Significance of relativistic wave equations for bound states

Wolfgang Lucha
Institut für Hochenergiephysik, Österreichische Akademie der Wissenschaften, A-1050 Wien, Austria

Heinz Rupprecht and Franz F. Schöberl
Institut für Theoretische Physik, Universität Wien, A-1090 Wien, Austria
(Received 12 December 1991)

We scrutinize the relevance of relativistic wave equations for the description of quark-antiquark bound states. By comparing the predictions of the nonrelativistic Schrödinger equation (with only the lowest-order relativistic corrections of the famous Breit-Fermi Hamiltonian), the spinless Salpeter equation, and a new semirelativistic wave equation (which incorporates relativistic kinematics and the complete relativistic corrections to the static interaction potential) for light and heavy quarkonia within three different potential models, we discuss the extent to which the use of relativistic wave equations is reasonable or necessary in order to reproduce the experimentally observed meson mass spectra. We are forced to conclude that—contrary to one's physical intuition—a relativistic treatment of bound states in a potential model provides no improvement at all compared to the corresponding nonrelativistic description.

PACS number(s): 11.10.St, 03.65.Ge, 11.10.Qr, 12.40.Qq

I. INTRODUCTION

One of the most popular approaches to hadrons is to describe them by means of a nonrelativistic Schrödinger equation as bound states of (constituent) quarks which interact via some effective potential. The overwhelming success of these nonrelativistic potential models not only for heavy quarkonia but also for light mesons remains up to now a miracle in hadron spectroscopy [1]. In principle, at least bound states consisting of light constituents should be dealt with in a relativistic framework. In this paper we would like to address the question of to what extent the employment of a relativistic equation of motion for the description of fermion-antifermion bound states is meaningful or even unavoidable.

The strategy of our investigation (and simultaneously the outline of this paper) is the following. We solve the nonrelativistic Schrödinger equation, the spinless Salpeter equation, and a new semirelativistic wave equation, all of them introduced in Sec. II, with the help of the numerical method briefly sketched in Sec. III for some typical interquark potentials presented in Sec. IV. Comparing in Sec. V the output of our fits to the experimentally observed meson mass spectra, we are, in Sec. VI, unambiguously led to the conclusion that, as far as the confrontation with experiment is concerned, an increase of the relativistic consistency of the bound-state wave equations makes things worse.

II. WAVE EQUATIONS

We are interested in Schrödinger-type eigenvalue equations of the form

$$H\psi(\mathbf{x})=M\psi(\mathbf{x}) , \tag{1}$$

where H is the Hamiltonian governing the dynamics of the bound state under consideration and $\psi(\mathbf{x})$ the configuration-space representation of the corresponding state vector. In the center-of-momentum system of the bound state, the energy eigenvalue arising from this equation is, of course, nothing else but the mass M of the composite particle.

We shall consider bound states consisting of fermions with equal masses $m_1=m_2=m$ and spins $\mathbf{S}_1,\mathbf{S}_2$, respectively, which interact via a spherically symmetric potential $V(r)$, $r\equiv|\mathbf{x}|$, where $\mathbf{x}$ denotes the relative coordinate of the bound-state constituents.

A. The nonrelativistic Schrödinger equation

The Hamiltonian containing the relativistic corrections up to order $1/c^2$ is called the generalized Breit-Fermi Hamiltonian (for a recent review see, for instance, Ref. [1]):

$$H=2m+\frac{\mathbf{p}^2}{m}-\frac{\mathbf{p}^4}{4m^3}+V(r), \tag{2}$$

where the potential $V(r)$ contains, in addition to the static interaction $V_{\text{stat}}(r)$, all relativistic corrections,

$$V(r)=V_{\text{stat}}(r)+H_{\text{SI}}+H_{\text{LS}}+H_{\text{SS}}+H_{\text{T}} . \tag{3}$$

The interaction between the fermionic bound-state constituents may be viewed as being generated by a (maybe only effective) exchange of a boson after integrating out the degrees of freedom corresponding to this virtual particle. The static potential $V_{\text{stat}}(r)$ may be decomposed according to the spin of this boson. For bound states consisting of quarks (and antiquarks) there is very strong evidence that the static interquark potential—originating from quantum chromodynamics—receives, at least within a nonrelativistic analysis, predominantly a vector and a scalar contribution (for a very recent review on the

phenomenological aspects of the forces acting within bound states of quarks see, e.g., also Ref. [1]):

$$V_{\rm stat}(r)=V_{\rm V}(r)+V_{\rm S}(r)\ . \tag{4}$$

The relativistic corrections to the static potential $V_{\rm stat}(r)$ discriminate between different spin structures. They consist of (1) the spin-independent term

$$H_{\rm SI}=\frac{1}{8m^2}\left[2\Delta V_{\rm V}(r)-*\nabla V_{\rm V}(r)\nabla*+*\nabla\cdot\mathbf{r}\frac{1}{r}\left[\frac{dV_{\rm V}(r)}{dr}\right]\mathbf{r}\cdot\nabla*+2*\nabla V_{\rm S}(r)\nabla*\right], \tag{5}$$

where for a generic operator $\mathcal{O}$ the expression between stars is the shorthand notation for

$$*\nabla\mathcal{O}\nabla*\equiv\Delta\mathcal{O}+2\nabla\mathcal{O}\cdot\nabla+\mathcal{O}\Delta\ , \tag{6}$$

