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Trapped surfaces in expanding open universes

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Assume we have an open $k=0$ Friedmann-Lemaître universe with spherically symmetric inhomogeneities on a spacelike slice that do not change (initially) the rate of expansion of the volume of the slice. We give a set of necessary and sufficient criteria for the formation of trapped surfaces due to those inhomogeneities on the initial surface. A bound for the size of a perturbed trapped region is found, which depends on the cosmological energy density.

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To a certain degree of accuracy the large-scale features of the Universe can be described by a homogeneous and isotropic model, the Friedmann solution. On the other hand, on smaller scales the Universe is neither homogeneous nor isotropic and it is widely believed that some inhomogeneities have led or still lead to the formation of very compact objects such as *white dwarfs*, *neutron stars*, or even *black holes* (either during the collapse of a star or even more spectacularly in the very early Universe [1]). The purpose of this Rapid Communication is to find out under which circumstances those inhomogeneities can lead to the formation of black holes.

According to their *mathematical* definition black holes are global objects, whose identification requires knowledge of the whole structure of spacetime. Thus, the natural and direct attack on the problem of the existence of black holes would be the investigation of the long-time evolution (the global Cauchy problem) of a gravitational system. In this way one could identify initial data that lead to the formation of black holes. That strategy was applied by Christodoulou [2] in his analysis of self-gravitating massless scalar fields, but, at the present stage, it is difficult to implement in sufficiently general situations. We adopt a different approach and will study the geometry of initial data of the Einstein equations coupled with a matter field, to diagnose the presence of trapped surfaces. These surfaces are two-surfaces with the property that narrow beams of light orthogonal to it at any point decrease in area, at least initially, when propagating outwards. (Note that this definition corresponds to the *outer* trapped surface of Hawking and Ellis [3].) Thus the intensity of light increases, when moving outwards; that property is of sufficient interest to motivate the study of trapped surfaces, but there are also indications—via the singularity theorems [3] and cosmic censorship [4,5]—that trapped surfaces imply the existence of black holes.

We denote the (positive-definite) three-dimensional metric of the hypersurface Σ by g_{ab}. The trace (tr$K$) of the second fundamental form K_{ab} is equal to the (positive) rate of change of the 3D volume, $(d/dt)(dV)=g^{ab}K_{ab}dV$.

The initial data [g_{ab},K_{ab},ρ (energy density) and J_b (matter current)] cannot be given freely but must satisfy the Hamiltonian and momentum constraints

$$^{(3)}R[g]-K_{ab}K^{ab}+(K^a_a)^2=16\pi\rho\,, \tag{1}$$

$$D_aK^a_b-D_bK^a_a=-8\pi J_b\,. \tag{2}$$

In (1) $^{(3)}R([g])$ is the scalar curvature of Σ and we put $c=1,G=1$.

A trapped surface is defined as a compact two-dimensional (smooth) spacelike surface S having the property that the expansion θ of *outgoing* future-directed null geodesics which are orthogonal to S is everywhere negative.

θ is related to the initial data g_{ab}, K_{ab} of the Einstein equations by

$$\theta=D_an^a-K_{ab}n^an^b+g_{ab}K^{ab}\,, \tag{3}$$

where D_a is the three-dimensional covariant (metric) derivative and n_a is the outward normal unit vector to S.

As has already been mentioned we want to consider a perturbed cosmological Friedmann-Lemaître model (with $k=0$) with a spherically symmetric perturbation.

We relate the background quantities (denoted by a caret) and perturbed ones as follows:

$$g_{ab}=\phi^4(r)\hat{g}_{ab},\ \hat{g}_{ab}=a(t)\delta_{ab}\,, \tag{4}$$

$$\hat{K}_{ab}=(da/dt)\delta_{ab}/2=\beta(t)\hat{g}_{ab}\,, \tag{5}$$

$$K_{ab}=\phi^4(r)\hat{K}_{ab}+\delta K_{ab}=g_{ab}\beta+\delta K_{ab}\,. \tag{6}$$

No linearization is involved; we are just describing a general solution in terms of its (finite) deviation from a homogeneous-isotropic background. The line element reads, in isotropic coordinates, $ds^2=a\phi^4(dr^2+r^2d\Omega^2)$. All indices of quantities without carets are raised and lowered with the metric g_{ab}. In (4), $a(t)$ is a scalar function determined by the Friedmann equation and in (5) $\beta(t)$ is a scalar function (the Hubble function) describing the rate of change of the three-metric.

