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Abelianization of non-Abelian lattice gauge theories

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(2+1)-dimensional pure SU(2) lattice gauge theory on a square lattice is shown to be exactly equivalent to an Abelian gauge theory on a Kagomé lattice. The new dynamical variables create or annihilate a unit of an additive, color-invariant electric flux. They provide a complete (and not overcomplete) basis for physical states in the form of nonoriented "allowed" closed loops of unit flux. This also establishes 't Hooft's conjecture that a gauge theory of the Abelian subgroup is relevant for the confinement mechanism.

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Quantum chromodynamics is formulated by using a multitude of massless color gauge bosons which are not in the spectrum of particles (in the confinement phase). This is analogous to working with the nonlinear σ model (which uses the Goldstone bosons) to describe the unbroken phase of a ferromagnet. Much of the difficulty involved in understanding and calculating hadron dynamics could result from describing the theory in a wrong language. Lattice gauge theory in the strong-coupling expansion [1] has the correct ingredients to describe the confinement phase. Here the basic object is the color-electric flux line. Unfortunately this picture has not been developed beyond the strong-coupling expansion, and even there one is soon lost in the quagmire of non-Abelian algebra. Also there have been conjectures that SU(N) gauge theory formulated [2] in loop space (which is related to the flux-line description) may be exactly solvable, at least in the $N\to\infty$ limit, but at present this has only remained as a fond hope. With regard to the confinement mechanism, 't Hooft [3] has conjectured that the monopoles of the $U(1)^{N-1}$ subgroup of SU(N) gauge theory may be relevant. But this sounds as if the various colors are not treated equally. Moreover, the formulation is not precise enough to establish the conjecture either analytically or by numerical calculations.

In this paper we make progress in these directions. We give a reformulation of (2+1)-dimensional, pure SU(2) lattice gauge theory which works with color-invariant local degrees of freedom and, therefore, is better suited to describe and handle the confinement phase. We rewrite the theory on a square lattice as an Abelian gauge theory on a "Kagomé" lattice, thereby doing away with the non-Abelian algebra. This involves new dynamical variables that create or annihilate a unit of an additive color-electric flux without reference to the color content. Using them we are able to obtain a complete basis for gauge-invariant states in the loop space without the redundancies and the constraints that the collection of all Wilson loops have [4]. This also establishes the precise sense in which a U(1) gauge theory is relevant for the confinement mechanism in SU(2) gauge theory.

These results are based on an earlier paper [5] by one of us where it was shown that it is possible to obtain an explicit labeling, in terms of gauge-invariant local fields, of the physical subspace of the Hilbert space of lattice gauge theory. For the (2+1)-dimensional, pure SU(2) case [6], the basis for physical states consists of all triangulations with sides which are half integers and with a coordination number 6. Each triangle satisfies the triangle inequalities. In Ref. [6], these triangles are locally embedded in a metric space in order to solve the triangle inequality and get the dual theory. Here we solve the constraints in a different way. One could say that we now have an algebraic solution of the constraints in contrast with the geometric solution of Ref. [6].

Given any triangle with half-integer sides j_1, j_2, and j_3 corresponding to the addition of angular moments, the combinations

$$N_1=j_2+j_3-j_1,\ N_2=j_3+j_1-j_2,\ N_3=j_1+j_2-j_3 \tag{1}$$

are always integers and non-negative. In fact, three arbitrary independent non-negative integers,

$$N_I\geq 0,\ I=1,2,3\,, \tag{2}$$

uniquely generate all such triangles. We will associate these integers with the sides of an inscribed triangle as in Fig. 1. Of course, the inscribed triangle does not satisfy the triangle inequality constraints in general, and is for figurative purposes only.

This permits us an alternative way of characterizing all triangulations. The j's have been associated [6] with the links of a triangular lattice obtained by drawing one set of diagonals on the lattice dual to the original lattice. For the convenience of representation, we deform this lattice into a regular triangular lattice (Fig. 1). Furthermore, we inscribe equilateral triangles in each triangle of this dual lattice. These inscribed triangles form a Kagomé lattice.