(2) the spin-orbit term

$$H_{\rm LS}=\frac{1}{2m^2r}\left[3\frac{dV_{\rm V}(r)}{dr}-\frac{dV_{\rm S}(r)}{dr}\right]\mathbf{L}\cdot\mathbf{S}\ , \tag{7}$$

where $\mathbf{S}=\mathbf{S}_1+\mathbf{S}_2$ is the total spin of the bound state and $\mathbf{L}=\mathbf{x}\times\mathbf{p}$ is the relative orbital angular momentum of its constituents; (3) the spin-spin term

$$H_{\rm SS}=\frac{2}{3m^2}\mathbf{S}_1\cdot\mathbf{S}_2\Delta V_{\rm V}(r)\ , \tag{8}$$

and (4) the tensor term

$$H_{\rm T}=\frac{1}{m^2}\left[\frac{1}{r}\frac{dV_{\rm V}(r)}{dr}-\frac{d^2V_{\rm V}(r)}{dr^2}\right]\times\left[\frac{(\mathbf{S}_1\cdot\mathbf{x})(\mathbf{S}_2\cdot\mathbf{x})}{r^2}-\frac{1}{3}\mathbf{S}_1\cdot\mathbf{S}_2\right]. \tag{9}$$

For bound-state constituents of spin $S_1=S_2=\frac{1}{2}$, the scalar product of their spins, $\mathbf{S}_1\cdot\mathbf{S}_2$, is given by

$$\mathbf{S}_1\cdot\mathbf{S}_2=\begin{cases}-\frac{3}{4} & \text{for spin singlets, } S=0,\\ +\frac{1}{4} & \text{for spin triplets, } S=1\ .\end{cases} \tag{10}$$

B. The spinless Salpeter equation

The spinless Salpeter equation may be regarded as the Schrödinger equation generalized to relativistic kinematics. For the case of two particles with equal masses m interacting via a spherically symmetric potential $V(r)$, the configuration-space representation of the spinless Salpeter equation in the center-of-momentum system of the two particles reads

$$[2\sqrt{-\Delta+m^2}+V(r)]\psi(\mathbf{x})=M\psi(\mathbf{x})\ . \tag{11}$$

In spite of the—hopefully not misleading—designation customary for this wave equation, the potential $V(r)$ may contain all the relativistic corrections given in Eqs. (5)–(9), some of which depend, of course, on the spins of the involved bound-state constituents.

C. The semirelativistic wave equation

In principle, there is no obstacle to taking into account the complete relativistic corrections to the static potential $V_{\rm stat}(r)$. Introducing the kinetic-energy operator in coordinate space,

$$E_l\equiv\sqrt{-\Delta_l+m^2}\ , \tag{12}$$

where

$$\Delta_l\equiv\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{l(l+1)}{r^2} \tag{13}$$

denotes the action of the Laplacian on states of definite orbital angular momentum l, the semirelativistic equation of motion for the radial part $R(r)$ of the bound-state wave function $\psi(\mathbf{x})=R(r)\mathcal{Y}_{lm}(\theta,\phi)$ with a static potential consisting only of vector and scalar contributions is given by [2–4]

$$\begin{aligned}(M-2E_l)R(r)=\frac{1}{4E_l}\Bigg\{&(E_l+m)[V_{\rm V}(r)+V_{\rm S}(r)](E_l+m)-2\left[2\frac{dV_{\rm V}(r)}{dr}-\frac{dV_{\rm S}(r)}{dr}\right]\frac{d}{dr}-2[2V_{\rm V}(r)-V_{\rm S}(r)]\Delta_l\\&-(E_l+m)V_{\rm V}(r)\frac{\Delta_l}{E_l+m}-\frac{\Delta_l}{E_l+m}V_{\rm V}(r)(E_l+m)\\&+\frac{1}{E_l+m}\Bigg[\left[\frac{d^2V_{\rm V}(r)}{dr^2}+\frac{d^2V_{\rm S}(r)}{dr^2}\right]\frac{d^2}{dr^2}\\&\quad+\left[\frac{dV_{\rm V}(r)}{dr}+\frac{dV_{\rm S}(r)}{dr}\right]\left[\frac{d}{dr}\Delta_l+\Delta_l\frac{d}{dr}+\frac{l(l+1)}{r^3}\right]\\&\quad+[V_{\rm V}(r)+V_{\rm S}(r)]\Delta_l\Delta_l\Bigg]\frac{1}{E_l+m}\Bigg]\frac{1}{E_l}R(r)\\&+\frac{1}{4E_l}\left\{-4\left[\frac{dV_{\rm V}(r)}{dr}\frac{d}{dr}+V_{\rm V}(r)\Delta_l\right]+2(E_l+m)V_{\rm V}(r)\frac{\Delta_l}{E_l+m}+2\frac{\Delta_l}{E_l+m}V_{\rm V}(r)(E_l+m)\right.\end{aligned}$$