Because of the spherical symmetry the general form of

K_{ab} is

$$K_{ab}=a(r)n_a n_b+b(r)g_{ab}\,. \tag{7}$$

We demand in all that follows that although the extrinsic curvature may be perturbed the value of its trace is not changed, $K_a^a=g^{ab}K_{ab}=\hat{g}^{ab}\hat{K}_{ab}$; from that we can conclude that $g^{ab}\delta K_{ab}=0$, that is,

$$\delta K_{ab}=(n_a n_b-g_{ab}/3)K(r)\,. \tag{8}$$

Intuitively, the motivation for (8) is that we do not wish to produce caustics among timelike curves orthogonal to Σ, but we have to admit that this condition is important also from the technical point of view. Later on we have to use the maximum principle, and without (8) we would not be able to do it.

We denote the perturbation of ρ by $\delta\rho$ and of the current J_b by δJ_b:

$$\rho=\hat{\rho}+\delta\rho,\quad J_b=\delta J_b\,. \tag{9}$$

In (9) the background current vanishes.

With the metrics (4), n^a and $D_a n^a$ read

$$n^a=(\phi^{-2}/\sqrt{a},0,0,),$$
$$D_a n^a=(\sqrt{a}r^2\phi^6)^{-1}\frac{d}{dr}(r^2\phi^4)\,. \tag{10}$$

The constraints (1) and (2) become

$${}^{(3)}R-\delta K^{ab}\delta K_{ab}=16\pi\delta\rho\,, \tag{11}$$

$$D_a\delta K_b^a=-8\pi\delta J_b\,. \tag{12}$$

To get (11) and (12) one should take into account the formulas (4)-(6) and (8) and the fact that the cosmological background satisfies $(K_a^a)^2-K_{ab}K^{ab}=6\beta^2=16\pi\hat{\rho}$. The metric $\hat{g}_{ab}$ is flat; hence, the scalar curvature of the conformally related metric g_{ab} reads

$${}^{(3)}R[g]=-8\phi^{-5}\Delta\phi\,. \tag{13}$$

Inserting this into the Eq. (11) we obtain

$$\phi^{-5}\Delta\phi=-2\pi\delta\rho-\delta K_{ab}\delta K^{ab}/8\,. \tag{14}$$

If the perturbation $\delta\rho$ is non-negative, the right-hand side of (14) is nonpositive, and from the maximum principle the function ϕ is decreasing. Integration of (14) over a ball V of a radius r_0 yields

$$\int_V\phi\Delta\phi d^3x=-2\pi\int_V\left[\delta\rho+\frac{\delta K^{ab}\delta K_{ab}}{16\pi}\right]dV\,. \tag{15}$$

In (15) d^3x denotes the volume element of the background metric and $dV=\phi^6d^3x$. Integrating (15) by parts, remembering that perturbations are spherically symmetric, rearranging in a way analogous to what was done in [6], and using Eq. (10), we obtain the identity

$$(a/2)r_0^2\phi^4D_an^a=-\delta M-\int_V(\delta K^{ab}\delta K_{ab}/16\pi)dV$$
$$+2\int_0^{r_0}\sqrt{a}r^2(d\phi/dr)^2dr+\sqrt{a}r_0\phi^2\,, \tag{16}$$

where $\delta M=\int_V\delta\rho\,dV$ is the mass of the inhomogeneity.

It is natural to consider two cases, corresponding to two ways of perturbing the homogeneous geometry. In the first case no current is produced, $\delta J_b=0$. That implies that $\delta K_{ab}=0$, as will be shown below. Thus the perturbations are *time symmetric* in the sense that the momentum is unchanged. In the second case the current is perturbed, $\delta J_b\neq 0$.

Now let us assume that $\delta J_b=0$. We claim this to imply that $\delta K_{ab}=0$. Inserting (8) into Eq. (12) yields $dK/dr+3\{[2(d\phi/dr)\phi+1]/r\}K=0$. The general solution is $K=K_0\phi^{-6}r^{-3}$. Therefore, because ϕ is regular, the only regular solution is $K=0$, as claimed above.

The condition that a sphere S is trapped now reads

$$D_an^a-\hat{K}^{ab}n_an_b+\hat{g}^{ab}\hat{K}_{ab}=D_an^a+2\beta<0\,. \tag{17}$$

Taking (17) into account we write, instead of (16) (remember that $\delta K_{ab}=0$),

$$(a/2)r_0^2\phi^4\theta=(a/2)r_0^2\phi^4(D_an^a+2\beta)$$
$$=-\delta M+2\int_0^{r_0}\sqrt{a}r^2(d\phi/dr)^2dr$$
$$+\sqrt{a}r_0\phi^2+r_0^2\phi^4\beta a\,. \tag{18}$$

From an inequality proven in [7] we know that if, in the Lichnerowicz equation (14), $\delta\rho\geq 0$, then

$$2\int_0^{r_0}r^2(d\phi/dr)^2dr+r_0\phi^2\leq\int_0^{r_0}\phi^2dr=L/\sqrt{a}\,, \tag{19}$$

where L is the proper radius of S, $L=\int_0^{r_0}\sqrt{g_{rr}}\,dr$.

