PHYSICAL REVIEW D VOLUME 45, NUMBER 3 1 FEBRUARY 1992

$NN \rightarrow \Delta N$ transition: Bohr's rules and optimal formalism

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(Received 8 July 1991)

The polarization structure of the $NN \rightarrow \Delta N$ transition is presented in a very compact way using two kinds of formalisms. First, the spin-space decomposition adapted for describing Bohr's rules is recalled. Second, the optimal formalism is applied to the transition. It is adapted for taking into account Bohr's relations and explicitly developed in helicity and transversity frames. Tables of transformation are given which simply relate observables defined in different formalisms.

PACS number(s): 13.75.Cs, 13.88.+e

I. INTRODUCTION

In view of the importance of the isobaric resonance in intermediate energy physics [1–5], a precise knowledge of the $NN \rightarrow \Delta N$ transition is more and more necessary. Experimentally, information can be extracted from the $NN \rightarrow NN\pi$ reactions [6]. Furthermore, data from Argonne [7] provide us with the first set of spin observables of the Δ production on a wide energy range. It is useful and opportune to discuss the spin structure of this transition in order to interpret the data already obtained and to advisedly plan future experiments in a framework appropriate to interface experimental and theoretical programs aiming at the exploration of this transition.

The description of the transition matrix and observables can be given in many ways. Any formalism presents observables in terms of bilinear combinations of amplitudes ("bicoms"), on the one hand, and yields linear and nonlinear relationships between observables, on the other hand. However, in general the matrix connecting observables and bicoms is far from diagonal and, hence, a given measurement depends on many bicoms and vice versa.

In this paper, we investigate two kinds of formalism. The first one, which we proposed [3,8] a few years ago, uses a spin-space decomposition of the transition matrix analogous to the Wolfenstein representation in N-N elastic scattering. The 16 complex spin amplitudes $f_i(\theta_\Delta)$ and $g_i(\theta_\Delta)$ are somewhat similar to the spin-nonflip and spin-flip amplitudes of pion-nucleon scattering. According to the polarization states of the four particles involved in the reaction, the spin observable definition is extended for the Δ production following Bystricky, Lehar, and Winternitz [9] for N-N elastic scattering. The Δ spin-space operators are constructed as an orthonormal basis adapted for emphasizing Bohr's rules. This formalism is convenient for studying nuclear reactions at intermediate energy physics. Use is made of this spin-space decomposition for tackling problems such as nucleon-nucleus scattering [5] and nuclear Δ production [4], by eikonal models. The iterated pion-exchange model of Kloet and Silbar [1] gives theoretical prediction for spin amplitudes, which are tested [10] with Argonne experimental data. The second formalism has been developed by Golstein and Moravcsik [11–16] during the past 15 years. It optimally diagonalizes the matrix connecting observables and bicoms and consequently is well adapted to the phenomenological determination of amplitudes. Each one of these formalisms allows one to express all observables by means of compact formulas. Note that the optimal formalism is in reality a multiple set of formalisms. It has been shown [13] that parity conservation reduces the set of optimal formalisms to those in which the orientation of the quantization direction of each particle is either in the reaction plane or perpendicular to it. Among all the possibilities for quantization directions, helicity and transversity frames play an important role and are particularly studied in the framework of the optimal formalism.

Without constraint, the number of amplitudes needed to describe the studied reaction is 32 and the number of bicoms is $32^2 = 1024$, which is also the number of observables. When parity conservation is imposed, the number of independent complex amplitudes reduces to 16 and consequently only 31 observables are needed to determine magnitudes and relative phases of these amplitudes. Special emphasis is put on the search for relationships between observables. Particularly, derivation is made of linear relations due to invariance under reflection with respect to the scattering plane, equivalent to the so-called "Bohr's rule" in nucleon-nucleon elastic scattering [9].

The paper is organized as follows. Section II is devoted to a brief summary of the spin-space decomposition developed in Refs. [3,8], specifying the properties of the Δ spin-space operators with respect to the eight Bohr's rules of the Δ-production reaction. Section III presents the optimal formalism of Refs. [11-16] and its applications to the Δ production in helicity and transversity frames, as well as in the fixed basis used in Sec. II. Section IV gives transformations which relate the various amplitudes of the transition matrix and explicits transformations between observables.

II. SPIN-SPACE DECOMPOSITION AND BOHR'S RULES

A convenient spin-space decomposition of the reaction matrix of $N_1N_2 \to \Delta_1 N_2'$ is given by [3]

$$M = \sum_{i=1}^{8}[f_i(\theta_\Delta)Q_i + g_i(\theta_\Delta)Q_i(\boldsymbol{\sigma}_2\cdot\hat{\boldsymbol{n}})], \tag{2.1}$$

where $\boldsymbol{\sigma}_2$ stands for the usual Pauli operator acting on nucleon 2, assuming nucleon 1 to undergo the transition and become the Δ. The unit vectors $(\hat{\boldsymbol{l}},\hat{\boldsymbol{m}},\hat{\boldsymbol{n}})$ of the right-handed orthonormal basis used as the reference frame, and common to the four particles involved in the reaction, are defined by

$$\hat{\boldsymbol{l}} = \hat{\boldsymbol{k}}, \quad \hat{\boldsymbol{n}} = \frac{\boldsymbol{k}\times\boldsymbol{k}_\Delta}{|\boldsymbol{k}\times\boldsymbol{k}_\Delta|}, \quad \hat{\boldsymbol{m}} = \hat{\boldsymbol{n}}\times\hat{\boldsymbol{l}}, \tag{2.2}$$