$$-\frac{1}{E_l+m}\left[\left[\frac{d^2V_V(r)}{dr^2}+\frac{d^2V_S(r)}{dr^2}\right]\left[\frac{2}{r}\frac{d}{dr}-\frac{l(l+1)}{r^2}\right]\right.$$
$$\left.+\frac{1}{r}\left[\frac{dV_V(r)}{dr}+\frac{dV_S(r)}{dr}\right]\left[\frac{d^2}{dr^2}+\Delta_l\right]\right]\frac{1}{E_l+m}\left.\right\}\frac{1}{E_l}\tfrac{4}{3}\mathbf{S}_1\cdot\mathbf{S}_2R(r)+\cdots . \quad (14)$$

Here the dots denote terms which do not contribute for spin-singlet states, i.e., $S=0$, or states with vanishing orbital angular momentum, i.e., $l=0$ (see, for instance, [1]), and hence can be omitted from the equation of motion for the subsequent discussion.

This semirelativistic bound-state wave equation follows from the Bethe-Salpeter equation [5] under the assumption of a static integral kernel—which corresponds to an instantaneous interaction between the bound-state constituents—and after dropping one term—which describes particle-antiparticle annihilation and their subsequent production and thus has no clear interpretation in terms of a potential—in the resulting Salpeter equation [6] for the suitably defined equal-time wave function (for more details see, for instance, Ref. [7]).

III. THE METHOD OF ORTHOGONAL COLLOCATION

For the numerical solution of the (semi)relativistic wave equations in Secs. II B and II C, we use a matrix method [8] which is closely related to the method of orthogonal collocation. The basic idea of this procedure is to approximate the nonlocal square-root differential operator $\sqrt{-\Delta+m^2}$, which enters in the spinless Salpeter equation (11) or in the semirelativistic wave equation (14), and all functions of this expression, by suitably defined matrix representations.

These matrices give the action of all operators showing up in the equations of motion on some set of basis functions for $\mathcal{L}_2(R^+)$, evaluated at some points $\{r_m, m=1,2,\ldots\}$. Following [8], we choose as our basis functions $\{f_n(r),\ n=0,1,\ldots,N\}$ the (complete, if $N=\infty$) orthonormal system

$$f_n(r):=\left[\frac{n!\lambda^{2l+1}}{(n+2l)!}\right]^{1/2}r^l\exp\left[-\frac{\lambda r}{2}\right]L_n^{(2l)}(\lambda r)\ , \quad (15)$$

where $L_n^{(2l)}(r)$ are the generalized Laguerre polynomials with parameter $2l$ [9]. The points r_m where these matrices are defined are conveniently chosen as the N zeros $\{r_m,\ m=1,2,\ldots,N\}$ of $f_N(r)$: $f_N(r_m)=L_N^{(2l)}(\lambda r_m)=0$. The parameter λ serves to accumulate the values of r_m in the region where the wave functions we are interested in are significantly different from zero. Empirically, it is roughly given by [10]

$$\lambda=\frac{\rho_N^{(l)}}{7m\langle r\rangle}\ , \quad (16)$$

where $\rho_N^{(l)}$ is the largest zero of $L_N^{(2l)}(r)$.

With the help of this set of basis functions, to every operator $\mathcal{O}$, including the potential $V(r)$ and derivatives such as

$$\frac{d}{dr},\Delta_l,\ldots,$$

an $N\times N$ matrix $\mathbf{O}$ may be associated by the prescription

$$(\mathbf{O})_{mn}=\sum_{k=1}^{N}[\mathcal{O}f_k(r_m)](\mathbf{f}^{-1})_{kn}\ , \quad (17)$$

where $\mathbf{f}$ is the $N\times N$ matrix $\mathbf{f}_{mn}:=f_n(r_m)$.

In order to find an (approximate) matrix representation of the square-root differential operator $E_l\equiv\sqrt{-\Delta_l+m^2}$, introduced in Eq. (12), consider the operator

$$Q_l\equiv E_l^2=-\Delta_l+m^2 \quad (18)$$

and its associated matrix $\mathbf{Q}_l$. This matrix is equivalent to the diagonal matrix $\mathbf{q}_l=\mathrm{diag}(q_1,q_2,\ldots,q_N)$ of its corresponding eigenvalues $\{q_m,\ m=1,2,\ldots,N\}$,

$$\mathbf{Q}_l=\mathbf{U}\mathbf{q}_l\mathbf{U}^{-1}\ . \quad (19)$$

Since the operator Q_l is the formal square of E_l, the matrix representation $\mathbf{E}_l$ of the operator E_l is defined by the square root of $\mathbf{Q}_l$,

$$\mathbf{E}_l:=\mathbf{Q}_l^{1/2}=\mathbf{U}\mathbf{q}_l^{1/2}\mathbf{U}^{-1}\ . \quad (20)$$

