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Simple effective Lagrangian for hard thermal loops

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We derive an alternative effective Lagrangian for the hard thermal loops between gluons in QCD at high temperature or chemical potential. This effective Lagrangian is elementary in form and manifestly gauge invariant.

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In gauge theories at high temperature, a consistent perturbative expansion requires the resummation of a set of diagrams termed "hard thermal loops" [1]. These amplitudes have several properties which greatly simplify their calculation [1,2]. They arise from one-loop diagrams, but are independent of the gauge-fixing condition used to compute them [1-3]. There are hard thermal loops in all multigluon amplitudes and in the amplitudes between a quark pair and any number of gluons, but not, for example, in amplitudes involving ghost lines. Consequently, hard thermal loops obey Ward identities similar to those of the tree amplitudes. These properties imply that the generating functional of hard thermal loops is a gauge-invariant functional of the quark and gluon fields [4].

Taylor and Wong [4] used gauge invariance to derive an effective action for gluonic hard thermal loops. Their action is rather involved in form, and is written in such a way that the gauge invariance is not manifest. In this Rapid Communication we present an equivalent effective action [5] in a form that it is relatively simple and manifestly gauge invariant. We deduce this effective action by first obtaining an expression which reproduces the hard thermal loop in the gluon self-energy. Once written properly, the general effective action follows immediately by invoking gauge invariance.

We consider an SU(N) gauge theory with coupling constant g in equilibrium at a temperature T. The quark fields are taken as N_f flavors of massless Dirac fermions in the fundamental representation of the gauge group. A chemical potential μ is introduced to incorporate a net excess of baryon number. With Euclidean conventions, the Lagrangian for the theory is

$$\mathcal{L}=\tfrac{1}{2}\,\mathrm{tr}(G^{\mu\nu}G^{\mu\nu})+\bar{\psi}\gamma^{\mu}D^{\mu}\psi\,. \qquad (1)$$

The gauge potential is $A^{\mu}=A^{\mu,a}t^{a}$, with the generators t^a of the fundamental representation normalized so that $\mathrm{tr}(t^a t^b)=\delta^{ab}/2$. The field strength tensor is $G^{\mu\nu}=\partial^{\mu}A^{\nu}-\partial^{\nu}A^{\mu}-ig[A^{\mu},A^{\nu}]$. In the quark term, $D^{\mu}=\partial^{\mu}-igA^{\mu}$ is the covariant derivative for the fundamental representation.

Hard thermal loops are the dominant loop diagrams in the limit of high temperature or high chemical potential [1,2]. In gluon amplitudes explicit calculations show that hard thermal loops are always proportional to the square of the thermal gluon mass m_g, where

$$m_g^2=\frac{1}{9}Ng^2T^2+\frac{1}{18}N_f g^2\left(T^2+\frac{3}{\pi^2}\mu^2\right). \qquad (2)$$

Physically, m_g is the smallest frequency at which time-dependent gluon fields propagate in the plasma. We discuss the case of nonzero temperature and zero chemical potential, with the understanding that the extension to a nonzero chemical potential is just a matter of changing the value of m_g.

Considering the temperature as held fixed, hard thermal loops are the dominant diagrams over "soft" momenta. A momentum $P^\mu=(p^0,\mathbf{p})$ is defined to be soft if each component is of order gT. (In general, when $\mu\neq 0$ soft momenta are defined to be those whose components are of order m_g.) In the imaginary-time formalism, the Euclidean p^0 is an integral multiple of $2\pi T$, and so soft momenta are those that are static, $p^0=0$, with spatial momentum $p=|\mathbf{p}|$ of order gT. The only hard thermal loop which does not vanish in the static limit is in the gluon self-energy. Denoting the hard thermal loop in the self-energy as $\delta\Pi^{\mu\nu}$, its only nonzero component at soft momentum is

$$\delta\Pi^{00}(p^0=0,\mathbf{p})=3m_g^2 . \tag{3}$$

This nonzero value of $\delta\Pi^{00}$ represents the screening of static electric fields with a screening mass $\sqrt{3}m_g$. This mass is represented in a Lagrangian just by adding a mass term for the A^0 field:

