First-order phase transitions with more than one conserved charge: Consequences for neutron stars

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We consider how first-order phase transitions in systems having more than one conserved charge (multicomponent systems) differ from those in systems having only one. In general, the properties of the transition are quite different in the two cases. Perhaps most importantly the pressure varies continuously with the proportion of phases in equilibrium, and is not a constant in the mixed phase as in the example of the gas-liquid transition in familiar one-component systems. We identify the microphysics responsible for the difference. In the case that one of the conserved charges is the electric charge, a geometrical structure in the mixed phase is expected. As an example, possible consequences are developed for the structure of a neutron star in which the transition to quark matter in the core occurs. It is also pointed out that the general results pertain to relativistic nuclear collisions in the so-called stopping or baryonrich domain where there are three conserved charges (baryon, electric, and strangeness), and impact the expected phase transition from confined hadronic matter to quark matter as regards signals that are supposedly driven by pressure. The physics discussed here is also relevant to the subnuclear gas-liquid transition that is under study in lower-energy nuclear collisions.

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I. INTRODUCTION

The most notable features of a system that possesses a single conserved quantity and that is in a state consisting of two phases in equilibrium, sometimes referred to as the mixed or coexistence phase, are the constancy of the pressure and densities of each of the two phases and the constant density discontinuity between them that hold at all proportions of the phases. The transition from one homogeneous phase to the other through the mixed phase is referred to as a first-order phase transition. The gas-liquid transition, in which the number of H₂O molecules is conserved, is the textbook example [1]. This phase transition is used as the paradigm for many others. However, these well-known properties of first-order phase transitions in simple bodies are all modified for systems possessing more than one conserved charge (meaning additive attribute of the constituents or in some contexts the number of molecules of an independent component). For brevity, we shall refer to a body with more than one conserved charge as complex.

Generally, a body or system contains conserved charges, sometimes called *independent components*, whose numbers were fixed in the preparation of the body. The preparation may have taken place in the laboratory for a ...nary mixture or in some natural process as in the formation of a star. We are interested in first-order phase transitions in such bodies and, in particular, in those that are complex.

The essentially different character of a first-order transition in a simple body as compared with one in a complex body originates in the possibility that in the latter the conserved charges can be shared by the two phases in equilibrium in different *concentrations* in each phase than

those with which the body was prepared, consistent with the conservation laws. If this is energetically favored by the internal forces and Fermi energies, then these degrees of freedom will be exploited by the system. For example, in the subsaturation nuclear liquid-vapor transition, the number of neutrons and protons is certainly conserved by the body at all times, and in either the pure liquid or vapor phase the number of each, and therefore their ratio, is fixed by the conditions of the preparation. But their ratio can be different in the vapor and liquid regions of the mixed phase: It is only their sum in all regions of the body that is conserved. Because of the isospin symmetry energy, it is favorable for the denser phase to become more symmetric. Such a degree of freedom is not available to a one-component system. We can go further: It is apparent that when the body is almost all in one phase or the other, the added degrees of freedom are of little importance; conservation laws impose almost as stringent conditions as in either pure phase. So, clearly, the degree to which the body can exploit the added freedom when in the mixed phase is a function of the proportion of the phases. Consequently, the energy density is not a linear function of the proportion, as it is for a simple substance, but varies as the proportion. Therefore the internal pressure does not remain constant while the body converts from one pure phase to the other. Of course, at each proportion of phases, the pressure is common in each phase in equilibrium. As a corollary, we may conclude also that all other properties in the mixed phase are nonlinear functions of the proportion. In the presence of gravity or other external fields, the differences may be accentuated between first-order phase transitions in systems that possess only one conserved quantity and those that have more than one. This theme will be taken up again in Sec. IV.

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Since many treatments of phase transitions are modeled after the textbook example involving a single chemical potential corresponding to a single component, our discussion may be relevant to many astrophysical problems, including phase transitions in the early Universe and the condensation of galaxies and other structures. It applies to multicomponent mixtures in chemistry. It is also relevant to accelerator experiments on the nuclear gas-liquid transition, where the neutron and proton numbers are conserved, and to the expected phase transition from confined hadronic to quark matter in laboratory collisions at relativistic energies between heavy nuclei, where the conserved charges are baryon, electric, and strangeness. It also may apply to the similar phase transition in neutron stars.

In previous studies of the transition between the confined and deconfined phases in neutron stars, most have treated the star as automatically charge neutral by ignoring β equilibrium and assuming that the confined phase is purely neutron [2]. In this approximation the transition is described in terms of only the neutron chemical potential. Several other authors have allowed for β equilibrium, but have approximated the phase transition by assuming that both phases in equilibrium are separately charge neutral and imposing this on the solution [3]. Both of the approximations have prevented the star from exploiting a degree of freedom that is available to a system with two chemical potentials. Most importantly, for the structure of the star, in both approximations the pressure is constant at all densities in the mixed phase. In the presence of a gravitational field, this means that the two phases will be separated by gravity, the denser sinking to the center. As a consequence, there can be no region of mixed phase and there is a discontinuity in the density of the star at the phase boundary which occurs at a definite radius, the discontinuity being that between the two phases in equilibrium. This is the situation so far treated in the literature [2,3]. Therefore, following a general discussion of phase transitions in complex bodies, we open up for the first time a discussion of aspects of neutronstar structure in which a first-order phase transition occurs, which have not previously been considered. This part of our paper is necessarily tentative and incomplete since at the present time we have no estimate of the surface interface energy between hadronic and quark matter and so are not able to estimate quantitatively the geometric structure of the mixed phase, which is an important aspect of the problem.

Our discussion of phase transitions and the mixed phase follows the usual practice of considering the bulk energy and neglecting finite-size effects such as surface and Coulomb *displacement* energies. Whether this is valid in any particular example has to be examined on a case-by-case basis. Some of the results on phase transitions developed in Sec. III were anticipated and correctly treated in Refs. [4-9], but not in the completeness or generality derived here. We note also that some, if not all, of the properties of phase transitions of complex bodies that we *derive* from the conditions of chemical, thermal, and mechanical equilibrium have been observed in macroscopic bodies [10]. Whereas the discussion of the treatise cited, and others on multicomponent systems, is phenomenological in nature and eschews a discussion of the microphysics in favor of the thermodynamic approach, we identify degrees of freedom available in the coexistence of two phases in equilibrium in a complex system and discuss how the internal forces exploit them, thus causing the different behavior of the mixed phase in a complex system as compared with a simple one.

Whether or not geometric structure or texture in the mixed phase is energetically important on the scale of the bulk energy, it is an interesting facet of the problem and could be of prime importance for transport properties. Several authors have studied the geometric structure (blobs, rods, slabs) associated with the surface and Coulomb energies when one phase is immersed in the other at various proportions for the nuclear subsaturation gas-liquid transition at zero temperature and zero net electric charge [11,12]. In principle, this phase transition involves two conserved charges and so should exhibit the phenomena we discuss in this paper. However, because of the approximation that the vapor phase has a density that is negligibly small, the system is explicitly denied the degree of freedom discussed in the second paragraph of this section. We do not question here the merit of the approximation for zero- or low-temperature subnuclear matter. We simply point out that these papers on the geometric structure have not implemented or discussed the physics that is the subject of this paper by reason of the approximation made.

The plan of the paper is as follows. In Sec. II we briefly review the properties of a first-order phase transition in a simple body, but do so without reference to a figure of pressure as a function of volume, as is usually done; rather, we introduce the notion of internal variables which are the solutions of the equations of motion presumed to describe the microphysics of the state of the body and define what a first-order phase transition means in terms of these and the thermodynamic variables. This is a useful preparation and contrast for a discussion of the general case of a body possessing several conserved charges, called a complex body, which we take up in Sec. III. Generally, there are forces associated with the conserved charges. We discuss the different role played by long- and short-range forces and how they affect the properties and possible geometric structure in the mixed or coexistence phase in Sec. IV, with special reference to the structure of neutron stars with quark matter cores. Under the assumption that the bulk energy dominates surface, Coulomb, and lattice energies, we illustrate the general results by considering the transition in equilibrium from confined hadronic to quark matter that may take place in neutron stars, if the central pressure is high enough, in Sec. V.

