Quantitative evaluation of first-order retardation corrections to the quarkonium spectrum

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We evaluate numerically first-order retardation corrections for some charmonium and bottomonium masses under the usual assumption of a Bethe-Salpeter purely scalar confinement kernel. The result depends strictly on the use of an additional effective potential to express the corrections (rather than to resort to Kato perturbation theory) and on an appropriate regularization prescription. The kernel has been chosen in order to reproduce in the instantaneous approximation a semirelativistic potential suggested by the Wilson loop method. The calculations are performed for two sets of parameters determined by fits in potential theory. The corrections turn out to be typically of the order of a few hundred MeV and depend on an additional scale parameter introduced in the regularization. A conjecture existing in the literature on the origin of the constant term in the potential is also discussed.

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I. INTRODUCTION

In this paper we want to evaluate quantitatively first-order corrections to the instantaneous approximation for a quark-antiquark Bethe-Salpeter (BS) kernel with a purely scalar confinement part.

As is well known in the positronium case [1,2], very-high-order corrections to the zero-order instantaneous approximation have been systematically evaluated. To our knowledge, no similar result exists in the literature for a confining kernel. In fact, only very rough order-of-magnitude estimates are considered in this case [3].

Theoretical analyses of heavy quarkonium are usually given in terms of an effective relativistic or semirelativistic two-body Hamiltonian:

$$\begin{aligned} H &= m_1+m_2+\sqrt{m_1^2+p_1^2}+\sqrt{m_2^2+p_2^2}+V \\ &\simeq m_1+m_2+\frac{p_1^2}{2m_1^2}+\frac{p_2^2}{2m_2^2}-\frac{p_1^4}{8m_1^3}-\frac{p_2^4}{8m_2^3} \\ &\quad +V_{\rm stat}+V_{\rm SD}+V_{\rm VD}\ , \end{aligned} \tag{1.1}$$

where the second line is intended as a first-order expansion in $1/m^2$ of the first one, $V_{\rm stat}$ denotes a zero-order purely static term, $V_{\rm SD}$ and $V_{\rm VD}$ are $1/m^2$ order spin-dependent and velocity-dependent terms.

In principle, the potential V should be obtained in the framework of QCD. In practice, starting from a pioneering work by Wilson [4], a method has been developed which works in terms of a $1/m^2$ expansion and gives the various contributions in the form of functional integrals in the gauge field only. Explicit analytical expressions can be obtained in the two limit situations $r\to 0$ and $r\to\infty$, admitting the significance of the weak-coupling expansion in the first case and of the strong-coupling expansion in the second one and making some additional plausible assumptions.

Adding simply the two contributions and restricting to the equal-mass case, one obtains [5–7], in the center-of-mass system ($\mathbf{p}_1=-\mathbf{p}_2=\mathbf{q}$),

$$V_{\rm stat}=-\frac{\kappa}{r}+C+\sigma r\ , \tag{1.2a}$$

$$\begin{aligned} V_{\rm SD} &= \frac{3}{2m^2r^2}(\mathbf{S}_1+\mathbf{S}_2)\cdot\mathbf{L}\left[\frac{\kappa}{r}-\frac{1}{3}\sigma r\right] \\ &\quad +\frac{3\kappa}{m^2r^3}S_1^h\left[\frac{r^hr^k}{r^2}-\frac{1}{3}\delta^{hk}\right]S_2^k \\ &\quad +\frac{8\pi\kappa}{3m^2}\mathbf{S}_1\cdot\mathbf{S}_2\delta_3(\mathbf{r})\ , \end{aligned} \tag{1.2b}$$

$$\begin{aligned} V_{\rm VD} &= \frac{\pi\kappa}{m^2}\delta(\mathbf{r})-\frac{\kappa}{m^2}q^h\frac{1}{r}q^h-\frac{C}{2m^2}q^2+\frac{5}{18m^2}\frac{\sigma}{r} \\ &\quad +\frac{1}{2m^2r^2}\left[\frac{\kappa}{r}-\frac{1}{3}\sigma r\right]L^2\ , \end{aligned} \tag{1.2c}$$

where $\kappa=\frac{4}{3}\alpha_s$, σ denotes the string tension, and the terms in C are perimeter terms which have been essentially introduced to make a smooth connection between the two asymptotic regions $r\to 0$ and $r\to\infty$. More elaborate contributions could obviously be introduced on phenomenological ground to describe better the intermediate region.

Equation (1.2c) has been derived in Refs. [6,7]. Notice that an alternative form which has also been proposed for $V_{\rm VD}$ is the so-called Barnes-Ghandour potential [8]

$$\begin{aligned} V_{\rm VD} &= \frac{\kappa\pi}{m^2}\delta^3(\mathbf{r})-\frac{\kappa}{m^2}q^h\frac{1}{r}q^h+\frac{1}{2m^2}\frac{\kappa L^2}{r^3} \\ &\quad -\frac{C}{m^2}q^2+\frac{1}{m^2}\frac{\sigma}{r}-\frac{\sigma}{m^2}q^hrq^h\ . \end{aligned} \tag{1.2c'}$$

This differs from Eq. (1.2c) for the terms in σ and in C; it has been suggested by considerations on the BS equation [see discussion following (1.8)] but cannot be derived in the context of the systematic Wilson loop approach.

By the use of potentials, remarkable successes have been obtained in understanding the spectrum and many properties of heavy quarkonia. However, in the context of QCD, the most appropriate tool for a complete treat-

ment of the problem should be the Bethe-Salpeter equation. In particular, it becomes necessary to resort to such an equation in order to extend the analysis to the case of the light mesons. Actually, a number of attempts have been made in this direction recently [9].

In this order of ideas, in analogy with the potential, we can write the BS kernel as the sum of a short-range perturbative term, a confinement term, and an intermediate one:

$$I=I_{\text{pert}}+I_{\text{conf}}+I_{\text{inter}} .$$

The structure of the perturbative part is obvious and in the Coulomb gauge at the lowest order it is given by

$$I_{\text{pert}}=(2\pi)^3\frac{4}{3}\alpha_s\frac{1}{2\pi^2}\left[-\frac{\gamma_0^{(1)}\gamma_0^{(2)}}{\mathbf{Q}^2}-\frac{1}{t}[\boldsymbol{\gamma}^{(1)}\cdot\boldsymbol{\gamma}^{(2)}-(\boldsymbol{\gamma}^{(1)}\cdot\hat{\mathbf{Q}})(\boldsymbol{\gamma}^{(2)}\cdot\hat{\mathbf{Q}})]\right], \tag{1.3}$$

where Q denotes the four-dimensional momentum transfer $p_1'-p_1$ and t stands for Q^2. On the contrary, in the lack of a sufficient understanding of the confinement mechanism, no general criterion exists to determine I_{conf} and I_{inter}. The only possibility seems to resort to phenomenological expressions and/or to a comparison with the potential (1.2).

The usual tool to establish a connection between the BS equation and the potential formalism is the so-called instantaneous approximation on the kernel. Introducing the center-of-mass variables

$$\begin{aligned}&\mathbf{p}_1=-\mathbf{p}_2=\mathbf{q}\ ,\quad \mathbf{p}_1'=-\mathbf{p}_2'=\mathbf{q}'\ ,\\ &p_{10}=\omega+\tfrac{1}{2}\sqrt{s}\ ,\quad p_{10}'=\omega'+\tfrac{1}{2}\sqrt{s}\ ,\\ &p_{20}=-\omega+\tfrac{1}{2}\sqrt{s}\ ,\quad p_{20}'=-\omega'+\tfrac{1}{2}\sqrt{s}\ ,\end{aligned} \tag{1.4}$$

the BS kernel can be written as $I(\omega',\mathbf{q}';\omega,\mathbf{q};s)$. Then the instantaneous approximation consists in setting $\omega=\omega'=0$ in I and in the off-shell amplitude $\bar{G}$. If one also replaces the quark propagator with its free particle expression and makes a comparison between the BS equation and the corresponding Lippmann-Schwinger (LS) equation for the Hamiltonian (1.1), it turns out

$$\begin{aligned}\langle \mathbf{q}'\sigma_1'\sigma_2'|V(s)|\mathbf{q}\sigma_1\sigma_2\rangle=&\frac{1}{(2\pi)^3}\frac{m^2}{w(\mathbf{q})w(\mathbf{q}')}\bar{u}^{(1)}_{\sigma_1'}(\mathbf{q}')\\ &\times\bar{u}^{(2)}_{\sigma_2'}(-\mathbf{q}')I^{(0)}(\mathbf{q}',\mathbf{q};s)\\ &\times u^{(1)}_{\sigma_1}(\mathbf{q})u^{(2)}_{\sigma_2}(-\mathbf{q})\end{aligned} \tag{1.5}$$

with

$$I^{(0)}(\mathbf{q}',\mathbf{q};s)=I(\omega'\mathbf{q}',\omega\,\mathbf{q};s)|_{\omega=\omega'=0}\ ,$$
$$w(\mathbf{q})=\sqrt{m^2+\mathbf{q}^2}\ .$$

For an appropriate choice of the kernel, potential (1.2) should be obtained from (1.5) by an expansion in $1/m^2$. In fact, if we replace the perturbative kernel (1.3) in (1.5) we obtain the correct terms in κ.