$$\delta\mathcal{L}_{\text{static}}=3m_g^2\,\text{tr}(A^0A^0) . \tag{4}$$

In the imaginary-time formalism dynamical processes are given by first computing amplitudes as functions of the discrete Euclidean energies p^0 and then analytically continuing to continuous Minkowski values: $p^0\to -i[\omega+i\,\text{sgn}(\omega)\epsilon]$, with ω a continuous energy. (The infinitesimal parameter $\epsilon>0$ is required to uniquely specify the direction of analytic continuation, and is left implicit.) For dynamical processes the requirement of soft momenta implies that both the energy ω and the spatial momentum p are of order gT.

While the gluon self-energy is the only static amplitude with a hard thermal loop, there are infinitely many dynamical amplitudes with hard thermal loops, including n-gluon amplitudes for all n. These hard thermal loops were first derived diagrammatically in Refs. [1,2]. Their existence can be understood by considering the effect of gauge transformations on (4). An infinitesimal gauge transformation of A^0 has the form $A^0\to A^0+\partial^0\lambda-ig[A^0,\lambda]$. The Lagrangian in (4) is obviously invariant under static gauge transformations, $\partial^0\lambda=0$, but not under time-dependent gauge transformations. Taylor and Wong [4] derived their effective Lagrangian by asking what functional must be added to $\delta\mathcal{L}_{\text{static}}$ in order to make the sum invariant under arbitrary gauge transformations.

We take a different tack for deriving the effective Lagrangian. We first find an expression which gives the complete hard thermal loop in the gluon self-energy. By modifying this expression to make it gauge invariant, we obtain the complete effective Lagrangian in a particularly elegant form. The hard thermal loop in the gluon self-energy was computed by Silin, Klimov, and Weldon [6]. It can be expressed in terms of two independent functions, the longitudinal and transverse self-energies $\delta\Pi_l$ and $\delta\Pi_t$:

$$\delta\Pi^{00}(P)=\delta\Pi_l(P) , \tag{5}$$

$$\delta\Pi^{0i}(P)=-\frac{i\omega p^i}{p^2}\delta\Pi_l(P) , \tag{6}$$

$$\delta\Pi^{ij}(P)=(\delta^{ij}-\hat{p}^i\hat{p}^j)\delta\Pi_t(P)-\hat{p}^i\hat{p}^j\frac{\omega^2}{p^2}\delta\Pi_l(P) , \tag{7}$$

where $\hat{\mathbf{p}}=\mathbf{p}/p$. The explicit expressions for these functions are

$$\delta\Pi_l(P)=3m_g^2Q_1\left[\frac{\omega}{p}\right] , \tag{8}$$

$$\delta\Pi_t(P)=\frac{3}{5}m_g^2\left[Q_3\left[\frac{\omega}{p}\right]-Q_1\left[\frac{\omega}{p}\right]-\frac{5}{3}\right] ; \tag{9}$$

Q_1 and Q_3 are Legendre functions of the second kind. In the limit of vanishing three-momentum, the transverse self-energy reduces to $\delta\Pi_t(\omega,p=0)=m_g^2$, so that for $p=0$ the gluon propagator has a pole at $\omega=m_g$.

As we wish eventually to construct a gauge-invariant Lagrangian, it is natural to start with powers of the field strength tensor $G^{\mu\nu}$. Assuming that two powers of $G^{\mu\nu}$ enter, and that the entire functional is proportional to m_g^2, the simplest term that can be added to the Lagrangian is

$$\delta\mathcal{L}_{\text{KRS}}=\frac{1}{2}m_g^2\,\text{tr}\left[G^{\mu\nu}\frac{1}{-\partial^2}G^{\mu\nu}\right] . \tag{10}$$

The inverse of the differential operator $\partial^2=(\partial^0)^2+\boldsymbol{\partial}^2$ is defined by Fourier transformation:

$$\frac{1}{-\partial^2}G^{\mu\nu}(x)=\int d^4y\,e^{iK\cdot(x-y)}\frac{1}{K^2}G^{\mu\nu}(y) . \tag{11}$$

