta,\phi)}{\sqrt{2\pi}r}\tilde{\mu}_{\sigma\lambda}e^{-i\lambda\omega u}\ . \tag{A9}$$

The helicity states $\tilde{\mu}_{\sigma\lambda}$ are the same as in (A6). To find the normalization, one notes that, on a $t=$const hypersurface,

$$\langle\psi_1,\psi_2\rangle=\int\bar{\psi}_1\gamma^\mu\psi_2\cdot d\mathbf{\Sigma}^{(3)}\ ,\quad \bar{\psi}_1=\psi_1^\dagger\gamma^0\ , \tag{A10a}$$

where $d\mathbf{\Sigma}^{(3)}$ is the volume element of a $t=$const hypersurface:

$$d\mathbf{\Sigma}^{(3)}=\hat{n}\sqrt{g^{(3)}}dr\,d\theta\,d\phi\ . \tag{A10b}$$

The unit normal is

$$\hat{n}=\frac{1}{(g^{tt})^{1/2}}\frac{\partial}{\partial t} \tag{A10c}$$

and the volume measure in the hypersurface is

$$\sqrt{g^{(3)}}=\sqrt{g_{rr}}r^2\sin\theta\ . \tag{A10d}$$

In the local representation of the γ matrices of (A2):

$$\gamma^t=\alpha^{-2}[\gamma^0-(V^r)^2\gamma^1]\ , \tag{A11a}$$

$$\gamma^0\gamma^t=\alpha^{-2}\begin{bmatrix}1&0&V^r&0\\0&1&0&-V^r\\V^r&0&1&0\\0&-V^r&0&1\end{bmatrix}\ . \tag{A11b}$$

The relevant inner products of the helicity states in the static frames are tabulated in (A26). Using (A26), (A10), and (A11), a normalization analogous to (4.19) for the physically interesting states ($\omega>0$, $\tilde{P}^r>0$ or $\omega<0$, $\tilde{P}^r<0$) is

$$\tilde{N}_\lambda=\alpha^{-1}\ . \tag{A12}$$

In analogy with (A7),

$$[\tilde{\psi}_{lmn'}^{(+)}(\tilde{P}^r>0)]_a\underset{r\to 2M}{\sim}\frac{\sqrt{4M}}{\sqrt{2\pi}r}\frac{\mathcal{Y}_{lm}(\theta,\phi)}{[X^1-X^0-c]^{1/2}}\tilde{\mu}_a \times\exp\left\{i4M\omega_{n'}\ln\left[\frac{X^1-X^0-c}{4M}\right]\right\}\ , \tag{A13a}$$

$$[\tilde{\psi}_{lmn'}^{(-)}(\tilde{P}^r<0)]_a\underset{r\to 2M}{\sim}\frac{\sqrt{4M}}{\sqrt{2\pi}r}\frac{\mathcal{Y}_{lm}^*(\theta,\phi)}{[X^1-X^0-c]^{1/2}}\tilde{\nu}_a \times\exp\left\{-i4M\omega_{n'}\times\ln\left[\frac{X^1-X^0-c}{4M}\right]\right\}\ . \tag{A13b}$$

The representation of the Dirac field as viewed by static observers at asymptotic infinity is

$$\tilde{\Psi}=\sum_a\sum_{lm}\int_{-\infty}^{\infty}d\tilde{P}_{n'}^r[(\tilde{\psi}_{lmn'}^{(+)})_a(\tilde{b}_{lmn'})_a +(\tilde{\psi}_{lmn'}^{(-)})_a(\tilde{d}_{lmn'}^\dagger)_a]\ . \tag{A14}$$

4. The local Fourier decomposition

To streamline the notation, the l and m quantum numbers are dropped in the expressions. In analogy with (4.14), one can write

$$(\tilde{\psi}_{m'}^{(+)})_a=\frac{\sqrt{4M}}{\sqrt{2\pi}r}R_{m'}(X^0,X^1)\ , \tag{A15a}$$

$$R_{m'}(X^0,X^1) =\tilde{\mu}_a(\omega_{m'})\frac{\exp\{i4M\omega_{m'}\ln[(X^1-X^0-c)/4M]\}}{[X^1-X^0-c]^{1/2}}\ . \tag{A15b}$$

