Differential luminosity under multiphoton beamstrahlung

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For the next generation of e^+e^- linear colliders in the TeV range, the energy loss due to *beamstrahlung* during the collision of the e^+e^- beams is expected to be substantial. One consequence is that the center-of-mass energy between the colliding particles can be largely degraded from the designed value. The knowledge on the differential luminosity as a function of the center-of-mass energy is essential for particle physics analysis on the interesting events. On the other hand, the beamstrahlung photon spectrum provides useful information on the low-energy backgrounds and high-energy $\gamma\gamma$ luminosity. In this paper, we derive analytic formulas for the e^+e^- and γ energy spectra under multiple beamstrahlung process, and the e^+e^- and $\gamma\gamma$ differential luminosities. Major characteristics of these formulas are discussed.

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I. INTRODUCTION

It is known that *beamstrahlung* [1], the synchrotron radiation from the colliding e^+e^- beams, will carry away a substantial fraction of the primary beam energy E_0 in future linear colliders. This, for one thing, will result in a degradation of the center-of-mass energy of the colliding beams. From the high-energy-physics point of view, it is important to know the luminosity as a function of the effective e^+e^- center of mass, so as to unfold, e.g., the energy dependence of particle production processes. In addition, the low energy end of the e^+e^- and γ spectra are also important for background analysis.

When the average number of beamstrahlung photons radiated per beam particle is much less than unity, the energy spectrum for the final e^+ or e^- beams is simply the well-known Sokolov-Ternov spectrum [2] for the radiated photons with the fractional photon energy, y ($\equiv E_\gamma/E_0$), replaced by the corresponding final electron (or positron) energy, $x=1-y$. When the condition is such that the average number of photons radiated is not much less than unity, the effect of successive radiation becomes important. Previously, the multiphoton beamstrahlung process has been studied by Blankenbecler and Drell [3] and independently by Yokoya and Chen [4]. In this paper, we shall adopt the formulation developed in Ref. [4] as the basis for our derivation of the differential luminosity. In Sec. II, we will review the electron spectrum under multiphoton beamstrahlung. Sec. III will be devoted to the derivation of the e^+e^- differential luminosity. In Sec. IV, we derive the photon spectrum, and in Sec. V, the $\gamma\gamma$ luminosity. The characteristic feature of our formula is discussed and a comparison to computer simulations is presented in the last section. Unless expressed explicitly, the convention $e=\hbar=c=1$ is assumed throughout this paper.

II. ELECTRON ENERGY SPECTRUM

Let $\psi(x,t)$ be the energy spectral function of the electron for energy $x\equiv E/E_0$ at time t normalized as $\int\psi(x,t)dx=1$. We assume that the emission of the photon takes place in an infinitesimally short time interval. Then the interference between successive radiation processes is negligible, and the evolution of the spectral function can be described by the rate equation

$$\frac{\partial\psi}{\partial t}=-\int_0^x dx''\,F(x'',x)\psi(x,t)+\int_x^1 dx'\,F(x,x')\psi(x',t)\ , \tag{1}$$

where the first term corresponds to the *sink,* and the second term to the *source,* for the evolution of $\psi(x,t)$. F is the spectral function of radiation, i.e., $F(x,x')dx'$ is the transition probability of an electron from energy x' to the energy interval $(x,x+dx)$ per unit time. Obviously, $F(x,x')=0$ if $x\geq x'$. Notice, however, that F does not include the probability for electrons to remain at the same energy without photon emission. Pulling out $\psi(x,t)$ from the first term, which is independent of x'', the remaining integral represents the average number of photon radiated per unit time by the electron with an instantaneous energy x:

$$\nu(x)=\int_0^x dx''\,F(x'',x)\ . \tag{2}$$

The spectral function of radiation can be characterized by the beamstrahlung parameter Υ, defined as

$$\Upsilon=\gamma_0\frac{B}{B_c}\ , \tag{3}$$

where $\gamma_0=E_0/mc^2$, B is the effective field strength in the beam, and $B_c=m^2c^3/e\hbar\sim4.4\times10^{13}$ G is the Schwinger critical field. High-energy e^+e^- beams generally follow Gaussian distributions in the three spatial dimensions. Thus the local-field strength varies inside the beam volume. It can be shown [5], however, through integrating over the impact parameter and the longitudinal variations, that the overall beamstrahlung effect can be simply described as if all particles experience, during an effective collision time $\tau=l/2=\sqrt{3}\sigma_z$, a uniform *mean field*

