RAPID COMMUNICATIONS

PHYSICAL REVIEW D VOLUME 45, NUMBER 10 15 MAY 1992

Obtaining the metric of our Universe

Mark W. Jacobs*
Department of Physics and Center for Space Science and Astrophysics, Stanford University, Stanford, California 94305-4060

Eric V. Linder†
Department of Physics and Center for Space Science and Astrophysics, Stanford University, Stanford, California 94305-4060, Steward Observatory, University of Arizona, Tucson, Arizona 85721, and Department of Astronomy, New Mexico State University, Las Cruces, New Mexico 88003‡

Robert V. Wagoner§
Department of Physics and Center for Space Science and Astrophysics, Stanford University, Stanford, California 94305-4060

(Received 4 November 1991)

We formulate a method for solving the gravitational field equations for perturbations to a Friedmann-Robertson-Walker metric, which does not depend on any kind of averaging procedure or make any *a priori* assumptions about the magnitude of fluctuations in the matter variables. We present a Green function for obtaining the effective potential which characterizes the metric perturbations directly from the (possibly large) density fluctuations, and describe the application to astrophysical observations, for example, the angular-diameter distance-versus-redshift relation. The results do not assume a particular model for the formation of structure in the matter distribution, and are valid everywhere in our Universe outside of strong-field regions (e.g., black holes).

PACS number(s): 98.80.Dr, 04.20.Jb

I. INTRODUCTION

The cosmological model that most simply describes the largest-scale properties of our Universe (its observed isotropy, assumed homogeneity, and evolution) is the Friedmann-Robertson-Walker (FRW) model. However, it is clear that there is indeed structure in the Universe continually observed on larger and larger scales. One question, then, is how well the ideal, perfectly smooth and symmetric FRW model approximates our cosmology.

The principle of equivalence tells us that all the information about a spacetime is in its metric, so the question becomes how close is the FRW metric to the realistic one. In general relativity, though, there is no general goodness-of-fit criterion for this problem. We employ a modified post-Newtonian formalism, analogous to that used for perturbations around Schwarzschild spacetime in neutron-star astrophysics, for example, that is able to generate a realistic metric along with a criterion showing the limits of validity of the FRW model.

The approach is like that of Futamase [1], but differs by solving the field equations through the use of scalar harmonics as spatial basis functions, while avoiding the use of any averaging procedure for the metric perturbations. The analysis is restricted to scalar perturbations because of their apparent dominance, and the results are valid assuming only that the metric perturbations are small, in the sense described below in Sec. II. No *a priori* restrictions are placed on the size of perburbations to the matter variables; in particular, the density fluctuations may be large. We do assume that the matter distribution and its evolution are given by theory and/or observation; i.e., we study the effects of structure but not its formation. Thus the results apply in conjunction with any model of structure consistent with our assumptions. This paper outlines the formalism and presents illustrative results and astrophysical applications, giving order-of-magnitude estimates for observational effects of the inhomogeneities. Complete and detailed treatments of the problem will appear in a future paper by Jacobs, Linder, and Wagoner (JLW) [2].

II. THE METRIC

The metric describing a homogeneous and isotropic Universe has the Robertson-Walker form

$$ds^2_{\rm RW} = a_F^2 \gamma_{ab}\, dx^a dx^b = a_F^2(\eta)[-d\eta^2 + (1+\tfrac{1}{4}kr^2)^{-2}(dx^2+dy^2+dz^2)]\,, \tag{1}$$

where a_F is the Friedmann expansion factor, η the conformal time ($d\eta = dt/a$), $r^2 = x^2+y^2+z^2$, and $k=0, \pm 1$ the spatial curvature parameter. (All variables except the expansion factor are dimensionless.) To include inhomogeneities in the cosmology we adopt the form

$$ds^2 = a^2(\eta)[\gamma_{ab} + h_{ab}]dx^a dx^b\,, \tag{2}$$

where h_{ab} are the metric perturbations representing the inhomogeneities and where a is not assumed to be the Friedmann form initially.