The matrix representations of other nonlocal operators involving E_l may be defined in a similar manner, for instance, for E_l^{-1} by

$$\mathbf{E}_l^{-1}:=\mathbf{U}\mathbf{q}_l^{-1/2}\mathbf{U}^{-1}\ , \quad (21)$$

for (E_l+m) by

$$(\mathbf{E}_l+\mathbf{m}):=\mathbf{U}(\mathbf{q}_l^{1/2}+\mathbf{m})\mathbf{U}^{-1}\ , \quad (22)$$

or, for $(E_l+m)^{-1}$ by

$$(\mathbf{E}_l+\mathbf{m})^{-1}:=\mathbf{U}(\mathbf{q}_l^{1/2}+\mathbf{m})^{-1}\mathbf{U}^{-1}\ . \quad (23)$$

In this way any relativistic equation of motion is converted to a simple matrix eigenvalue problem for the wave-function vector $\mathbf{R}_m\equiv R(r_m)$.

We use the method of orthogonal collocation since the analytical method developed in Ref. [11], although somewhat faster in rate of convergence, is not well suited for the treatment of potentials involving transcendental functions like, e.g., the error function.

IV. SOME SAMPLE POTENTIALS

Let us apply the previously developed ideas to hadrons, considered as bound states of (constituent) quarks which

are bound by the strong interaction. We have at our disposal a generally accepted quantum field theory of the strong interaction, namely, quantum chromodynamics (QCD), a non-Abelian gauge theory for quarks and gluons. Nevertheless, because of the intrinsically nonperturbative nature of the bound-state problem in non-Abelian gauge theories, it is, up to now, not possible to derive the forces acting between the quarks inside a hadron from first principles, that is, QCD. Consequently, the corresponding interquark potential has to be determined phenomenologically.

A. Funnel potential

The reasoning leading to the various proposed potentials $V(r)$ is rather simple. For small distances between the quarks, one expects from one-gluon exchange—by analogy to one-photon exchange in quantum electrodynamics—a Coulomb-like contribution to the potential, that is, $V(r) \propto 1/r$. For large distances, in order to be able to describe confinement, the potential has to rise to infinity. From lattice-gauge-theory computations there are hints that this rise is a linear one, that is, $V(r) \propto r$. The most reasonable possibility to construct an interquark potential which satisfies both of the above constraints is to simply add these two contributions. This leads to the so-called funnel (or Cornell) potential [12–14]:

$$V(r) = -\frac{\kappa}{r} + ar \ , \tag{24}$$

which depends on just two parameters, viz., on the Coulomb-like coupling constant κ and on the slope a of the linear term. This funnel-shaped potential represents the prototype of all of the proposed realistic interquark potentials. A closer inspection reveals that all phenomenologically acceptable "QCD-inspired" potentials are only variations around the funnel potential [1].

As should be evident from its above-mentioned origin, the Coulomb-like part of the funnel potential is of vectorial spin structure. In contrast to this, the majority of all investigations of this question points towards a scalar spin structure of the (linear) confining part. Consequently the splitting of the funnel potential, according to the respective spin structure, into vector and scalar contribution reads

$$V_{\rm V}(r) = -\frac{\kappa}{r}, \quad V_{\rm S}(r) = ar + V_0 \ . \tag{25}$$

Usually, a constant V_0 is added to the confining contribution. The origin of this constant may be traced back to the infrared divergence of the momentum-space expression of this part of the potential, which makes a regularization of the involved integral necessary [1,15]. This constant has to be regarded as an additional arbitrary parameter in the potential, its arbitrariness being induced by the arbitrariness in the choice of the renormalization point. In the case of a linear confining potential, and for a special choice of the renormalization point, the constant V_0 is related to the slope a of the linear potential by [1,15]

$$V_0 = -2\sqrt{a}\exp(-\gamma_E + \tfrac{1}{2}) \ , \tag{26}$$

where $\gamma_E = 0.577\,215\ldots$ is the Euler-Mascheroni constant.

The appearance of a Coulomb-like contribution $\sim 1/r$ in the vector part of the potential causes, however, some problems since in this case, because of the involved derivatives of the potential, the relativistic-correction terms in the Breit-Fermi Hamiltonian (2) become singular. For instance, due to the relation $\Delta 1/r = -4\pi\delta^{(3)}(\mathbf{x})$, the spin-spin interaction (8) derived from a pure Coulomb potential involves a δ function,

$$H_{\rm SS} = \frac{8\pi\kappa}{3m^2}\mathbf{S}_1\cdot\mathbf{S}_2\delta^{(3)}(\mathbf{x}) \ , \tag{27}$$

which makes, for one of the two possible spin alignments in Eq. (10), the Breit-Fermi Hamiltonian unbounded from below. This unlucky circumstance would prevent a nonperturbative treatment of the Breit-Fermi Hamiltonian. Of course, these singularities are only a consequence of the nonrelativistic approximation inherent to the Breit-Fermi Hamiltonian. They are not present in the completely relativistic form of the bound-state wave equation discussed in Sec. II C.