where $\boldsymbol{k}$ and $\boldsymbol{k}_\Delta$ are the initial-beam-nucleon and final-Δ center-of-mass three-momenta, respectively. For purposes of implementing parity conservation, note that $\hat{\boldsymbol{n}}$ is a pseudovector, while $\hat{\boldsymbol{l}}$ and $\hat{\boldsymbol{m}}$ are true polar vectors. All the dynamic is contained in the 16 complex spin amplitudes $f_i(\theta_\Delta)$ and $g_i(\theta_\Delta)$, analogous to the spin-nonflip and spin-flip amplitudes of pion-nucleon scattering. The eight Q_i in Eq. (2.1) are spin-space operators which transform as true scalar because of parity conservation. We recall them for sake of completeness

$$\begin{aligned}
&Q_1 = (\boldsymbol{S}\cdot\hat{\boldsymbol{l}})(\boldsymbol{\sigma}_2\cdot\hat{\boldsymbol{l}}), \quad Q_2 = \frac{2i}{\sqrt{3}}(\hat{\boldsymbol{m}}\cdot\overleftrightarrow{\mathrm{T}}\cdot\hat{\boldsymbol{n}})(\boldsymbol{\sigma}_2\cdot\hat{\boldsymbol{l}}),\\
&Q_3 = (\boldsymbol{S}\cdot\hat{\boldsymbol{m}})(\boldsymbol{\sigma}_2\cdot\hat{\boldsymbol{m}}), \quad Q_4 = -\frac{2i}{\sqrt{3}}(\hat{\boldsymbol{l}}\cdot\overleftrightarrow{\mathrm{T}}\cdot\hat{\boldsymbol{n}})(\boldsymbol{\sigma}_2\cdot\hat{\boldsymbol{m}}),\\
&Q_5 = (\boldsymbol{S}\cdot\hat{\boldsymbol{n}})(\boldsymbol{\sigma}_2\cdot\hat{\boldsymbol{n}}), \quad Q_6 = \frac{2i}{\sqrt{3}}(\hat{\boldsymbol{l}}\cdot\overleftrightarrow{\mathrm{T}}\cdot\hat{\boldsymbol{m}})(\boldsymbol{\sigma}_2\cdot\hat{\boldsymbol{n}}),\\
&Q_7 = (\hat{\boldsymbol{l}}\cdot\overleftrightarrow{\mathrm{T}}\cdot\hat{\boldsymbol{l}}), \quad Q_8 = \frac{1}{\sqrt{3}}[(\hat{\boldsymbol{m}}\cdot\overleftrightarrow{\mathrm{T}}\cdot\hat{\boldsymbol{m}}) - (\hat{\boldsymbol{n}}\cdot\overleftrightarrow{\mathrm{T}}\cdot\hat{\boldsymbol{n}})].
\end{aligned} \tag{2.3}$$

The $\boldsymbol{S}$ and $\overleftrightarrow{\mathrm{T}}$ quantities in Eq. (2.3) are the rank 1 and 2 irreducible tensorial operators which link the nucleon 1 spin space to the Δ spin space, respectively. We recall that using dyadic notation

$$(\hat{\boldsymbol{a}}\cdot\overleftrightarrow{\mathrm{T}}\cdot\hat{\boldsymbol{b}}) = \tfrac{1}{2}[(\boldsymbol{S}\cdot\hat{\boldsymbol{a}})(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{b}}) + (\boldsymbol{S}\cdot\hat{\boldsymbol{b}})(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{a}})]. \tag{2.4}$$

Two years ago, we developed [8] a systematic formalism for the spin observables of the $NN \to \Delta N$ transition, according to the polarization states of the four baryons involved in the reaction. The notation introduced by Bystricky, Lehar, and Winternitz [9] in nucleon-nucleon scattering is extended to the Δ transition. The spin observables are denoted $X_{(I,p)\alpha_2'\alpha_1\alpha_2}$ where the (I,p) symbol refers to the Δ polarization states and α indices to nucleon polarization states, α_1 standing for the beam, α_2 for the target and α_2' for the recoil nucleon. The index α is equal to 0 in the case of an unpolarized initial particle or if the polarization of final nucleon is not detected, and is equal to $\hat{\boldsymbol{l}},\hat{\boldsymbol{m}},\hat{\boldsymbol{n}}$ according to polarization along one of these directions. The four corresponding $\mathcal{P}(\alpha)$ nucleon spin operators are $1,(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{l}}),(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{m}})$, and $(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}})$, respectively. The 16 Δ spin states are referred by symbol (I,p) where $I = 1,2,3,4$ and $p = 0,\hat{\boldsymbol{l}},\hat{\boldsymbol{m}},\hat{\boldsymbol{n}}$.

In terms of the M transition amplitude of Eq.(2.1), the spin observables take the form

$$\sigma X_{(I,p)\alpha_2'\alpha_1\alpha_2} = \tfrac{1}{4}\mathrm{Tr}[M^\dagger\Omega_I(p)\mathcal{P}_2(\alpha_2')M\mathcal{P}_1(\alpha_1)\mathcal{P}_2(\alpha_2)], \tag{2.5}$$

where σ is the differential cross section for unpolarized particles up to phase-space factors, so that $X_{(1,0)000} = 1$.