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$$B_{\text{mean}} \simeq \frac{5}{6} \frac{eN}{\sigma_z(\sigma_x+\sigma_y)} , \quad (4)$$

where N is the total number of particles in a bunch, σ_x, σ_y, σ_z are the rms sizes of the Gaussian beam, and $l=2\sqrt{3}\sigma_z$ is the effective length of the oncoming bunch in our model. Thus in the following calculations we will assume, for the entire beam,

$$\Upsilon \equiv \Upsilon_{\text{mean}} \simeq \frac{5}{6} \frac{r_e^2\gamma_0 N}{\alpha\sigma_z(\sigma_x+\sigma_y)} , \quad (5)$$

where r_e is the classical electron radius, and α is the fine structure constant. For $\Upsilon \ll 1$, the radiation is in the *classical* regime, such as that in the SLAC Linear Collider (SLC), where $\Upsilon \sim 0.004$. In contrast, for the next-generation linear colliders, $\Upsilon \sim 0.1$ to 1, and it starts to enter into the *quantum* regime. Notice, however, that the typical number of photons radiated per beam particle is of the order unity. Thus even in the classical regime, such as that in SLC, the discrete nature of beamstrahlung should not be overlooked.

The transition probability F derived by Sokolov and Ternov [2] is

$$F(x,x') = \frac{v_{\text{cl}}\kappa}{xx'} f(\xi,\eta) ,$$

$$f(\xi,\eta) = \frac{3}{5\pi} \frac{1}{1+\xi\eta} \left[\int_\eta^\infty du\, K_{5/3}(u) + \frac{\xi^2\eta^2}{1+\xi\eta} K_{2/3}(\eta) \right] , \quad (6)$$

where $\xi \equiv 3x'\Upsilon/2, \eta = \kappa[(1/x)-(1/x')]$, and for convenience, $\kappa \equiv 2/(3\Upsilon)$. To be sure, while Υ (and therefore κ) is a global parameter in beamstrahlung, the parameter ξ as defined here is not. For any given Υ, ξ ranges from 0 to $3\Upsilon/2$, according to the instantaneous energy carried by the individual particle between successive radiation processes. K_ν's are the modified Bessel functions and ν_{cl} is the number of photons per unit time (or length, with $c=1$), calculated by the classical theory of radiation. By definition, this is also the limiting case for $\nu(x)$ where $x \to 0$:

$$\nu_{\text{cl}} = \nu(x=0) = \frac{5}{2\sqrt{3}} \frac{\alpha^2}{r_e\gamma_0} \Upsilon . \quad (7)$$

Note that for a given field strength ν_{cl} is independent of the particle energy. In general, however,

$$\nu(x) \equiv \nu_{\text{cl}} U_0(x\Upsilon) , \quad (8)$$

where

$$U_0(v) = \begin{cases} 1, & v \to 0 , \\ (28\sqrt{3}/45)\Gamma(2/3)(3v)^{-1/3} = 1.012 v^{-1/3}, & v \to \infty \end{cases}$$
$$\approx [1+v^{2/3}]^{-1/2} .$$

To look for a compact analytic solution for ψ in Equation (1), the exact Sokolov-Ternov spectral function in Eq. (6) is somewhat cumbersome. One can instead invoke an approximate expression [4], which is independent of ξ, to replace $f(\xi,\eta)$ in Eq. (6):

$$\tilde{f}(\eta) = \frac{1}{\Gamma(1/3)} \eta^{-2/3} e^{-\eta} . \quad (9)$$

With this approximation, Eq. (1) can be solved by proper Laplace transformations. The details can be found in Ref. [4]. The solution is

$$\psi(x,t) = e^{-\nu_{\text{cl}}t} \left[\delta(1-x) + \frac{e^{-\eta_x}}{1-x} h(\eta_x^{1/3}\nu_{\text{cl}}t) \right] \qquad \Upsilon \ll 1 , \quad (10)$$

where $\eta_x \equiv k[(1/x)-1]$, and

$$h(u) = \frac{1}{2\pi i} \int_{\lambda-i\infty}^{\lambda+i\infty} \exp(up^{-1/3}+p)dp = \sum_{n=1}^{\infty} \frac{u^n}{n!\Gamma(n/3)} , \quad (11)$$

with $\lambda > 0$ and $0 \le u \le \infty$. The first term in Equation (10) represents the electron population that suffers no radiation. The nth term in the Taylor expansion of the second term corresponds to the process of n-photon emissions.