The last term in (18) is related to the area of the sphere S by

$$S=4\pi ar_0^2\phi^4\,. \tag{20}$$

Thus, using (19), (20), and the relation between $\hat{\rho}$ and β [see the formula below Eq. (12)] we conclude from (18) that the expansion θ is estimated from above as

$$(a/2)r_0^2\phi^4\theta\leq-\delta M+L+S\sqrt{\hat{\rho}/6\pi}\,. \tag{21}$$

Thus we have proven the following.

Theorem 1. (A sufficient condition.) Assume that spherical perturbations of homogeneous cosmological ($k=0$) Cauchy data satisfy the conditions that (i) $\delta\rho$ is non-negative, (ii) $K_a^a=$const, i.e., the rate of expansion of the volume is not perturbed, and (iii) $\delta J_b=0$.

If at a sphere S its radius L and the mass of the inhomogeneity satisfy

$$\delta M>L+S\sqrt{\hat{\rho}/6\pi}\,, \tag{22}$$

then S is trapped.

A necessary condition easily follows if one uses the following inequality proven in [6] (valid for any smooth function):

$$L/2\leq 2\sqrt{a}\int_0^{r_0}r^2(d\phi/dr)^2dr+\sqrt{a}r_0\phi^2\,. \tag{23}$$

Proceeding in exactly the same way as for the sufficient condition one finds, from the identity (18) and (23), the following.

Theorem 2. (A necessary condition.) Assume that perturbed initial data satisfy conditions (ii) and (iii) of theorem 1. If

$$\delta M < L/2 + S\sqrt{\hat{\rho}/6\pi}\,, \tag{24}$$

then S is not trapped.

Now we make a few remarks.

(1) Note that here we do *not* require, in contrast with theorem 1, that $\delta\rho \geq 0$. This is because in the proof of the inequality (23) it is not required that the energy density is positive [6].

(2) Another form of the necessary condition for S to be trapped is $\delta M > R_0 + S(\hat{\rho}/6\pi)^{1/2}$, where R_0 is the areal radius, $R_0 = \sqrt{S/4\pi}$. The proof is analogous to that of [6].

(3) The sufficient condition is saturated by perturbations of the form of a massive spherical shell (see an explicit solution in [7]) while examples saturating the necessary condition can be found in the first reference of [6].

Thus the estimates in both theorems are sharp. From Eq. (14) we conclude that

$$-\int_V \phi\Delta\phi\, d^3x \geq 2\pi\delta M\,. \tag{25}$$

The left-hand side of (25) is shown in [7] to be smaller than $2\pi(R_0+L)$, assuming that the right-hand side of (14) is nonpositive. That is guaranteed if $\delta\rho$ is nonnegative. Thus, the following is true.

Theorem 3. Under the above conditions, the mass of the inhomogeneity $\delta M = \int_V \delta\rho\, dV$ inside a sphere S cannot exceed the sum of its proper radius L and the areal radius R_0:

$$\delta M = \int_V \delta\rho\, dV \leq L + R_0\,. \tag{26}$$

Hence, we have proved a result that supports Einstein's view that "matter cannot be concentrated arbitrarily" [8].

From the necessary condition and from (26) we conclude that if

$$L/2 + S\sqrt{\hat{\rho}/6\pi} \geq R_0 + L\,, \tag{27}$$

then S cannot be trapped, because δM cannot exceed the expression on the right-hand side of (27). Subtracting $L/2$, and dividing both sides of (27) by S, we arrive at (note that $S = 4\pi R_0^2$)

$$\sqrt{\hat{\rho}/6\pi} \geq 1/4\pi R_0 + L/8\pi R_0^2\,. \tag{28}$$

Thus, we arrive at the following theorem.

Theorem 4. (Absence of large trapped surfaces.) Assume the conditions of theorem 1. If the background energy density $\hat{\rho}$ satisfies (28) at a sphere S of proper radius L and areal radius R_0, then S cannot be trapped.

Let us remark that large $\hat{\rho}$ means that the rate of expansion of the Universe is large [see the relation below (12)]. Therefore a part of theorem 3 is intuitively obvious—the quicker the geometry is expanding, the more difficult it should be to have negative expansion, i.e., to enforce initially parallel photons to create caustics. It comes as a surprise, however, that there is a bound from above for the size of largest trapped spheres while there is no lower bound; in a sense, it is easier to create small trapped surfaces than large ones.