Let us pay attention instead to the terms in σ and take into account that

$$\langle \mathbf{q}'|\sigma r|\mathbf{q}\rangle=-\frac{\sigma}{\pi^2}\frac{1}{\mathbf{Q}^4}\ . \tag{1.6}$$

The simplest assumption on I_{conf} which reproduces (1.2a) and (1.2b) turns out to be

$$I_{\text{conf}}=(2\pi)^3\left[\frac{-\sigma}{\pi^2}\frac{1}{t^2}\right]. \tag{1.7}$$

What suggests the choice of a purely scalar confinement kernel is the structure of the σ-dependent terms in Eq. (1.2b); alternative possibilities, however, are given in Ref. [10]. Of course, being infrared divergent, Eq. (1.7) has to be supplied with an appropriate regularization prescription (cf. Sec. II).

We notice now that the form of the σ-dependent term in V_{VD} which derives from the kernel (1.7) coincides with Eq. (1.2c') and not with (1.2c). To obtain formally Eq. (1.2c) from (1.5), it is necessary to add a $1/m^2$ term to (1.7) [11]:

$$I_{\text{conf}}=(2\pi)^3\left\{-\frac{\sigma}{\pi^2}\left[\frac{1}{t^2}+\frac{1}{3m^2}\left[-\frac{2}{3}\frac{1}{t}+\frac{1}{2}\frac{u}{t^2}+\frac{1}{2}\frac{v^2}{t^3}\right]\right]\right\}, \tag{1.8}$$

where

$$u=(p_2'-p_1)^2\ ,\quad v=(p_1-p_2')(p_2-p_2')\ .$$

The unsatisfactory aspect of Eq. (1.8) is that it is given in terms of a $1/m^2$ expansion rather than in a closed form. For definiteness in this paper we shall assume (1.8). Because of the approximation involved, however, the results do not depend critically on such a choice.

Coming to the terms in C occurring in (1.2), we observe that they should require an expression of the form

$$I^{(0)}_{\text{inter}}=(2\pi)^3\frac{C}{2\pi^2}\delta^3(\mathbf{Q})\left[1+\frac{\mathbf{q}^2}{2m^2}\right] \tag{1.9}$$

for the instantaneous limit of I_{inter}. It is clear that such a term cannot be obtained from a function of the variable t alone. Then we are left with various alternatives.

(1) We may notice that in Eq. (1.2) a variation δC of C in the term $C-(C/2m^2)q^2$ can be compensated, to the order $1/m^2$, by a variation of the mass $\delta m=-\frac{1}{2}\delta C$ in the kinetic term. So, we could conjecture that, in a full relativistic treatment, it can be taken $C=0$.[1]

(2) We could replace C everywhere in (1.2) with a more regular expression, e.g., $Ce^{-\eta r}$. In fact, this choice would correspond to take

[1]Using potential (1.2), the fit of the $b\bar{b}$ spectrum is practically unmodified changing C from the usual phenomenological value $C=-0.8$ GeV to $C=0$ (cf. Ref. [7]); this is obviously no longer completely true in the $c\bar{c}$ case.

$$I_{\text{inter}}=(2\pi)^3\frac{C}{\pi^2}\frac{\eta}{(t-\eta^2)^2}\left[1+\frac{u}{8m^2}\right]. \tag{1.10}$$

(3) According to a suggestion by Gromes [12], the constant C could arise automatically from I_{conf} as a combined effect of the infrared regularization prescription and the instantaneous approximation. As we shall discuss later, in our framework this would amount to stating that C represents a leading contribution to the retardation corrections.

In practice, we shall assume (1.3) and (1.8) for I_{pert} and I_{conf} and we shall put I_{inter} equal to zero or alternatively take it directly in the instantaneous form (1.9), thinking of it possibly as the limit of a more elaborate expression. Then we shall evaluate first-order corrections to the instantaneous form of the term (1.7) alone, while we shall neglect not only the corrections to (1.9) but also those to the spin part in (1.3) and to the $1/m^2$ part in (1.8), considering them as higher-order ones. In this connection we may observe that the leading term of the vectorial kernel in Eq. (1.3) is already instantaneous while retardation effects appear only in the spin-dependent part [13].

Since, even in the direct applications of the BS equation, only instantaneous confining kernels have been employed for numerical practicability, it should be interesting to work out retardation corrections in full relativistic kinematics. Here, however, we restrict ourselves to the semirelativistic perspectives of potential (1.2). Furthermore, rather then referring to usual Kato perturbation theory [2], we have found it convenient to work in terms of an additional effective potential $V_{\text{ret}}^{(1)}$. In this philosophy we think of the mass M_a of the meson a as obtained from

$$M_a=2m+W_a^{(0)}+W_a^{\text{rel}}+W_a^{\text{ret}}, \tag{1.11}$$

where $W_a^{(0)}$ is the eigenvalue obtained by the numerical resolution of the ordinary Schrödinger equation for the static potential (1.2a), W_a^{rel} equals the expectation value of $-p^4/4m^3+V_{\text{SD}}+V_{\text{VD}}$ on the numerical "unperturbed" eigenfunction Φ_a (and possibly includes the pair creation effects) and the retardation correction W_a^{ret} is expressed by

$$W_a^{\text{ret}}=\langle\Phi_a|V_{\text{ret}}^{(1)}|\Phi_a\rangle. \tag{1.12}$$

Our basic result is an evaluation of (1.12). Actually, again for numerical practicability, we have replaced in this equation the numerical Φ_a with eigenfunctions of the harmonic-oscillator type and we have restricted ourselves to a limited number of states. We have considered only the first three S states both for the $b\bar{b}$ and the $c\bar{c}$ systems.

We have performed our calculations for two assignments of the parameters coming from phenomenological fits previously obtained with potential (1.2). In the first assignment it was taken $C=0$ [7] and in the second one $C\neq 0$ [14]. The value of W_{nS}^{ret} depends on a scale parameter μ introduced in the regularization prescription. For $\mu=50$ MeV it comes out typically of few hundreds MeV and so it cannot be said to be small. Because of the methodological character of the present work and of the small number of states taken into account, we have made no attempt to readjust the parameters to fit the data again. Therefore, more work would be necessary in order to draw a really definite conclusion about the phenomenological consistency of the kernel. However, for the first choice of the parameters, the results do not seem to support our simple assumption on the kernel while the situation becomes better for the second choice. In this second case for $\mu\simeq 135$ MeV, all states considered but $3Sc\bar{c}$ differ from the data by less than 50 MeV without any readjustment of the parameters and the agreement can be improved if m_c is reduced by a few tens of MeV. To test the conjecture of Gromes [12], we have also repeated the calculations with the second choice of the parameters but taking $C=0$. The results do not seem to support the conjecture.

As we mentioned, nothing essentially changes if kernel (1.7) is used instead of (1.8).

It should be stressed that our success in obtaining a quantitative evaluation of the retardation corrections depends strictly on the particular regularization prescription we have adopted and on the use of the effective potential $V_{\text{ret}}^{(1)}$.

The plan of paper is the following one. In Sec. II, we discuss the regularization procedure and Gromes' conjecture; in Sec. III, we perform the explicit calculation of W_{nS}^{ret} for the two choices of parameters and present the results; in Sec. IV we discuss the results and make some additional remarks. In the Appendixes we derive the expression of $V_{\text{ret}}^{(1)}$ and discuss some important analytic matter.

II. THE REGULARIZATION PROCEDURE

As we mentioned, the right-hand sides of Eqs. (1.6)–(1.8) have to be explicitly regularized so that they correctly define well-behaved distributions, respectively, in ordinary space and in Minkowski space. An obvious regularization of $1/\mathbf{Q}^4$ is provided by

$$\left[\frac{1}{\mathbf{Q}^4}\right]_{\text{reg}}=-\frac{1}{2}\frac{\partial^2}{\partial\varepsilon^2}\frac{1}{\mathbf{Q}^2+\varepsilon^2}=\frac{1}{(\mathbf{Q}^2+\varepsilon^2)^2}-\frac{4\varepsilon^2}{(\mathbf{Q}^2+\varepsilon^2)^3}, \tag{2.1}$$

where the limit $\varepsilon\to 0$ is implied. Equation (2.1) amounts simply to consider σr as the limit of $\sigma r e^{-\varepsilon r}$. A possible definition of $1/t^2$ consistent with (2.1) is

$$\left[\frac{1}{t^2}\right]_{\text{reg}}=\frac{1}{2}\left[\frac{\partial^2}{\partial\varepsilon^2}-\varepsilon\ln\frac{\varepsilon}{\mu}\frac{\partial^3}{\partial\varepsilon^3}\right]\frac{1}{t-\varepsilon^2+i0}, \tag{2.2}$$

where μ is a constant with the dimension of a mass that, in practice, we shall treat as an additional fitting parameter. Notice that the second term appearing in (2.2) has no effect in the instantaneous limit,

$$\varepsilon\ln\frac{\varepsilon}{\mu}\frac{\partial^3}{\partial\varepsilon^3}\frac{e^{-\varepsilon r}}{r}=-\varepsilon\ln\frac{\varepsilon}{\mu}r^2e^{-\varepsilon r}\to 0, \tag{2.3}$$

and it has no counterpart in (2.1). It is, however, essential to make I_{conf} finite in the coordinate space and to

subtract residual logarithmic divergences otherwise appearing in (1.12). With definitions (2.1) and (2.2), some of the integrals occurring in the expression $V_{\rm ret}^{(1)}$ can be analytically performed and the right-hand side of (1.12) becomes numerically calculable.