II. PHASE TRANSITIONS WITH ONE CONSERVED CHARGE

Here we recall the features of a first-order phase transition in a system that carries one conserved charge, as, for example, the number of H_2O molecules in the case of the gas-liquid transition. The Gibbs condition for phase equilibrium is that the system be in chemical, thermal, and mechanical equilibrium, meaning that for the two phases, we must have

$$\mu_1 = \mu_2 = \mu , \quad T_1 = T_2 = T ,$$

$$p_1 \equiv p(\{\phi_1\}, \mu, T) = p(\{\phi_2\}, \mu, T) \equiv p_2 ,$$
(1)

where the subscripts 1,2 denote the two phases, respectively, and $\{\phi_i\}$ denotes the internal variable that characterize a solution to the equations of motion of the system, exclusive of the chemical potential, in phase *i* at temperature *T*. If there are *m* internal variables

$$\{\phi\} = \phi^1 \dots, \phi^m , \qquad (2)$$

they satisfy the equations of motion denoted by

$$\mathcal{D}_{j}(\{\phi\},\mu,T)=0, \quad j=1,\ldots,m$$
 (3)

If and only if a solution μ to Eqs. (1) exists for $\{\phi_1\} \neq \{\phi_2\}$ does the system possesses a first-order phase transition. The density of the charge whose chemical potential is μ is different in the two phases since the function $\rho(\{\phi\},\mu)$ is a different function of μ because the internal variables $\{\phi\}$ have different values in the two phases. We call these constant densities ρ_1 and ρ_2 :

$$\rho_1 \equiv \rho(\{\phi_1\}, \mu, T) , \quad \rho_2 \equiv \rho(\{\phi_2\}, \mu, T) . \tag{4}$$

This is true of other functions of μ , such as the energy density. The only properties that are necessarily common to the two phases that are in equilibrium are those indicated in Gibbs relations (1), and these are constant for all densities or, in other words, all proportions of the phases in the "mixed phase" that lies in the interval $\rho_1 \leq \rho \leq \rho_2$. The pure (i.e., homogeneous) phases lie above and below this range, respectively. The mixed phase is some mixture of the two phases that occupy different volumes of space. Because the densities of the conserved charge are different and fixed in the two phases in equilibrium [Eqs. (4)], the phases contribute to the average density in the proportion of the phases themselves; thus,

$$\rho = (1 - \chi)\rho_1 + \chi\rho_2 , \qquad (5)$$

where $0 \le \chi \le 1$. This may seem unremarkable, but it is the essential difference between a transition between two phases in simple and complex systems, because in the complex system the internal forces have the freedom to arrange the most energetically favored charge densities in each phase at each proportion, consistent with conservation laws, as we shall see. In contrast, as we just saw above, for the simple system, once the Gibbs relation is satisfied [Eqs. (1)], the density of each phase in equilibrium remains fixed at all proportions of the phases and they are determined by Eqs. (4).

Since the pressure is independent of the proportion of phases in a simple body, the phases will be separated by any external field that distinguishes between them, such as gravity, which distinguishes their different densities.

We now illustrate with a concrete example the transition from neutron to quark matter in a compact star and the consequences for the structure of compact stars that stem from treating the transition as if it occurred in a simple body. This is the mold into which all discussions of this phase transition have been fitted heretofore, apparently unconsciously, usually by the seemingly innocuous neglect of β equilibrium (i.e., by assuming a purely neutron star) [2]. For the purpose of illustration, we adopt, for the description of the confined hadronic phase, the Lagrangian of Zimanyi and Moszkowski [13], solved in the mean-field approximation, and, for the quark matter phase, the bag model [14]. Of course, none of the general properties outlined above depend on these model choices. To correspond with the earlier discussions of quarks in neutron stars [2], we assume that the only baryons present are neutrons. For the quarks we assume that the u, d, s quarks are massless. Then both phases are automatically charge neutral and there is only one chemical potential: that for the baryon number. We note that this situation is arrived at through assumptions that are seemingly innocuous, but have the effect of reducing a complex system to a simple one, as is done in one way or another in Ref. [2].

Later in this section, we will explore briefly a different assumption with respect to the quarks which also reduces the system to a simple one. There are therefore two contrasts that are made: (1) The effect on neutron-star structure of assumptions that approximate it as a simple system is compared with the structure that apparently would follow if treated as the complex system it really is. This latter case is developed in subsequent sections, and of course, the *two* conserved charges in a star, baryon and electric, are what make a first-order phase transition complex. (2) The other contrast, which is made entirely in this section, will be with respect to the *difference* in the conclusions one would draw as to the presence of quarks in neutron stars, depending on *how* the reduction of a complex system to a simple one is accomplished.

The pressure and neutron chemical potential at phase equilibrium are given by the crossing of the curves in Fig. 1. The dotted line in this particular instance shows



FIG. 1. Pressure as function of chemical potential for firstorder phase transition (solid lines). Crossing is point of phase equilibrium. Dotted line is schematic. See text for description of dashed lines.

schematically the way the pressure evolves as a function of chemical potential when there is a first-order phase transition in a body for which one has a complete theory for the internal variables describing the state of the body. (In our example this is not so. We cannot solve QCD through the confined-deconfined transition, and so we use separate models for each phase. Hence, in this instance, the dotted line is schematic.) Figure 2 shows the constancy of the pressure and chemical potential and the linear dependence of density of the energy density in the mixed phase. The constant pressure of the mixed phase in the presence of a gravitational field has the consequence that gravity separates the two phases, and one is left with only the pure phases. Said another way, the mixed phase at one density cannot support the mixed phase at another one since the pressure exerted by each is the same for a simple substance in equilibrium between its two phases. Said still another way, the equations of star structure assure that the pressure is a continuous function of the Schwarzschild radius and decreases monotonically from the center of the star; therefore, the two equal pressure points at the opposite ends of the mixed phase (Fig. 2) are mapped onto the same radial point in the star. Consequently, for a simple substance, there are no stable stars with central densities falling in the range of the mixed phase, and for those stars with central densities that lie above that range, the distribution of matter as a function of the radial coordinate suffers a discontinuity, falling from ρ_2 to ρ_1 at the radius where the pressure equals that of the phase equilibrium. These aspects are illustrated in Figs. 3 and 4 by the dashed lines. The solid lines correspond to stars in β equilibrium with a first-order phase transition in which the baryon number and electric charge are conserved, the case that we discuss next.

All of the above discussion was based on the neglect of β equilibrium in the *hadronic* sector, as in Ref. [2]. Under the assumption of equal u, d, s quark masses, the quark phase is *trivially* in equilibrium and charge neutral. Alternately, one could suppose that the assumed purely neutron core was converted to the corresponding number



FIG. 2. Pressure, energy density, and chemical potential when there is only one conserved charge. Flat region in p and μ is mixed phase.



FIG. 3. As a function of central density, the dashed line interrupted by dots is the family of stars for the idealized case having one chemical potential. Regions are n (pure neutron stars) and n-q (quark core and neutron matter mantle stars). Solid curve is family of compact stars in β equilibrium (two chemical potentials); stars below m are made of neutron-star matter; those above have mixed phase cores.

of u,d quarks and that none of these weakly decayed into strange quarks, the assumption made by Baym and Chin [2]. They imposed the corresponding constraint that $n_d = 2n_u$, which again is trivially charge neutral, so that the star, by approximation, again has only one chemical potential, that for the baryon number. However, this



FIG. 4. Dashed line is profile of an idealized star with one chemical potential at mass limit of previous figure. Core is pure quark matter and exterior is pure neutron matter. Solid curve is star at mass limit which is in β equilibrium. The core is in the mixed phase of hadronic and quark matter. The exterior is neutron-star matter. Dividing point is marked by *m*.

choice for the number of u quarks compared with d quarks is extremely β unstable. It has the consequence that the computed energy per added baryon, $\mu = \partial \epsilon / \partial \rho$, is very large compared with an equilibrium state, so that the predicted phase transition is at a correspondingly high baryon density. Said another way, the neutron chemical potential at which the pressure in the quark phase vanishes is a factor $[(1+2^{4/3})/3]^{3/4}$ larger than in the equilibrium model of massless u, d, s quarks. The chemical potential at which the pressure curve crosses that of the hadronic phase is correspondingly higher. This case is also shown in Fig. 1 by the dashed lines. The number density at which conversion of hadronic matter begins is a very high density of 1.03 fm⁻³ in this case as compared with 0.306 fm⁻³ in the equilibrium case of u, d, s quark matter. We thus see how a seemingly inno-

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cuous approximation, the neglect of β equilibrium in the quark sector, led to a major overestimate of the transition density in the model adopted to explore the question.