The 16 orthonormalized Δ spin-space operators $\Omega_I(p)$ where $p = 0,\hat{\boldsymbol{l}},\hat{\boldsymbol{m}},\hat{\boldsymbol{n}}$ and $I = 1, 2, 3, 4$ satisfy

$$[\Omega_I(p)]^2 = 1 \tag{2.6}$$

and

$$\tfrac{1}{4}\mathrm{Tr}[\Omega_I^\dagger(p)\Omega_{I'}(p')] = \delta_{II'}\delta_{pp'}. \tag{2.7}$$

They are constructed in a way to simplify the writing of relations obtained by invariance under reflection in the scattering plane. These eight relations involving operators are expressed in terms of $\Omega_I(p)$ as

$$\begin{aligned}
&\Omega_I(0)1_2M1_11_2 = \Omega_I(\hat{\boldsymbol{n}})(\boldsymbol{\sigma}_2\cdot\hat{\boldsymbol{n}})M(\boldsymbol{\sigma}_1\cdot\hat{\boldsymbol{n}})(\boldsymbol{\sigma}_2\cdot\hat{\boldsymbol{n}}),\\
&\Omega_I(\hat{\boldsymbol{l}})(\boldsymbol{\sigma}_2\cdot\hat{\boldsymbol{l}})M(\boldsymbol{\sigma}_1\cdot\hat{\boldsymbol{l}})(\boldsymbol{\sigma}_2\cdot\hat{\boldsymbol{l}})\\
&\qquad = \Omega_I(\hat{\boldsymbol{m}})(\boldsymbol{\sigma}_2\cdot\hat{\boldsymbol{m}})M(\boldsymbol{\sigma}_1\cdot\hat{\boldsymbol{m}})(\boldsymbol{\sigma}_2\cdot\hat{\boldsymbol{m}}),
\end{aligned} \tag{2.8}$$

for $I = 1,2,3,4$ and are called Bohr's rules in reference to the so-called Bohr's rule in nucleon-nucleon elastic scattering [9]. Equations (2.8) are verified for

$$\begin{aligned}
&\Omega_I(\hat{\boldsymbol{n}}) = \Omega_I(0)\Sigma(\hat{\boldsymbol{n}}),\\
&\Omega_I(\hat{\boldsymbol{l}}) = \Omega_I(0)\Sigma(\hat{\boldsymbol{l}}),\\
&\Omega_I(\hat{\boldsymbol{m}}) = \Omega_I(0)\Sigma(\hat{\boldsymbol{m}}),
\end{aligned} \tag{2.9}$$

where $\Sigma(\hat{\boldsymbol{a}})$ is defined by

$$\Sigma(\hat{\boldsymbol{a}}) = \frac{1}{6}(T_3\hat{\boldsymbol{a}}\hat{\boldsymbol{a}}\hat{\boldsymbol{a}})_0 - \frac{1}{5}(\boldsymbol{\sigma}_\Delta\cdot\hat{\boldsymbol{a}}), \tag{2.10}$$

in terms of rank-3 tensor $(T_3\hat{\boldsymbol{a}}\hat{\boldsymbol{a}}\hat{\boldsymbol{a}})_0$ (Eqs. (2.11b) and (2.13) of Ref. [8]) and rank-1 generalized Pauli spin operator $\boldsymbol{\sigma}_\Delta$. This magic $\Sigma(\hat{\boldsymbol{a}})$ spin-space operator satisfies the relation

$$\Sigma(\hat{\boldsymbol{a}})\Sigma(\hat{\boldsymbol{b}}) = (\hat{\boldsymbol{a}}\cdot\hat{\boldsymbol{b}}) + i\Sigma(\hat{\boldsymbol{a}}\times\hat{\boldsymbol{b}}), \tag{2.11}$$

analogous to the well-known Pauli operators relation

$$(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{a}})(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{b}}) = (\hat{\boldsymbol{a}}\cdot\hat{\boldsymbol{b}}) + i(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{a}}\times\hat{\boldsymbol{b}}). \tag{2.12}$$

By virtue of Eq.(2.9), it is sufficient to specify the four $\Omega_I(0)$ to reach a complete knowledge of the orthonormal basis of Δ spin-space operators. We take

$$\begin{aligned}
&\Omega_1(0) = 1, \quad \Omega_2(0) = \frac{1}{4}(\hat{\boldsymbol{n}}\cdot\overleftrightarrow{\mathrm{T}}_\Delta\cdot\hat{\boldsymbol{n}}),\\
&\Omega_3(0) = \frac{1}{4\sqrt{3}}[(\hat{\boldsymbol{l}}\cdot\overleftrightarrow{\mathrm{T}}_\Delta\cdot\hat{\boldsymbol{l}}) - (\hat{\boldsymbol{m}}\cdot\overleftrightarrow{\mathrm{T}}_\Delta\cdot\hat{\boldsymbol{m}})],\\
&\Omega_4(0) = \frac{1}{2\sqrt{3}}(T_3\hat{\boldsymbol{l}}\hat{\boldsymbol{m}}\hat{\boldsymbol{n}})_0,
\end{aligned} \tag{2.13}$$

where $(\hat{\boldsymbol{a}}\cdot\overleftrightarrow{\mathrm{T}}_\Delta\cdot\hat{\boldsymbol{b}})$ is the dyadic notation for rank 2 tensors defined in Eqs. (2.11a) and (2.12) of Ref. [8]. Note that, under space reflection, $\Omega_I(p)$ operators have specified parity characters, $\Omega_I(p)$ being scalar for $p = 0, \hat{n}$ and pseudoscalar for $p = \hat{l}, \hat{m}$, for all I. This is analogous to nucleon case, $\mathcal{P}(\alpha)$ being scalar for $\alpha = 0, \hat{n}$ and pseudoscalar for $\alpha = \hat{l}, \hat{m}$.