For finite values of Υ, the rate equation cannot be solved exactly since $\nu(x)$ is not constant in time anymore. However, in the intermediate regime where $\Upsilon \lesssim 10$, $\nu(x)$ should not deviate from ν_{cl} too significantly. This suggests a solution based upon minor perturbation from the above classical result. It is found [4] that

$$\psi(x,t) = e^{-\nu_\gamma t} \left[\delta(1-x) + \frac{e^{-\eta_x}}{1-x} h(\eta_x^{1/3}\bar{\nu}t) \right] , \quad \Upsilon \lesssim 10 , \quad (12)$$

for the intermediate regime, where

$$\nu_\gamma \equiv \nu(x=1) = U_0(\Upsilon)\nu_{\text{cl}} , \quad \bar{\nu} \equiv x\nu_{\text{cl}} + (1-x)\nu_\gamma . \quad (13)$$

In effect, $\bar{\nu}$ is a linear interpolation between the two extrema ν_{cl} and ν_γ. We see that $\bar{\nu} \to \nu_{\text{cl}}$ as $x \to 1$, since for the electron to remain at high energy after an n-photon process, it can only have radiated classically. On the other hand, $\bar{\nu} \to \nu_\gamma$ as $x \to 0$. This indicates that the low energy electron spectrum is mostly contributed by quantum radiations.

III. CENTER-OF-MASS e^+e^- LUMINOSITY

To find the differential luminosity $\mathcal{L}(x)$ as a function of the effective center-of-mass energy squared, s, one needs to convolute the energy spectrum of one beam, $\psi(x_1,t)$, with the other $\psi(x_2,t)$. Let $t=0$ when the e^+e^- bunches first meet. In addition, let the longitudinal coordinate z along the beam be defined such that $z=0$ at the front of each beam. Then the first z slice in beam 1 will always encounter a "fresh" beam 2:

$$\frac{d^3\mathcal{L}_{e^+e^-}(x_1,x_2,0)}{dx_1dx_2dz} \propto \frac{2}{l} \int_0^{l/2} dt\, \psi(x_1,t)\psi(x_2,0) , \quad (14)$$

where l is the total length of each bunch. As explained in the preceding section, our model assumes a uniform field within an effective bunch length $l=2\sqrt{3}\sigma_2$, in relating to the Gaussian distribution. The total collision time is $l/2$ because both beams move at the speed of light against each other. A slice at z in beam No. 1, however, will always see a beam No. 2 which has evolved for a time $t=z/2$:

$$\frac{d^3\mathcal{L}_{e^+e^-}(x_1,x_2,z)}{dx_1dx_2dz}\propto\frac{2}{l}\int_0^{l/2}dt\,\psi(x_1,t)\psi(x_2,z/2)\ . \quad (15)$$

Adding all z slices in beam no. 1 together, we have

$$\frac{d^2\mathcal{L}_{e^+e^-}(x_1,x_2)}{dx_1dx_2}\propto\frac{4}{l^2}\int_0^{l/2}dt\,\psi(x_1,t)\int_0^{l}dz\,\psi(x_2,z/2)$$
$$=\frac{4}{l^2}\int_0^{l/2}dt\,\psi(x_1,t)\int_0^{l/2}dz\,\psi(x_2,z)\ . \quad (16)$$

Note that the two integrals in the last expression are functionally identical. Inserting the spectral function in Eq. (10), we find, for $\Upsilon\ll 1$,

$$\psi(x)\equiv\frac{2}{l}\int_0^{l/2}dt\,\psi(x,t)$$
$$=\frac{1}{N_{\rm cl}}\left[(1-e^{-N_{\rm cl}})\delta(1-x)+\frac{e^{-\eta_x}}{1-x}\bar{h}(x)\right]\ ,\quad \Upsilon\ll 1\ , \quad (17)$$

where $\eta_x=\kappa[(1/x)-1]$, and $N_{\rm cl}=\nu_{\rm cl}l/2$ is the average number of photons radiated per particle during the entire collision of the e^+e^- beams. The function $\bar{h}(x)$ in the second term is

$$\bar{h}(x)\sum_{n=1}^{\infty}\frac{\eta_x^{n/3}}{n!\Gamma(n/3)}\gamma(n+1,N_{\rm cl})\ , \quad (18)$$

where $\gamma(n+1,N_{\rm cl})$ is the incomplete gamma function.