Chemical potentials are used to express the conditions of equilibrium of a system of several substances which are composed of independent components or carry conserved charges. In general, transformations among the substances are possible. The transformations define relations among the chemical potentials. The conservation laws refer to the amount of each of the independent components or conserved charges with which the system was initially prepared. We denote these n charges or numbers of each of the independent components by Q_{α} $(\alpha = a, b, \ldots, n)$. Then the chemical potentials of all substances can be written as a linear combination of as many independent chemical potentials as there are conservation laws. We call these $\mu_a, \mu_b, \ldots, \mu_n$. The coefficients in the linear combination refer to the number of the conserved quantities that the component possesses. The numbers need not be integers. The baryon and electric charges possessed by quarks are fractional.

A body has two distinct phases, which we denote by the subscripts 1 and 2, and can undergo a first-order phase transition between them if and only if (i) the *m* equations of motion for the internal structure $\mathcal{D}_j(\{\phi\},\mu_a,\ldots,\mu_n,\ldots,T\}=0, j=1,\ldots,m$, have two distinct simultaneous solutions for the internal variables,

$$\{\phi\} \equiv \phi^1, \phi^2, \dots, \phi^m , \qquad (6)$$

such that

$$\{\phi_1\} \not\equiv \{\phi_2\} , \tag{7}$$

where

$$\mathcal{D}_i(\{\phi_1\},\mu_a,\ldots,\mu_n,T)=0, \qquad (8)$$

$$\mathcal{D}_{i}(\{\phi_{2}\},\mu_{a},\ldots,\mu_{n},T)=0 \ (j=1,\ldots,m);$$
 (9)

(ii) the solutions are subject to the relevant conservation laws as discussed below; and (iii) the Gibbs conditions can be satisfied. Note that when a comprehensive theory of the internal structure of the body is available, the equations of motion and all other functions of the internal and thermodynamic variables are the same functions in the two phases. The distinction between phases is that the internal variables $\{\phi\}$ describing the body occupy different regions of the internal space, which is what we mean by the inequality (7). These internal variables, which are solutions of the equations of motion, are, of course, functions of the chemical potentials (and temperature), but it is useful to retain their explicit appearance so as to distinguish phases. The Gibbs condition for phase equilibrium at fixed T when there are several conserved charges, say, n, is

$$\mu_{1,a} = \mu_{2,a} = \mu_a ,$$
...,
$$\mu_{1,n} = \mu_{2,n} = \mu_n ,$$

$$T_1 = T_2 = T ,$$

$$p(\{\phi_1\}, \mu_a, \dots, \mu_n, T) = p(\{\phi_2\}, \mu_a, \dots, \mu_n, T) .$$
(10)

The pressure is the same function on the two sides of the last equation, but is evaluated in *different* regions of the space of internal variables, so that the left and right sides are different functions of the chemical potentials. Clearly, this equation, unlike the example of one chemical potential, does not suffice to specify the pressure and chemical potentials at phase equilibrium. If there is a first-order phase transition, the pressure as a function of one of the chemical potentials behaves locally in the general manner of Fig. 1. Let this one be μ_a . Generally, one thinks of varying the volume V of the body by external means at a constant temperature. This, of course, corresponds to varying the *n* densities of conserved charges, Q_{α}/V , $\alpha = a, b, \ldots, n$, and hence their chemical potentials, among them μ_a . Of course, the ratios of the densities (concentrations) remain unchanged while the body is in a pure phase and is equal to the values established in the preparation. From the theoretical and calculational point of view, it is a matter of convenience as to which is used as the independent variable when the body is in a pure phase.

Let us write the n conservation laws for the charge densities in either of the pure phases 1 or 2 as

$$q_{\alpha}(\{\phi_1\}\mu_a,\ldots,\mu_n,T) = Q_{\alpha}/V$$
, (11)

$$q_{\alpha}(\{\phi_2\},\mu_a,\ldots,\mu_n,T)=Q_{\alpha}/V \quad (\alpha=a,\ldots,n).$$
(12)

For concreteness, let phase 1 be the low-density phase. We could find the properties of the system in this pure phase by choosing a volume or a density, say, $q_a \equiv Q_q / V$ (with all other densities having ratios equal to those established in the preparation of the system) and solving the set of Eqs. (8) and (11) for the *m* internal variables $\{\phi_1\}$ and $n \mu$'s at each volume, and then computing the pressure $p(\{\phi_1\}, \mu_a, \ldots, \mu_n, T)$ and any other quantities of interest. There are m + n equations in that many unknowns.

The equality of the chemical potentials at phase equilibrium requires a careful examination of how phase equilibrium is to be found. We study first the end points of the mixed phase, where the proportion of one or other of the phases is very small. Then we turn to the intermediate region. Imagine that we begin with the body in phase 1, [therefore defined by the solution $\{\phi_1\}$ of Eq. (8)] and increase the pressure by decreasing the volume or increasing the density q_a . At some pressure and corresponding volume V_1 as yet unknown, parts of the body will begin to convert to the other phase. The body is said to be in the mixed phase. At pressure below and just to the point where this conversion begins, the conservation conditions are still expressed as in the pure phase 1 by Eq. (11) because there is vanishing amount of phase 2. At this point these conservation conditions, the equations of motion (8) and Gibbs condition for equilibrium pressure expressed in Eqs. (10), provide the m + n + 1 equations that specify the volume V_1 (or, equivalently, the charge density $q_{a,1} \equiv Q_a / V_1$), the values of the *n* chemical potentials, and the m internal variables corresponding to the state of the body at the onset of the phase transition. Similarly, if the mixed phase is approached from the pure phase 2, at the point where the system is about to begin conversion, the conservation laws as expressed in phase 2 still apply. A similar set of equations defines this boundary between pure and coexistence phases. Equation (9) replaces (8) and Eq. (12) replaces (11), and these, together with the Gibbs condition, define the volume V_2 or density $q_{a,2} \equiv Q_a / V_2$, corresponding to the upper end of the phase coexistence region, together with the internal variables and chemical potentials at this boundary. We have two different sets of equations that define the state of the system at the extremes of the coexistence phase, respectively, and their solutions $V, \{\phi\}, \mu_a, \ldots, \mu_n$ will in general be different. Accordingly, the chemical potentials and pressure, though common in each phase in equilibrium, are different at the two ends of the coexistence region, as are the densities of the conserved charges. This is in marked contrast to the corresponding properties of a simple system.