The present formalism allows one to express all spin observables, denoted hereafter by IFE, which stands for "initial-frame experiments," by means of a compact formula

$$\sigma X_{(I,p)\alpha_2'\alpha_1\alpha_2} = \sum_{i,j=1}^{8} [\mathrm{Re}C_0\mathrm{Re}(f_i^* f_j + g_i^* g_j Z_{\alpha_2}) + \mathrm{Im}C_0\mathrm{Im}(f_i^* f_j + g_i^* g_j Z_{\alpha_2})][K_I s(p)v(p)v(\alpha_1)s(\alpha_2)]_{ij}$$
$$+[2\mathrm{Re}C_1\mathrm{Re}(f_i^* g_j) + 2\,\mathrm{Im}C_1\mathrm{Im}(f_i^* g_j)][K_I s(p)v(p)v(\alpha_1)s(\alpha_2)s(\hat{n})]_{ij}, \qquad (2.14)$$

for $I = 1,2,3$ and, for $I = 4$,

$$\sigma X_{(4,p)\alpha_2'\alpha_1\alpha_2} = \sum_{i,j=1}^{8} [\mathrm{Re}C_0\mathrm{Im}(f_i^* f_j + g_i^* g_j Z_{\alpha_2}) - \mathrm{Im}C_0\mathrm{Re}(f_i^* f_j + g_i^* g_j Z_{\alpha_2})][K_4 s(p)v(p)v(\alpha_1)s(\alpha_2)]_{ij}$$
$$+[2\,\mathrm{Re}C_1\,\mathrm{Im}(f_i^* g_j) - 2\,\mathrm{Im}C_1\mathrm{Re}(f_i^* g_j)][K_4 s(p)v(p)v(\alpha_1)s(\alpha_2)s(\hat{n})]_{ij}. \qquad (2.15)$$

All the quantities needed to make these equations explicit are defined in Ref. [8]: the C_0 and C_1 coefficients are displayed in Table II as functions of p, α_2', α_1, and α_2; Z_{α_2} is given by Eq. (2.16) and the K, s and v matrices are displayed in Appendix C of this reference. Products of s and v matrices can be simplified by taking into account their properties given by Eqs. (2.18) and (2.20) of the same reference.

Each Δ spin operator $\Omega_I(p)$ carries 64 observables, each of the three nucleons being unpolarized or polarized along $\hat{l}, \hat{m}$, or $\hat{n}$. Then, with 16 $\Omega_I(p)$, 1024 IFEs can be defined. Accounting the parity invariance, this number is reduced by a factor 2, the nonvanishing experiments corresponding to an even number of $\hat{l}$ and $\hat{m}$ indices among the four subscripts p, α_2', α_1, and α_2. Taking into account Bohr's rules divides also the number of independent IFE by a factor 2, so we are left with 256 of them. According to Eq. (2.8), the Bohr's relations hold only between operators with $p = 0$ and $\hat{n}$, on the one hand, and $p = \hat{l}$ and $\hat{m}$, on the other hand. They are translated into observables by

$$X_{(I,p)bcd} = \pm X_{(I,p')b'c'd'}. \qquad (2.16)$$

The indices p', b', c', d' are related to indices p, b, c, d by transformations $0 \leftrightarrow \hat{n}$ and $\hat{l} \leftrightarrow \hat{m}$, respectively. The $\pm$ sign appearing in Eq.(2.16) is due to the use of Eq.(2.12) in Bohr's rules of Eqs.(2.8).

Finally, with parity conservation and the so-called Bohr's rules, the study of the spin observables of the $NN \to \Delta N$ transition may be reduced to 256 IFEs where $I = 1,2,3,4$ and $p = 0, \hat{l}$, for example, once the polarization states of Δ are expanded on the $\Omega_I(p)$ orthonormal basis.

We recall that taking account of the Pauli principle reduces the number of IFEs to 160, by relations between observables at θ_Δ and $(\pi - \theta_\Delta)$. It is still more than enough to determine the 16 complex spin amplitudes f_i and g_i, the role of quadratic and more generally nonlinear relations being to reduce further the number of independent IFEs.

III. OPTIMAL FORMALISM

Following Goldstein and Moravcsik *et al.* [11–16], let us denote the reaction matrix of $N_1N_2 \to \Delta_1 N_2'$ by

$$M = \sum_{\lambda l \Lambda L} D(\lambda, l; \Lambda, L)\delta^{\lambda l} \otimes \delta^{\Lambda L}, \qquad (3.1)$$

where the $D(\lambda, l; \Lambda, L)$'s are the amplitudes and the $\delta^{\lambda l}$ and $\delta^{\Lambda L}$ the spin-momentum tensors referring to particles Δ_1 and N_1, N_2', and N_2, respectively. The indices λ, l, Λ, and L are the magnetic projections along the $\hat{z}$ quantization axis of each particle, Δ_1, N_1, N_2', and N_2, respectively. The reaction is completely described by a set of 32 amplitudes $D(\lambda, l; \Lambda, L)$, each of which is a function of energy and scattering angle.