For completeness we remark that the term in $1/t^3$ appearing in (1.8) should also be regularized; a possible choice would be

$$\left[\frac{1}{t^3}\right]_{\rm reg}=-\frac{1}{24}\left[\frac{\partial^4}{\partial\varepsilon^4}+\frac{1}{2}\varepsilon\left[1-3\ln\frac{\varepsilon}{\mu}\right]\frac{\partial^5}{\partial\varepsilon^5}-\frac{1}{2}\varepsilon^2\ln\frac{\varepsilon}{\mu}\frac{\partial^6}{\partial\varepsilon^6}\right]\frac{1}{t-\varepsilon^2}\ .$$

As we mentioned, however, we neglect the noninstantaneous corrections to this term and we do not need to use explicitly this equation.

To show that the prescription (2.2) is correct, it is sufficient to check that the kernel in the coordinate representation is a locally integrable function in the limit $\varepsilon\to 0$. Indeed, one has

$$\begin{aligned}\tilde{I}(x)&=-(2\pi)^3\frac{\sigma}{\pi^2}\int\frac{d^4Q}{(2\pi)^4}e^{-iQ\cdot x}\left[\frac{1}{t^2}\right]_{\rm reg}\\&=-8\pi\sigma D_\varepsilon^\mu\int\frac{d^4Q}{(2\pi)^4}e^{-iQ\cdot x}\frac{1}{Q^2-\varepsilon^2+i0}\\&=-8\pi\sigma D_\varepsilon^\mu\Delta_F(x,\varepsilon)\ ,\end{aligned}\tag{2.4}$$

where $x=x_1-x_2$ and

$$D_\varepsilon^\mu=\frac{1}{2}\left[\frac{\partial^2}{\partial\varepsilon^2}-\varepsilon\ln\left[\frac{\varepsilon}{\mu}\right]\frac{\partial^3}{\partial\varepsilon^3}\right]\ .\tag{2.5}$$

Recalling the explicit expression

$$\Delta_F(x)=\frac{1}{4\pi}\delta(x^2)+\frac{i\varepsilon}{4\pi^2}\frac{K_1(\varepsilon\sqrt{-x^2+i0})}{\sqrt{-x^2+i0}}\tag{2.6}$$

(cf. [15]), one obtains

$$\frac{\partial^2}{\partial\varepsilon^2}\Delta_F(x,\varepsilon)\to\ln(\varepsilon\sqrt{x^2})+\text{const}\ ,$$

$$\frac{\partial^3}{\partial\varepsilon^3}\Delta_F(x,\varepsilon)\to\frac{1}{\varepsilon}\ ,$$

and, hence,

$$\tilde{I}(x)=\sigma\ln(\mu^2x^2)\ .\tag{2.7}$$

Equation (2.7) contains correctly only a logarithmic singularity in x^2 and so is of the desired type.

Let us now briefly discuss the conjecture by Gromes [12,3] in our framework. This author starts from the analytic regularization

$$\left[\frac{1}{t^2}\right]_{\rm reg}=\frac{\partial}{\partial\varepsilon}\frac{\varepsilon\tilde{\rho}^\varepsilon}{t^{2-\varepsilon/2}}\tag{2.8}$$

($\tilde{\rho}$ being a constant with the dimension of a length) and obtains the same equation (2.7) with $\mu=1/\rho=(1/2\tilde{\rho})e^{\gamma-1/2}$ ($\gamma=0.577\cdots$ being the Euler constant). Then he rewrites this equation in the form[2]

$$\tilde{I}(x)=\frac{\sigma}{2\pi}\ln\left|\frac{x_0^2-r^2-i0}{x_0^2-\rho^2}\right|+\frac{\sigma}{2\pi}\ln\left|\frac{x_0^2-\rho^2-i0}{\rho^2}\right|\ ,\tag{2.9}$$

he argues that the second term can be neglected for a sufficiently large ρ and takes the instantaneous approximation on the first one.

The result is a contribution to the static potential of the form

$$\sigma(r-\rho)\ ,\tag{2.10}$$

which corresponds to $C=-\sigma\rho$ (Gromes also suggests $\tilde{\rho}=1/\sqrt{\sigma}$ and so $C=-2e^{-(\gamma-1/2)}\sqrt{\sigma}$, a formula which is phenomenologically successful). The procedure is motivated by the fact that the instantaneous limit defined as $\int_{-\infty}^{+\infty}dx_0\tilde{I}(x)$ is not finite for the whole (2.7). In our framework, the difficulty does not arise since, in the spirit of the distribution theory, we perform the limit $\varepsilon\to 0$ only after the instantaneous approximation has been taken. Furthermore, using prescription (2.2) rather than (2.8), as we mentioned, we become able to evaluate analytically certain integrals which are essential for an actual calculation of the retardation correction. By a comparison of the two procedures, it is apparent that, if the two approximations in (2.9) were legitimate for some value of ρ, the quantity C would correspond to a dominant contribution in such retardation corrections. That is to say, the correction should turn out approximately independent of the state.

The above statement becomes particularly clear if we rewrite Eq. (2.9) in momentum space using our regularization and not taking the limit $\varepsilon\to 0$ in a first moment. We have

$$-4\pi\sigma D_\varepsilon^\mu\frac{1}{Q_0^2-\mathbf{Q}^2-\varepsilon^2+i0}=D_\varepsilon^\mu\left[\frac{1}{Q_0^2-\mathbf{Q}^2-\varepsilon^2+i0}-\delta^3(\mathbf{Q})\int d^3\mathbf{Q}'\frac{e^{-i\mathbf{Q}'\rho}}{Q_0^2-\mathbf{Q}'^2-\varepsilon^2+i0}\right]+\delta^3(\mathbf{Q})D_\varepsilon^\mu\int d^3\mathbf{Q}'\frac{e^{-i\mathbf{Q}'\rho}}{Q_0^2-\mathbf{Q}'^2-\varepsilon^2+i0}\ .\tag{2.11}$$

[2]Actually Gromes used a Euclidean metric.

In fact, if, in the right-hand side of this equation, we neglect the second term, put $Q_0=0$, and now actually perform the limit $\varepsilon\to 0$, we obtain again $\langle \mathbf{q}'|\sigma(r-\rho)|\mathbf{q}\rangle$. Consequently, such a quantity should equal the corresponding one provided by the whole left-hand side in (2.11), i.e., $\langle \mathbf{q}'|\sigma r+V^{(1)}_{\rm ret}|\mathbf{q}\rangle$, and we should have $W^{\rm ret}_{nl}=-\sigma\rho=C$.

III. CORRECTION TO THE INSTANTANEOUS APPROXIMATION

As we mentioned, we have found it convenient to express the corrections to the instantaneous approximation in terms of additional contributions to the potential. Such contributions can be obtained by an iterative procedure based on a comparison between the BS and the LS equations (cf. Appendix A).