One way out is to smear the δ function in the spin-spin term by replacing it by a function which converges weakly towards the δ function, like, e.g.,

$$\delta^{(3)}(\mathbf{x}) \rightarrow \frac{1}{r_0^3}\exp\left[-\pi\frac{r^2}{r_0^2}\right] \ , \tag{28}$$

which entails the modified spin-spin interaction

$$H_{\rm SS} = \frac{8\pi\kappa}{3m^2 r_0^3}\mathbf{S}_1\cdot\mathbf{S}_2\exp\left[-\pi\frac{r^2}{r_0^2}\right] \ . \tag{29}$$

In Sec. V, for the numerical applications of the funnel potential with the regularization (28) of the δ function, the characteristic length scale r_0 will be parametrized in terms of the quark mass m by two constants K_1, K_2:

$$r_0 = K_1 m^{-K_2} \ . \tag{30}$$

B. The erf potential

Another possibility is to modify the (potentially dangerous) Coulomb interaction by introducing a factor which provides some sort of damping, like, e.g., the error function erf(r) [16]:

$$V_{\rm V}(r) = -\frac{\kappa}{r}\mathrm{erf}(\sqrt{\pi}\mu r), \quad V_{\rm S}(r) = ar + V_0 \ . \tag{31}$$

Then, because of

$$\Delta\frac{\mathrm{erf}(r)}{r} = -\frac{4}{\sqrt{\pi}}\exp(-r^2) \ , \tag{32}$$

the Laplacian acting on the vector part of the potential generates no singularity at all in the spin-spin interaction,

$$H_{\rm SS} = \frac{8\pi}{3m^2}\kappa\mu^3\mathbf{S}_1\cdot\mathbf{S}_2\exp(-\pi\mu^2 r^2) \ . \tag{33}$$

As before, for the numerical applications of this potential in Sec. V, the scale factor μ will be parametrized in terms of the quark mass m by two constants K_1, K_2:

$$\mu = K_1 m^{K_2} . \tag{34}$$

C. Smooth-transition potential

Still more sophisticated is the potential [10]

$$V_V(r) = \left[-\frac{\kappa}{r} f(\mu r) + d - ar \right] F(r) - br \exp\left[-\frac{r}{c} \right] ,$$
$$V_S(r) = ar + V_0 . \tag{35}$$

Here, the singularity of the Coulomb part is regularized by

$$f(x) = 1 - \left[1 + \frac{x}{2} \right] e^{-x} . \tag{36}$$

The transition from the vector to the scalar part of the potential is described by the Fermi-Dirac function

$$F(r) = \frac{1}{1 + \exp[-4k(r - R_1)]} , \tag{37}$$

where $k = F'(R_1)$ characterizes the width of the transition region. In the limit $k \to -\infty$, $F(r)$ reduces to the Heaviside function,

$$\lim_{k \to -\infty} F(r) = \theta(R_1 - r) = \begin{cases} 1 & \text{for } r < R_1 , \\ 0 & \text{for } r > R_1 . \end{cases} \tag{38}$$

The potential parameters d and R_1 may be determined by requiring the short-range part of the vector potential [the term in parentheses on the right-hand side of Eq. (35)] and its first derivative to vanish at the point R_1:

$$d = aR_1 + \frac{\kappa}{R_1} f(z) \tag{39}$$

and

$$e^{-z} \left[\frac{1}{z^2} + \frac{1}{z} + \frac{1}{2} \right] - \frac{1}{z^2} = \frac{a}{\kappa \mu^2} , \tag{40}$$

with the dimensionless variable z defined by

$$z \equiv \mu R_1 . \tag{41}$$

Again, for the numerical applications of this potential in Sec. V, the scale factor μ will be parametrized in terms of the quark mass m by two constants K_1, K_2:

$$\mu = K_1 m^{K_2} . \tag{42}$$

V. RESULTS AND CONCLUSIONS

A. General procedure

In this section we investigate the relativistic equations of motion, in particular, on the one hand, the spinless Salpeter equation of Sec. II B and, on the other hand, the semirelativistic wave equation of Sec. II C, with respect to their significance for the description of bound states of fermion-antifermion pairs. To this end we compare the predictions of these relativistic wave equations for the energy levels of quark-antiquark bound states with those resulting from the nonrelativistic Schrödinger equation, amended by only the lowest-order relativistic corrections of the generalized Breit-Fermi Hamiltonian, as given in Sec. II A. We shall simultaneously consider bound states of light quarks, like $(u\bar{u})$ and $(s\bar{s})$, as well as the charmonium $(c\bar{c})$ and bottomonium $(b\bar{b})$ systems. The corresponding energy spectra are obtained by performing in each case a fit which minimizes the quantity

$$\chi^2 \equiv \sum_i \left[\frac{M_i - M_i^{\exp}}{\Delta M_i} \right]^2 , \tag{43}$$

where M_i and $M_i^{\exp}$ are the theoretically predicted and experimentally observed [17] values for the meson masses, respectively, and ΔM_i the corresponding experimental error [17].