The center-of-mass energy squared for the system of two particles with energies x_1 and x_2, normalized to the reference center-of-mass energy squared, $s_0=4$, is $s\equiv x_1x_2$. The differential luminosity as a function of s is therefore

$$\frac{d\mathcal{L}_{e^+e^-}(s)}{ds}=\mathcal{L}_0\int_s^1\int_0^1dx_1\,dx_2\times\delta(s-x_1x_2)\psi(x_1)\psi(x_2)\ , \quad (19)$$

where $\mathcal{L}_0$ is the nominal luminosity of the collider, including the *enhancement factor* due to the beam-beam disruption effect [6]. It is straightforward to find that

$$\frac{d\mathcal{L}_{e^+e^-}(s)}{ds}=\frac{\mathcal{L}_0}{N_{\rm cl}^2}\Big\{(1-e^{-N_{\rm cl}})^2\delta(1-s)+2(1-e^{-N_{\rm cl}})\frac{e^{-\eta_s}}{1-s}\bar{h}(s)+\int_s^1\frac{dx}{x}\frac{e^{\eta_x-\eta_{s/x}}}{(1-x)(1-s/x)}\bar{h}(x)\bar{h}(s/x)\Big\}, \quad (20)$$

where $\eta_s=\kappa[(1/2)-1]$. It can be shown that in the classical regime the last term is much smaller than unity, and is negligible. Thus

$$\frac{d\mathcal{L}_{e^+e^-}(s)}{ds}=\frac{\mathcal{L}_0}{N_{\rm cl}^2}\Big\{[1-e^{-N_{\rm cl}}]^2\delta(1-s)+2[1-e^{-N_{\rm cl}}]\frac{e^{-\eta_s}}{1-s}\bar{h}(s)\Big\}\ ,\quad \Upsilon\ll 1\ . \quad (21)$$

For the intermediate regime, the spectral function of Eq. (10) should be replaced by Eq. (12). The derivation is essentially the same, and we find

$$\psi(x)=\frac{1}{N_\gamma}[(1-e^{-N_\gamma})\delta(1-x)+\frac{e^{-\eta_x}}{1-x}\tilde{h}(x)]\ ,\quad \Upsilon\lesssim 10\ , \quad (22)$$

where $N_\gamma=\nu_\gamma l/2$, and

$$\tilde{h}(x)=\sum_{n=1}^{\infty}\left[\frac{\bar{\nu}}{\nu_\gamma}\right]^n\frac{\eta_x^{n/3}}{n!\Gamma(n/3)}\gamma(n+1,N_\gamma)\ . \quad (23)$$

When the average energy loss per electron is becoming substantial, which is possible in the transition regime, the integral term in Eq. (20) should be retained. The differential luminosity in this regime is therefore

$$\frac{d\mathcal{L}_{e^+e^-}(s)}{ds}=\frac{\mathcal{L}_0}{N_\gamma^2}\Big\{[1-e^{-N_\gamma}]^2\delta(1-s)+2[1-e^{-N_\gamma}]\frac{e^{-\eta_s}}{1-s}\tilde{h}(s)+\int_s^1\frac{dx}{x}\frac{e^{\eta_x-\eta_{s/x}}}{(1-x)(1-s/x)}\tilde{h}(x)\tilde{h}(s/x)\Big\}\ ,\quad \Upsilon\lesssim 10\ , \quad (24)$$

where , in addition to η_s, the x dependence of $\bar{\nu}$ in $\tilde{h}(s)$ is also replaced by s.