Proceeding in direct analogy to (4.14)–(4.18), we find, noting that

$$\mu_a(-\vec{P})=\nu_a(\vec{P})\ ,$$

$$(\tilde{\psi}_{m'}^{(+)})_a|_{\mathcal{V}_+}=\int_0^\infty dP_m[A_{mm'}(\psi_m^{(+)})_a+B_{mm'}(\psi_m^{(-)})_a]|_{\mathcal{V}_+}\ , \tag{A16a}$$

$$A_{mm'}=(1-i)\frac{e^{2\pi M\omega_{m'}}}{2\pi\sqrt{P_m}}e^{iP_mc}P_m^{-i4M\omega_{m'}} \times 4M^{1/2-i4M\omega_{m'}}\Gamma(\tfrac{1}{2}+i4M\omega_{m'})\ , \tag{A16b}$$

$$B_{mm'}=(i-1)\frac{e^{-2\pi M\omega_{m'}}}{2\pi\sqrt{P_m}}e^{-iP_m c}P_m^{-i4M\omega_{m'}}$$
$$\times 4M^{1/2-i4M\omega_{m'}}\Gamma(\tfrac{1}{2}+i4M\omega_{m'}) . \quad \text{(A16c)}$$

5. Inverting the Fourier decomposition

As in (4.21), one can define a normalized spinor valued function on the abstract manifold M:

$$(\Psi^{(+)}_{lmn})_a(X^0,X^1,\theta,\phi)$$
$$=\mathcal{Y}_{lm}(\theta,\phi)\mu_a\frac{e^{-iP_n(X^0-X^1)}}{(2M)^{1/4}r^{3/4}\sqrt{2\pi}\sqrt{2}} , \quad \text{(A17a)}$$

$$\langle(\Psi^{(+)}_m)_a,(\Psi^{(+)}_n)_b\rangle=\delta_{ab}\delta(P_m-P_n) , \quad \text{(A17b)}$$

$$\langle(\Psi^{(-)}_m)_a,(\Psi^{(-)}_n)_b\rangle=\delta_{ab}\delta(P_m-P_n) , \quad \text{(A17c)}$$

and

$$(\psi^{(+)}_m)_a|_{\mathcal{V}_+}\simeq\left[\frac{2M}{r}\right]^{1/4}(\Psi^{(+)}_m)_a . \quad \text{(A17d)}$$

The spinor valued function $(\tilde{\psi}^{(+)}_{m'})_a$ is extended as in (4.23a):

$$(\tilde{\Psi}^{(+)}_{m'})_a=\int_0^\infty[A_{mm'}(\Psi^{(+)}_m)_a+B_{mm'}(\Psi^{(-)}_m)_a]dP_m \quad \text{(A18a)}$$

and

$$(\tilde{\psi}^{(+)}_{m'})_a|_{\mathcal{V}_+}\simeq\left[\frac{2M}{r}\right]^{1/4}(\tilde{\Psi}^{(+)}_{m'})_a . \quad \text{(A18b)}$$

One can show by direct integration that the useful normalization exists on spacelike hypersurfaces in M:

$$\langle(\tilde{\Psi}^{(+)}_{m'})_a,(\tilde{\Psi}^{(+)}_{n'})_b\rangle=\delta_{ab}\delta(\omega_{m'}-\omega_{n'}) , \quad \text{(A18c)}$$

$$\langle(\tilde{\Psi}^{(-)}_{m'})_a,(\tilde{\Psi}^{(-)}_{n'})_b\rangle=\delta_{ab}\delta(\omega_{m'}-\omega_{n'}) . \quad \text{(A18d)}$$

Rewriting (A18a) as

$$(\tilde{\Psi}^{(+)}_{m'})_a=\sum_b\int_0^\infty[(\Psi^{(+)}_m)_b\langle(\Psi^{(+)}_m)_b,(\tilde{\Psi}^{(+)}_{m'})_a\rangle$$
$$+(\Psi^{(-)}_m)_b\langle(\Psi^{(-)}_m)_b,(\tilde{\Psi}^{(+)}_{m'})_b\rangle]dP_m , \quad \text{(A19a)}$$

one establishes from (A18a) and (A19a) that

$$\langle(\Psi^{(+)}_m)_b,(\tilde{\Psi}^{(+)}_{m'})_a\rangle=A_{mm'}\delta_{ab} , \quad \text{(A19b)}$$

$$\langle(\Psi^{(-)}_m)_b,(\tilde{\Psi}^{(+)}_{m'})_a\rangle=B_{mm'}\delta_{ab} . \quad \text{(A19c)}$$

Using the symmetries of the inner product,

$$\langle(\tilde{\Psi}^{(-)}_{m'})_a,(\Psi^{(+)}_m)_b\rangle=B_{mm'}\delta_{ab} , \quad \text{(A19d)}$$

$$\langle(\tilde{\Psi}^{(-)}_{m'})_a,(\Psi^{(-)}_m)_b\rangle=\langle(\Psi^{(+)}_m)_a,(\tilde{\Psi}^{(+)}_{m'})_b\rangle$$
$$=A^*_{mm'}\delta_{ab} . \quad \text{(A19e)}$$