In the following, the "center of gravity" (COG) denotes the average mass of the $(S=1,\ l=1)$ states, for heavy quarkonia usually called χ_q $(q = c, b, \ldots)$,

$$\mathrm{COG}(^3P_j) \equiv \tfrac{1}{9}[5M(^3P_2) + 3M(^3P_1) + M(^3P_0)] , \tag{44}$$

where we recall the usual spectroscopic notation $^{2S+1}l_J$ for a state with orbital angular momentum l, spin S, and total angular momentum J; $S, P, D, F, \ldots$ corresponds to orbital angular momentum $l = 0, 1, 2, 3, \ldots$, respectively.

Since the spin-singlet $l=1$ states of the heavy quarkonia, i.e., $c\bar{c}(^1P_1)$ and $b\bar{b}(^1P_1)$, have not yet been unambiguously confirmed by experiment [17], we use as a first estimate of their experimental masses the center of gravity of the corresponding spin-triplet $l=1$ states, as defined above. Our justification for this is that (at least within a perturbative treatment) the center of gravity $\mathrm{COG}(^3P_j)$, for which perturbatively the contributions of the spin-orbit and tensor interactions separately add up to zero, equals the mass $M(^1P_1)$ of the $(S=0,\ l=1)$ state, which does not receive a contribution at all from the spin-orbit and tensor interactions because of $S=0$ (see, e.g., Ref. [1]).

For the remainder of this section our aim will be a simultaneous description, by one and the same interquark potential, of light mesons, such as π and ρ, as well as heavy quarkonia, such as the charmonium and bottomonium systems. According to the concept of a running gauge coupling constant, in these potentials we allow for different values of the Coulomb-like parameter κ for light (κ_l) and heavy (κ_h) quarkonia, or even for charmonium (κ_c) and bottomonium (κ_b) systems. However, in view of the well-known difficulties one encounters when taking into account also the spin-independent relativistic corrections (5) of the Breit-Fermi Hamiltonian, we take the liberty to ignore these terms for the investigations based on the Schrödinger and spinless Salpeter equations.

TABLE I. Quarkonium mass spectra (in units of GeV) predicted for the funnel potential by the Schrödinger and spinless Salpeter equation, respectively, as well as the corresponding values of χ^2 obtained by the fit.

State	Experiment [17]	Nonrelativistic Schrödinger equation	Spinless Salpeter equation
$s\bar{s}(1S)$	1.004	1.004	1.004
$s\bar{s}(2S)$	1.620	1.782	1.676
$\eta_c(1S)$	2.980	2.998	2.985
$J/\psi(1S)$	3.097	3.097	3.097
$c\bar{c}(1^1P_1)$	3.525	3.507	3.522
$\psi(2S)$	3.686	3.686	3.686
$\Upsilon(1S)$	9.460	9.459	9.460
$\Upsilon(2S)$	10.023	10.028	10.024
χ^2		1831	78.3

TABLE III. Quarkonium mass spectra (in units of GeV) predicted for the erf potential by the Schrödinger and spinless Salpeter equation, respectively, as well as the corresponding values of χ^2 obtained by the fit.

State	Experiment [17]	Nonrelativistic Schrödinger equation	Spinless Salpeter equation
π	0.138	0.138	0.139
$\rho(770)$	0.768	0.769	0.768
$\eta_c(1S)$	2.980	2.976	2.965
$J/\psi(1S)$	3.097	3.097	3.097
$c\bar{c}(1^1P_1)$	3.525	3.524	3.522
$\psi(2S)$	3.686	3.686	3.686
$\psi(4040)$	4.04	4.021	4.05
$\Upsilon(1S)$	9.460	9.460	9.460
$\Upsilon(2S)$	10.023	10.023	10.022
$\Upsilon(3S)$	10.355	10.355	10.357
χ^2		16.3	160.9

B. Spinless Salpeter equation versus nonrelativistic Schrödinger equation

As a first step, we investigate the effect of the incorporation simply of relativistic kinematics into the bound-state equation of motion. To this end we compare the predictions of, on the one hand, the Schrödinger equation of Sec. II A and, on the other hand, the spinless Salpeter equation of Sec. II B, obtained from a separate numerical fit for each of the potentials sketched in Sec. IV. The results of these fits are presented in Tables I, III, and V, respectively, with the corresponding parameter values given in Tables II, IV, and VI. In all cases, the additive constant V_0 in the confining part $V_S(r)=ar+V_0$ of the potential is assumed to be related to the slope a of the linear rise by Eq. (26) and is thus not varied independently.

For the funnel potential (25) with regularization (28), it is not possible to obtain a satisfactory fit which also includes the light mesons π and ρ. Nevertheless, in order to get at least an idea of the range of validity of this potential model, we consider, in addition to the heavy quarkonia ($c\bar{c}$) and ($b\bar{b}$), the (spin-averaged) $1S$ and $2S$ states of ($s\bar{s}$).