To lowest order one finds

$$\begin{aligned}\langle \mathbf{q}',\sigma_1'\sigma_2'|V^{(1)}_{\rm ret}|\mathbf{q},\sigma_1\sigma_2\rangle=&-\frac{m^2}{(2\pi)^3 w(\mathbf{q})w(\mathbf{q}')}\\ &\times{\rm Re}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{m^2}{w(k)^2}\bar u^{(1)}_{\sigma_1'}(\mathbf{q}')\bar u^{(2)}_{\sigma_2'}(-\mathbf{q}')\\ &\times\left\{i\int\frac{d\omega_k}{2\pi}\left[I_0(\mathbf{q}',\mathbf{k})\right.\right.\\ &\times\frac{\Lambda^{(1)}_+(\mathbf{k})\Lambda^{(2)}_+(-\mathbf{k})}{[\omega_k-\sqrt{s}/2+w(\mathbf{k})-i0][\omega_k+\sqrt{s}/2-w(\mathbf{k})+i0]}I_1(\omega_k;\mathbf{k},\mathbf{q})\\ &+\bar I_1(\omega_k;\mathbf{q}',\mathbf{k})\\ &\left.\left.\times\frac{\Lambda^{(1)}_+(\mathbf{k})\Lambda^{(2)}_+(-\mathbf{k})}{[\omega_k-\sqrt{s}/2+w(\mathbf{k})-i0][\omega_k+\sqrt{s}/2-w(\mathbf{k})+i0]}I_0(\mathbf{k},\mathbf{q})\right]\right.\\ &\left.+I_0(\mathbf{q}',\mathbf{k})\frac{\Lambda^{(1)}_-(-\mathbf{k})\Lambda^{(2)}_-(\mathbf{k})}{\sqrt{s}+2w(\mathbf{k})}I_0(\mathbf{k},\mathbf{q})\right\}u^{(1)}_{\sigma_1}(\mathbf{q})u^{(2)}_{\sigma_2}(-\mathbf{q})\ .\end{aligned}\tag{3.1}$$

Here

$$\begin{aligned}I_1(\omega;\mathbf{q}',\mathbf{q})&=I(0,\mathbf{q}';\omega,\mathbf{q})-I_0(\mathbf{q}',\mathbf{q})\ ,\\ \bar I_1(\omega';\mathbf{q}',\mathbf{q})&=I(\omega',\mathbf{q}';0,\mathbf{q})-I_0(\mathbf{q}',\mathbf{q})\ ,\end{aligned}\tag{3.2}$$

$\Lambda^{(j)}_+$ and $\Lambda^{(j)}_-$ are the Dirac spinor positive- and negative-energy projectors and the prescription Re stays properly for the *self-adjoint part* of the following matrix element.[3]

For the kernel and in the approximation discussed in the Introduction, we have

$$I_1(\omega';\mathbf{q}',\mathbf{q})=-(2\pi)^3\frac{\sigma}{\pi^2}D^\mu_\varepsilon\left[\frac{1}{\omega^2-\mathbf{Q}^2-\varepsilon^2}+\frac{1}{\mathbf{Q}^2+\varepsilon^2}\right]\ .\tag{3.3}$$

Then, performing explicitly the integration in ω_k in (3.1) and neglecting all contributions of order $1/m^2$ (included the third term containing $\Lambda_-\Lambda_-$), we end up with

$$\begin{aligned}\langle \mathbf{q}',\sigma_1'\sigma_2'|V^{(1)}_{\rm ret}|\mathbf{q},\sigma_1\sigma_2\rangle=\delta_{\sigma_1'\sigma_1}\delta_{\sigma_2'\sigma_2}{\rm Re}&\left\{\int d^3\mathbf{k}\left[\frac{\sigma}{2\pi^2}\frac{\partial^2}{\partial\varepsilon^2}-\frac{\kappa}{2\pi^2}\right]\frac{1}{(\mathbf{q}'-\mathbf{k})^2+\varepsilon^2}\right.\\ &\times\frac{\sigma}{\pi^2}D^\mu_{\varepsilon'}\frac{1}{-\sqrt{s}+2w(\mathbf{k})+2\sqrt{(\mathbf{k}-\mathbf{q})^2+\varepsilon'^2}-i0}\\ &\left.\times\frac{1}{(\mathbf{k}-\mathbf{q})^2+\varepsilon'^2}+(\mathbf{q}'\rightleftarrows\mathbf{q})\right\}\end{aligned}\tag{3.4}$$

(obviously, in the first factor inside the integral D^μ_ε has been replaced by $\partial^2/\partial\varepsilon^2$). The resulting leading retardation

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[3]As is well known, prescriptions different from $\omega,\omega'=0$ have been proposed as zero-order approximations in the BS equations. Such prescriptions, which consist in taking ω and ω' as appropriate functions of the momentum (see, e.g., [8]), should amount to taking into account some part of the retardation effect. The contact with our approach would be made by simply substituting our current variable ω_k with the assigned function of $\mathbf{k}$.

correction to the energy levels can be expressed as [cf. (1.12)]

$$W_a^{\rm ret}=\langle \Phi_a | V_{\rm ret}^{(1)} | \Phi_a \rangle \tag{3.5}$$

after setting $s=s_a=(2m+W_a^{(0)})^2$.

Using the Schrödinger equation,

$$\int d^3\mathbf{q}\left[\frac{\sigma}{2\pi^2}\frac{\partial^2}{\partial\varepsilon^2}-\frac{\kappa}{2\pi^2}\right]\frac{1}{(\mathbf{k}-\mathbf{q})^2+\varepsilon^2}\tilde{\Phi}_a(\mathbf{q})=[2w(\mathbf{k})-\sqrt{s_a}]\tilde{\Phi}_a(\mathbf{k})\ , \tag{3.6}$$

Eq. (3.5) becomes

$$W_a^{\rm ret}=\mathrm{Re}\int d^3\mathbf{q}\int d^3\mathbf{k}\left\{\tilde{\Phi}_a^*(\mathbf{k})\left[w(\mathbf{k})-\frac{\sqrt{s_a}}{2}\right]\frac{\sigma}{2\pi^2}D_\varepsilon^\mu\frac{1}{-\sqrt{s_a}/2+w(\mathbf{k})+\sqrt{(\mathbf{k}-\mathbf{q})^2+\varepsilon^{2\prime}}-i0}\right.$$

$$\left.\times\frac{1}{(\mathbf{k}-\mathbf{q})^2+\varepsilon^{2\prime}}\tilde{\Phi}_a(\mathbf{q})+\phi(\mathbf{k})\rightleftarrows\phi(\mathbf{q})\right\}\ . \tag{3.7}$$

This equation can be further simplified if we introduce both the momentum and the coordinate-space wave functions $\tilde{\Phi}_a(\mathbf{k})$ and $\Phi_a(\mathbf{r})$:

$$\Phi_{nlm}(\mathbf{r})=\phi_{nl}(r)Y_{lm}(\hat{r})\ ,\quad \tilde{\Phi}_{nlm}(\mathbf{k})=\tilde{\phi}_{nl}(k)Y_{lm}(\hat{k})\ ,$$

use the nonrelativistic approximation for $w(\mathbf{k})$, and perform the trivial angular integration. Restricting to the S-wave case, we are left with

$$W_{nS}^{\rm ret}=2\frac{\sigma}{\pi^3}D_\varepsilon^\mu\int_0^\infty dk\,k\int_0^\infty dp\,p\int_0^\infty dr\,\tilde{\phi}_{nS}(k)\left[\frac{k^2}{2m}-\frac{W_{nS}^{(0)}}{2}\right]\sin(kr)\sin(pr)\phi_{nS}(r)$$

$$\times\mathcal{P}\frac{1}{-W_{nS}^{(0)}/2+k^2/2m+\sqrt{p^2+\varepsilon^2}}\frac{1}{p^2+\varepsilon^2}\ , \tag{3.8}$$

where now $\mathcal{P}$ stays for the *principal value* prescription.

In order to carry on the analytical calculation as far as possible, getting rid of some integrals, we replace ϕ_{nS} with the radial eigenfunction of the harmonic tridimensional oscillator

$$\phi_{nS}(r)=\frac{1}{\lambda_{nS}^{3/2}}\left[\frac{2(n-1)!}{\Gamma(n+1/2)}\right]^{1/2}L_{n-1}^{1/2}\left[\frac{r^2}{\lambda_{nS}^2}\right]e^{-r^2/2\lambda_{nS}^2}\ , \tag{3.9a}$$

$$\tilde{\phi}_{nS}(k)=\lambda_{nS}^{3/2}\left[\frac{2(n-1)!}{\Gamma(n+1/2)}\right]^{1/2}L_{n-1}^{1/2}(\lambda_{nS}^2k^2)e^{-k^2\lambda_{nS}^2/2}\ , \tag{3.9b}$$

where the standard notation for the Laguerre polynomials has been used and the parameters λ_{nS} have been determined variationally by minimizing the expectation value of the zero-order Hamiltonian. Inserting (3.9) in Eq. (3.8), we can perform analytically the integration in r and are left with the double integral

$$W_{nS}^{\rm ret}=2C_{nS}\int_0^\infty dk\,k\left[\frac{k^2}{2m}-\frac{W_{nS}^{(0)}}{2}\right]e^{-\lambda_{nS}^2}\int_0^\infty dp\,pe^{-\lambda_{nS}^2p^2/2}g_{nS}(p,k)$$

$$\times D_\varepsilon^\mu\mathcal{P}\frac{1}{-W_{nS}^{(0)}/2+k^2/2m+\sqrt{p^2+\varepsilon^2}}\frac{1}{p^2+\varepsilon^2}=2C_{nS}D_\varepsilon^\mu I_{nS}(\varepsilon)\ , \tag{3.10}$$

where C_{nS} and $g_{nS}(p,k)$ are a coefficient and a function that depend on the particular n. For $n=1$ we have

$$g_{1S}(p,k)=\sinh(kp\lambda_{1S}^2)\ ,\quad C_{1S}=4\sqrt{2}\frac{\sigma\lambda_{1S}}{\pi^3}\ .$$

The analogous expressions for $n=2$ and 3 are reported in Appendix B and Table I.