Then, expanding $(\Psi^{(+)}_m)_a$ as in (A19a), one can use (A19d) and (A19e) to find the inverse of (A16a):

$$(\psi^{(+)}_m)_a|_{\mathcal{V}_+}=\int_0^\infty d\omega_{m'}[A^*_{mm'}(\tilde{\psi}^{(+)}_{m'})_a+B_{mm'}(\tilde{\psi}^{(-)}_{m'})_a]|_{\mathcal{V}_+} . \quad \text{(A20)}$$

6. The Bogoliubov transformation

By setting

$$\Psi|_{\mathcal{V}_+}=\tilde{\Psi}|_{\mathcal{V}_+} \quad \text{(A21)}$$

as in (4.37), one finds

$$\tilde{b}_{m'}=\int_0^\infty[A^*_{mm'}b_m+B^*_{mm'}d^\dagger_m]dP_m , \quad \text{(A22a)}$$

$$\tilde{d}^\dagger_{m'}=\int_0^\infty[A_{mm'}d^\dagger_m+B_{mm'}b_m]dP_m . \quad \text{(A22b)}$$

7. The stress-energy tensor

The stress-energy tensor for the Dirac field is [26]

$$T_{\mu\nu}=\frac{i}{4}[\bar{\psi}\gamma_\mu\nabla_\nu\psi+\bar{\psi}\gamma_\nu\nabla_\mu\psi$$
$$-(\nabla_\mu\bar{\psi})\gamma_\nu\psi-(\nabla_\nu\bar{\psi})\gamma_\mu\psi] , \quad \text{(A23)}$$

where ∇_ν is the covariant derivative defined in (B1) and (B2). The connection terms in (B7) are such that their contributions to the stress-energy tensor vanish in the calculations of the dynamic components, T_{00}, $T_{0\rho}$, and $T_{\rho\rho}$.

By direct substitution, one finds, in analogy with (5.8),

$$\langle O_{\rm loc}|T_{tt}(\psi_m,\psi_m)|O_{\rm loc}\rangle|_{\mathcal{V}_+}=-\frac{1}{4\pi^2r^2}\sum_{lm}\int_{-\infty}^{\infty}\omega\, d\tilde{P}^r \quad \text{(A24)}$$

and

$$\langle O_\infty|T_{tt}(\tilde{\psi}_{m'},\tilde{\psi}_{m'})|O_\infty\rangle=-\frac{1}{4\pi^2r^2}\sum_{lm}\int_{-\infty}^{\infty}\omega\, d\tilde{P}^r . \quad \text{(A25)}$$

Note that these zero-point energies are negative.

Using (A16) and (A22), one can calculate in parallel with (5.10) to find a result such as (5.11). The result is complicated by the spin sums, which are tabulated for $r\simeq 2M$:

$$\bar{\mu}_+\gamma^0\gamma_t\bar{\mu}_+=\begin{cases}-\alpha^2, & \tilde{P}^r>0,\\ -4, & \tilde{P}^r<0,\end{cases} \quad \text{(A26a)}$$

$$\bar{\mu}_-\gamma^0\gamma_t\bar{\mu}_-=\begin{cases}-\alpha^2, & \tilde{P}^r>0,\\ -4, & \tilde{P}^r<0,\end{cases} \quad \text{(A26b)}$$

$$\bar{\nu}_+\gamma^0\gamma_t\bar{\nu}_+=\begin{cases}-4, & \tilde{P}^r>0,\\ -\alpha^2, & \tilde{P}^r<0,\end{cases} \quad \text{(A26c)}$$

$$\bar{\nu}_-\gamma^0\gamma_t\bar{\nu}_-=\begin{cases}-4, & \tilde{P}^r>0,\\ -\alpha^2, & \tilde{P}^r<0.\end{cases} \quad \text{(A26d)}$$

One also needs [27]

$$\Gamma(\tfrac{1}{2}+iy)\Gamma(\tfrac{1}{2}-iy)=\frac{\pi}{\cos\pi y} \quad \text{(A27)}$$

to facilitate the calculation analogous to (5.10) and (5.11):

$$\langle O_{\rm loc}|T_{tt}(\tilde{\psi}_{m'},\tilde{\psi}_{m'})|O_{\rm loc}\rangle|_{\mathcal{V}_+}=-\frac{1}{4\pi^2r^2}\sum_{lm}\left[\int_{\tilde{P}^r<0}\omega\, d\tilde{P}^r+\int_{\tilde{P}^r>0}\omega\tanh(4\pi M\omega)d\tilde{P}^r\right] . \quad \text{(A28)}$$

Normal ordering this result with respect to the static observers at infinity as in (5.12) yields

$$\langle O_{\rm loc}|:T_{tt}(\tilde{\psi}_{m'},\tilde{\psi}_{m'}):|O_{\rm loc}\rangle|_{\mathcal{V}_+} \simeq \frac{1}{2\pi^2r^2}\sum_{lm}\int_{m_e}^{\infty}\frac{\omega\, d\omega}{e^{8\pi M\omega}+1} . \quad \text{(A29)}$$