It remains to describe the mixed phase at arbitrary proportion of the two phases. Choose a volume V in the range $V_1 \ge V \ge V_2$. Let $1-\chi$ denote the fraction of the volume occupied by phase 1 and χ by phase 2. The conservation laws written above in Eqs. (11) and (12) for the pure phases now must be written as a *single* equation for each charge that involves *both* phases:

$$(1-\chi)q_{\alpha}(\{\phi_1\},\mu_a,\ldots,\mu_n,T) + \chi q_{\alpha}(\{\phi_2\},\mu_a,\ldots,\mu_n,T) = Q_{\alpha}/V \quad (\alpha = a,\ldots,n) .$$
(13)

These reduce to the separate laws for the pure phases for $\chi = 0$ or 1. As noted below, we must allow the phases in equilibrium the freedom to satisfy these less restrictive conditions and not impose conditions separately on each phase in equilibrium, or else the Gibbs condition on the equality of the chemical potentials would be violated. The equations governing the body in the mixed phase are the equations of motion [Eqs. (8) and (9)], of which there are 2m, namely, *m* equations for the solution of the *m*

internal variables $\{\phi_i\}$ in each phase (i = 1, 2), the *n* conservation laws [Eq. (13)], and the Gibbs condition [Eqs. (10)] for the *n* chemical potentials and proportion χ . There are therefore 2m + n + 1 simultaneous equations that describe the state of the body in the mixed phase, including the proportion χ of phases. If a solution exists at the chosen volume or, equivalently, densities of charges, Q_{α}/V , $\alpha = a, \ldots, n$, with $0 < \chi < 1$ and $\{\phi_1\} \neq \{\phi_2\}$, it represents a state of the body in the mixed phase.

We emphasize that in the mixed phase Eq. (13) is the only way of expressing the conservation laws that is consistent with Gibbs phase equilibrium. To apply Eqs. (11) and (12) separately to the volumes of each phase $(1-\chi)V$ and χV would in general ensure that the chemical potentials are unequal in the two phases. Moreover, there are n more conditions than variables.

To understand the special nature of a first-order phase transition in a system of more than one conserved charge, it is important to note that the ratios of the charge densities, Q_{α}/V , in the *pure* phases are forever fixed once the system has been prepared with specified number of each type of charge. However, the ratios of charges or components may be different in each of the phases in phase equilibrium; only the sum of charges in the two phases [Eq. (13)] must retain the fixed values established in the preparation. Thus we understand the role that is opened up for the internal forces, including the Fermi energy, to optimize the proportions of charges that the phases in equilibrium carry, so as to minimize the free energy of the system. In our formulation this is implicit in the simultaneous solution of Eqs. (8)-(10) and (13) described above. For a system with n conserved charges, there are n-1 concentrations or degrees of freedom that the internal forces can exploit when two phases are in equilibrium. In particular, there is no such freedom for a system carrying but one conserved charge or one independent component, that is to say, a simple system. Thus as we have noted, the two systems of equations described above in general have different solutions, so that the pressure does not remain constant while the body converts from one pure phase to the other, and likewise all chemical potentials change, though, of course, they are common to the two phases in equilibrium at any particular proportion of the phases. The common pressure and chemical potentials vary continuously as the proportion of phases in equilibrium changes in a complex body.

Once the 2m + n + 1 equations [Eqs. (8)-(10) and (13)] are solved simultaneously, all other additive quantities can be computed from the rule, expressed here for the charge density corresponding to the conservation law labeled a:

$$q_{a} = (1 - \chi)q_{a}(\{\phi_{1}\}, \mu_{a}, \dots, \mu_{n}, T) + \chi q_{a}(\{\phi_{2}\}, \mu_{a}, \dots, \mu_{n}, T) .$$
(14)

As a consequence of the variation of the chemical potentials with proportion of the phases, all other additive properties such as densities and energy density do so too, but they do not necessarily vary linearly with proportion χ . For example, compare Eqs. (5) and (14), where in the first of these equations the density is a fixed value in each phase, corresponding to the unique value of the chemical potential at phase equilibrium, whereas in the second all the charge densities themselves vary as the proportion of the phases. All of these aspects of a first-order phase transition in a body with more than one conserved charge are at variance with those corresponding to a body with only one. The feature that is common in both cases is the discontinuity in the densities across the interface between the two phases in equilibrium; this is what distinguishes a first- from a second-order phase transition. However, the magnitude of the discontinuity in the general case varies with the proportion of phases in equilibrium.

From the above discussion, we see that a system would have to be prepared with very particular concentrations of conserved charges to evade the above behavior, if it is possible at all. In particular, it would have to be so that at all proportions of the phases in equilibrium, χ , the internal forces favored the particular concentrations with which the system was prepared over any other in both phases. The points 0 and 1 in Fig. 5 are coincident in this circumstance. For most systems such a choice of concentrations does not exist, because the number of conditions that must be satisfied exceeds the number of variables, so that it is accidental if there is such a choice. Indeed, even to satisfy the lesser conditions that the pressure and chemical potentials at the extremes of the mixed phase be equal, it would have to be so that, for the particular set of concentrations of charges, the overdetermined set of Eqs. (8)-(10) and the 2*n* equations,

$$q_{\alpha}(\{\phi_1\},\mu_a,\ldots,\mu_n,T) = \frac{Q_{\alpha}}{V_1} ,$$

$$(\alpha = a,\ldots,n) , \quad (15)$$

$$q_{\alpha}(\{\phi_2\},\mu_a,\ldots,\mu_n,T)=\frac{Q_{\alpha}}{V_2}$$

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have a simultaneous solution. But there are 2m + 2n + 1 equations in only 2m + n variables: the 2m internal variables $\{\phi_1\} \neq \{\phi_2\}$ and the *n* chemical potentials μ_a, \ldots, μ_n . Much less it is possible to find a simultane-



ous solution for *each* proportion $0 < \chi < 1$ of Eqs. (8)–(10) and the 2n equations:

$$q_{\alpha}(\{\phi_1\},\mu_a,\ldots,\mu_n,T) = \frac{Q_{\alpha}}{(1-\chi)V},$$

$$(\alpha = a,\ldots,n). \quad (16)$$

$$Q_{\alpha}$$

$$q_{\alpha}(\{\phi_2\},\mu_{\alpha},\ldots,\mu_n,T)=\frac{\mathcal{Q}_{\alpha}}{\chi V}$$
,

The system is overdetermined at a continuous infinity of proportions. These facts serve to emphasize the generality of properties that have been derived for complex systems.

It is perhaps instructive to have a picture in mind to illustrate the general situation. In the case of one conserved charge (such as the number of H_2O molecules), the pressure as a function of chemical potential for this charge has the familiar form of a curve that turns in direction, crossing itself at some pressure and chemical potential, as in Fig. 1. The crossing point corresponds to the equilibrium values of pressure and chemical potential at the given temperature. Now imagine a third axis corresponding to an additional chemical potential as in Fig. 5. Pull the curve as an elastic sheet out into this new direction. Of course, it need not be by parallel displacement and indeed the surface so created may have convolutions. The *point* of intersection, A, of the pressure in the two pure phases has become a *curve* labeled AA'; the two phases at each point on the curve have equal chemical potentials and pressure, as required for equilibrium, but in general they vary along the curve. However, the entire curve does not correspond to physically accessible states of the body, given the conservation laws. To understand this consider the pure homogeneous phases. The conservation laws [Eqs. (11) and (12)] for each pure phase describe a curve in the chemical potential plane parametrized by the volume V (or, equivalently, any one of the charge densities, the ratios being fixed in the preparation). The projection of each of these curves onto the pressure surface of the corresponding phase describes a curve on each surface. Segments of these in the maximum-pressure phases are labeled a0 and 1b. On these curves the body is in the pure phases 1 and 2, respectively, and satisfies the conservation laws, different points on the curve corresponding to different volumes or densities of a given charge. Phase equilibrium is possible only when each of these curves crosses the curve of pressure intersection, AA'. We must realize that the figure represents the pressure surface only locally in the neighborhood of a phase transition, and there may be other regions of the chemical potentials where there is only one pressure surface and only one phase. When there is a distinction of phases, the points at which the two curves defined by the conservation laws cross the pressure intersection curve at 0 and 1 will not generally be coincident, as we proved above, and they define the extremes of the mixed phase. The proportion of phase 2 at the point 0 is zero and at 1 it is unity. The proportion varies between these values along the curve 01. This is the curve of phase equilibrium. In general, the pressure and both chemical potentials vary along it, as do the corresponding charge densities. Of course, at each point on the intersection curve, which corresponds to some definite proportion of phases, χ , all of these quantities are equal throughout the body, as required by the conditions of equilibrium. This construction serves to illustrate why a first-order phase transition with more than one conserved charge is so different from a transition with exactly one. When there is more than one conserved charge, phase equilibrium is possible when the pressure as a function of the independent chemical potentials has the topology of the generic form illustrated.

