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Asymptotic behavior of perturbation theory for the electromagnetic current-current correlation function in QCD

Lowell S. Brown and Laurence G. Yaffe
Department of Physics, University of Washington, Seattle, Washington 98195
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A simple and direct approach is used to examine the constraints imposed by asymptotic freedom and analyticity on the large-order behavior of perturbation theory for the current-current correlation function and its imaginary part which gives the R ratio in high-energy e^+-e^- annihilation.

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The time-ordered product of two electromagnetic current operators defines a correlation function whose Fourier transform is given by

$$K^{\mu\nu}(q) = \int (d^4x)\, e^{-iqx}\, \langle 0|\mathcal{T}\{j^\mu(x) j^\nu(0)\}|0\rangle \,. \qquad (1)$$

Current conservation implies the structure

$$K^{\mu\nu}(q) = (g^{\mu\nu}q^2 - q^\mu q^\nu)\, K(-q^2)\,. \qquad (2)$$

The scalar function $K(t)$ is analytic in the entire $t = -q^2$ plane save for a cut along the positive real axis. The discontinuity across this cut is related to the high-energy limit of the total e^+-e^- hadronic cross section if one neglects the Z^0-exchange contribution. In terms of the R ratio, defined as the ratio of the total cross sections for e^+-$e^- \to$ hadrons to e^+-$e^- \to$ muon pairs, we have

$$R(s) = 12\pi\, \mathrm{Im}\, K(s + i0^+)\,. \qquad (3)$$

Our work concerns the large-$|t|$ behavior of $K(t)$ in an asymptotically free theory such as QCD. For convenience in presenting the results, we will explicitly discuss a theory in which the perturbative expansion of the β function,

$$\mu^2 \frac{d}{d\mu^2}\, g^2(\mu^2) \equiv \beta(g^2) = -b_0\, g^4 - b_1\, g^6 - \cdots \,, \qquad (4)$$

has a vanishing second term, $b_1 = 0$. By making a suitable redefinition of the coupling, all higher terms in the expansion of the β function may be chosen to vanish, and this we shall do. Assuming that $b_1 = 0$ simplifies the analysis, but does not alter the general character of the results. The general case of a β function whose first two terms are nonvanishing will be presented elsewhere [1].

In the deep Euclidean region, $t \to -\infty$, $K(t)$ has an asymptotic expansion in powers of the running coupling $g^2(-t)$,

$$K(t) \sim C(\mu^2) + \tilde{c}_{-1}\, g^2(-t)^{-1} + \tilde{c}_0 \ln g^2(-t) + \sum_{m=1}^{\infty} c_m\, g^2(-t)^m \,, \qquad (5)$$

with real coefficients $\{\tilde{c}_k, c_m\}$ computable in renormalized perturbation theory. (The origin of the first two nonanalytic pieces will be reviewed below.) Similarly, the imaginary part of $K(s)$ has an expansion,

$$\mathrm{Im}\, K(s + i0^+) \sim \sum_{n=0}^{\infty} a_n\, g^2(s)^n \,, \qquad (6)$$

whose coefficients $\{a_n\}$ determine the perturbative expansion of the R ratio.

In this paper, we examine the relation between the absorptive coefficients $\{a_n\}$ and the dispersive coefficients $\{c_m\}$, and the resulting implications on their possible large-order behavior. The renormalization group and analyticity are the only ingredients in our analysis. We find that the absorptive coefficients may be easily expressed in terms of a sum over the dispersive coefficients or, conversely, that the $\{c_m\}$ may be expressed in terms of the $\{a_n\}$. The explicit relations are presented below in Eqs. (27), (28), and (38). If one defines the Borel transforms of the perturbative series (5) and (6) as

$$C(z) \equiv \tilde{c}_0 + \sum_{m=1}^{\infty} c_m\, z^m/(m-1)!\,, \qquad (7)$$

$$A(z) \equiv \sum_{n=1}^{\infty} a_n\, z^n/(n-1)!\,, \qquad (8)$$

then we find that the relation between the two sets of coefficients is summarized in the remarkably simple result

$$C(z) = A(z)/\sin(\pi b_0\, z)\,. \qquad (9)$$

Recently West has also considered the implications of the renormalization group combined with analyticity for the large-order behavior of perturbation theory [2]. He argues that from these ingredients alone one may deduce a unique large-order behavior for the coefficients $\{a_n\}$ of the R ratio. We find, however, that these assumptions are only sufficient to yield the relation between the absorptive and dispersive coefficients. As discussed below, Eq. (9) has implications for the domain of convergence of $C(z)$ and hence on the large-order behavior of the dispersive coefficients. However, it places no nontrivial restric-

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tions on $A(z)$. Consequently, the renormalization group and analyticity alone do not determine the large-order behavior of the absorptive coefficients, contrary to the claim of [2].