Because of the delicate interference between the principal part and the regularization prescriptions, a direct numerical evaluation of the integral in the second member of (3.10) is possible only if W_{nS} is negative. In the general case, however, we can obtain the same result drawing the simpler quantity I_{nS} vs ε and taking advantage of the ex-

TABLE I. Analytical expression of the constant C_{nS} and of the coefficients A_1^{nS}, B_2^{nS}, A_3^{nS} for the first three S states. Here $z=\lambda\sqrt{-mW_{nl}^{(0)}}$ and $\phi(x)=(2/\sqrt{\pi})\int_0^x dt\, e^{-t^2}$.

State	C_{nS}	A_1^{nS}	B_2^{nS}	A_3^{nS}
$1S$	$4\sqrt{2}\sigma\lambda/\pi^3$	$-\frac{\pi^{3/2}}{8\lambda}$	$-m\lambda\frac{\sqrt{\pi}}{2}\mathrm{Re}\{1-\sqrt{\pi}ze^{z^2}[1-\Phi(z)]\}$	$-\frac{1}{32}\pi^{3/2}\lambda$
$2S$	$8\sqrt{2}\sigma\lambda/(3\pi^3)$	$\frac{3\pi^{3/2}}{16\lambda}$	$m\lambda\frac{\sqrt{\pi}}{2}\mathrm{Re}\{\sqrt{\pi}e^{z^2}[1-\Phi(z)](-\frac{9}{4}z-3z^3-z^5)+\frac{3}{2}+\frac{5}{2}z^2+z^4\}$	$\frac{7}{64}\pi^{3/2}\lambda$
$3S$	$\frac{15}{2\pi^3\sqrt{2}}\sigma\lambda$	$-\frac{2\pi^{3/2}}{15\lambda}$	$-m\lambda\sqrt{\pi}\mathrm{Re}\{\sqrt{\pi}e^{z^2}[1-\Phi(z)][-z-\frac{8}{3}z^3-\frac{104}{45}z^5-\frac{32}{45}z^7-\frac{16}{225}z^9]+\frac{8}{15}+\frac{86}{45}z^2+\frac{452}{225}z^4+\frac{152}{225}z^6+\frac{16}{225}z^8\}$	$-\frac{11}{90}\pi^{3/2}\lambda$

pansion (Appendix C)

$$I(\varepsilon)=A_0+A_1\varepsilon+A_2\varepsilon^2+A_3\varepsilon^3+\cdots+(B_2\varepsilon^2+B_4\varepsilon^4+\cdots)\ln\frac{\varepsilon}{\mu}\ . \tag{3.11}$$

In fact, replacing (3.11) in (3.10) and taking explicitly the limit $\varepsilon\to 0$, one obtains

$$W_{nS}^{\mathrm{ret}}=C_{nS}(2A_2^{nS}+3B_2^{nS})\ . \tag{3.12}$$

Notice that, from (3.11), the need of the logarithmic term in D_ε^μ is again apparent.

The coefficients $A_1, A_3, \ldots$ and B_2 in (3.11) can be obtained in a closed form and are reported in Table I; A_0 coincides with $I(0)$; the remaining coefficients have to be determined by a fit of the numerical $I(\varepsilon)$ for some definite value μ_0 of μ. Notice that, since $I(\varepsilon)$ does not depend on μ, we must have

$$A_2(\mu)=A_2(\mu_0)+B_2\ln\frac{\mu}{\mu_0}$$

and so, for an arbitrary μ,

$$W_{nS}^{\mathrm{ret}}(\mu)=W_{nS}^{\mathrm{ret}}(\mu_0)+C_{nS}B_2^{nS}\ln\frac{\mu}{\mu_0}\ . \tag{3.13}$$

In practice, we have performed the calculation for the states $1S$, $2S$, and $3S$ of the $b\bar{b}$ and $c\bar{c}$ systems for the three assignments of the parameters reported in Table II. Here CH1 coincides with choice II of Ref. [7], CH2 with column 2 of Table III of Ref. [14], CH3 is considered to test Gromes' conjecture and it coincides with CH2 but with I_{inter} set equal to 0. In the CH3 case, the value of C in CH2 should be supplied by a dominant contribution in the retardation correction; i.e., we should find $W_{nl}^{\mathrm{ret}}\simeq -0.7$. In the cases CH2 and CH3, even the pair-creation effects have been added to W_{nl}^{rel} in Eq. (1.11).

In the fits we have put $\mu_0=0.05$ GeV, fixed A_1, A_3, and B_2 on their "analytical" values and treated A_2 or A_2, A_4, and B_4 as free parameters; all other coefficients have been neglected. An example is illustrated in Fig. 1, while the value obtained for A_2 are reported for CH1 in Table III together with the variational values of λ_{nS}, the corresponding "analytical" values of B_2, A_1, A_3, and the "numerical" value of A_1, obtained treating also this quantity as a free parameter. The difference between A_1 and A_1^{num} tests the reliability of the calculation and it is of the order of a few percent in all the cases with the exception of $3Sb\bar{b}$. For this last state the discrepancy is about 30% for CH1, and even 300% for CH2, making meaningless the evaluation along this way in this second case [however, $W_{3S}^{(0)}(b\bar{b})<0$ and so the calculation can be performed by direct evaluation of the integral in (3.10)].

The error reported for A_2 has been estimated as a combination of two different quantities, $\delta_1 A_2$ and $\delta_2 A_2$, related mainly to the numerical error in the absolute value of $I(\varepsilon)$ and to the uncertainty in the fit, respectively. The first quantity $\delta_1 A_2$ is defined as $|A_2|\cdot|A_1-A_1^{\mathrm{num}}|/|A_1|$; the second one $\delta_2 A_2$ is set equal to the difference between the values obtained for A_2 including or not including terms in B_4 and A_4 in the fit.

The final expressions for W_{nS}^{ret} are reported in Table IV as function of $\xi=\ln\mu/\mu_0$; in parentheses there are also reported the values of $W_{nl}^{\mathrm{ret}}(\mu_0)$ obtained by direct calculation of the integral (3.10) where available. A comparison between the results obtained by the two methods shows that they are essentially consistent. This was not obvious *a priori* because of the complexity of the calculations. The discrepancies could suggest, however, that the errors reported in Table IV have been underestimated for some states.

A plot of the resulting masses M_{nS} is shown in Figs. 2–5 together with the error bands as resulting from Table III. In Figs. 2–5, wider error bands are also reported for the $1S$ states related to the use of the varia-

TABLE II. The three assignments of the parameters CH1, CH2, CH3 as explained in the text. The energy is expressed in GeV.

	$\frac{4}{3}\alpha_s$	σ	C	m_c	m_b
CH1	0.5	0.189	0	1.357	4.770
CH2	0.587	0.146	−0.707	1.913	5.268
CH3	0.587	0.146	0	1.913	5.268

TABLE III. Values of λ_{nS} (in GeV^{-1}) and A_1, A_2, B_2, A_3, as an example for the first three $b\bar{b}$ and $c\bar{c}$ S states for choice CH1 of the parameters. The coefficients A_1, A_3, B_2 are calculated according to Table I, A_2 is obtained from the fit. $A_1^{\rm num}$ denotes the values obtained for A_1 leaving even this quantity as a free parameter in the fit for a comparison with the exact value. The errors reported for A_2 are estimated as described in the text (energy in GeV).

	λ	A_1	$A_1^{\rm num}$	A_2	B_2	A_3
$1S$ $b\bar{b}$	0.9	−0.773	−0.779	3.44±0.08	−1.501	−0.157
$2S$ $b\bar{b}$	1.4	0.746	0.727	−2.40±0.06	−0.697	0.853
$3S$ $b\bar{b}$	1.6	−0.464	−0.603	−2.82±0.84	0.800	−1.080
$1S$ $c\bar{c}$	1.8	−0.387	−0.369	0.60±0.10	0.567	−0.313
$2S$ $c\bar{c}$	2.3	0.454	0.450	0.39±0.08	−0.597	1.400
$3S$ $c\bar{c}$	2.6	−0.285	−0.238	0.24±0.07	0.270	−1.770

tional wave function. Again, this additional error is estimated by the formula

$$\delta_3 W_{1S}^{\rm ret} = W_{1S}^{\rm ret}[(W_{1S}^{(0)})_{\rm var} - (W_{1S}^{(0)})_{\rm num}]/(W_{2S}^{(0)} - W_{1S}^{(0)}) .$$

Notice that $(W_{nS}^{(0)})_{\rm var} - (W_{nS}^{(0)})_{\rm num}$ turns out to be a few MeV for the states $2S$ and $3S$, while it is larger for $1S$. For instance, in the CH1 case, it amounts to 17 MeV for $1Sc\bar{c}$ and to 47 MeV for $1Sb\bar{b}$. This corresponds to an estimated error on $W_{1S}^{\rm ret}$ of about 5% for $c\bar{c}$ and 7% for $b\bar{b}$.