The results obtained with relativistic kinematics are significantly better than the ones following from the nonrelativistic Schrödinger equation (Table I). For the latter equation of motion, however, the relative magnitude of the Coulomb-like parameter κ is in accordance with the ideas of asymptotic freedom as is expected for the strong gauge coupling constant of quantum chromodynamics, that is, $\kappa_l > \kappa_h$. In contrast to this (and to one's physical intuition), this is no longer the case for the spinless Salpeter equation (Table II). The incorporation of relativistic kinematics thus improves the description of the empirically found meson spectrum, but at the price of requiring unacceptable values for the parameters of the potential.

For the erf potential (31), in spite of reasonable fits obtained from both nonrelativistic and relativistic kinematics (Table III), a similar behavior is found for the Coulomb-like parameter: $\kappa_l < \kappa_c < \kappa_b$ in both cases, which is again in clear conflict with asymptotic freedom (Table IV). It is highly unlikely that, within this potential model, hadronic features such as decay rates or masses of excited states may be reproduced.

For the smooth-transition potential (35), the incorporation of relativistic kinematics reduces the quality of the fit drastically (Table V) and entails parameter values the relative magnitude of which is once more in conflict with the expectations of quantum chromodynamics (Table VI).

TABLE II. Quark masses and potential parameters obtained from a fit of the quarkonium mass spectra with the Schrödinger and spinless Salpeter equation, respectively, for the funnel potential.

State	Nonrelativistic Schrödinger equation	Spinless Salpeter equation
m_s (GeV)	0.573	0.551
m_c (GeV)	1.797	1.800
m_b (GeV)	5.187	5.176
κ_l	0.606	0.364
κ_h	0.506	0.456
a (GeV^2)	0.187	0.211

TABLE IV. Quark masses and potential parameters obtained from a fit of the quarkonium mass spectra with the Schrödinger and spinless Salpeter equation, respectively, for the erf potential.

State	Nonrelativistic Schrödinger equation	Spinless Salpeter equation
m_c (GeV)	1.996	1.959
m_b (GeV)	5.387	5.325
κ_l	0.136	0.092
κ_c	1.293	1.063
κ_b	1.600	1.227
a (GeV^2)	0.095	0.130

TABLE V. Quarkonium mass spectra (in units of GeV) predicted for the smooth-transition potential by the Schrödinger and spinless Salpeter equation, respectively, as well as the corresponding values of χ^2 obtained by the fit.

State	Experiment [17]	Nonrelativistic Schrödinger equation	Spinless Salpeter equation
π	0.138	0.138	0.138
$\rho(770)$	0.768	0.768	0.767
$\eta_c(1S)$	2.980	2.987	2.975
$J/\psi(1S)$	3.097	3.097	3.097
$c\bar{c}(1^1P_1)$	3.525	3.505	3.495
$\psi(2S)$	3.686	3.686	3.685
$\psi(4040)$	4.04	4.028	4.12
$\Upsilon(1S)$	9.460	9.460	9.461
$\Upsilon(2S)$	10.023	10.022	10.017
$\Upsilon(3S)$	10.355	10.358	10.375
χ^2		70.3	2163.7

TABLE VII. Quarkonium mass spectra (in units of GeV) predicted for the funnel potential by the spinless Salpeter and semirelativistic wave equation, respectively, as well as the corresponding values of χ^2 obtained by the fit.

State	Experiment [17]	Spinless Salpeter equation	Semirelativistic wave equation
π	0.138	0.138	0.451
$\rho(770)$	0.768	0.768	0.474
$\eta_c(1S)$	2.980	2.957	3.094
$J/\psi(1S)$	3.097	3.097	3.100
$c\bar{c}(1^1P_1)$	3.525	3.525	3.519
$\psi(2S)$	3.686	3.686	3.686
$\psi(4040)$	4.04	4.11	4.10
$\Upsilon(1S)$	9.460	9.461	9.462
$\Upsilon(2S)$	10.023	10.018	10.019
$\Upsilon(3S)$	10.355	10.362	10.368
χ^2		759.0	744 663

C. Semirelativistic wave equation versus spinless Salpeter equation

As the next step towards full relativistic consistency, by employing the semirelativistic wave equation (14) for the description of fermion-antifermion bound states, we take into account the complete relativistic corrections to the static potential $V_{\text{stat}}(r)$. Since the semirelativistic wave equation is not plagued by the (spurious) singularities generated by a nonrelativistic expansion in powers of $1/c$, no regularization of the Coulomb part is required and one may deal with the pure funnel potential (25).

The results of the fits for the predicted energy levels and the corresponding sets of parameters are summarized in Tables VII and VIII, respectively. While in the previous subsection the additive constant V_0 in the linear part of the funnel potential was determined by the slope a, we now regard it as an independent parameter. Its variation may be used to optimize the numerical fit.