The large-order behavior of the coefficients $\{a_n\}$ or $\{c_m\}$ is directly related to the nature and location of the singularities in the corresponding Borel transforms $A(z)$ or $C(z)$ nearest to the origin. For example, if

$$a_n \sim \kappa\, b^n\, \Gamma(n+\gamma)/\Gamma(1+\gamma) \tag{10}$$

as $n \to \infty$, then $A(z)$ will be analytic in the domain $|bz| < 1$ and have the singularity

$$A(z) \sim \kappa\,(bz)\,(1-bz)^{-1-\gamma} \tag{11}$$

as $b\,z \to 1$. Conversely, if $A(z)$ has a singularity of this form, and no other singularities for $|bz| \leq 1$, then the coefficients $\{a_n\}$ will have the large-order behavior (10).

The existence of zeros in the $\sin(\pi b_0 z)$ denominator in the relation between the Borel transforms (9) implies that $C(z)$ will have singularities at all nonzero integer values of $b_0\, z$ unless $A(z)$ has compensating zeros. Hence, one of the following possibilities for the large-order behavior must occur.

(1) If $A(z)$ has a radius of convergence greater than $1/b_0$ [so that the absorptive coefficients $\{a_n\}$ grow slower than $(b_0-\epsilon)^n\, n!$ for some $\epsilon > 0$], then $C(z)$ will have simple poles at $z = \pm 1/b_0$. Hence, if the residues $A_\pm \equiv A(\pm 1/b_0)$ are not both zero, the dispersive coefficients will have large-order behavior which is determined by these residues,

$$c_m \sim \frac{1}{\pi}\,[A_+\, b_0^m - A_-\,(-b_0)^m]\,(m-1)! \tag{12}$$

as $m \to \infty$.

(2) If $A(z)$ has a radius of convergence equal to $1/b_0$, so will $C(z)$. The dispersive coefficients will grow faster than the absorptive coefficients by a single power of m. For example, if the absorptive coefficients behave for large n as

$$a_n \sim A_+\, b_0^n\, \Gamma(n+\gamma_+) + A_-\,(-b_0)^n\, \Gamma(n+\gamma_-) \tag{13}$$

for some constants $A_\pm$ and $\gamma_\pm$, then the Borel transform $A(z)$ will have singularities $A(z) \sim A_\pm\,(\pm b_0 z)\,(1 \mp b_0 z)^{-1-\gamma_\pm}\,\Gamma(1+\gamma_\pm)$, as $b_0 z \to \pm 1$. Dividing by $\sin(\pi b_0 z)$ gives the Borel transform $C(z)$ the singularities $C(z) \sim A_\pm (b_0 z/\pi)\,(1 \mp b_0 z)^{-2-\gamma_\pm}\,\Gamma(1+\gamma_\pm)$ and thus the dispersive coefficients will grow like

$$c_m \sim \frac{A_+}{\pi\,(\gamma_+ + 1)}\, b_0^m\, \Gamma(m+\gamma_+ + 1) - \frac{A_-}{\pi\,(\gamma_- + 1)}\,(-b_0)^m\, \Gamma(m+\gamma_- + 1) \tag{14}$$

as $m \to \infty$.

(3) If $A(z)$ has a radius of convergence less than $1/b_0$, so will $C(z)$. In this case, the dispersive coefficients will have the same large-order behavior as the absorptive coefficients, with both growing faster than $(b_0+\epsilon)^n\, n!$ as $n \to \infty$.

These constraints on the possible large-order behavior are independent of the specific dynamics of the asymptotically free theory and follow solely from the existence of renormalized perturbation theory (with a one-term β function). Explicit studies of perturbation theory in QCD show the following [3].