The variation of pressure with proportion of phases in equilibrium is perhaps the most consequential difference and may lead to structurally different compact stars, in which the confined hadronic-to-quark matter phase transition occurs, than would be computed in approximations that treat the star as a simple body having a single conserved charge (baryon number) rather than a complex one. All previous discussions of the phase transition in neutron stars have been forced into this mold, usually by treating the star as if it were purely neutron and, so, a single-component system [2]. The variation of pressure also impacts expectations that have been developed for signals of the phase transition in relativistic nuclear collisions in the "stopping" or "baryon-rich domain" that have been built around the notion of a "plateau" in pressure and, hence, in observables that are driven by pressure.

IV. ROLE OF THE FORCE RANGE ASSOCIATED WITH A CONSERVED CHARGE

Clearly, in any application of the general theory, a distinction exists between systems for which all charges are associated with short-range forces and those systems for which one or more of the forces is a long-range one since in the first case regions occupied by one phase will not interact with the other except at their boundaries, while in the second they will interact as well through the longrange force. We discuss an example of each.

We have designated conserved quantities by the word charges. In some cases these are components of a mixture, as in ethane-heptane, which for certain ranges of concentration of the components possess a gas-liquid phase transition. When there is a phase transition, the pressure is observed to exhibit the variation through the mixed phase discussed above [10,15]. Presumably, it is the molecular forces that are involved in determining that, as the proportion of the two phases changes, the energy is lowered by optimizing the ratio of the two substances in the gas and liquid, constrained, of course, by the conservation of the total number of molecules of heptane and ethane, rather than each phase in equilibrium retaining the same ratio as in the pure phases, which is, of course, fixed at the time of preparation of the mixture. As explained earlier, the energy in consequence will not be a linear function of the proportion of the phases, and hence the pressure will vary as the proportion of phases in equilibrium. In such a system, there are no long-range forces, and in the bulk approximation, the two phases in equilibrium, which, of course, occupy separate volumes, can each be broken up, moved about, or merged, at no

cost in energy. As a weaker effect, the surface interface energy will play a role in determining the size and shapes of the volumes occupied by the two phases. In fact, all regions of like phase will tend to coalesce. We will not discuss this case further here since it does not correspond to the circumstances of a compact star.

Let us turn now to a contrasting situation where longrange interactions will act between regions of one phase and the other. An example of interest is the so-called neutron star. These compact objects are, of course, not purely neutron; the Fermi energy of neutrons in the denser regions will exceed the energy of the proton and electron and eventually other baryon species, and the energy will be lowered by conversion of neutrons into protons and hyperons. Of course, strangeness is not conserved on a time scale much in excess of the weak-interaction scale. Therefore the conserved quantum numbers are the baryon number and electric charge. The star is essentially charge neutral because particles contributing to net charge could not be bound by gravity. If in the dense interior of the star the pressure surpasses that for converting hadrons to quark matter, we have a phase transition of a substance with two conserved charges, and the general theory described above applies if it turns out that the bulk energy dominates the surface interface energy and any Coulomb displacement energy.

Because of the constraint of charge neutrality, a neutron star is highly isospin asymmetric. Even below the phase-transition density, the isospin part of the strong interaction has scope to reduce the isospin within the constraint of charge neutrality by populating hyperons of favored isospin (opposite to the neutron) [16-18]. If the pressure is reached somewhere at a depth in the star that some baryons are converted to quark matter in equilibrium with the remaining baryon matter, the isospin component of the strong interaction has still further scope to reduce the isospin of the hadronic phase since now excess (positive) charge built up in this phase can be neutralized overall by the quark matter, which carries some of the conserved baryon charge as well. Of course, chemical equilibrium in the lowest-energy state of the combined system will prescribe the limit to which this tendency can go at each proportion of the phases. In this example it is the isospin interaction in the hadronic phase, mediated at least in part by the coupling of the ρ meson to the hadronic isospin current, which is the driving force in reducing the isospin asymmetry of the hadronic phase when it is in equilibrium with the quark phase. We are, of course, assuming that the quark matter phase does not have an explicit isospin interaction which is repulsive for isospin asymmetric mixture of quarks, certainly not in the limit of asymptotic freedom, aside from the ever present Fermi energies. In the situation just described, regions of hadronic matter in equilibrium with quark matter will have a net positive charge, neutralized by the negative charge of quark matter and to a much lesser degree by electrons. The reason that electrons will play a minor role in charge neutrality, once densities are reached where neutrality can be achieved in large part among baryons, is that baryons alone carry that conserved quantum number and must be present, but electrons must be present only to the extent of neutralizing any excess positive charge not already neutralized among particles carrying baryon charge [16-18]. So the mixed phase of hadronic and quark matter consists of regions of opposite electric charge. This charge will prevent the regions from growing too large, and the surface energy will act in the opposite sense in preventing them from becoming too small. We expect therefore that the mixed phase will form a Coulomb lattice, in the same sense as subsaturation charge-neutral matter is expected to do [11,12]. However, whereas in this latter case the size of nuclei provides a natural length scale and the semiempirical mass formula provides the surface energy, and whereas the nuclear phase is dense and the gas phase is dilute and sometimes taken to have vanishing density, none of these applies to the hadron-quark phase transition. Both phases are dense at supernuclear densities in the star, the surface interface energy between them is not known or easy to estimate, and neither phase is self-bound, certainly not at the densities in the star, so that there is no firstorder guess possible for the size of the regions of hadronic and quark matter phase. Thus it is not possible at this time to compare a screening length with the structure size to justify the treatment of each phase as uniform.

In the example just discussed, surface interface and Coulomb energies will come into play, as described qualitatively above, to form a lattice of regions of one phase immersed in the other. According to the proportion of the two phases in equilibrium, the common pressure, chemical potentials, and geometric structure will vary. We expect that the outer region of the neutron star will consist of hadronic matter. At some depth where the pressure is higher, some of this will be converted to quark matter, which will form a lattice immersed in hadronic matter. Much deeper in the star, the roles will be reversed in accordance with the associated pressure. In between, other geometric phases may occur corresponding to intermediate pressures. Perhaps, in the inner core, the pressure will be so high as to convert all of the matter to quark matter. The density profile of the star in this picture is continuous and in general consists of three regions: the outer hadronic region, the intermediate mixed-phase region of varying proportion of phases and geometric texture with depth, and the interior quark matter core. The picture that emerges is quite different than when the degree of freedom corresponding to the presence of the second conserved charge is suppressed, the situation discussed in Sec. II. In such a case, where the pressure is a constant in the coexistence phase, a large density discontinuity occurs at the depth in the star that separates the two phases, the denser quark phase having sunk to the central region, both phases being assumed to be charge neutral by fiat (Fig. 3). The extent of the discontinuity is that between pure hadronic matter at the density just above which conversion will commence and the density of pure quark matter at the completion of conversion.

We have discussed the role of long-range forces acting in the mixed phase between different regions occupied by one phase or the other. These refer to forces associated with the conserved charges and not to the external gravitational field. In a star gravity plays its usual role in accordance with Einstein's equations, arranging matter according to the relationship between energy density and pressure and implicitly selecting the geometrical texture of the mixed phase in accordance with its relation to the pressure, thereby selecting the texture as a function of depth in the star.

In the absence of information on the surface energy, we cannot provide any details on the texture of the mixed phase, nor can we compare sizes of structure with a screening length. For the same reason, we cannot estimate the electrostatic potential in the Coulomb lattice and, hence, the shift in electric charge chemical potentials [19]. These are outstanding problems. However, we expect that the bulk energy which is supernuclear and of the order of 1 GeV/fm³, including as it does the mass and Fermi energies, dominates the surface and Coulomb lattice energies. If this is true and if the lattice structure does form in the mixed phase, then we can calculate the macroscopic structure of the star and its composition to some accuracy without knowing the details of the texture of the mixed phase.