The procedure is now straightforward. We write out

*Electronic address: mark@relgyro.stanford.edu.
†Electronic address: elinder@charon.nmsu.edu.
‡Present address.
§Electronic address: wagoner@leland.stanford.edu.

the Einstein field equations, expanding in an infinite series of powers and derivatives of the metric perturbations. Following Futamase [1] we denote the typical magnitude of the metric perturbations h_{ab} by a small quantity ϵ^2 (in the manner of the post-Newtonian formalism—see Will [3] for a comprehensive discussion), and the ratio of their scale length to that of the FRW particle horizon by κ (a two-length-scale separation). We then truncate the expansion of the field equations by retaining only terms $O(\epsilon^2)$ or larger. (The next set of terms are $O(\epsilon^4\kappa^{-2})$ [or $O(\epsilon^4\kappa^{-1})$ for the time-space components], known as the pseudotensor order [4]. Retaining these terms would include the energy density of the perturbations as well as their nonlinear interactions and back reactions on the FRW component. We neglect these for reasons mentioned below.) The consistency conditions for this truncation are simply $\epsilon^2 \ll 1$, $\epsilon^2 \ll \kappa$.

The details of the solution are dealt with by JLW [2]. Briefly, if the perturbations are expanded in scalar harmonics, the field equations can be solved by taking their spatial projections against different scalar modes, without the use of any averaging procedure. This differs from [1] but gives the same result for the form of the metric: if the parameters satisfy the condition

$$\epsilon^2 \ll \kappa \tag{3}$$

(which we like to call the shear condition, for reasons discussed in Sec. IV) then the metric (2) takes the form

$$ds^2 = a_F^2(\eta)[-(1+2\phi)d\eta^2 + (1-2\phi)\gamma_{ij}dx^i dx^j]\,. \tag{4}$$

We work in longitudinal gauge, where i,j run over spatial coordinates and $\phi(x^a)$ is the effective (quasi-Newtonian) potential of the inhomogeneities. Together, Eqs. (3) and (4) give corrections to the FRW case along with a goodness-of-fit criterion. In deriving this form for the metric, one obtains field equations relating the metric perturbations and expansion factor to the inhomogeneous matter variables—density, pressure, and velocity. Denoting the FRW background density as ρ_0, these have the order of magnitude

	Galaxy	Cluster of galaxies	
ϵ^2,κ	10^{-6}, 10^{-5}	10^{-6}, 10^{-3}	
$\delta\rho/\rho_0 \sim \epsilon^2(\kappa^{-2}+1)$	10^4	1	(5)
$\delta p/\rho_0 \sim \epsilon^2,\ \epsilon^4\kappa^{-2}$	10^{-2}	10^{-6}	
$v \sim \epsilon^2\kappa^{-1},\ \epsilon$	10^{-3}	10^{-3}	

for each scalar mode. The density fluctuations may be small or large, depending on whether $\epsilon \ll \kappa$ or $\epsilon \gg \kappa$ (bound system). This is reflected in the pressure and velocity perturbations, which take two forms correspondingly, but which are always small themselves. Physically, ϵ^2 and κ correspond to the gravitational potential and size of an inhomogeneity; values for a typical galaxy (cluster) of mass $10^{12}M_\odot$ ($10^{14}M_\odot$) and size 30 kpc (3 Mpc) are shown above.

Strictly speaking, the homogeneous expansion factor $a(\eta)$ is perturbed from its Friedmann value by the back-reaction effect (pseudotensor energy density) of the inhomogeneities, mentioned above. By order of magnitude, $a = a_F[1+O(\langle\epsilon^4\kappa^{-2}\rangle)]$. That is, should the shear condition $\epsilon^2 \ll \kappa$ be violated over a significant portion of the Universe, the resulting cosmology will differ noticeably from FRW even on large scales; however, observations indicate that this is not the case. Thus we use the Friedmann expansion in Eq. (4), which should be a realistic metric throughout most of the Universe.

III. GREEN FUNCTION

A particularly interesting result is that the component of the Einstein field equations relating the metric perturbations and density fluctuations resembles a diffusion equation. It is possible to obtain a formal solution which can then be integrated over all scalar modes, giving a Green-function relation between the density fluctuations $\delta\rho = \rho(\eta,\mathbf{x}) - \rho_0(\eta)$ and the quasi-Newtonian potential $\phi(\eta,\mathbf{x})$. In general, we find [with $G=c=1$ and $a'(\eta) = da/d\eta$]

$$\phi(\eta,\mathbf{x}) = \frac{3}{4\pi}\int_\infty \phi(\eta_0,\mathbf{y})\mathcal{G}(\eta_0,\eta,\mathbf{x},\mathbf{y})d^3\mathbf{y} - \int_{\eta_0}^{\eta}\int_\infty \delta\rho(u,\mathbf{y})\mathcal{G}(u,\eta,\mathbf{x},\mathbf{y})a^3(u)d^3\mathbf{y}\frac{du}{a'(u)}\,, \tag{6}$$