IV. CONCLUSIONS

Let us make some final remarks and try to draw some conclusions. First of all, we stress again that the success of our calculation depends strictly on the use of the effective potential (3.1) and on the possibility of obtaining a simple result such as Eq. (3.10). Had we applied the usual Kato perturbation theory for the BS equation [2], it would have been impossible to obtain a simplified equation of the form (3.7) involving only two three-dimensional integrals. We also notice that, as it is apparent from Table IV, the magnitude of the retardation corrections depends linearly on the value of ξ and it turns out typically of the order of few hundreds MeV. It is evident that, as an effect of the regularization, the naive estimate $Q_0/|\mathbf{Q}| \simeq q/m$ is not correct. For certain values of ξ, the corrections to the $b\bar{b}$ states can even exceed the $c\bar{c}$ ones.

Furthermore, as we already mentioned, no attempt has been made in this paper to fit again the experimental data by readjusting the quark masses and the potential parameters after the inclusion of the retardation corrections. Such an attempt would have not been significant because of the small number of the states which we have been able to handle. We can, however, gain some indications by trying to reconcile the results with the data by adjusting the scale parameter μ alone.

In the CH1 case, as it is apparent from Figs. 2 and 3, all the $c\bar{c}$ curves cut the data nearly $\xi \simeq -2.15$. For such value of ξ, however, all values of the $b\bar{b}$ masses and separations differ from the data by more than 200 MeV. It does not seem possible, therefore, to restore the con-

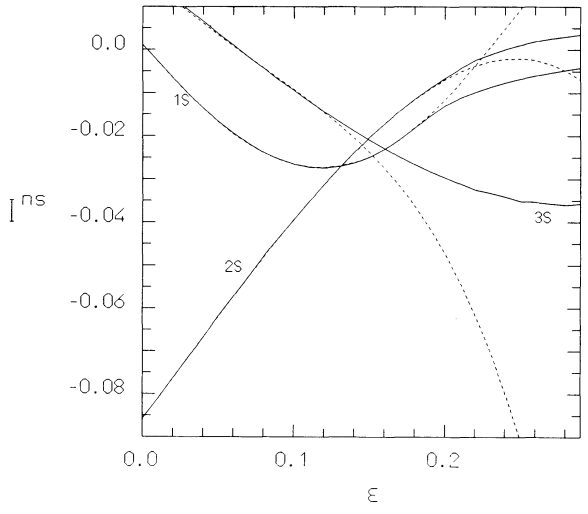

FIG. 1. Fit of I_{nS} vs ε for the first three $c\bar{c}$ S states and parameter assignment CH1 as an example. Solid lines represent the values of $I(\varepsilon)$ as obtained by numerical integration; dotted lines represent the same quantity obtained by interpolation by Eq. (3.11) at small ε. The coefficients A_1, A_3, and B_2 are calculated according to Table I, A_2, A_4, B_4 are introduced as free parameters in the fit. The energy is expressed in GeV.

TABLE IV. Values of the retardation corrections $W_{nS}^{\rm ret}$ as obtained by Eq. (3.12) as a function of $\xi = \ln\mu/\mu_0$ ($\mu_0 = 50$ MeV) for the three different assignments of parameters considered. In parentheses are reported the values of $W_{nS}^{\rm ret}$ for $\xi = 0$ obtained by direct calculation of (3.10) where available. The result reported for the $3Sb\bar{b}$ state in the CH2 case is obtained only by the second method. In this case energy is expressed in MeV.

		CH1	CH2	CH3
$b\bar{b}$	$1S$	74±5−93ξ (80)	51±8−17ξ (44)	60±4−37ξ
	$2S$	−222±4−45ξ	−68±6+33ξ (−90)	−23±12+81ξ
	$3S$	−168±87+83ξ	57−75ξ	
$c\bar{c}$	$1S$	180±12+70ξ	62±6−27ξ (79)	274±36−47ξ
	$2S$	−53±9−63ξ	−33±8+129ξ (−40)	−73±4−67ξ
	$3S$	109±12+45ξ	161±16+5ξ	42±8+54ξ

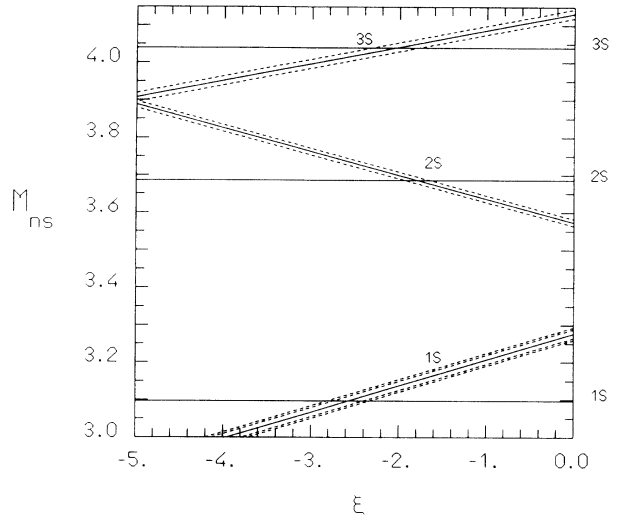

FIG. 2. Masses (in GeV) of the $c\bar{c}$ S states as obtained by Eq. (1.11) and the value of W_{nS}^{ret} reported in Table IV for choice CH1 of the parameters. The solid horizontal straight lines represent the experimental data; the dashed lines represent the error bands as given in the text.

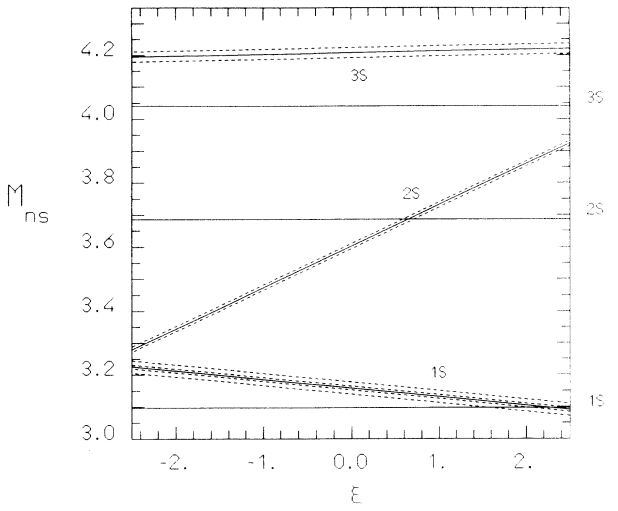

FIG. 4. Masses (in GeV) of $c\bar{c}$ S states as above for choice CH2 of the parameters.

sistency with the data by a small readjustment of the potential parameters.

In the CH2 case, on the contrary, for $\xi=1$ ($\mu=136$ MeV) all states differ from the data by less than 50 MeV except the $3S$ $c\bar{c}$ state, which differs by about 150 MeV. As is apparent from Fig. 4, the disagreement of this last state can be reduced to less than 80 MeV without disturbing the agreement of the other five states by diminishing the mass of the quark c of about 30–40 MeV. Taking into account that the retardation correction to the $c\bar{c}$ $3S$ mass is presumably less accurate than those for the other states (cf. Fig. 1), the situation seems satisfactory enough.

Finally, in the CH3 case, on the basis of Gromes' conjecture, we should expect W_{nl}^{ret} to turn out close to -700 MeV for every nl. In fact, such a result would restore the consistency with the data which was destroyed by setting $C=0$ at the potential level. However, a look at Table IV shows that this is not the case. Indeed, the best agreement among the various W_{nl}^{ret} occurs nearby $\xi=0$ and for such a ξ they do not exceed 100 MeV in modulus but for one state and they occur with opposite signs.

We notice that the above results do not change essentially if we use kernel (1.7) rather than (1.8). As a matter of fact, for choice CH1 with $C=0$, the differences in W_{nl}^{ret} turn out to be of few MeV.

We also notice that, having chosen the intermediate kernel directly in the instantaneous form (1.9), rather than, e.g., in the form (1.10), we have actually neglected

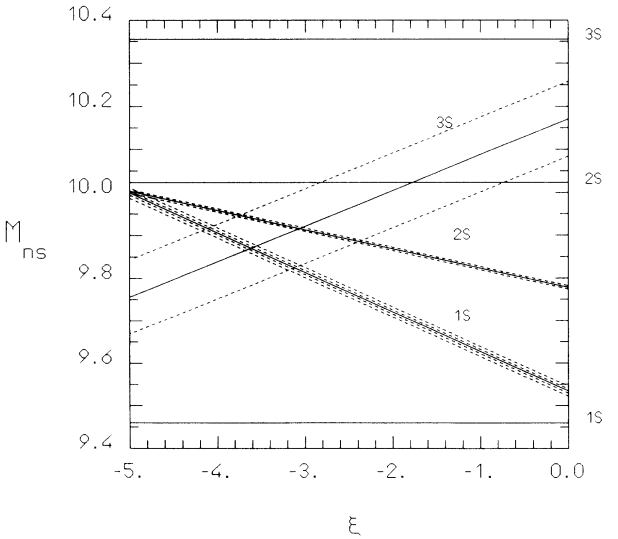

FIG. 3. Masses (in GeV) of the $b\bar{b}$ states as above for choice CH1 of the parameters.