The spin observables are defined by

$$\mathcal{L}(uvH_p, UVH_P; \xi\omega H_q, \Xi\Omega H_Q) = \mathrm{Tr}(Q^{\xi\omega H_q}\rho^{\Xi\Omega H_Q} M \rho^{uvH_p}\rho^{UVH_P} M^\dagger), \qquad (3.2)$$

where u and v characterize the spin-space matrix of particle N_1, U, and V that of N_2, ξ, and ω being for Δ_1, Ξ, and Ω for N_2'. Each of the two indices, for a given particle of spin S, takes $(2S+1)$ values from 1 to $(2S+1)$, which are related to magnetic projections along the quantization axis. The H's can be either "real"(R) or "imaginary"(I), for off-diagonal elements of the density matrix. For diagonal elements, H is only "real" and the label (R) may be omitted for the sake of simplicity. The indices p, P, q, Q are equal to $+1$ or -1, H_1 standing for R and H_{-1} for I. The ρ and Q operators, describing initial polarizations and measured final polarizations, denote all the spin-space operators required to generate spin observables of the reaction. As usual, putting Eq. (3.1) into Eq. (3.2), it is easy to see that spin observables are given by bilinear combinations of amplitudes, called "bicoms."

In the optimal formalism, the choice of observables and amplitudes provides observable-bicom relations as simple as possible. For this, the δ's in Eq. (3.1) are chosen to have only one nonzero element and the ρ and Q operators to be "minimally Hermitian," so that the corresponding matrices have a minimal number of nonzero elements compatible with the Hermiticity requirement. For each particle, ρ and Q associated matrix elements are defined by

$$(\rho^{uvH_p})_{l'l} = \frac{1}{2}[(1+p)+i(1-p)](\delta_{\{u\}l}\delta_{\{v\}l'}+p\delta_{\{u\}l'}\delta_{\{v\}l}), \tag{3.3}$$

where, for a spin S particle, l and l' correspond to $+S,\ldots,-S$ magnetic components along its quantization axis. The symbol $\{u\}$ is related to u index by

$$\{u\} = S(2u-1) \mod (2S+1). \tag{3.4}$$

All spin-space matrices corresponding to the $N_1N_2 \to \Delta_1 N_2'$ transition are given in the Appendix. Note that this appendix contains also relationships between ρ and Q operators and Cartesian tensors, defined in the $\hat{x}, \hat{y}, \hat{z}$ basis and written with dyadic notation, analogous to the $\Omega_I(p)$ operators presented in Sec. II.

Then, spin observables of Eq. (3.2) may be written

$$\begin{aligned}\mathcal{L}(uvH_p, UVH_P; \xi\omega H_q, \Xi\Omega H_Q) = \pm 2H_{pqPQ}[\, & D(\{\xi\},\{v\};\{\Xi\},\{V\})D^*(\{\omega\},\{u\};\{\Omega\},\{U\}) \\ & +pD(\{\xi\},\{u\};\{\Xi\},\{V\})D^*(\{\omega\},\{v\};\{\Omega\},\{U\}) \\ & +qD(\{\omega\},\{v\};\{\Xi\},\{V\})D^*(\{\xi\},\{u\};\{\Omega\},\{U\}) \\ & +pqD(\{\omega\},\{u\};\{\Xi\},\{V\})D^*(\{\xi\},\{v\};\{\Omega\},\{U\}) \\ & +PD(\{\xi\},\{v\};\{\Xi\},\{U\})D^*(\{\omega\},\{u\};\{\Omega\},\{V\}) \\ & +pPD(\{\xi\},\{u\};\{\Xi\},\{U\})D^*(\{\omega\},\{v\};\{\Omega\},\{V\}) \\ & +qPD(\{\omega\},\{v\};\{\Xi\},\{U\})D^*(\{\xi\},\{u\};\{\Omega\},\{V\}) \\ & +pqPD(\{\omega\},\{u\};\{\Xi\},\{U\})D^*(\{\xi\},\{v\};\{\Omega\},\{V\})],\end{aligned} \tag{3.5}$$

where the plus sign is for $\sum I = 0, 3, 4$ and the minus sign for $\sum I = 1, 2$, $\sum I$ being number of I indices among H_p, H_P, H_q, H_Q or also number of -1 among p, q, P, Q. The symbol H_{pqPQ} is equal to "real" or "imaginary," if the product $pqPQ$ is $+1$ or -1, respectively.

We recall that for spin-$\frac{1}{2}$ particles, we have two diagonal states 11 and 22 and two off-diagonal ones $12R$ and $12I$. For spin-$\frac{3}{2}$ particles, the possibilities are somewhat numerous. There are four diagonal states 11, 22, 33, and 44 and twelve off-diagonal ones $12R$, $13R$, $14R$, $23R$, $24R$, and $34R$ and six analogous ones with I replacing R.