with a time-dependent Green function $\mathcal{G}$, which for $k=0$ has the form

$$\mathcal{G}(u,\eta,\mathbf{x},\mathbf{y}) = \frac{4\pi}{3}\frac{a(u)}{a(\eta)}\frac{1}{(4\pi)^{3/2}C_\text{f}^3(u,\eta)} \times \exp\left[-\frac{|\mathbf{y}-\mathbf{x}|^2}{4C_\text{f}^2(u,\eta)}\right], \tag{7}$$

$$C_\text{f}^2(u,\eta) \equiv \frac{1}{3}\int_u^\eta (a/a')dw\,.$$

This result may be used in formulas for gravitational light deflection (lensing) and other effects of the metric on the propagation of photons, giving predictions for observational effects directly in terms of the matter distribution $\rho_0 + \delta\rho$, and possibly allowing one to discriminate among various models of structure formation, as expressed in the form of $\delta\rho(\eta,\mathbf{x})$.

With $a/a' \sim \eta$, we find that for distances $a\Delta x \equiv a|\mathbf{x}-\mathbf{y}|$ well within the horizon ($\Delta x/\eta \ll 1$), the Green function $\mathcal{G}(u,\eta,\mathbf{x},\mathbf{y})$ peaks at a value of look-back conformal time $\eta - u \equiv \Delta u \sim \Delta x^2/\eta \ll \eta$, with a width of the same magnitude Δu. If the density fluctuations $\delta\rho(u,\mathbf{y})$ evolve on a conformal time scale $\gg \Delta u$, and if the initial time $\eta_0 \ll \eta$, then this Green function produces the usual Newtonian result

$$\phi(\eta,\mathbf{x}) \simeq -\int_\infty \frac{\delta\rho(\eta,\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}a^2(\eta)d^3\mathbf{y}\,. \tag{8}$$

IV. PHOTON PROPAGATION

Two basic astrophysical applications exist for the formalism of Secs. II and III. Either we can consider problems connected with the behavior of the inhomogeneities, such as the growth of structure, nonlinear interactions, or back reaction on the expansion, or we can look at the effects of the inhomogeneities on observations, through their influence on photon propagation. Photon propagation is particularly interesting for two reasons. For one, the vast majority of our observations of the Universe come to us via photons traveling over cosmological distances, so it behooves us to understand their propagation in order to be able to interpret their information correctly. Second, light probes many scales of the Universe, from its total path length to its closest impact parameter. In this sense light propagation gives many one-dimensional samples of the Universe, as opposed to the volume sampling of inhomogeneity evolution.

In investigating the effects of the matter distribution on light we simply apply the usual machinery of general relativity to the metric (4). Since the geodesic equation, geodesic deviation equation, etc., are well known, here we just summarize the results in terms of the order-of-magnitude parametrization of ϵ and κ. Further details are available in the work of JLW [2]. Quantities of interest include the perturbations to the point expansion (area change) and shear (distortion) of a bundle of rays (see Sachs [5] for general-relativistic geometric optics), changes in the distance-redshift relation, and the ray deflection angle on the sky (gravitational lensing). For instance, in a spatially flat Universe with nonrelativistic matter an integral equation for the angular-diameter distance versus redshift (with $y=1+z$) is [6]

$$r(y)=r_{\rm FRW}(y)-\frac{1}{H_0^2}\int_1^y du\, u^{-1/2}(\nabla^2\phi)r(u)r_{\rm FRW}(u,y)\,, \tag{9}$$

where $r_{\rm FRW}(u,y)$ is just the angular-diameter distance relation generalized to an observer at redshift u and $\phi(\eta,\mathbf{x})$ is the quasi-Newtonian potential of Eq. (6). Spatially, ϕ can be represented as a sum of plane waves, and their oscillation over the path length will dilute the seemingly dominant second term ($\sim\epsilon^2/\kappa^2>1$ in the nonlinear density regime). Indeed, calculation shows that the fractional deviation from the FRW result is reduced to order $\epsilon^2/\kappa\ll 1$ at appreciable redshifts and to $\epsilon^2\ll 1$ at $z<\kappa\ll 1$. So the integration along the line of sight effectively provides an averaging procedure that produces the FRW result.