The Lagrangian in (10) was proposed by Kreuzer, Rebhan, and Schulz [7] to implement a partial resummation of hard thermal loops. The Abelian version of (10) is identical to the mass term in the Schwinger model in 1+1 dimensions (at $T=0$); there m_g is replaced by the gauge coupling e, which has dimensions of mass in 1+1 dimensions. The self-energy generated by (10) is

$$\delta\Pi^{\mu\nu}_{\text{KRS}}(P)=m_g^2\left[\delta^{\mu\nu}-\frac{P^\mu P^\nu}{P^2}\right] , \tag{12}$$

$P^2=-\omega^2+p^2$. Thus (10) is not the correct effective Lagrangian; instead of giving rise to the hard thermal loops of (8) and (9), it generates a Lorentz-invariant gluon mass. To see this in another way, observe that while (10) appears nonlocal, if we choose the gauge where $\partial^\mu A^\mu=0$, then to quadratic order in the gluon field, $\delta\mathcal{L}_{\text{KRS}}$ reduces to a local mass term

$$\delta\mathcal{L}_{\text{KRS}}\simeq m_g^2\,\text{tr}(A^\mu A^\mu) . \tag{13}$$

The resulting self-energy is constant. In contrast, the self-energies of (8) and (9) are not only nontrivial functions of ω/p, they are not even analytic in ω. Because of Landau damping, hard thermal loops have cuts running from $\omega=0$ to $\omega=p$. This nonanalyticity can only be produced by a Lagrangian which is nonlocal in any gauge.

To construct such a Lagrangian we use our experience gained in computing hard thermal loops. By definition, in a hard thermal loop every external momentum is soft. The one-loop diagrams which generate these amplitudes, however, are those in which the field running around the loop has a "hard" momentum, whose components are of order T. This hard field propagates on its mass shell, and its emission from, and absorption into, the thermal distribution generates the cuts of Landau damping. Hence we introduce a four-vector $K^\mu = k(i,\hat{\mathbf{k}})$ to represent the momentum of the hard field in the loop. Since the particle is massless and on its mass shell, K is a null vector: $K^2=0$. In a loop diagram one averages over all possible directions of the loop momenta, for which we introduce the notation

$$\langle f(K)\rangle \equiv \int \frac{d\Omega_{\hat{k}}}{4\pi} f(K) . \tag{14}$$

Only the angular integral will matter in the following, as the energy k of the hard field drops out. Alternatively, we could introduce an integral over the energy k in such a form that it gives the thermal gluon mass in (2).

With these definitions, the Lagrangian which generates the hard thermal loops in the gluon self-energy is

$$\delta\mathcal{L}_2 = \frac{3}{2} m_g^2 \,\mathrm{tr}\left[G^{\mu\alpha} \left\langle \frac{K^\alpha K^\beta}{-(K\cdot\partial)^2} \right\rangle G^{\mu\beta} \right] , \tag{15}$$

where $K\cdot\partial = k(i\partial^0 + \hat{\mathbf{k}}\cdot\boldsymbol{\partial})$. The subscript 2 on $\delta\mathcal{L}_2$ indicates that it only generates the hard thermal loops in the gluon two-point function. To show that $\delta\mathcal{L}_2$ is correct it is easiest to work in momentum space, making the replacement $-(K\cdot\partial)^2 \to (K\cdot P)^2$. We need the identities

$$\left\langle \frac{K^0K^0}{(K\cdot P)^2} \right\rangle = \frac{1}{P^2} , \tag{16}$$

$$\left\langle \frac{K^0K^i}{(K\cdot P)^2} \right\rangle = \hat{p}^i \left[\frac{1}{p^2} Q_0\left(\frac{\omega}{p}\right) - \frac{\omega}{pP^2} \right] , \tag{17}$$

$$\left\langle \frac{K^iK^j}{(K\cdot P)^2} \right\rangle = \delta^{ij} \frac{1}{p^2} Q_1\left(\frac{\omega}{p}\right) - \hat{p}^i\hat{p}^j \left[\frac{3}{p^2} Q_1\left(\frac{\omega}{p}\right) + \frac{1}{P^2} \right] . \tag{18}$$

Using these identities, and the definitions of the Legrendre functions Q_0, Q_1, and Q_3, it is straightforward to show that $\delta\mathcal{L}_2$ correctly generates the hard thermal loops of (8) and (9).