(i) The ultraviolet behavior of individual m-loop diagrams can generate contributions behaving as $c_{m+1} \sim (-b_0/k)^m\, m!$, for $k = 1, 2, \ldots$, leading to singularities in the Borel transform $C(z)$ at the points $z = -k/b_0$ on the negative real axis. Near the first singularity [4], $C(z) \sim (b_0\, z+1)^{-1+\gamma}$, where γ is related to the anomalous dimension of local operators of dimension 6. These contributions are referred to as *ultraviolet renormalons.*

(ii) The infrared behavior of m-loop diagrams can generate contributions behaving as $c_{m+1} \sim (b_0/k)^m\, m!$, for $k = 2, 3, \ldots$, corresponding to singularities in $C(z)$ at the points $z = k/b_0$ on the positive real axis. The absence of a singularity at $b_0\, z = 1$ is related to the lack of any physical gauge-invariant local operator of dimension 2. Near the first singularity [5], $C(z) \sim (b_0\, z-2)^{-1-2b_1/b_0^2}$. These contributions are referred to as *infrared renormalons.*

(iii) Instantons generate singularities in the Borel transform on the positive real axis (starting at $z = 4\pi$) to the right of the leading infrared renormalon singularity.

(iv) No other sources of singularities in the Borel transform are known.

The ultraviolet renormalon behavior just described is entirely consistent with the allowed behavior in Eq. (14). However, our results show that the absence of an infrared renormalon singularity at $b_0\, z = 1$ is only possible if the Borel transform of the absorptive coefficients has a zero at $b_0\, z = 1$. This constraint does not appear to have been previously noted. Whether or not a singularity in $C(z)$ at $b_0\, z = 1$ is actually present does not seem to be clearly established; no convincing argument demonstrating its absence is known to the authors.[1] We turn now to the details of our work.

For the leading large momentum behavior, one may neglect all mass parameters as they lead to corrections in $K(t)$ suppressed by powers of t, and hence vanishing faster than any power of $g^2(-t)$ as $t \to \infty$. A mass-independent renormalization scheme with renormalization point μ will be assumed. Therefore the dimensionless function $K(t)$ depends on μ^2 and the renormalized coupling $g^2(\mu^2)$ in the form

$$K(t) = K(t/\mu^2, g^2(\mu^2))\,. \tag{15}$$

Because the electromagnetic currents are conserved they

[1]This question is related to the possibility of nonperturbative corrections of order $1/q^2$ in the coefficient functions of the operator product expansion [6]. Unlike $1/q^4$ contributions (which are connected to the infrared renormalon at $b_0 z = 2$) such $1/q^2$ terms cannot be removed by a redefinition of the local operators appearing in the expansion.

acquire no anomalous dimension. However, the product of two current operators is singular and one subtraction proportional to the unit operator is required for the proper definition of the time-ordered product in Eq. (1). Consequently, $K(t)$ satisfies an inhomogeneous renormalization group equation,

$$\mu^2 \frac{d}{d\mu^2} K(t/\mu^2, g^2) = \left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(g^2)\frac{\partial}{\partial g^2}\right) K(t/\mu^2, g^2) = D(g^2)\,, \tag{16}$$

where the function $D(g^2)$ has a perturbative expansion

$$D(g^2) = d_0 + d_1 g^2 + d_2 g^4 + \cdots\,. \tag{17}$$

The renormalization group equation may be used to transfer the momentum dependence into a running coupling $g^2(-t)$ defined by

$$\int_{g^2(\mu^2)}^{g^2(-t)} \frac{dg^2}{\beta(g^2)} = \ln(-t/\mu^2)\,. \tag{18}$$

This defines a coupling $g^2(-t)$ which is independent of the renormalization point μ but which obeys Eq. (4) with μ^2 replaced by $-t$. With this definition in hand, the general solution of the renormalization group equation (16) may be written as

$$K(t/\mu^2, g^2(\mu^2)) = K(-1, g^2(-t)) - \int_{g^2(\mu^2)}^{g^2(-t)} dg^2 \frac{D(g^2)}{\beta(g^2)}\,. \tag{19}$$

The presence of the additional term involving $D(g^2)$, coming from the subtraction needed to renormalize the product of currents, alters the perturbative expansion of $K(t)$ in powers of $g^2(-t)$. To see this, note that

$$-\int_{g^2(\mu^2)}^{g^2(-t)} dg^2 \frac{D(g^2)}{\beta(g^2)} = \frac{d_0}{b_0}\left[\frac{1}{g^2(\mu^2)} - \frac{1}{g^2(-t)}\right] + \left(\frac{d_1}{b_0} - \frac{d_0 b_1}{b_0^2}\right) \ln\left[\frac{g^2(-t)}{g^2(\mu^2)}\right] + \cdots\,, \tag{20}$$

where the ellipsis stands for a power series in $g^2(-t)$ minus the same series in $g^2(\mu^2)$. All the terms involving $g^2(\mu^2)$ may be absorbed in a single μ^2-dependent parameter $C(\mu^2)$. The series in $g^2(-t)$ combines with the perturbative expansion of $K(-1, g^2(-t))$ to yield a modified expansion in powers of $g^2(-t)$. Hence $K(t)$ has a large-t asymptotic expansion of the previously stated form

$$K(t) \sim C(\mu^2) + \tilde{c}_{-1}\, g^2(-t)^{-1} + \tilde{c}_0 \ln g^2(-t) + \sum_{m=1}^{\infty} c_m\, g^2(-t)^m\,. \tag{21}$$