IV. PHOTON ENERGY SPECTRUM

Next we look for the companion formulas for the beamstrahlung photons. Let us ignore the loss of photons due to beamstrahlung pair creation [7], which constitutes only a fraction $\sim\alpha$ (fine structure constant) of the total photon population. Then the time evolution of the spectrum is dominated by the beamstrahlung process alone:

$$\frac{\partial\phi}{\partial t}=\int_y^1dx\,F(x-y,x)\psi(x,t)\ , \quad (25)$$

where $y \equiv E_\gamma / E_0$ is the photon fractional energy. Therefore

$$\phi(y,t)=\int_0^t dt' \int_y^1 dx\, F(x-y,x)\psi(x,t') , \quad (26)$$

Note that while $\int \psi(x,t)dx=1$, which conserves the electron (or positron) number, the photon number accumulates along the course of collision, and in general $\int \phi(y,t)dy \neq 1$. Combining Eqs. (6), (9), and (10), we have, for $\Upsilon \ll 1$,

$$\phi(y,t)=\frac{\nu_{\rm cl}\kappa^{1/3}}{\Gamma(1/3)} y^{-2/3} \int_0^t dt'\, e^{-\nu_{\rm cl}t'} \{ (1-y)^{-1/3} e^{-\kappa y/(1-y)} + I(y,t') \} , \quad (27)$$

where

$$I(y,t')=\sum_{n=1}^{\infty} \frac{\kappa^{n/3}(\nu_{\rm cl}t')^n}{n!\Gamma(n/3)} e^{\kappa} \int_y^1 dx\, x^{-(n+1)/3} (x-y)^{-1/3} (1-x)^{n/3-1} e^{-\kappa/(x-y)} .$$

The above integrand is exponentially suppressed when $x \to y$ for any value of y. On the other hand, when $x \to 1$, it is dominated by the term $(1-x)^{n/3-1}$. So it is a reasonable approximation by setting $x^{-(n+1)/3} \approx 1$. Under this approximation, we find

$$\int_y^1 dx\, (x-y)^{-1/3} (1-x)^{n/3-1} e^{-\kappa/(x-y)} = \Gamma(n/3)\kappa^{-1/6} (1-y)^{n/3-1/6} e^{-\kappa/2(1-y)} W_{-n/3+1/6,1/3} \left[\frac{\kappa}{1-y} \right] , \quad (28)$$

where $W_{\mu,\nu}(z)$ is the Whittaker function:

$$W_{\mu,\nu}(z)=\frac{z^{\mu} e^{-z/2}}{\Gamma(\nu-\mu+\frac{1}{2})} \int_0^{\infty} u^{\nu-\mu-1/2} e^{-u} \left[1+\frac{u}{z} \right]^{\nu+\mu-1/2} du \equiv z^{\mu} e^{-z/2} [1-w_{\mu,\nu}(z)] , \quad (29)$$

where $w_{\mu,\nu}(z) \to 0$ as $z \to \infty$. In the classical limit, $\kappa \gg 1$. Thus $\kappa/(1-y) \gg 1$ for all y, and the Whittaker function takes the asymptotic form $W_{\mu,\nu}(z)=z^{\mu} e^{-z/2}$. We therefore have

$$I(y,t')=(1-y)^{-1/3} e^{-\kappa y/(1-y)} \sum_{n=1}^{\infty} \frac{[(1-y)^{2/3}\nu_{\rm cl}t']^n}{n!} . \quad (30)$$

Inserting Equation (30) into Equation (27), we find

$$\phi(y,t)=\frac{\nu_{\rm cl}\kappa^{1/3}}{\Gamma(1/3)} y^{-2/3} (1-y)^{-1/3} e^{-\kappa y/(1-y)} \times \int_0^t dt'\, e^{-[1-(1-y)^{2/3}]\nu_{\rm cl}t'} . \quad (31)$$

The integration over time is straightforward, and we finally obtain

$$\phi(y,t)=\frac{\kappa^{1/3}}{\Gamma(1/3)} y^{-2/3} (1-y)^{-1/3} e^{-\kappa y/(1-y)} G(y) , \quad \Upsilon \ll 1 , \quad (32)$$

where

$$G(y)=\frac{1}{g(y)}[1-e^{-g(y)\nu_{\rm cl}t}] , \quad g(y)=1-(1-y)^{2/3} . \quad (33)$$

Note that in the limit $\nu_{\rm cl}t \ll 1$, the terms in the square brackets can be replaced by $g(y)\nu_{\rm cl}t$. This recovers the known expression for the beamstrahlung photon spectrum using single-photon (i.e., disregarding the loss of e^- energy between successive radiation processes) picture:

$$\lim_{\nu_{\rm cl}t \to 0} \phi(y,t)=\frac{\kappa^{1/3}}{\Gamma(1/3)} y^{-2/3} (1-y)^{-1/3} e^{-\kappa y/(1-y)} \nu_{\rm cl}t . \quad (34)$$

In the $y \ll 1$ limit, the y dependence is approximately $\propto y^{-2/3}$.