The 32 observables referring to diagonal arguments only are simply related to one bicom and more particularly to the magnitude of one amplitude by

$$\mathcal{L}(uu, UU; \xi\xi, \Xi\Xi) = 16|D(\{\xi\},\{u\};\{\Xi\},\{U\})|^2. \tag{3.6}$$

The observables referring to three, two, one, or zero diagonal arguments are related to one, two, four, or eight bicoms, respectively (the number of these various observables being 192, 384, 320, and 96, respectively). For instance, we have

$$\mathcal{L}(uvH_p, UU; \xi\xi, \Xi\Xi) = 16H_p[D(\{\xi\},\{u\};\{\Xi\},\{U\})D^*(\{\xi\},\{v\};\{\Xi\},\{U\})], \tag{3.7}$$

or

$$\begin{aligned}\mathcal{L}(uvH_p, UVH_P; \xi\xi, \Xi\Xi) = \pm 8H_{pP}[\, & D(\{\xi\},\{u\};\{\Xi\},\{U\})D^*(\{\xi\},\{v\};\{\Xi\},\{V\}) \\ & +pD(\{\xi\},\{u\};\{\Xi\},\{V\})D^*(\{\xi\},\{v\};\{\Xi\},\{U\})],\end{aligned} \tag{3.8}$$

or

$$\begin{aligned}\mathcal{L}(uvH_p, UVH_P; \xi\omega H_q, \Xi\Xi) = \pm 4H_{pqP}[\, & D(\{\xi\},\{v\};\{\Xi\},\{V\})D^*(\{\omega\},\{u\};\{\Xi\},\{U\}) \\ & +pD(\{\xi\},\{u\};\{\Xi\},\{V\})D^*(\{\omega\},\{v\};\{\Xi\},\{U\}) \\ & +qD(\{\omega\},\{v\};\{\Xi\},\{V\})D^*(\{\xi\},\{u\};\{\Xi\},\{U\}) \\ & +pqD(\{\omega\},\{u\};\{\Xi\},\{V\})D^*(\{\xi\},\{v\};\{\Xi\},\{U\})].\end{aligned} \tag{3.9}$$

Note that we have not said anything so far about the quantization direction of each particle. Indeed, these directions are arbitrary. Parity invariance reduces the number of independent amplitudes by a factor 2 and limits the choices for quantization direction of each particle to be along the normal of the scattering plane or in any direction in the scattering plane [13]. If the quantization direction of each particle is its own momentum (see Fig. 1), we obtain the helicity formalism developed by Jacob and Wick [17]. For invariance under parity, helicity amplitudes satisfy

$$D^h(\lambda, l; \Lambda, L) = (-)^{\lambda+l+\Lambda+L+1} D^h(-\lambda, -l; -\Lambda, -L). \tag{3.10}$$

The transversity formalism presented by Kotanski [18] is

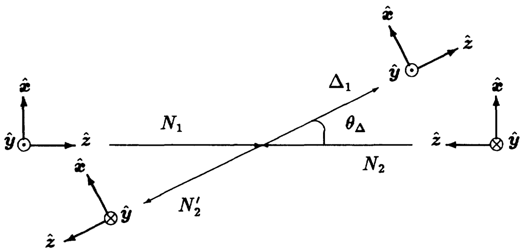

FIG. 1. Helicity frame.

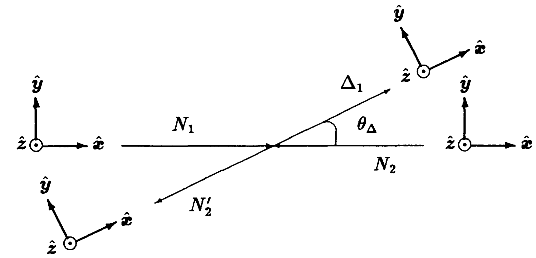

FIG. 2. Transversity frame.

obtained with directions for each particle as in Fig. 2, where the quantization direction for all particles is the normal to the reaction plane. Another particular transversity formalism is defined with the same fixed basis for all particles, more precisely with a common $(\hat{x}, \hat{y}, \hat{z})$ basis equal to the fixed basis $(\hat{l}, \hat{m}, \hat{n})$ of Eq. (2.2). Parity invariance gives for transversity amplitudes

$$D^t(\lambda, l; \Lambda, L) = (-)^{\lambda+l+\Lambda+L} D^t(\lambda, l; \Lambda, L), \qquad (3.11)$$

and set equal to zero half of these.

It is not the purpose of this paper to make relations between bicoms and helicity or transversity optimal observables explicit. But, it is interesting to show how these optimal formalisms become quite different from each other when parity conservation and Bohr's rules are taken into account.

In the transversity frame, for the purpose of implementing parity conservation, note that $\hat{z}$ is a pseudovector, while $\hat{x}$ and $\hat{y}$ are true polar vectors (see Fig. 2). Consequently, nucleon spin-space operators ρ and Δ operators Q have a determined parity character (see the Appendix). Under space reflection, their transformations may be written

$$\rho^{uvH_p} \to (-)^{u+v} \rho^{uvH_p},$$
$$Q^{\xi\omega H_q} \to (-)^{\xi+\omega} Q^{\xi\omega H_q}. \qquad (3.12)$$

Hence, with parity conservation, all transversity optimal observables are either even or odd, and the relation

$$\mathcal{L}^t(uvH_p, UVH_P; \xi\omega H_q, \Xi\Omega H_Q)$$
$$= (-)^W \mathcal{L}^t(uvH_p, UVH_P; \xi\omega H_q, \Xi\Omega H_Q), \qquad (3.13)$$

with $W = u + v + U + V + \xi + \omega + \Xi + \Omega$, eliminates half, or 512, of these observables. In addition, among the remaining 512 even observables, 32 vanish because of nullity of the operator product $Q^{\xi\omega H_q} \rho^{\Xi\Xi} M \rho^{uu} \rho^{UU}$, and we have

$$\mathcal{L}^t(uu, UU; \xi\omega H_q, \Xi\Xi) = 0, \quad \text{for } (u+U+\xi+\Xi) \text{ even}. \qquad (3.14)$$