The presence of the $1/g^2(-t)$ and $\ln g^2(-t)$ terms in Eq. (21) may, at first glance, appear odd. However, Eq. (18) implies that

$$1/g^2(-t) = b_0 \ln(-t/\mu^2) + 1/g^2(\mu^2) + O(g^2)\,, \tag{22}$$

where the $O(g^2)$ remainder vanishes in the special case of a one-term β function. Hence, the $1/g^2(-t)$ term is precisely what is required to generate the $\ln(q^2/\mu^2)$ behavior of the free-field correlation function. Similarly, the $\ln g^2(-t)$ term reflects the presence of $\ln[\ln(-t)]$ terms in the large momentum behavior of $K(t)$.

Several different approaches may be used to deduce the relation between the dispersive coefficients $\{c_m\}$ and the absorptive coefficients $\{a_n\}$. Among them are the following.

(1) Analytically continue the expansion (21) from t real and negative to $t = s + i0^+$, with s real and positive, reexpress $g^2(se^{-i\pi})$ in terms of $g^2(s)$, and reexpand the result in powers of $g^2(s)$.

(2) Expand K along an arbitrary ray in the complex t plane,

$$K(te^{i\phi}) \sim C(\mu^2) + \tilde{c}_{-1}\, g^2(-t)^{-1} + \tilde{c}_0 \ln g^2(-t) + \sum_{m=0}^{\infty} c_m(\phi)\, g^2(-t)^m\,,$$

then derive, and solve, the renormalization-group equation for the ϕ dependence of the coefficients $\{c_m(\phi)\}$.

(3) Given the asymptotic form of $\operatorname{Im} K(s)$ displayed in Eq. (6), compute the large-t asymptotic behavior of the once-subtracted dispersion relation:

$$K(t) = K(0) + \frac{t}{\pi}\int_0^\infty \frac{ds}{s}\, \frac{\operatorname{Im} K(s + i0^+)}{s - t}\,.$$

We will use the first approach as this is by far the most convenient for the special case of a one-term β function. The latter two approaches are employed in [1], as the first approach cannot be easily applied with a general β function. One may worry that analytically continuing the asymptotic expansion for the Euclidean correlation function (21) will not yield a valid expansion along other rays in the complex t plane. However, using the last approach sketched above this may be shown not to be a problem.[2]

Analytically continuing the expansion (21) back to the positive real axis gives

[2]Strictly speaking, this requires that $K(t)$ be polynomially bounded in the complex t plane, or equivalently that $\operatorname{Im} K(s)$ have the smooth asymptotic expansion of Eq. (6) without, for example, oscillatory behavior like $\sin\sqrt{s/\Lambda^2}$ persisting to arbitrarily large s. Such behavior is certainly unexpected and is absent in perturbation theory; however, we know of no rigorous proof excluding this pathology. We will assume that (6) holds, in which case, one may prove that the expansion (21) is valid throughout the cut t plane [1].

$$K(s+i0^+) \sim C(\mu^2) + \tilde{c}_{-1}\, g^2(se^{-i\pi})^{-1} + \tilde{c}_0 \ln g^2(se^{-i\pi}) + \sum_{m=0}^{\infty} c_m\, g^2(se^{-i\pi})^m . \quad (23)$$

For a one-term β function, Eq. (22) with $\mu^2 = t = s$ implies that

$$g^2(se^{-i\pi}) = \frac{g^2(s)}{1 - i\pi b_0\, g^2(s)} . \quad (24)$$

Reexpanding $\ln g^2(se^{-i\pi})$ and $g^2(se^{-i\pi})^n$ in powers of $g^2(s)$ yields

$$K(s+i0^+) \sim C(\mu^2) + \tilde{c}_{-1}\, g^2(s)^{-1} + \tilde{c}_0 \ln g^2(s) - \tilde{c}_{-1}\,(i\pi b_0) + \tilde{c}_0 \sum_{n=1}^{\infty} \frac{1}{n}\,(i\pi b_0)^n g^2(s)^n + \sum_{n=1}^{\infty} c_n\, g^2(s)^n \sum_{m=0}^{\infty} \frac{(n+m-1)!}{(n-1)!\, m!}\,(i\pi b_0)^m g^2(s)^m . \quad (25)$$