V. NEUTRON-QUARK MATTER STARS: A DIFFERENT VIEW

Having stated certain caveats in the previous section, we now illustrate the theory developed in Sec. III for a first-order phase transition in complex systems by treating a compact star in general β equilibrium in the uniform bulk approximation. It has two conserved charges: baryon and electric. The overall charge of the star is zero. Charge-neutral matter in equilibrium we call neutron-star matter. We contrast the structure of a star made from it to the fictional case of purely neutron star. Matter which is purely neutron is not the ground state of dense charge-neutral matter, but it is the model most often studied in connection with the possible phase transition to quark matter in the core of a neutron star [2]. It has only one conserved charge, the baryon number, since charge neutrality is automatic, and so by fiat such a formulation of neutron-star structure evades the interesting physics that is engaged when the system is allowed to seek the lowest-energy state in the mixed phase by sharing the charges in proportions that are different from the pure phases.

Ideally, one would have a single theory that describes both phases; the confined hadronic phase and locally deconfined quark matter phase. Presumably, this theory is lattice QCD. However, QCD has not been solved for finite chemical potentials, nor has a description of dense hadronic and quark matter, the two phases in which we are interested in, been developed as a function of baryon density or volume at zero temperature. (Neutron stars become cold on the nuclear scale within seconds of birth.) If the phase transition is a first-order one, then in its neighborhood the pressure as a function of the two chemical potentials for the conserved charges, baryon and electric, behaves as in Fig. 5, with μ_a the baryon chemical potential. We describe the hadronic phase by a nuclear field theory and the quark phase as a free gas of massive quarks whose vacuum energy is displaced from that of the hadronic phase by the "bag" constant B.

We adopt as a model of the confined phase a generalization of the Zimanyi-Moszkowski [13] nuclear-fieldtheory Lagrangian. With the addition of the ρ meson, which introduces an isospin-symmetry force, the model describes the bulk properties of equilibrium isospinsymmetric uniform matter rather well. We make no special case for it as a description of the confined phase other than this normalization to measured nuclear properties and the fact that it is a covariant theory, a fact of some merit in a theory that is to be extrapolated to high density. The generalized Lagrangian is

$$\mathcal{L} = \sum_{B} \left\{ (1 + g_{\sigma B} \sigma / m_{B}) \overline{\psi}_{B} [i \gamma_{\mu} \partial^{\mu} - g_{\omega B} \gamma_{\mu} \omega^{\mu} - \frac{1}{2} g_{\rho B} \gamma_{\mu} \tau \cdot \rho^{\mu}] \psi_{B} - \overline{\psi}_{B} m_{B} \psi_{B} \right\}$$

+
$$\frac{1}{2} (\partial_{\mu} \sigma \partial^{\mu} \sigma - m_{\sigma}^{2} \sigma^{2}) - \frac{1}{4} \omega_{\mu\nu} \omega^{\mu\nu} + \frac{1}{2} m_{\omega}^{2} \omega_{\mu} \omega^{\mu} - \frac{1}{4} \rho_{\mu\nu} \cdot \rho^{\mu\nu} + \frac{1}{2} m_{\rho}^{2} \rho_{\mu} \cdot \rho^{\mu} + \sum_{\lambda} \overline{\psi}_{\lambda} (i \gamma_{\mu} \partial^{\mu} - m_{\lambda}) \psi_{\lambda} .$$
(17)

Here the sum on B is over the baryon species, neutrons, protons, and all charge states of hyperons to convergence. The sum on λ is over e^- and μ^- . The mesons are the scalar, vector, and isovector, σ, ω, ρ .

As a first step in making the problem tractable, we transform all baryon fields by

$$\psi_B = \left[1 + \frac{g_{\sigma B} \sigma}{2m_B} \right]^{-1/2} \Psi_B . \tag{18}$$

The resulting Lagrangian appears as before except that the factor $(1+g_{\sigma B}/m_B)$ is absent and the baryon masses m_B are replaced by effective masses:

$$m_B^* = \left(1 + \frac{g_{\sigma B}\sigma}{2m_B}\right)^{-1} m_B . \tag{19}$$

We solve the field equations in the frequently used meanfield approximation (Hartree) [13,16,20]. In this approximation the theory is regarded as an effective one in which coupling constants are adjusted to properties of nuclear matter or finite nuclei, but not to the vacuum interaction between nucleons. The baryon source currents in the Euler-Lagrange equations for the mesons are replaced by their ground-state expectation values. The ground state is defined as having the single-particle momentum eigenstates of the Dirac equations filled to the top of the Fermi sea for each baryon species in accordance with the conditions of chemical equilibrium.

The equations for the mean meson fields in uniform static matter, in which space and time derivatives can be dropped, which we now denote by σ for the scalar meson and ω_0 and ρ_{03} for the timelike and isospin three component of the vector and vector-isovector mesons, are

$$\sigma = \sum_{B} \frac{g_{\sigma B}}{m_{\sigma}^{2}} \left[1 + \frac{g_{\sigma B}\sigma}{2m_{B}} \right]^{-2} \langle \overline{\Psi}_{B} \Psi_{B} \rangle$$

$$= \sum_{B} \frac{g_{\sigma B}}{m_{\sigma}^{2}} \left[1 + \frac{g_{\sigma B}\sigma}{2m_{B}} \right]^{-2} \frac{2J_{B} + 1}{2\pi^{2}}$$

$$\times \int_{0}^{k_{B}} \frac{m_{B}^{*}}{(k^{2} + m_{B}^{*2})^{1/2}} k^{2} dk , \qquad (20)$$

$$\omega_0 = \sum_B \frac{g_{\omega B}}{m_{\omega}^2} (2J_B + 1) k_B^3 / (6\pi^2) , \qquad (21)$$

$$\rho_{03} = \sum_{B} \frac{g_{\rho B}}{m_{\rho}^{2}} I_{3B} (2J_{B} + 1) k_{B}^{3} / (6\pi^{2}) . \qquad (22)$$

The angular brackets denote ground-state expectation. Through the dependence of the effective mass on σ , the first of these equations is a transcendental equation for this field variable. The spacelike components of both vector fields and all but the isospin three-component of the ρ field vanish, as can be shown explicitly [16]; they do so for the physical reasons that the ground state is isotropic and has definite charge.

The Dirac equations for the baryons are

$$(i\gamma_{\mu}\partial^{\mu} - m_{B}^{*} - g_{\omega B}\gamma_{\mu}\omega^{\mu} - \frac{1}{2}g_{\rho_{B}}\gamma_{\mu}\tau\cdot\boldsymbol{\rho}^{\mu})\Psi_{B} = 0.$$
(23)

Taking account of the vanishing of the spacelike components of the vector fields, the eigenvalues can be found as

$$e_B(k_B) = g_{\omega B}\omega_0 + g_{\rho_B}\rho_{03}I_{3B} + (k_B^2 + m_B^{*2})^{1/2} .$$
(24)

In the above equations, I_{3B} is the isospin projection of baryon charge state B.

Chemical equilibrium can be imposed through the two independent chemical potentials μ_n, μ_e for the conserved baryon and electron charge. Strangeness is not conserved on any macroscopic time scale. For the baryon species *B* we have $\mu_B = b_B \mu_n - q_B \mu_e$, where b_B and q_B are the baryon and electron charge (in units of *e*) of particles *B*. The Fermi momenta for the baryons are the positive real solutions of

$$e_B(k_B) = \mu_B \quad (N \text{ equations}) ,$$
 (25)

where N is the number of different baryon species, including their charge states, that are listed in Ref. [16]. The lepton Fermi momenta are the positive real solutions of

$$(k_e^2 + m_e^2)^{1/2} = \mu_e ,$$

$$(k_\mu^2 + m_\mu^2)^{1/2} = \mu_\mu = \mu_e .$$
(26)

Contact with empirical data is made in the following way. When the above equations are solved at a chosen value of the conserved density of baryons,

$$\rho = \sum_{B} (2J_{B} + 1)k_{B}^{3} / (6\pi^{2}) , \qquad (27)$$

and with the conservation condition that the density of hadronic isospin vanishes,

$$\sum_{B} I_{3B}(2J_{B}+1)k_{B}^{2}/(6\pi^{2})=0 , \qquad (28)$$

there is an equilibrium state consisting of an equal number of neutrons and protons. The coupling constants are adjusted so that the properties for this state correspond closely to those of symmetric nuclear matter having the saturation density, binding energy, symmetry energy, compression modulus, and effective nucleon mass at saturation:

$$\rho_0 = 0.16 \text{ fm}^{-3}, \quad B / A = 16 \text{ MeV},$$

 $a_{\text{sym}} = 32.5 \text{ MeV}, \quad (29)$

 $K = 225 \text{ MeV}, \quad m_{\text{sat}}^* / m = 0.855.$

The coupling constants are

~

$$(g_{\sigma}/m_{\sigma})^2 = 7.487$$
, $(g_{\omega}/m_{\omega})^2 = 2.615$,
 $(g_{\rho}/m_{\rho})^2 = 4.774$ fm². (30)

In the lowest-energy state, the leptons are absent since they are not needed to satisfy the conservation conditions [Eqs. (27) and (28)], and their presence would only increase the energy.