In the remaining part of this Rapid Communication we will discuss spherically symmetric perturbations that change momenta, but still preserve the trace of the second fundamental form. In that case the expansion θ [see Eq. (3)] is given by the expression

$$\begin{aligned}\theta &= D_a n^a - \hat{K}^{ab} n_a n_b - \delta K^{ab} n_a n_b + \hat{g}^{ab}\hat{K}_{ab} = D_a n^a + 2\beta - \delta K^{ab} n_a n_b \\ &= (\phi^2 + 2r\phi\, d\phi/dr + r\phi^4\beta\sqrt{a} - \sqrt{a}\, r\phi^4 K/3)[(\sqrt{a}/2) r_0^2 \phi^4]^{-1}\,.\end{aligned} \tag{29}$$

In addition to the Hamiltonian constraint, we have to deal also with the momentum constraints (12). Multiplying Eq. (12) by the normal n^b, integrating it over V, using the spherical symmetry, and subtracting the resulting identity from Eq. (16), one finds, after routine manipulations analogous to those of [6],

$$\begin{aligned}\delta M - \int_V \delta J_b n^b\, dV = &-(a r_0^2 \phi^4/2)(D_a n^a + 2\beta - 2K/3) + r_0^2\phi^4\beta a + \sqrt{a}\int_0^{r_0}\{\tfrac{7}{6}\phi^2 - [\phi + 6r\phi\, d\phi/dr - \sqrt{a}K\phi^3 r]^2/6 \\ &+ 4r(d\phi/dr)[\phi + 2r\, d\phi/dr + \sqrt{a}\beta r\phi^3 - \sqrt{a}\, rK\phi^3/3] - 4\sqrt{a}\, r^2\phi^3(d\phi/dr)\beta\}\, dr\,.\end{aligned} \tag{30}$$

The third term in the integral is negative if *trapped surfaces are absent inside* S and the perturbations $\delta\rho$ are nonnegative. Indeed, then $d\phi/dr$ is nonpositive, while the sign of the second set of square brackets is equal to that of θ and is positive; hence their product is negative.

Thus, under the above assumptions (including the absence of trapped surfaces inside S), we estimate the left-hand side of (30) in the following way:

$$\begin{aligned}\delta M - \int_V \delta J_b n^b\, dV &\leq \frac{7}{6}\int_0^{r_0}\sqrt{a}\,\phi^2\, dr - \int_0^{r_0} 4a r^2\phi^3 (d\phi/dr\,\beta)\, dr + r_0^2\phi^4\beta a \\ &= 7L/6 - a\int_0^{r_0}\left(\frac{d}{dr}\phi^4 r^2\beta\right) dr + \beta a\int_0^{r_0} 2r\phi^4\, dr + r_0^2\phi^4\beta a = 7L/6 + 2\beta a\int_0^{r_0} r\phi^4\, dr\,.\end{aligned} \tag{31}$$

We use the fact that $R_0 \leq L$ to estimate the second term in (31). Namely,

$$\int_0^{r_0} 2a r\phi^2(\phi^2\, dr) = 2\sqrt{a}\int_0^{r(L)} r\phi^2\, dL \leq 2\int_0^L L\, dL = L^2\,.$$

Hence, finally we obtain, from (31),

$$\delta M - \int_V \delta J_b n^b dV \leq 7L/6 + \beta L^2 = 7L/6 + L^2\sqrt{\tfrac{8}{3}\pi\hat{\rho}}\,. \tag{32}$$

In the last inequality we have used the relation between β and $\hat{\rho}$ expressed below the formula (12).

Thus, we may conclude the following.

Theorem 5. (A sufficient condition.) Assume that spherically symmetric perturbations satisfy the conditions (i) $\delta\rho \geq 0$ and (ii) $K_a^a =$const. If at a sphere S

$$\delta M - \int_V \delta J_b n^b dV \geq 7L/6 + L^2\sqrt{\tfrac{8}{3}\pi\hat{\rho}}\,, \tag{33}$$

then there exists a trapped surface inside S.

Theorem 5 implies that the influx of matter into S makes it easier to form trapped surfaces, while the large outflux from S can make the formation of trapped surfaces impossible, even if the energy δM of a perturbation is very large.

Let us summarize the whole discussion. Theorem 5 gives a sufficient condition for the formation of trapped surfaces by *moving* perturbations, when the initial momentum of the gravitational field is changed. Theorems 1 and 2 give a sufficient and a necessary condition, respectively, for the formation of trapped surfaces in the case when only the energy density is perturbed, while initial momenta are unchanged. Theorem 3 says that in a sphere of a fixed radius only a finite amount of perturbed energy can be packed. Theorem 4 is the consequence of Theorems 2 and 3; it states that if the rate of expansion of the Universe is very large, then there is an upper limit for the size of trapped surfaces. These results are also valid for nonspherical perturbations, assuming that deviations from spherical symmetry are small. A precise meaning of "smallness" can be given by using techniques of, e.g., [9].

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