As in the scalar case, this represents the energy density of the Hawking pairs. This interpretation is a consequence of the expectation value of the number operator which is of the same form as (5.16):

$$\langle O_{\rm loc}|\tilde{N}(\omega_{m'})|O_{\rm loc}\rangle=\frac{\|\tilde{\psi}_{m'}\|^2}{e^{8\pi M\omega_{m'}}+1} . \quad \text{(A30)}$$

Finally, the renormalized stress-energy tensor of the Dirac field in the freely falling vacuum is

$$\langle O_{\rm loc}|T_{tt}|O_{\rm loc}\rangle_{\rm ren}|_{\mathcal{V}_+}=-\frac{1}{2\pi^2r^2}\sum_{lm}\int_{m_e}^{\infty}\frac{\omega\, d\omega}{e^{8\pi M\omega}+1} . \quad \text{(A31)}$$

APPENDIX B: THE DIRAC EQUATION IN THE FREELY FALLING FRAMES

The covariant derivative of a spinor in curved space is defined by [28]

$$\nabla_\nu\psi=(\partial_\nu+iS^{ab}\omega_{ab\nu})\psi , \quad \text{(B1)}$$

$$\nabla_\nu\bar{\psi}=\partial_\nu\bar{\psi}-i\bar{\psi}S^{ab}\omega_{ab\nu} , \quad \text{(B2)}$$

where S^{ab} is the generator of local Lorentz transformations

$$S^{ab}=-\frac{i}{4}[\gamma^a,\gamma^b] \quad \text{(B3)}$$

and $\omega_{ab\nu}$ is the connection.

To calculate the connection, one can apply the structure equations of the method of moving frames to the orthonormal tetrads carried by PFF observers. In particular,

$$d\omega^\alpha=\omega^\alpha{}_\beta\wedge\omega^\beta , \quad \text{(B4a)}$$

$$\omega^\alpha{}_\beta=\omega^\alpha{}_{\beta\gamma}\omega^\gamma , \quad \text{(B4b)}$$

where ω^α are the basis covectors of (2.5). One finds

$$\omega^\rho{}_u=\left[\frac{M}{r^2V^r}\right]\omega^\rho , \quad \text{(B5a)}$$

$$\omega^\theta{}_\phi=\frac{1}{r}\cot\theta\omega^\phi , \quad \text{(B5b)}$$

$$\omega^u{}_\phi=-\frac{V^r}{r}\omega^\phi , \quad \text{(B5c)}$$

$$\omega^\rho{}_\phi=-\frac{1}{r}\omega^\phi , \quad \text{(B5d)}$$

$$\omega^u{}_\theta=-\frac{V^r}{r}\omega^\theta , \quad \text{(B5e)}$$

$$\omega^\rho{}_\theta=-\frac{1}{r}\omega^\theta . \quad \text{(B5f)}$$

If one defines the spinor connection as

$$\Gamma_\nu=S^{ab}\omega_{ab\nu} , \quad \text{(B6)}$$

then from (B5), (B3), and (A2) one finds that

$$\Gamma_0=0 , \quad \text{(B7a)}$$

$$\Gamma_\rho=+\frac{iM}{r^2V^r}\begin{bmatrix}0 & \sigma_z\\ \sigma_z & 0\end{bmatrix} , \quad \text{(B7b)}$$

$$\Gamma_\phi=\frac{1}{r}\begin{bmatrix}+\sigma^2 & -iV^r\sigma^1\\ -iV^r\sigma^1 & +\sigma^2\end{bmatrix}+\frac{1}{r}\cot\theta\begin{bmatrix}\sigma_z & 0\\ 0 & \sigma_z\end{bmatrix} , \quad \text{(B7c)}$$

$$\Gamma_\theta=\frac{1}{r}\begin{bmatrix}-\sigma^1 & -iV^r\sigma^2\\ -iV^r\sigma^2 & -\sigma^1\end{bmatrix} . \quad \text{(B7d)}$$

The Dirac equation is

$$i(\gamma^\nu\partial_\nu\psi+\gamma^\nu\Gamma_\nu\psi)-m\psi=0 , \quad \text{(B8)}$$

$$i\not{\partial}\psi+\left[-\frac{M}{r^2V^r}\gamma^0+\frac{2}{r}[V^r\gamma^0-\gamma^\rho]+\frac{\cot\theta\gamma^\theta}{r}\right]\psi=m\psi . \quad \text{(B9)}$$

For ultrarelativistic outgoing modes (i.e., $P^\rho\gg 1/4M$), the Dirac equation near the horizon reduces to its flat-space form with high accuracy. This justifies the exponential behavior in (A6a).

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