When the same theory is solved with the same coupling constants but with the conservation condition of isospin symmetry replaced by that of the condition of charge neutrality expressed by

$$q_{H} + q_{e} = 0 ,$$

$$q_{H} \equiv \sum_{B} (2J_{B} + 1)q_{B}k_{B}^{3}/(6\pi^{2}) ,$$

$$q_{e} \equiv -\sum_{\lambda} k_{\lambda}^{3}/(3\pi^{2}) = 0 ,$$
(31)

where the first sum is over the baryons whose electric charges are denoted by q_B , Fermi momenta by k_B , and the second sum is over the leptons e^- and μ^- , we get the corresponding solution for β -stable charge-neutral matter called neutron-star matter at the chosen baryon density ρ . The solution of the above coupled equations provides the values for

$$\sigma, \omega_0, \rho_{03}, \mu_n, \mu_e, k_e, k_\mu, k_n, k_p, k_\Lambda, \dots,$$
(32)

of which there are (7+N).

Note that from Eqs. (24)-(26) the Fermi momenta depend on the internal field variables σ , ω_0 , ρ_{03} and the two independent chemical potentials $\mu_{n,}\mu_{e}$ for conserved baryon and electric charges, so that the expressions for the baryon and charge density, [Eqs. (27) and (31)] in the hadronic phase are in accordance with the dependences

quoted in Eq. (11). This completes a description of the equations that define the solution of the above Lagrangian for charge-neutral matter in equilibrium, which is called neutron-star matter.

Once the solution has been found, the equation of state can be calculated from

$$\epsilon = \frac{1}{2}m_{\sigma}^{2}\sigma^{2} + \frac{1}{2}m_{\omega}^{2}\omega_{0}^{2} + \frac{1}{2}m_{\rho}^{2}\rho_{03}^{2}$$

$$+ \sum_{B} \frac{2J_{B} + 1}{2\pi^{2}} \int_{0}^{k_{B}} (k^{2} + m_{B}^{*2})^{1/2}k^{2}dk$$

$$+ \sum_{\lambda} \frac{1}{\pi^{2}} \int_{0}^{k_{\lambda}} (k^{2} + m_{\lambda}^{2})^{1/2}k^{2}dk , \qquad (33)$$

which is the energy density, while the pressure is given by

$$p = -\frac{1}{2}m_{\sigma}^{2}\sigma^{2} + \frac{1}{2}m_{\omega}^{2}\omega_{0}^{2} + \frac{1}{2}m_{\rho}^{2}\rho_{03}^{2} + \frac{1}{3}\sum_{B}\frac{2J_{B}+1}{2\pi^{2}}\int_{0}^{k_{B}}k^{4}dk\,/(k^{2}+m_{B}^{*2})^{1/2} + \frac{1}{3}\sum_{\lambda}\frac{1}{\pi^{2}}\int_{0}^{k_{\lambda}}k^{4}dk\,/(k^{2}+m_{\lambda}^{2})^{1/2}.$$
(34)

To describe quark matter, we use a simple version of the bag model for which the pressure, energy density, and baryon number and charge density at T=0 are given by

$$\begin{split} p &= -B + \sum_{f} \frac{1}{4\pi^{2}} \left[\mu_{f} (\mu_{f}^{2} - m_{f}^{2})^{1/2} (\mu_{f}^{2} - \frac{5}{2}m_{f}^{2}) \right. \\ &+ \frac{3}{2} m_{f}^{4} \ln \left[\frac{\mu_{f} + (\mu_{f}^{2} - m_{f}^{2})^{1/2}}{m_{f}} \right] \right] , \\ \epsilon &= B + \sum_{f} \frac{3}{4\pi^{2}} \left[\mu_{f} (\mu_{f}^{2} - m_{f}^{2})^{1/2} (\mu_{f}^{2} - \frac{1}{2}m_{f}^{2}) \right] , \end{split}$$
(35)
$$&- \frac{1}{2} m_{f}^{4} \ln \left[\frac{\mu_{f} + (\mu_{f}^{2} - m_{f}^{2})^{1/2}}{m_{f}} \right] , \\ \rho &= \sum_{f} \frac{(\mu_{f}^{2} - m_{f}^{2})^{3/2}}{3\pi^{2}} , \end{cases}$$
$$&q = \sum_{f} q_{f} \frac{(\mu_{f}^{2} - m_{f}^{2})^{3/2}}{3\pi^{2}} , \end{split}$$

where the sum f is over flavors. In this simple model of quark matter, there are no internal variables since the quarks are assumed to form a free Fermi gas. Pressure, densities, etc., are specified simply by the chemical potentials. We take, for the masses, $m_u = m_d = 0$, $m_s = 150$ MeV, and $m_c = 1500$ MeV. Because of the long time scale, strangeness is not conserved in a star. The quark chemical potentials for a system in chemical equilibrium are therefore related to those for the baryon number and electron by

$$\mu_{u} = \mu_{c} = \frac{1}{3}(\mu_{n} - 2\mu_{e}) , \quad \mu_{d} = \mu_{s} = \frac{1}{3}(\mu_{n} + \mu_{e}) . \quad (36)$$

Charge neutrality in this pure phase can be expressed by $q+q_e=0$, with q given as above and q_e as defined in Eqs. (31).

In Fig. 6 the behavior of the pressure, the energy densi-



FIG. 6. Pressure, energy density, and chemical potentials as function of weighted baryon density [cf. Eq. (37)] when there is more than one conserved charge.

ty weighted according to the proportion of phases,

$$\boldsymbol{\epsilon} = (1 - \chi)\boldsymbol{\epsilon}_{H}(\boldsymbol{\mu}_{n}, \boldsymbol{\mu}_{e}) + \chi\boldsymbol{\epsilon}_{O}(\boldsymbol{\mu}_{n}, \boldsymbol{\mu}_{e}) , \qquad (37)$$

and chemical potentials are shown as a function of the weighted baryon density. The monotonic increase of pressure in the mixed phase is in sharp contrast to the behavior in the description involving a single chemical potential, which was shown in Fig. 2. It is interesting also to compare the density of hadronic and quark matter in the mixed phase in the two cases. In the present instance, this is shown in Fig. 7. The densities are different in the two phases in equilibrium, but only by a few percent, except at the low-density end of the mixed phase, and they vary with the proportion of the phases. In contrast, when only one charge is conserved, as in Fig. 2, the density of each phase is a constant for all proportions and the density of the quark phase is about a factor 2 larger than



FIG. 7. Baryon and energy densities of each phase as a function of weighted baryon density. The proportion of the phases varies from pure hadronic at the low end of the mixed phase to pure quark matter at the high end.

the confined hadronic phase. In a word, a first-order phase transition in a system with more than one conserved charge is *smooth* compared with one in which there is a single charge. The populations in the star at the mass limit are shown as a function of Schwarzschild radius in Fig. 8. The entire core of the star out to 7.5 km is in the mixed phase. Populations of quarks are very high within 5 km. The quark matter phase is seen to be negatively charged, while the hadronic matter phase is positively charged. This has been explained in Sec. IV as a consequence of the isospin-symmetry force of the strong interaction. Their charges are almost equal and opposite through most of the star, the lepton populations which complete the charge neutrality being very small.