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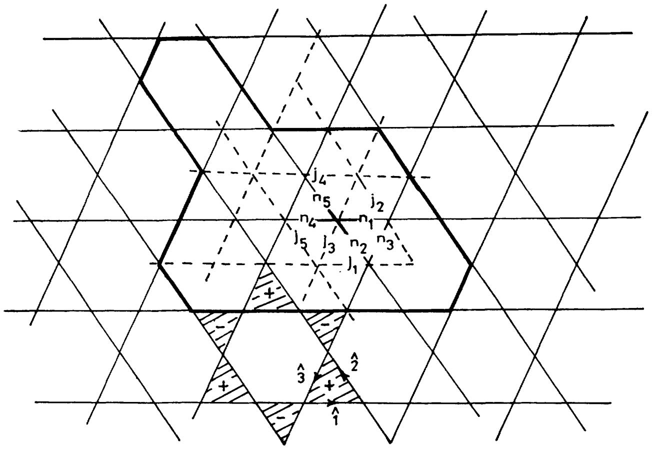

FIG. 1. The dual lattice with one set of diagonals drawn is shown by the dashed lines. Axes 1 and 2 have been distorted to make an angle of 60°, giving a regular triangular lattice. The triangles inscribed into the triangles of this dual lattice form a Kagomé lattice, represented by the solid lines. j_i and N_l are associated with the links of the triangular lattice and the Kagomé lattice, respectively. The triangles of the Kagomé lattice can be given the unique signatures plus and minus as shown. The conservation of N_l's at the vertices of these triangles is also represented. An example of an "allowed" closed loop is shown by the heavy lines.

We associate the integers N_l with the links of this Kagomé lattice. For any inscribed triangle, the sum of N_l on any two sides gives the j on the link of the triangular lattice on which they impinge. Since each such dual link is common to two inscribed triangles, we get a constraint (Fig. 1) such as

$$N_1+N_2=N_4+N_5 ; \tag{3}$$

i.e., the sum of the weights on two sides of any triangle of the Kagomé lattice should equal the sum on the two sides of the other triangle meeting at the common vertex. By assigning arbitrary non-negative integers to the links of the Kagomé lattice, subject to such a constraint at every vertex, all triangulations are uniquely generated. This, therefore, is an alternative for specifying a basis for the physical states.

We now describe the dynamics in this basis. The effect of a plaquette operator is to independently change by $\pm\frac{1}{2}$ the j's associated with the six lines of the triangular lattice incident on the vertex dual to the plaquette. This corresponds to a change by ± 1 of certain N's on a "star" of the Kagomé lattice centered on this dual vertex. We will denote those links on which N increases (by 1) by a solid line and those on which N decreases by a jagged line. Then the plaquette operator corresponds to the diagrams of Fig. 2, and other diagrams obtained from them by rotating by multiples of 60° and/or by interchanging the solid and the jagged lines, in case the result is a distinct diagram.

The amplitudes for these various processes may be computed using the matrix element in the j basis [Eq. (13) of Ref. [6]] and will not be presented here. (The phase factor given there needs a correction.) Quite remarkably,

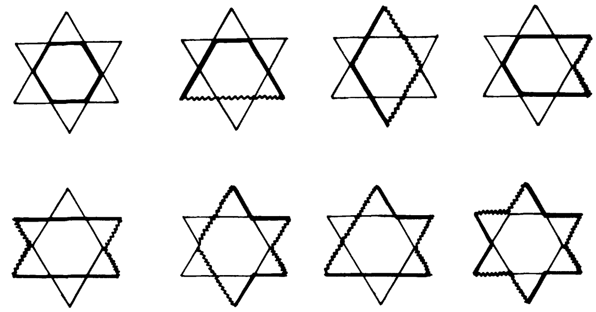

FIG. 2. An operator corresponding to a plaguette term in the Hamiltonian is represented by these diagrams and others obtained by rotating by multiples of 60° and/or the interchange of heavy and jagged lines in case the result is distinct. The links on which N_l increases (decreases) are represented by heavy (jagged) lines.

these amplitudes have a sixfold symmetry corresponding to the rotations of the star by multiples of 60° [in our representation where the $\hat{2}$ axis (Fig. 1) is inclined to the $\hat{1}$ axis at 60°]. This is in spite of the fact that the $\hat{3}$ links (on which the sum of the color spins on the $\hat{1}$ and $\hat{2}$ links are represented) are altogether on a different footing. The other term in the Hamiltonian, $\sum \mathbf{E}^2(n,i)$, which is diagonal in the $\{N_l\}$ basis, does not respect this symmetry because $i=1,2$ only.