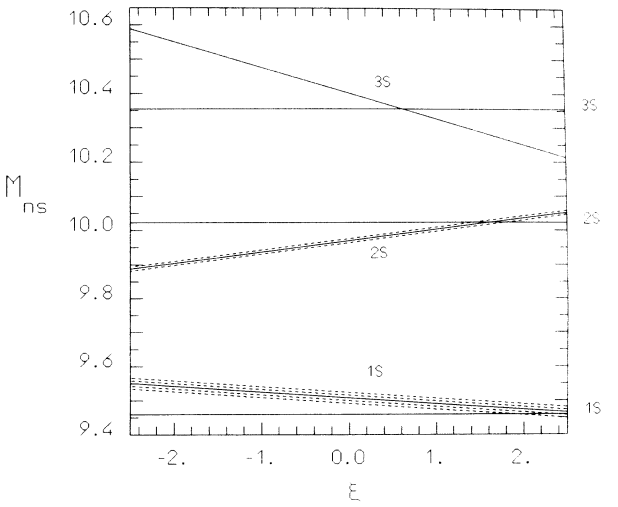

FIG. 5. Masses (in GeV) of the $b\bar{b}$ S states as above for choice CH2 of the parameters.

the corresponding retardation corrections. However, because of the less singular character of that term, such corrections are generally expected to be small. In fact, for CH2 assignment, taking, e.g., $\eta=0.1$ GeV, they turn out typically of about 20 MeV. A more detailed analysis does not seem appropriate in the present context.

ACKNOWLEDGMENTS

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APPENDIX A

In an abbreviated form, the BS equation for the off-shell amplitude $\mathcal{G}$ can be written ($\mathcal{G}$ equals the connected and amputated four-point function multiplied by i)

$$\mathcal{G}(\omega,\omega')=\mathcal{I}(\omega,\omega')-\int d\omega_k \mathcal{I}(\omega,\omega_k)S(\omega_k)\mathcal{G}(\omega_k,\omega') = \mathcal{I}(\omega,\omega')-\int d\omega_k \mathcal{G}(\omega,\omega_k)S(\omega_k)\mathcal{I}(\omega_k,\omega') . \tag{A1}$$

We use free one-particle propagators and set

$$S(\omega_k)=-\frac{i}{2\pi}\frac{\gamma_0^{(1)}[\omega_k+(1/2)\sqrt{s}\,]-\boldsymbol{\gamma}^{(1)}\cdot\mathbf{k}+m}{[\omega_k+(1/2)\sqrt{s}\,]^2-\mathbf{k}^2-m^2+i0} \times\frac{\gamma_0^{(2)}[-\omega_k+(1/2)\sqrt{s}\,]+\boldsymbol{\gamma}^{(2)}\cdot\mathbf{k}+m}{[-\omega_k+(1/2)\sqrt{s}\,]^2-\mathbf{k}^2-m^2+i0} . \tag{A1'}$$

The instantaneous approximation consists in replacing $\mathcal{I}(\omega,\omega')$ by $\mathcal{I}^{(0)}=\mathcal{I}(0,0)$ and $S(\omega_k)$ by $S^{(0)}(\omega_k)$,

$$\mathcal{G}^{(0)}=\mathcal{I}^{(0)}-\int d\omega_k \mathcal{I}^{(0)}S^{(0)}(\omega_k)\mathcal{G}^{(0)} = \mathcal{I}^{(0)}-\mathcal{I}^{(0)}\mathcal{R}^{(0)}\mathcal{G}^{(0)} , \tag{A2}$$

being

$$S^{(0)}(\omega_k)=\frac{i}{2\pi}\frac{m^2}{w(\mathbf{k})^2}\frac{\Lambda_+^{(1)}(\mathbf{k})\Lambda_+^{(2)}(-\mathbf{k})}{[\omega_k+\sqrt{s}/2-w(\mathbf{k})+i0](\omega_k-[\sqrt{s}/2-w(\mathbf{k})]-i0)} ,$$

$$\mathcal{R}^{(0)}=\int d\omega_k S_0(\omega_k)=-\frac{m^2}{w(\mathbf{k})^2}\frac{\Lambda_+^{(1)}(\mathbf{k})\Lambda_+^{(2)}(-\mathbf{k})}{\sqrt{s}-2w(\mathbf{k})} .$$

In order to obtain corrections to this approximation we can write

$$\mathcal{I}(\omega,\omega')=\mathcal{I}^{(0)}+\mathcal{I}^{(1)}(\omega)+\overline{\mathcal{I}}^{(1)}(\omega')+\mathcal{I}^{(2)}(\omega,\omega') , \qquad S(\omega_k)=S^{(0)}(\omega_k)+S^{(1)}(\omega_k) , \tag{A3}$$

having defined

$$\mathcal{I}^{(1)}(\omega)=\mathcal{I}(\omega,0)-\mathcal{I}(0,0) , \quad \overline{\mathcal{I}}^{(1)}(\omega)=\mathcal{I}(0,\omega)-\mathcal{I}(0,0) ,$$

$$\mathcal{I}^{(2)}(\omega,\omega')=\mathcal{I}(\omega,\omega')-\mathcal{I}(\omega,0)-\mathcal{I}(0,\omega')+\mathcal{I}(0,0) .$$

Then, by setting

$$\mathcal{G}(\omega)=\mathcal{G}(\omega,0)=\mathcal{G}^{(0)}+\mathcal{G}^{(1)}(\omega)+\cdots$$

and taking into account

$$\int d\omega_k S^{(1)}(\omega_k)=\mathcal{R}^{(1)}=\frac{m^2}{w(\mathbf{k})^2}\frac{\Lambda_-^{(1)}(-\mathbf{k})\Lambda_-^{(2)}(\mathbf{k})}{\sqrt{s}+2w(\mathbf{k})} ,$$

one finds

$$\mathcal{G}^{(1)}(\omega)=\mathcal{I}^{(1)}(\omega)-\int d\omega_k\{[\mathcal{I}^{(1)}(\omega)+\overline{\mathcal{I}}^{(1)}(\omega_k)]S^{(0)}(\omega_k)\mathcal{G}^{(0)}+\mathcal{I}^{(0)}S^{(1)}(\omega_k)\mathcal{G}^{(0)}+\mathcal{I}^{(0)}S^{(0)}(\omega_k)\mathcal{G}^{(1)}(\omega_k)\} . \tag{A4}$$

Iterating such an equation, taking $\omega=0$, and using (A2), one obtains finally, after suitable rearrangements,

$$\mathcal{G}^{(1)}(0)=-\int d\omega_k[\overline{\mathcal{I}}^{(1)}(\omega_k)S^{(0)}(\omega_k)\mathcal{I}^{(0)}+\mathcal{I}^{(0)}S^{(0)}(\omega_k)\mathcal{I}^{(1)}(\omega_k)]-\mathcal{I}^{(0)}\mathcal{R}^{(1)}\mathcal{I}^{(0)} + \left[\int d\omega_k[\overline{\mathcal{I}}^{(1)}(\omega_k)S^{(0)}(\omega_k)\mathcal{I}^{(0)}+\mathcal{I}^{(0)}S^{(0)}(\omega_k)\mathcal{I}^{(1)}(\omega_k)]+\mathcal{I}^{(0)}\mathcal{R}^{(1)}\mathcal{I}^{(0)}\right]\mathcal{R}^{(0)}\mathcal{G}^{(0)}-\mathcal{I}^{(0)}\mathcal{R}^{(0)}\mathcal{G}_1(0) . \tag{A5}$$

Comparing the Lippmann-Schwinger equation

$$T=V-VR^{(0)}T \quad \left[R^{(0)}=\frac{1}{2w(\mathbf{k})-\sqrt{s}}\right]$$

with (A2), one obtains the potential defined in (1.5). Setting, however,

$$V=V^{(0)}+V^{(1)}+\cdots ,$$

$$T=T^{(0)}+T^{(1)}+\cdots ,$$

one has

$$T^{(1)}=V^{(1)}-V^{(1)}R^{(0)}T^{(0)}-V^{(0)}R^{(0)}T^{(1)} , \tag{A6}$$

which, compared with (A5), provides the expression (3.1) for $s<4m^2$. Notice that, for $s>4m^2$, one must have $\mathrm{Im}\mathcal{G}(\omega,\omega')\to 0$ for $\varepsilon\to 0$ under the assumption of confinement. The Re prescription in (3.1) anticipates this fact. Such a prescription is essential to make the use of the free propagator [Eq. (A1′)] consistent with confinement. Above, Re and Im stay properly for the self-adjoint and anti-self-adjoint parts.