The Lagrangian $\delta\mathcal{L}_2$ is the only term proportional to m_g^2, involving two powers of the field strength tensor, which reproduces the hard thermal loop in the gluon self-energy. For example, one might try instead

$$\delta\mathcal{L}_{\mathrm{KRS}} = \frac{1}{2} m_g^2 \,\mathrm{tr}\left[G^{\mu\nu} \left\langle \frac{k^2}{-(K\cdot\partial)^2} \right\rangle G^{\mu\nu} \right] . \tag{19}$$

After integration over the direction of $\hat{\mathbf{k}}$, by the identity of (16) this reduces to the wrong effective Lagrangian, that of (10). We also note that any Lagrangian involving an angular average, such as (15), is only unique up to terms whose angular average vanishes.

Given the fact that $\delta\mathcal{L}_2$ generates the correct two-point function, it is then trivial to use gauge invariance to construct the effective Lagrangian which generates the hard thermal loops in all gluon amplitudes. As the field strength tensor transforms homogeneously under local gauge transformations, a gauge-invariant functional is formed merely by replacing the partial derivative ∂^μ in $\delta\mathcal{L}_2$ by the appropriate covariant derivative. As the derivative acts on the field strength tensor, for an SU(N) gauge theory the appropriate covariant derivative is that for the adjoint representation, $D^\mu = \partial^\mu - ig[A^\mu, \,]$ (here $[A^\mu, \,]$ denotes the adjoint operator). Thus the complete effective Lagrangian for gluonic hard thermal loops is

$$\delta\mathcal{L} = \frac{3}{2} m_g^2 \,\mathrm{tr}\left[G^{\mu\alpha} \left\langle \frac{K^\alpha K^\beta}{-(K\cdot D)^2} \right\rangle G^{\mu\beta} \right] . \tag{20}$$

Although tedious, we have used identities similar to (16)–(18) to verify that it does give the correct hard thermal loop in the three-gluon amplitude. We can then appeal to the treelike Ward identities satisfied by hard thermal loops to conclude that the hard thermal loops in gluon amplitudes between four or more gluons coincide with known results [1,2]. These Ward identities ensure that the generalization from $\delta\mathcal{L}_2$ and $\delta\mathcal{L}$ is correct. This can also be seen by playing with other functionals involving powers of the field strength tensor and covariant derivatives. So as not to spoil the mass dimensions, one could insert, say, powers of $(G^{\mu\nu})^2/(K\cdot D)^4$ into the expression. But the insertion of such a term will give amplitudes that clearly have many more energy denominators than are produced by one-loop diagrams. As for $\delta\mathcal{L}_2$ in (15), $\delta\mathcal{L}$ in (20) is unique only up to the addition of terms with vanishing angular average.

It is not at all trivial to show that the effective Lagrangian $\delta\mathcal{L}$ is equal to that of Taylor and Wong [4]. They computed the functional $W(gA)$, which must be added to $\delta\mathcal{L}_{\mathrm{static}}$ in order that the sum is gauge invariant. This construction is analogous [8] to how the Wess-Zumino term can be derived in 1+1 dimensions (at $T=0$) [9]. By construction, $W(gA)$ vanishes in the static limit. It is not immediately obvious how our effective Lagrangian (20) reduces to (4) in the static limit.

The effective Lagrangian $\delta\mathcal{L}$ has several advantages over that of Taylor and Wong [4]. Their result is a functional of the gauge-dependent vector potential A^μ; also, while the derivative $dW(gA)/dg$ is simple, $W(gA)$ is not [8]. The Lagrangian in (20) is both simple and manifestly gauge invariant, constructed from factors of $G^{\mu\nu}$ and D^μ. We note that these results can also be understood physically, in the spirit of Wilson's approach to the renormalization group [5]: Integrating over hard thermal particles produces an effective Lagrangian for soft fields, which is just $\delta\mathcal{L}$ to leading order in the coupling constant g.