All of these changes (to expansion, shear, etc.) from the FRW values work out to be of order $\epsilon^2\kappa^{-1}$—another reason why we say that if the "shear condition" were violated, we would see a Universe very different from the FRW one. Larger effects include image distortion and amplification and the related ray crossing or Jacobian parameter, which is the ratio of the differential deflection between two rays relative to their separation. (The latter also controls the creation of multiple images of a single source and photon mixing of the microwave background radiation, for example. See Linder [7] for a further general discussion of these effects.) These effects are found to be of order $\epsilon^2\kappa^{-2}$, where we have assumed κ^{-1} independent fluctuations along the line of sight, and so are possibly observable for nonlinear density fluctuations. One can generally write these variables in terms of the density power spectrum and calculate their numerical values either by appropriate integration or by numerical simulation.

Now, in fact, we have been a trifle quick with our order-of-magnitude estimates since there is no unique ϵ or κ; rather these will depend on the impact parameter of the light ray. The values we have been using merely correspond to the fiducial case of the impact parameter being of the order of the inhomogeneity size. In reality, we should integrate our results over the probability of encountering a given impact parameter. This then depends on the nature of the inhomogeneities and their distribution and so cannot generally be treated analytically. For the case of many randomly distributed point masses near the line of sight, however, we find a reduction in the estimated order of magnitudes by κ, coverting $\epsilon^2\kappa^{-2}$ effects into $\epsilon^2\kappa^{-1}$ ones, for example. This could explain why gravitational lensing effects are not observed everywhere there is a nonlinear density concentration; rather the probability of lensing would be a more reasonable 10^{-3} for cluster scales, using the numbers of Sec. II.

V. CONCLUSION

The method outlined here can include realistic inhomogeneities in a cosmological model while avoiding the use of what may be problematic averaging procedures. In this way the simple post-Newtonian form of the metric emerges naturally from the field equations. We use Futamase's [1] parameterization of the strength of the gravitational potential of the inhomogeneities by a small quantity ϵ^2 and their scale length by κ, providing a goodness-of-fit criterion relative to the FRW model.

Investigating the properties of photon propagation in such a Universe, we derive statistical measures of imaging effects such as amplification and distortion (gravitational lensing) in a convenient order-of-magnitude form. Detailed predictions can now be developed in two ways. Either an analytically amenable inhomogeneity field is used to give statistical results or an arbitrary mass distribution is treated by numerical simulation. Both have been carried out, the former by Linder [7] and the latter by Tomita and Watanabe [8,9], for example.

Most promising, however, is the development of the Green-function solution for the inhomogeneity potential. This allows us to take an arbitrary density field from any structure formation scenario and translate it directly into observables.

ACKNOWLEDGMENTS

This work was supported in part by NASA Grants No. NAGW-763 at Steward Observatory, and No. NAGW-299 and No. NAS8-36125 at Stanford University. E.V.L. gratefully acknowledges the astrophysics groups at Hirosaki University and the Yukawa Institute for Theoretical Physics for hospitality and useful discussions, and an AAS/NSF International Travel Grant which made such visits possible. M.W.J. and E.V.L. acknowledge the generous support of the Gravity Probe-B group and the Center for Space Science and Astrophysics at Stanford University.

[1] T. Futamase, Phys. Rev. Lett. **61**, 2175 (1988); Mon. Not. R. Astron. Soc. **237**, 187 (1989); T. Futamase and M. Sasaki, Phys. Rev. D **40**, 2502 (1989).

[2] M. W. Jacobs, E. V. Linder, and R. V. Wagoner (unpublished).

[3] C. M. Will, *Theory and Experiment in Gravitational Physics* (Cambridge Univ. Press, Cambridge, England, 1981).

[4] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1975).

[5] R. K. Sachs, Proc. R. Soc. London **A264**, 309 (1961).

[6] E. V. Linder, Astron. Astrophys. **206**, 190 (1988), Appendices B and C; Futamase and Sasaki (cf. [1], Sec. 4a) discuss a similar equation, in comparison with the Dyer-Roeder distance formula (Ref. [1] therein).

[7] E. V. Linder, Mon. Not. R. Astron. Soc. **243**, 353 (1990); **243**, 362 (1990); and (unpublished).

[8] K. Tomita and K. Watanabe, Prog. Theor. Phys. **82**, 563 (1989).

[9] K. Watanabe and K. Tomita, Astrophys. J. **355**, 1 (1990); **370**, 481 (1991).