To extend our result to the nonclassical regime, we find that a similar calculation as above but using Eq. (11) for the electron spectrum would be quite complex, due to the additional x dependence in $\bar{\nu}$. Instead, we shall follow the same philosophy as in Sec. II by adopting the form of Eq. (32) and replacing $\nu_{\rm cl}$'s by ν_γ and $\bar{\nu}$ in a similar fashion. An inspection of $I(y,t')$ in Eq. (27) suggests that, if one intends to extract $(\bar{\nu}t')^n$ out from the integrand such that a similar calculation for the nonclassical regime can follow, the x dependence in $\bar{\nu}$ should be properly averaged over the spectrum. Again, in the linear approximation, we find

$$\langle \bar{\nu} \rangle = \frac{1}{1-y} \int_y^1 dx [x\nu_{\rm cl}+(1-x)\nu_\gamma] = \frac{1}{2}[(1+y)\nu_{\rm cl}+(1-y)\nu_\gamma] . \quad (35)$$

In principle, one could then express $I(y,t')$ in terms of the Whittaker function. But if one wishes to further simplify $I(y,t')$ through the asymptotic expansion of Eq. (29), then it is necessary that the correction term $w_{\mu,\nu}(z)$ be retained. In the n-photon process, the leading order $n=1$ dominates, which gives $\mu=-\frac{1}{6}$ and $\nu=\frac{1}{3}$. Ignoring the y dependence in z, we find that

$$w_{\mu,\nu}\left[\frac{\kappa}{1-y} \right] \approx \frac{1}{6\sqrt{\kappa}} \equiv w , \quad \Upsilon \lesssim 5 . \quad (36)$$

We then have

$$\phi(y,t)=\frac{\kappa^{1/3}}{\Gamma(1/3)}y^{-2/3}(1-y)^{-1/3}e^{-\kappa y/(1-y)}\tilde{G}(y)\ , \qquad \Upsilon \lesssim 5\ , \tag{37}$$

where

$$\tilde{G}(y)=\frac{1-w}{\tilde{g}(y)}[1-e^{-\tilde{g}(y)\nu_\gamma t}]+w[1-e^{-\nu_\gamma t}]\ ,$$
$$\tilde{g}(y)=1-\frac{\langle\bar{\nu}\rangle}{\nu_\gamma}(1-y)^{2/3}\ . \tag{38}$$

V. CENTER-OF-MASS $\gamma\gamma$ LUMINOSITY

The $\gamma\gamma$ center-of-mass luminosity can be obtained in the same way we did in Sec. III. It amounts to looking for integration of $\phi(y,t)$ over the e^+e^- collision time. We find, for $\Upsilon \ll 1$,

$$\phi(y)=\frac{2}{l}\int_0^{l/2}dt\ \phi(y,t)$$
$$=\frac{\kappa^{1/3}}{\Gamma(1/3)}y^{-2/3}(1-y)^{-1/3}e^{-\kappa y/(1-y)}\bar{G}(y)\ , \tag{39}$$

where

$$\bar{G}(y)=\frac{1}{g(y)}\left[1-\frac{1}{g(y)N_{\rm cl}}(1-e^{-g(y)N_{\rm cl}})\right]\ . \tag{40}$$

For the nonclassical regime, the corresponding expression reads

$$\phi(y)=\frac{\kappa^{1/3}}{\Gamma(1/3)}y^{-2/3}(1-y)^{-1/3}e^{-\kappa y/(1-y)}\bar{\tilde{G}}(y)\ , \qquad \Upsilon \lesssim 5\ , \tag{41}$$

where

$$\bar{\tilde{G}}(y)=\frac{1-w}{\tilde{g}(y)}\left\{1-\frac{1}{\tilde{g}(y)N_\gamma}[1-e^{-\tilde{g}(y)N_\gamma}]\right\}$$
$$+w\left\{1-\frac{1}{N_\gamma}[1-e^{-N_\gamma}]\right\}\ . \tag{42}$$

The center-of-mass $\gamma\gamma$ luminosity is then

$$\frac{d\mathcal{L}_{\gamma\gamma}(s)}{ds}=\mathcal{L}_0\int_s^1\int_0^1 dy_1\,dy_2\,\delta(s-y_1y_2)\phi(y_1)\phi(y_2)\ . \tag{43}$$

The integration is quite involved, and since simple expression of $d\mathcal{L}_{\gamma\gamma}/ds$ for the whole range of $0\le s\le 1$ is not easily attainable, numerical calculations may be necessary.