Taking the imaginary part and collecting terms now gives the desired relation between the absorptive coefficients,

$$\operatorname{Im} K(s+i0^+) \sim \sum_{n=0}^{\infty} a_n\, g^2(s)^n , \quad (26)$$

and the dispersive coefficients, namely,

$$a_{2n} = \sum_{k=0}^{n-1} \frac{(2n-1)!}{(2k+1)!\,(2n-2k-2)!} \times (-)^k\,(\pi b_0)^{2k+1}\, c_{2(n-k)-1} \quad (27)$$

and

$$a_{2n+1} = \sum_{k=0}^{n-1} \frac{(2n)!}{(2k+1)!\,(2n-2k-1)!} \times (-)^k\,(\pi b_0)^{2k+1}\, c_{2(n-k)} + \frac{1}{(2n+1)}\,(-)^n\,(\pi b_0)^{2n+1}\,\tilde{c}_0 . \quad (28)$$

For $n = 0$, Eq. (27) is replaced by $a_0 = -\pi b_0\, \tilde{c}_{-1}$.

To simplify these results, it is convenient to introduce rescaled coefficients for $n > 0$,

$$\tilde{a}_n = \frac{(\pi b_0)^{-n}}{(n-1)!}\, a_n \quad (29)$$

and

$$\tilde{c}_n = \frac{(\pi b_0)^{-n}}{(n-1)!}\, c_n . \quad (30)$$

Both the results (27) and (28) (with the latter including the $\tilde{c}_0$ contribution) are encompassed in the single relation

$$\tilde{a}_{n+1} = \sum_{l=0}^{[n/2]} \frac{(-)^l}{(2l+1)!}\, \tilde{c}_{n-2l} , \quad (31)$$

where $[x]$ denotes the integer part of x. This relation may be easily inverted to express the dispersive coefficients in terms of the absorptive coefficients. To do this, first note that the coefficients $\{\tilde{c}_n\}$ and $\{\tilde{a}_n\}$ are simply the expansion coefficients of the rescaled Borel transforms (7) and (8):

$$A(z/\pi b_0) = \sum_{n=1}^{\infty} \tilde{a}_n\, z^n \quad (32)$$

and

$$C(z/\pi b_0) = \sum_{n=0}^{\infty} \tilde{c}_n\, z^n . \quad (33)$$

Inserting Eq. (31) into the definition of $A(z)$ and interchanging the two sums shows that the relation between the two sets of expansion coefficients is exactly equivalent to the simple result

$$A(z) = \sin(\pi b_0\, z)\, C(z) , \quad (34)$$

as stated earlier.

To solve for the dispersive coefficients one need only reexpand $C(z) = A(z)/\sin(\pi b_0\, z)$ in powers of z. This is easily done using the expansion

$$\frac{1}{\sin z} = \sum_{k=0}^{\infty} M_{2k}\, z^{2k-1} , \quad (35)$$

where

$$M_{2k} = \frac{1}{(2k)!}\, |(2^{2k}-2) B_{2k}| , \quad (36)$$

and B_n are the Bernoulli numbers. Identifying the coefficient of z^n yields

$$\tilde{c}_n = \sum_{k=0}^{[n/2]} M_{2k}\, \tilde{a}_{n-2k+1} , \quad (37)$$

or

$$n\, c_n = \sum_{k=0}^{[n/2]} \frac{n!}{(2k)!\,(n-2k)!}\, |(2^{2k}-2) B_{2k}| \times (\pi b_0)^{2k-1}\, a_{n-2k+1} . \quad (38)$$

Combined with the special case $\tilde{c}_{-1} = -(\pi b_0)^{-1} a_0$, this result shows that all terms in the asymptotic expansion of the Euclidean correlation function (21) except for the overall additive (and renormalization-point-dependent) constant, are uniquely determined by the knowledge of the asymptotic expansion of the spectral density (or the R ratio).

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[1] L. S. Brown, L. G. Yaffe, and C. Zhai (in preparation).
[2] G. B. West, Phys. Rev. Lett. **67**, 1388 (1991).
[3] See, for example, *Large Order Behavior of Perturbation Theory*, edited by J. C. Le Guillou and J. Zinn-Justin (North-Holland, Amsterdam, 1990), and references therein.
[4] G. Parisi, Phys. Lett. **76B**, 65 (1978).
[5] A. H. Mueller, Nucl. Phys. **B250**, 327 (1985).
[6] See, for example, F. David, Nucl. Phys. **B234**, 237 (1984).