The effective Lagrangian $\delta\mathcal{L}$ can also be applied to the hard thermal loops in an Abelian gauge theory coupled to massless fermions, such as in hot QED. Explicit calculation shows that the only hard thermal loop for photon amplitudes is in the two-point function [1,2]. While hard thermal loops do appear in individual diagrams with four, six, or any even number of external photons, they cancel in the total amplitude. This feature is obvious from $\delta\mathcal{L}$, for in an Abelian theory, the appropriate covariant derivative which acts on the Abelian field strength is the ordinary derivative. Hence in an Abelian gauge theory with gauge coupling e, the complete effective Lagrangian is given simply by $\delta\mathcal{L}_2$ in (15), after replacing m_g^2 by the thermal photon mass squared, $m_\gamma^2 = e^2N_f(T^2+3\mu^2/\pi^2)/9$, and the trace by a factor of $\frac{1}{2}$.

For completeness we also present the generating functional for the hard thermal loops between a quark pair and any number of gluons. The hard thermal loops are proportional to the square of the thermal quark mass, with

$$m_q^2 = \frac{C_F}{8} g^2 \left[T^2 + \frac{\mu^2}{\pi^2} \right] ; \tag{21}$$

$C_F = (N^2-1)/(2N)$ is the Casimir constant for the fundamental representation. Then the effective Lagrangian which generates the hard thermal loops in quark amplitudes is

$$\delta\tilde{\mathcal{L}} = m_q^2 \bar{\psi} \gamma^\mu \left\langle \frac{K^\mu}{K\cdot D} \right\rangle \psi , \tag{22}$$

with D^μ the covariant derivative for the fundamental representation. This expression is equivalent to the effective Lagrangian for quarks of Taylor and Wong [4,8]. In analogy to the derivation of the gluon Lagrangian in (20), the quark Lagrangian in (22) can be deduced by showing that $\delta\tilde{\mathcal{L}}$ gives the correct hard thermal loop in the quark self-energy [10]. Gauge invariance then dictates that the hard thermal loops in all other quark amplitudes are obtained by replacing the ordinary derivative by the appropriate covariant derivative.

In an Abelian theory the hard thermal loops of amplitudes between a fermion pair and any number of photons are nontrivial. These can be read off from (22) by replacing m_q^2 with the thermal fermion mass squared, $m_f^2 = e^2(T^2+\mu^2/\pi^2)/8$, and using the Abelian covariant derivative, $D^\mu = \partial^\mu - ieA^\mu$.

Lastly, one might inquire about the hard thermal loops of scalar fields. Calculation shows that even for dynamical processes the only hard thermal loop for a scalar field is a local mass term. In the present formalism this can be seen by assuming that the effective Lagrangian is constructed from powers of the covariant derivative and its contraction with the vector K. For instance, a possible effective Lagrangian for a complex scalar field ϕ in the fundamental representation of the gauge group is

$$\delta\mathcal{L}_s = m_s^2 \phi^\dagger \left\langle \frac{D^2}{(K\cdot D)^2} \right\rangle \phi , \tag{23}$$

where m_s is the thermal scalar mass. But by the identity of (16), this reduces to the mass term

$$\delta\mathcal{L}_s = m_s^2 \phi^\dagger \phi . \tag{24}$$

The simplicity of these effective Lagrangians might motivate new lines of inquiry. For dynamical processes, the dominant physics over large distances is controlled by an effective action that is an integral over $\mathcal{L} + \delta\mathcal{L} + \delta\tilde{\mathcal{L}}$, where the individual terms are given in (1), (20), and (22). Perhaps this effective action has new stationary points, representing novel nonperturbative phenomena.

As this manuscript was being written we received a paper by Frenkel and Taylor [11] who derive a generating functional equivalent to (20). We also thank Ulrich Heinz for pointing out that (2) and (21) are exact for arbitrary values of μ/T. The research of R.D.P. is supported in part by Contract No. DE-AC02-76CH0016 with the U.S. Department of Energy.

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