The masses of stars in the two cases were compared in Fig. 3. Unlike the situation where the star is artificially treated as having a single conserved charge with a consequent constant pressure in the mixed phase and therefore an absence of stable stars with central densities falling in the range of the mixed phase, in the case with two conserved charges, as corresponds to a star in β equilibrium, there is no unstable range until the normal Oppenheimer mass limit is reached. The density profile of the two cases was compared in Fig. 4.

We have solved both models in the limit of infinite uniform matter. Thus we assume that all significant regions of space occupied by either phase are large in the sense that the volume energy is large compared with the surface and Coulomb rearrangement energies associated with the nonuniform distribution of charge in the mixed phase. This is the spirit of beginning an investigation in the natural hierarchy of considering first the bulk energy. From dimensional arguments Coulomb and surface energy density is $\epsilon_C \sim (1/V)(\rho_1 V_1)(\rho_2 V_2)/r_{12} \sim r^2$, while $\epsilon_{surf} \sim r^2/V \sim 1/r$. Minimizing the sum with respect to dimension, we find $\epsilon_{surf} = 2\epsilon_C$.] We note that the volume energy against which the importance of these finite-size effects are to be assessed can be read from Fig. 7 and is



FIG. 8 Composition of a hybrid star at the mass limit. Exterior to about 7.6 km is neutron-star matter. Interior to this is a mixed phase of confined hadronic and quark matter which is overall charge neutral.

seen to be ~ 1 GeV/fm³. We also note that the differences in the energies densities of the two phases of the mixed phase are nearly the same, and so we expect the surface energy to be small. Therefore, although the geometric texture of the mixed phase is likely to be an important feature of the mixed phase, the energy associated with it may be small and is assumed to be so in this paper.

VI. SUMMARY

We extended the discussion of first-order phase transitions in bulk matter to the general case that there is more than one conserved charge in the system, called a complex system, and applied it for the first time to the possible phase transition in the core of neutron stars from hadrons to quark matter. In this paper we neglect finite-size effects that will be associated with the formation of geometric structure of one phase immersed in the other.

For a complex system, the conserved charges can be shared by the two phases in equilibrium in different concentrations in each phase than those with which the body was prepared, consistent with the conservation laws, and it will do so if there are internal forces and Fermi energies which would lower the energy. In fact, it is incorrect to insist that the phases in equilibrium satisfy the conservation laws individually, according, say, to their proportion; Gibbs conditions would be violated in general, as we showed. The condition that the phases in equilibrium satisfy the conservation laws, not individually but only overall, amounts to a weaker constraint. For two phases in equilibrium, there are n-1 independent ratios of charge densities (concentrations) that the internal forces can optimize. How much weaker the conservation conditions are in the mixed phase of a complex system compared with the pure phases obviously depends on the proportions of the phases in equilibrium. (At the extremes of the mixed phase, the constraint is as strong as in the neighboring pure phase, while it is weaker in between.) The energy will therefore vary in a nonlinear fashion with respect to the proportion (or density). Therefore the pressure will vary as the proportion and also the chemical potentials. Of course, this freedom is not available to the system when in a pure phase, nor is it available to a simple system, a single-component system. Compared with first-order phase transitions in single-component systems, such as water, the differences in properties of the mixed phase are dramatic. The constancy of the pressure at all proportions of phases in equilibrium in the latter case is well known, but is peculiar to there being but one conserved "charge." When there is more than one, the mixed phase has the degrees of freedom just written of which are otherwise not available.

Finding phase equilibrium in the general case is far from trivial. It requires solving simultaneously the equations of motion of the internal structure of the body, [Eqs. (8) and (9)] for *two* solutions $\{\phi_1\}, \{\phi_2\}$ that are coupled by the conservation laws [Eqs. (13)] and that satisfy the Gibbs conditions summarized in Eq. (10) for arbitrary proportions χ in the mixed phase.

Many systems in chemistry, physics, and astrophysics

that undergo phase transitions are complex in the sense of containing more than one conserved charge. There are many cases in the literature where a complex system has been fitted into the mold of a simple one by approximation. The consequences in the case that we have examined are drastic. For example, to assess whether neutron stars are likely to have quark cores, many authors approximated both phases as the equivalent of pure neutron matter [2]. This turns a complex system into a simple one and has unexpected and unintended consequences, both for the estimate of the density at which the transition is expected to occur, as well as for the structure of the resulting stars if the transition does occur. First, neutron matter is β unstable and the corresponding quark phase especially so. This latter fact leads to an estimate of the density at which neutron matter converts to quark matter, that is, $\sim 3-4$ times too high in comparison with a calculation that takes account of the equilibrium of the system and the conservation of the two charges of the resulting problem. Thus an estimate by Baym and Chin [2] that the transition density is at $\sim 10\rho_0$ and therefore likely beyond the density found in neutron stars becomes $\sim 3\rho_0$, a density that is expected in neutron stars. Furthermore, there are structural differences in the star model that result from the treatment of the phase transition as if it occurred in a simple body. The mixed phase in a sim*ple body* cannot exist in the presence of gravity, because the pressure in that phase is constant. As noted, this causes a discontinuity in the density distribution in the star occurring at the radius where Gibbs criteria are satisfied. Subject to the caveats discussed in Sec. IV in a compact star in β equilibrium, involving as it does both the conserved baryon and electric charge, the transition region is smoothed out; the density distribution is continuous; only its derivative is discontinuous at the boundaries of the mixed phase with the pure phases. The mixed phase occupies a finite region of the star. In our particular example, it occupied the inner sphere of 7 km radius.

The question naturally arises whether the differences in star structure attributable to our handling of the conservation laws compared with earlier approximations would cause the physics of the star to be different in an observable way. One cannot directly measure the density distribution, for example. What our finding does imply in this respect is that hybrid stars do not form a separate class from neutron stars, as they would appear to do if forced into the mold of bodies lacking the degree of freedom available to complex bodies. This is relevant as far as their ability to sustain fast rotation is concerned, since the energy distribution in a hybrid star, when treated as the complex body that it is, continuous, just as in neutron stars. Another way in which we expect differences to occur that are connected to observables is in viscosity, which influences stability to fast rotation [21], and transport properties, which, of course, influence the cooling rate and electrical conductivity of the core, which in turn influences the decay rate of the magnetic field on which the observation of pulsed radiation depends. These expectations are grounded on the finite width of the mixed phase and its expected varying topological structure

(blobs, rods, slabs) with depth, on the one hand, and the different transport properties of the two phases, on the other hand [22].

One important facet of the example that we studied has been neglected in this paper: the actual geometric structures in the mixed phase that evolve as a function of pressure and their contribution to the energy density and pressure. This involves the unsolved problem of the surface energy between dense nuclear and quark matter. For the treatment of our example to be reasonable, the bulk matter approximation that we have used to obtain an equation of state for each phase must be a good one. This means two things. (1) Bulk energy dominates surface and Coulomb displacement energies, the competition between which determines the sizes and shapes of the blobs of one phase immersed in the other. We have argued but not proved that this is so. (2) If there are proportions for which the size of the objects of one phase immersed in the other is too small for the (uniform) bulk approximation to be valid, the total energy must nonetheless be dominated by the bulk energy of the other phase. Since this situation is likely to arise when the mixed

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phase has a preponderance of one phase or the other, this condition is likely to be satisfied by the dominant phase, the total energy of the low-proportion phase being small by comparison.

The outstanding problems involved in an understanding of the hadron-to-quark matter phase transition in neutron stars, aside from the description of these phases of matter themselves, relate to the geometric structure of the mixed phase and its evolution with the proportion of phases. This involves the surface interface energy between the two phases and the degree to which the regions of one phase immersed in the other are uniform or nonuniform as a result of the Coulomb interaction.

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