In the helicity frame, $\hat{y}$ is a pseudovector, while $\hat{x}$ and $\hat{z}$ are polar vectors under space reflection (see Fig. 1). Then, most of spin-space operators have no determined parity character (see the Appendix). They transform themselves following

$$\rho^{uvH_p} \to p(-)^{u+v} \rho^{\widetilde{uv}H_p},$$
$$Q^{\xi\omega H_q} \to q(-)^{\xi+\omega} Q^{\widetilde{\xi\omega}H_q}, \qquad (3.15)$$

where the "mirror" arguments $\widetilde{uv}$ are related to uv arguments by $\widetilde{uv}$ = 22, 12 for uv = 11, 12 and similarly $\widetilde{\xi\omega}$ = 44, 33, 34, 24, 14, 23 for $\xi\omega$ = 11, 22, 12, 13, 14, 23, respectively. Then, helicity optimal observables are related by

$$\mathcal{L}^h(uvH_p, UVH_P; \xi\omega H_q, \Xi\Omega H_Q)$$
$$= pPqQ(-)^W \mathcal{L}^h(\widetilde{uv}H_p, \widetilde{UV}H_P; \widetilde{\xi\omega}H_q, \widetilde{\Xi\Omega}H_Q). \qquad (3.16)$$

From this equation, the equality between $\widetilde{uv}$ and uv for 12 and between $\widetilde{\xi\omega}$ and $\xi\omega$ for 14 and 23 yields 16 vanishing observables.

For technical reasons, instead of primary observables defined by Eq. (3.5), experimental programs of polarization measurements prefer a linear combination of them, which are called secondary observables [14]. In these observables, the polarization state of each particle is either averaged (i.e., the particle is unpolarized or its polarization is not measured) or satisfies the "null-sum" criterion, which requires that the sum of the coefficients of primary arguments vanishes. Secondary observables are labeled with secondary diagonal arguments A and Ψ for nucleons and with A, Ψ_1, Ψ_2, Ψ_3 for Δ, the off-diagonal arguments being unchanged.

The definition of secondary arguments in terms of primary ones is chosen as

$$A = \frac{1}{2}[(11) + (22)], \quad \Psi = \frac{1}{2}[(11) - (22)], \qquad (3.17)$$

for nucleon. For the Δ particle, we choose

$$A = \frac{1}{2}[(11) + (22) + (33) + (44)],$$
$$\Psi_1 = \frac{1}{2}[(11) - (22) - (33) + (44)],$$
$$\Psi_2 = \frac{1}{2}[-(11) + (22) - (33) + (44)],$$
$$\Psi_3 = \frac{1}{2}[-(11) - (22) + (33) + (44)]. \qquad (3.18)$$

The argument A, standing for averaged, is obtained by summing over all diagonal states of the particle and corresponds to unity operator. Consequently, the corresponding nucleon secondary spin-space operators are

$$\rho^A = 1, \quad \rho^\Psi = (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{z}}) \tag{3.19}$$

and, similarly for the Δ particle,

$$Q^A = 1, \quad Q^{\Psi_1} = \frac{1}{4}(\hat{\boldsymbol{z}}\cdot\overleftrightarrow{\mathrm{T}}_\Delta\cdot\hat{\boldsymbol{z}}),$$
$$Q^{\Psi_2} = \Sigma(\hat{\boldsymbol{z}}), \quad Q^{\Psi_3} = \frac{1}{4}(\hat{\boldsymbol{z}}\cdot\overleftrightarrow{\mathrm{T}}_\Delta\cdot\hat{\boldsymbol{z}})\Sigma(\hat{\boldsymbol{z}}). \tag{3.20}$$

For a spin-$\frac{3}{2}$ particle, the choice of the three Ψ_i arguments is not unique, since the "null-sum" criterion is not sufficient to fix coefficients. Here, we have an opportunity to deal with observables which conform more to laboratory realities or which simplify relationships between observables. Unfortunately, this choice of secondary observables, more adapted to experiments, increases the complexity of their relations with bicoms. The set chosen in Ref. [16] minimizes this complexity for some observables, but not for all. The choice advocated in Eqs. (3.18) and (3.20) is adapted for taking into account Bohr's rules presented in Sec. 2 and developed below .

For transversity optimal secondary observables, the constraints of parity conservation give

$$\mathcal{L}^t(\alpha,\beta;\gamma,\delta) = (-)^{W'}\mathcal{L}^t(\alpha,\beta;\gamma,\delta), \tag{3.21}$$

where $W' = [\alpha]+[\beta]+[\gamma]+[\delta]$ with

$$[\alpha] = \begin{cases} 0, & \text{if } \alpha = A,\Psi, \\ 1, & \text{if } \alpha = 12R, 12I, \end{cases} \tag{3.22}$$

and similarly for $[\beta]$ and $[\delta]$ nucleon arguments and for the Δ argument

$$[\gamma] = \begin{cases} 0, & \text{if } \gamma = A,\Psi_1,\Psi_2,\Psi_3, \\ \xi+\omega, & \text{if } \gamma = \xi\omega R, \xi\omega I. \end{cases} \tag{3.23}$$

Equation (3.21) eliminates half of secondary observables.