In order to express the Hamiltonian in this basis, we associate a harmonic-oscillator creation operator a^* and an annihilation operator a with each link of the Kagomé lattice. The weight N_l on the link is interpreted as the eigenvalue of the corresponding number operator. The conservation law at every vertex is interpreted as the Gauss-law constraint associated with a U(1) local gauge invariance defined as follows. The triangles of the Kagomé lattice can be consistently assigned signatures plus or minus according to whether they are pointing up or down, respectively (Fig. 1). This signature is inherited by the sides of these triangles. We assign a ± 1 local charge to both ends of a link according to whether the signature of the link is plus or minus, respectively. This way we get a U(1) gauge theory on the Kagomé lattice differing from the usual U(1) gauge theory in the following ways. On each link the variable is an oscillator instead of a planar rotator, the U(1) charge depends on the signature of the link, and the possible (gauge-invariant) interactions appear with very specific amplitudes.

As in the usual U(1) lattice gauge theory [7], we can solve the additive constraints at the vertices by using closed loops. Consider a closed loop of the Kagomé lattice which everywhere either goes straight or takes only 60° turns (Fig. 1). $N=1$ for the links of this loop is consistent with the constraints. We shall call such a loop an "allowed" loop. As N_l takes only non-negative values, the loops are not assigned any orientation. An arbitrary collection of such allowed loops generate all the allowed configurations (with suitable boundary conditions). The total number of transits along a link gives N for the link. In case loops intersect or (partially or completely) overlap, different ways of forming closed loops are not distinguished. All this is in contrast with the usual U(1) lattice gauge theory. This way we get a complete basis for physical states in the loop space without the redundancies and the constraints that the set of all Wilson loops have [4].

An allowed loop increases j on the links of the dual lattice it intersects by $\frac{1}{2}$. Therefore it increases the total color flux on the links of a closed loop of the original lattice (which closely follows the allowed loop except perhaps at the "corners") by $\frac{1}{2}$. In contrast, the usual Wilson loop operator has a complicated action, changing the total color flux on its links by both $+\frac{1}{2}$ and $-\frac{1}{2}$. In this sense, our allowed loops are more basic entities.

We now give a cursory discussion of the implications. A careful analysis will be developed elsewhere. In spite of the differences with the usual U(1) lattice gauge theory, most of the concepts and techniques used there [8] can be carried over. The commuting operators $u=aN^{-1/2}$ and $u^*=N^{-1/2}a^*$ are the analogues of the usual U(1) link variable $U=\exp(-i\theta)$ and the complex conjugate $U^*=\exp(i\theta)$, respectively. Also the number operator N is the analogue of the electric field E. In the phase where the compactness of the link variable, or equivalently the discreteness of the conjugate variable, is irrelevant, there are massless vector bosons. This can be seen by ignoring the fact that the spectrum of θ is compact and of N is discrete, and considering quadratic fluctuations about their expectation values. The calculation is now involved because, in place of the $UUUU$ term of the Hamiltonian, we now have a set of terms involving products of up to twelve u variables with coefficients depending on the conjugate variable N. Even though there is just one kind of link variable u, there can be three massless excitations of the SU(2) gauge theory because of the Kagomé lattice involved and the specific interactions. The effects of compactness of θ can be interpreted in terms of the monopole degrees of freedom. Confinement would be a consequence of a condensate of these monopoles.

In the usual U(1) lattice gauge theory, the Gauss law can be solved [9] using potentials for the electric field in contrast with the usual potentials for the magnetic field. The Hamiltonian expressed in these potentials is local and gives the dual formulation of the theory. These steps can be repeated for our case. Even though the flux lines do not have an orientation and are on the Kagomé lattice, it is possible to solve the Gauss law using certain potentials. This will be demonstrated elsewhere.

The techniques and ideas introduced in this paper can be extended to higher dimensions [10] and to other groups [11].

We conclude by summarizing the highlights of this paper. We have mapped non-Abelian lattice gauge theory exactly into an Abelian gauge theory on a Kagomé lattice. As the physics of Abelian lattice gauge theories is well understood [8], we now have a powerful tool for analyzing non-Abelian gauge theories. The conjecture of 't Hooft [3] that the topological excitations of the Abelian subgroup of SU(N) determine the confinement mechanism of SU(N) gauge theory is thereby placed on a firm footing. We also have a description of non-Abelian gauge theories in terms of color-invariant local degrees of freedom. All of this means that now there is the prospect of unambiguously locating the physics underlying hadron dynamics by both numerical and analytical techniques.

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