APPENDIX B

The explicit expressions for the functions g_{2S}, g_{3S} in Eq. (3.10) are

$$g_{2S}=[\tfrac{3}{4}+\lambda_{2S}^2(\tfrac{3}{2}p^2+k^2)-\lambda_{2S}^4k^2(k^2+p^2)]\sinh(kp\lambda_{2S}^2)+[-3\lambda_{2S}^2kp+2k^3p\lambda_{2S}^4]\cosh(kp\lambda_{2S}^2) ,$$

$$\begin{aligned} g_{3S}=&\{-\tfrac{2}{3}+\tfrac{8}{9}k^2\lambda_{3S}^2-\tfrac{8}{45}k^4\lambda_{3S}^4-2\lambda_{3S}^2(k^2+p^2)(-\tfrac{4}{3}+\tfrac{16}{9}k^2\lambda_{3S}^2-\tfrac{16}{45}k^4\lambda_{3S}^4)\\ &+(\tfrac{8}{15}-\tfrac{32}{45}k^2\lambda_{3S}^2+\tfrac{32}{225}k^4\lambda_{3S}^4)[3-6\lambda_{3S}^2(k^2+p^2)+\lambda_{3S}^4(p^4+6k^2p^2+k^4)]\}\sinh(kp\lambda_{3S}^2)\\ &+\{(\tfrac{8}{15}-\tfrac{32}{45}k^2\lambda_{3S}^2+\tfrac{32}{225}k^4\lambda_{3S}^4)[12\lambda_{3S}^2kp-4\lambda_{3S}^4(kp^3+k^3p)]\\ &+4\lambda_{3S}^2kp(-\tfrac{4}{3}+\tfrac{16}{9}k^2\lambda_{3S}^2-\tfrac{16}{45}k^4\lambda_{3S}^4)\}\cosh(kp\lambda_{3S}^2) . \end{aligned} \tag{B1}$$

APPENDIX C

As an example we derive expansion (3.11) in detail and obtain A_1, A_3, B_2 for the state $1S$. To this aim we shall introduce the quantity

$$J(\varepsilon,W)=\int_0^\infty dk\,k\left[\frac{k^2}{2m}-W\right]e^{-\lambda_{1s}^2k^2}\int_0^\infty dp\,pe^{-\lambda_{1s}^2p^2/2}\sinh(kp\lambda_{1s}^2)\frac{1}{-W/2+k^2/2m+\sqrt{p^2+\varepsilon^2}}\frac{1}{p^2+\varepsilon^2} \tag{C1}$$

to be studied as an analytical function in W. The quantity $I_{1S}(\varepsilon)$ is afterwards obtained as

$$I_{1S}(\varepsilon)=\mathrm{Re}J(\varepsilon,W)|_{W=W_{1S}^{(0)}} . \tag{C2}$$

We choose a $\mu>0$ and write

$$J(\varepsilon,W)=\int_0^\infty dk\left[\int_0^\mu dp+\int_\mu^\infty dp\right]\cdots=J_0(\varepsilon,W)+J_\infty(\varepsilon,W) .$$

Then we notice that $J_\infty(\varepsilon,W)$ is a regular function of ε^2 and can be expanded as

$$J_\infty(\varepsilon,W)=A_0^\infty+\varepsilon^2A_2^\infty+\frac{\varepsilon^4}{2}A_4^\infty+\cdots \tag{C3}$$

and gives contribution only to the coefficient $A_0, A_2, A_4, \ldots$.

To study $J_0(\varepsilon,W)$, let us make the position $\alpha_k=k^2/2m-W/2$, take $\mathrm{Re}W<-4\mu$, $\varepsilon<\mu$, and expand the integrand in p:

$$f(k,p):=pe^{-p^2\lambda^2/2}\sinh(kp\lambda^2)\frac{1}{\alpha_k+\sqrt{p^2+\varepsilon^2}}\frac{1}{p^2+\varepsilon^2}$$
$$\simeq\frac{1}{\alpha_k}\left\{\frac{kp^2\lambda^2}{p^2+\varepsilon^2}-\frac{kp^2\lambda^2}{\alpha_k\sqrt{p^2+\varepsilon^2}}+\left[\frac{-k\lambda^4}{2}+\frac{k^3\lambda^6}{6}\right]\frac{p^4}{p^2+\varepsilon^2}\right.$$
$$\left.+\frac{k\lambda^2p^2}{\alpha_k^2}-\frac{k\lambda^2}{\alpha_k^3}p^2\sqrt{p^2+\varepsilon^2}+\left[\frac{k\lambda^4}{2}-\frac{k^3\lambda^6}{6}\right]\frac{p^4}{\alpha_k\sqrt{p^2+\varepsilon^2}}+\frac{k^3\lambda^6}{6}\frac{p^4}{\alpha_k^2}+\cdots\right\}.$$

We have

$$\int_0^\mu dp\, f(k,p)=\left[\frac{k\lambda^2\mu}{\alpha_k}-\frac{k\lambda^2\mu^2}{2\alpha_k^2}-\frac{k\lambda^2\mu^4}{4\alpha_k^4}+\left[-\frac{k\lambda^4}{2\alpha_k}+\frac{k^3\lambda^6}{\alpha_k}\right]\frac{\mu^3}{3}+\frac{k\lambda^2\mu^3}{3\alpha_k^2}+\frac{k\lambda^4\mu^4}{8\alpha_k^2}+\cdots\right]$$
$$+\varepsilon\left[\frac{-k\lambda^2}{\alpha_k}\right]\frac{\pi}{2}+\varepsilon^2\left[\frac{k\lambda^2}{\alpha_k\mu}-\frac{1}{2}\frac{k\lambda^2}{\alpha_k^2}(\tfrac{1}{2}-\ln 2)-\frac{3}{8}\frac{k\lambda^2\mu}{\alpha_k^4}+\left[\frac{k\lambda^4}{2\alpha_k}-\frac{k^3\lambda^6}{6\alpha_k}\right]\mu-\frac{1}{8}\frac{k\lambda^4\mu^2}{\alpha_k^2}+\cdots\right]$$
$$+\varepsilon\ln\left[\frac{\varepsilon}{\mu}\right]\left[-\frac{k\lambda^2}{2\alpha_k^2}\right]+\varepsilon^3\left[-\frac{k\lambda^4}{2\alpha_k}+\frac{k^3\lambda^6}{6\alpha_k}\right]\frac{\pi}{2}$$
$$+\varepsilon^4\left[-\frac{k\lambda^2}{3\alpha_k\mu^3}+\frac{3}{16}\frac{k\lambda^2}{\alpha_k^2\mu^2}-\frac{3}{32}\frac{k\lambda^2}{\alpha_k^4}+\left[-\frac{k\lambda^4}{2\alpha_k}+\frac{k^3\lambda^6}{6\alpha_k}\right]\frac{1}{\mu}-\frac{7}{32}\frac{k\lambda^4}{2\alpha_k^2}+\cdots\right]. \tag{C4}$$

By integrating in k the terms proportional to ε, ε^3, and $\varepsilon^2\ln(\varepsilon/\mu)$, we arrive at the expressions reported in Table I for the state $1S$:

$$A_1^{1S}=-\frac{\pi}{2}\lambda^2\int_0^\infty dk\, k^2\exp(-\lambda^2k^2)=-\frac{\pi^{3/2}}{8\lambda},$$
$$B_2^{1S}=-m\lambda^2\int_0^\infty dk\, e^{-\lambda^2k^2}\frac{k^2}{k^2-W_{1S}^{(0)}m}$$
$$=-m\lambda\frac{\sqrt{\pi}}{2}\mathrm{Re}\{1-\lambda\sqrt{-\pi mW_{1S}^{(0)}}e^{(-\lambda^2W_{1S}^{(0)}m)}[1-\Phi(\lambda\sqrt{-W_{1S}^{(0)}m})]\}, \tag{C5}$$
$$A_3^{1S}=\frac{\pi}{2}\int_0^\infty dk\, e^{-\lambda^2k^2}\left[-\frac{k^2\lambda^4}{2}+\frac{k^4\lambda^6}{6}\right]=-\frac{\pi^{3/2}\lambda}{32},$$

where $\Phi(x)=2/\sqrt{\pi}\int_0^x dt\, e^{-t^2}$. Notice that, in the case $W_{1S}^{(0)}>0$, one can write explicitly

$$B_2^{1S}=-m\lambda\frac{\sqrt{\pi}}{2}[1+i\lambda\sqrt{\pi W_{1S}^{(0)}m}\,e^{(-\lambda^2W_{1S}^{(0)}m)}\Phi(i\lambda\sqrt{W_{1S}^{(0)}m})].$$

Notice also that the coefficients of the terms in $1,\varepsilon^2,\varepsilon^4,\ldots$ cannot be obtained in a closed form. Of course, a similar method with a little bit more technique can be applied to the states $2S,3S$; the second and the third line of Table I are obtained in this way.

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