VI. DISCUSSION

To confirm our theoretical formulas, we perform computer simulations using the code ABEL [8]. The parameters of a linear collider with a center-of-mass energy $\frac{1}{2}$ TeV designed by Palmer [9] (the Machine G in Table I in Ref. [9]) were used.

The parameter $\Upsilon=0.39$ in this example uses the nominal values of σ_x and σ_y. As is well known, the field intensity of a flat beam (i.e., $\sigma_x \gg \sigma_y$) is determined largely by σ_x. In the case when the *disruption* in the x dimension is not negligible, the effective σ_x during collision is different from the nominal value. This is indeed the case for Palmer's G machine. The disruption parameter is defined as

$$D_{x,y}=\frac{2Nr_e\sigma_z}{\gamma\sigma_{x,y}(\sigma_x+\sigma_y)}\ . \tag{44}$$

The effective σ_x can be deduced from the luminosity enhancement factor for round beams [10]:

$$H_D=1+\frac{2}{3\sqrt{\pi}}D\ , \quad D\ll 1\ . \tag{45}$$

Since the enhancement results from the reduction of the effective beam size, we can estimate the effective σ_x as

$$\bar{\sigma}_x\sim\frac{\sigma_x}{\sqrt{H_D}}\ . \tag{46}$$

In our case, $D_x=0.7$. Thus $\bar{\sigma}_x\sim 0.89\sigma_x$, and we find the effective $\Upsilon\sim 0.44$.

The simulation has the disruption effect included, but the beamstrahlung parameter as defined in Eq. (5) was not calculated in ABEL. Instead, for every photon radiated, there is a *critical energy* registered, using the local field strength and the instantaneous energy of the radiating electron prior to its radiation. The average of all the critical energies is then translated into an effective beamstrahlung parameter $\Upsilon\sim 0.43$, which is in very good agreement with what we estimated above. Note that this effective Υ from simulation has been weighted by the photon number, and does not have a fixed electron energy.

Using this effective value of $\Upsilon(=0.43)$, and with the bunch length $l=2\sqrt{3}\sigma_z=0.38$ mm, we calculate the number of photons $\phi(y,l/2)\Delta y$, with $\Delta y=0.02$, at the end of the collision using Eq. (37). Figure 1 shows the final photon spectrum from our formula and from simulations. We see that the agreement is quite good for a large part of the spectrum. Both high- and low-energy ends of the spectrum from our theory, however, tend to be softer

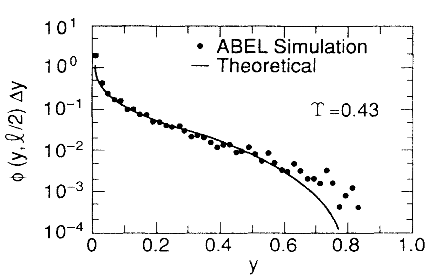

FIG. 1. Final beamstrahlung photon spectrum calculated by computer simulation and by the analytic formula equation (37). The number of photons $\phi(y,l/2)\Delta y$ is plotted against photon energy y, where $\Delta y=0.02$ in this case. Parameters from Palmer's G machine where $\Upsilon=0.43$ were used.

than that from the simulation. But the statistics from simulation is quite low at the high energy end, thus the discrepancy there should not be overemphasized. The average number of photons radiated per particle is obtained by integrating $\phi(y,l/2)$ over y. We find $\int \phi(y,l/2)dy \sim 3.27$. This agrees with the simulation result, ~ 3.55 photons per electron, to within 10%. Incidentally, however, the direct estimation $N_\gamma = \nu_\gamma l/2 \sim 3.55$ agrees almost perfectly with the simulation result. The discrepancy is due mainly to the slight underestimation of photon spectrum, Eq. (37), in the $y \ll 1$ limit.