While now the spinless Salpeter equation (with the additional free parameter V_0) produces a more or less reasonable description of the experimental data, the fit based on the semirelativistic wave equation has to be called simply a disaster (Table VII). The Coulomb-like parameter κ now follows only for the spinless Salpeter equation the trend demanded by asymptotic freedom, $\kappa_l > \kappa_c > \kappa_b$, whereas in the case of the semirelativistic wave equation no general tendency may be recognized (Table VIII). We conclude that the semirelativistic wave equation—which, in contrast to the Schrödinger and spinless Salpeter equation, incorporates also the spin-independent relativistic corrections to the static potential—is not able to describe successfully the spectra of quark-antiquark bound states, at least, not with such a simple form of the quark interaction as is represented by the funnel potential.

VI. SUMMARY

Motivated by the impressive success of nonrelativistic potential models for the description of hadrons as bound states of constituent quarks, we analyzed the changes brought about by a relativistic treatment of hadrons, where the nonrelativistic Schrödinger equation, on which the former investigations are based, is replaced by a relativistic equation of motion which incorporates relativistic kinematics, such as the so-called spinless Salpeter equation, or even the full relativistic corrections to the in-

TABLE VI. Quark masses and potential parameters obtained from a fit of the quarkonium mass spectra with the Schrödinger and spinless Salpeter equation, respectively, for the smooth-transition potential.

State	Nonrelativistic Schrödinger equation	Spinless Salpeter equation
m_c (GeV)	1.841	1.711
m_b (GeV)	5.214	5.099
κ_l	0.475	0.623
κ_h	0.475	0.723
b (GeV^2)	1.948	0.591
c (GeV^{-1})	0.866	0.790
k (GeV)	−3.335	−9.251
a (GeV^2)	0.121	0.228

TABLE VIII. Quark masses and potential parameters obtained from a fit of the quarkonium mass spectra with the spinless Salpeter and semirelativistic wave equation, respectively.

State	Spinless Salpeter equation	Semirelativistic wave equation
m_u (GeV)	0.336	0.336
m_c (GeV)	1.351	2.060
m_b (GeV)	4.751	5.430
κ_l	1.768	0.452
κ_c	0.692	0.718
κ_b	0.647	0.628
a (GeV^2)	0.189	0.201
V_0 (GeV)	0.036	−1.345

teraction potential, such as the semirelativistic wave equation of Sec. II C.

Our results are somewhat strange and contradict the naive feeling one might have when moving from a nonrelativistic to an (at least more) relativistic treatment of systems which by no means can be regarded as nonrelativistic. As may be seen from the achieved minimal values of χ^2, introducing into the bound-state equation of motion the relativistically correct kinetic term $\sqrt{-\Delta+m^2}$ either deteriorates the quality of the fit drastically or leads to an unrealistic dependence of the potential parameters on the relevant energy scale. Similarly, taking into account, in addition, the relativistically consistent form of the interaction between the bound-state constituents, as is done by the semirelativistic wave equation, enlarges the discrepancy between the theoretically predicted mass spectra and experiment still more. Our paradoxical findings thus confirm the conclusion of Refs. [3,4] without any restriction: A relativistic treatment of quark-antiquark bound states, by means of a semirelativistic wave equation or its static-interaction approximation, does not imply any improvement in the description of meson mass spectra. Rather, the opposite seems to be true: The nonrelativistic potential-model approach to bound states of quarks appears to be superior to its (semi)relativistic extensions which necessarily involve the troublesome spin-independent relativistic corrections to the static potential.

[1] W. Lucha, F. F. Schöberl, and D. Gromes, Phys. Rep. **200**, 127 (1991).
[2] W. Lucha, H. Rupprecht, and F. F. Schöberl, Phys. Rev. D **45**, 385 (1992).
[3] A. Gara, B. Durand, L. Durand, and L. J. Nickisch, Phys. Rev. D **40**, 843 (1989); **43**, 2447(E) (1991).
[4] A. Gara, B. Durand, and L. Durand, Phys. Rev. D **42**, 1651 (1990).
[5] E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951).
[6] E. E. Salpeter, Phys. Rev. **87**, 328 (1952).
[7] W. Lucha and F. F. Schöberl, Int. J. Mod. Phys. A (to be published).
[8] L. Durand and A. Gara, J. Math. Phys. **31**, 2237 (1990).
[9] *Handbook of Mathematical Functions,* edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1964).
[10] H. Rupprecht, Ph.D. thesis, University of Vienna, 1991.
[11] W. Lucha, H. Rupprecht, and F. F. Schöberl, Phys. Rev. D **45**, 1233 (1992).
[12] E. Eichten, K. Gottfried, T. Kinoshita, J. Kogut, K. D. Lane, and T.-M. Yan, Phys. Rev. Lett. **34**, 369 (1975).
[13] E. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane, and T.-M. Yan, Phys. Rev. D **17**, 3090 (1978); **21**, 313(E) (1980).
[14] E. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane, and T.-M. Yan, Phys. Rev. D **21**, 203 (1980).
[15] D. Gromes, Z. Phys. C **11**, 147 (1981); **14**, 94(E) (1982).
[16] P. Falkensteiner, D. Flamm, and F. Schöberl, Phys. Lett. **131B**, 450 (1983).
[17] Particle Data Group, J. J. Hernández *et al.*, Phys. Lett. B **239**, 1 (1990).