For the e^+e^- differential luminosity, a two-dimensional plot from the simulation results of $(d^2\mathcal{L}_{e^+e^-}/dx_1dx_2)\Delta x_1\Delta x_2$ per beam crossing as a function of x_1 and x_2 is shown in Fig. 2. The example used in this calculation was Palmer's F machine, the so-called *flat* beam design, for a 0.5 TeV collider. The beamstrahlung parameter is $\Upsilon \simeq 0.12$, considerably smaller than the G machine. Indeed, in this case the average number of photons per electron is of the order one, and the average energy loss is only $\sim 4\%$. We see that the most striking character of the e^+e^- luminosity spectrum in this particular case is that, aside from the sharp delta function at the nominal machine energy, other contribution to the e^+e^- luminosity comes essentially from the matching between a full energy particle and a beamstrahlung degraded particle. This is evidenced by the "walls" on the edges of the two-dimensional plot, which corresponds to the second term in Eq. (24). The last (integral) term in that equation is seen to be negligible in this case. However, because of the stronger beamstrahlung and larger number of photons per electron,

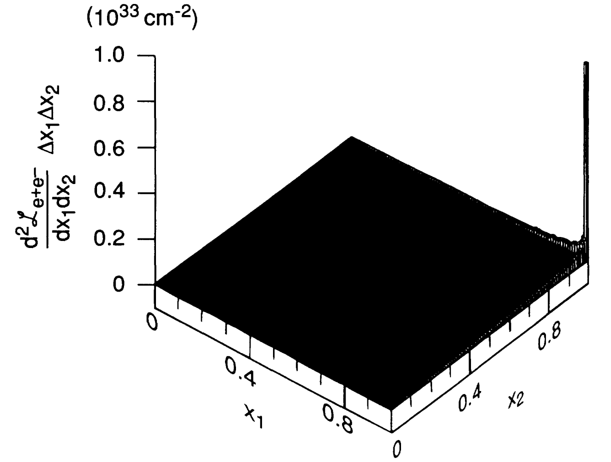

FIG. 2. Two-dimensional plot of the e^+e^- differential luminosity $(d^2\mathcal{L}_{e^+e^-}/dx_1\,dx_2)\Delta x_1\,\Delta x_2$ per beam crossing as a function of the e^+e^- fractional energies, x_1,x_2, from computer simulation. The width of the bins is $\Delta x_1 = \Delta x_2 = 0.02$. The example used is Palmer's F design for a 0.5 TeV linear collider, where $\Upsilon \simeq 0.12$.

there is a finite contribution from this integral term in the case of Palmer's G machine.

It goes without saying that the $e\gamma$ luminosity can also be derived by convoluting $\psi(x)$ and $\phi(y)$.

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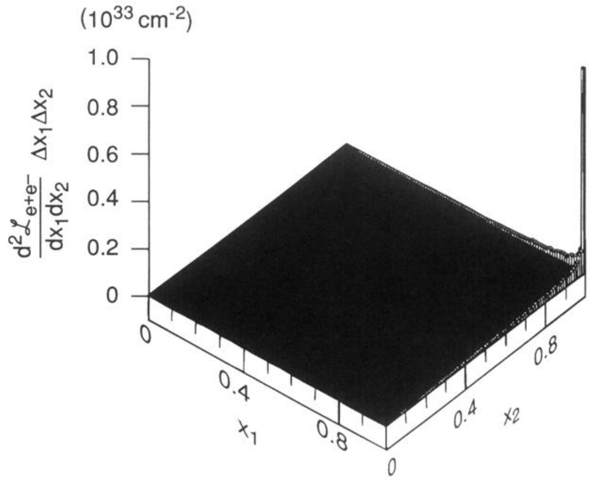

FIG. 2. Two-dimensional plot of the e^+e^- differential luminosity $(d^2\mathcal{L}_{e^+e^-}/dx_1\,dx_2)\Delta x_1\,\Delta x_2$ per beam crossing as a function of the e^+e^- fractional energies, x_1,x_2, from computer simulation. The width of the bins is $\Delta x_1 = \Delta x_2 = 0.02$. The example used is Palmer's F design for a 0.5 TeV linear collider, where $\Upsilon \simeq 0.12$.