The eight Bohr's relations of Eq. (2.8) are written in the transversity frame in terms of secondary spin-space operators of Eqs. (3.20) by

$$Q^A\rho^A M\rho^A\rho^A = Q^{\Psi_2}\rho^\Psi M\rho^\Psi\rho^\Psi, \tag{3.24a}$$
$$Q^{\Psi_1}\rho^A M\rho^A\rho^A = Q^{\Psi_3}\rho^\Psi M\rho^\Psi\rho^\Psi, \tag{3.24b}$$
$$Q^{13R(I)}\rho^A M\rho^A\rho^A = -Q^{13R(I)}\rho^\Psi M\rho^\Psi\rho^\Psi, \tag{3.24c}$$
$$Q^{24R(I)}\rho^A M\rho^A\rho^A = Q^{24R(I)}\rho^\Psi M\rho^\Psi\rho^\Psi, \tag{3.24d}$$

and

$$Q^{12R}\rho^{12R}M\rho^{12R}\rho^{12R} = -Q^{12I}\rho^{12I}M\rho^{12I}\rho^{12I},$$
$$Q^{14R}\rho^{12R}M\rho^{12R}\rho^{12R} = -Q^{14I}\rho^{12I}M\rho^{12I}\rho^{12I},$$
$$Q^{23R}\rho^{12R}M\rho^{12R}\rho^{12R} = Q^{23I}\rho^{12I}M\rho^{12I}\rho^{12I},$$
$$Q^{34R}\rho^{12R}M\rho^{12R}\rho^{12R} = -Q^{34I}\rho^{12I}M\rho^{12I}\rho^{12I}. \tag{3.25}$$

The preceding equations involving operators, except Eqs. (3.24c) and (3.24d), connect different Δ polarization states and each equation yields 32 relationships between corresponding secondary observables. In contrast, Eqs. (3.24c) and (3.24d) relate the same Δ polarization state with different nucleon polarization states. Each one gives 16 relations between the 32 observables of a defined Δ polarization state: namely, $13R$, $13I$, $24R$, or $24I$. These relations are translated into secondary observables by

$$\mathcal{L}^t(\alpha,\beta;\gamma,\delta) = \pm\mathcal{L}^t(\alpha',\beta';\gamma',\delta'), \tag{3.26}$$

where α,β,δ and α',β',δ' nucleon secondary arguments are interchanged as

$$A \leftrightarrow \Psi, \quad 12R \leftrightarrow 12I, \tag{3.27}$$

and where γ and γ' secondary Δ arguments are interchanged as

$$\begin{aligned} &A \leftrightarrow \Psi_2, \quad \Psi_1 \leftrightarrow \Psi_3, \\ &13R(I) \leftrightarrow 13R(I), \quad 24R(I) \leftrightarrow 24R(I), \\ &12R \leftrightarrow 12I, \quad 14R \leftrightarrow 14I, \\ &23R \leftrightarrow 23I, \quad 34R \leftrightarrow 34I. \end{aligned} \tag{3.28}$$

The sign $\pm$ in Eq. (3.26) is due to relations between nucleon Pauli operators [see Eq. (2.12)]. Combining Eqs. (3.21) and (3.26), we are left with 256 linearly independent observables; 16 for each of the four Δ arguments $13R$, $13I$, $24R$, $24I$ and 32 for each of the six Δ arguments A, Ψ_1, $12R$, $14R$, $23R$, $34R$, for example. Then, 10 of the 16 polarized states of the Δ are involved in the determination of the 256 linearly independent observables.

In the helicity frame, when secondary observables are concerned, nucleon spin-space operators have a determined parity character, ρ^A being scalar and ρ^Ψ pseudoscalar. For Δ, Q^A and Q^{Ψ_1} are scalar whereas Q^{Ψ_2} and Q^{Ψ_3} are pseudoscalar (see the Appendix). Parity conservation for secondary helicity observables yields

$$\mathcal{L}^h(\alpha,\beta;\gamma,\delta) = (-)^{W'}\mathcal{L}^h(\alpha,\beta;\widetilde{\gamma},\delta), \tag{3.29}$$

where W' is always equal to $[\alpha]+[\beta]+[\gamma]+[\delta]$ but with, for the nucleon argument,

$$[\alpha] = \begin{cases} 0, & \text{if } \alpha = A, 12I, \\ 1, & \text{if } \alpha = \Psi, 12R, \end{cases} \tag{3.30}$$

and similarly for $[\beta]$ and $[\delta]$ nucleon arguments and, for the Δ argument,

$$[\gamma] = \begin{cases} 0, & \text{if } \gamma = A,\Psi_1, \\ 1, & \text{if } \gamma = \Psi_2,\Psi_3, \\ \xi+\omega, & \text{if } \gamma = \xi\omega R, \\ \xi+\omega+1, & \text{if } \gamma = \xi\omega I. \end{cases} \tag{3.31}$$

Secondary Δ arguments γ and $\widetilde{\gamma}$ are equal for A, Ψ_1, Ψ_2, Ψ_3, $14R$, $14I$, $23R$, $23I$ or are related as $\widetilde{\gamma} = 34R$, $34I$, $24R$, $24I$ for $\gamma = 12R$, $12I$, $13R$, $13I$, respectively. Equation (3.29) makes secondary observables vanish for which γ is equal to $\widetilde{\gamma}$ and W' is odd.

The eight Bohr's rules are written in the helicity frame as

$$Q^A\rho^A M\rho^A\rho^A = (Q^{14I}-Q^{23I})\rho^{12I}M\rho^{12I}\rho^{12I},$$
$$Q^{\Psi_1}\rho^A M\rho^A\rho^A = (Q^{14I}+Q^{23I})\rho^{12I}M\rho^{12I}\rho^{12I},$$

(3.32)

$$(Q^{13R}+Q^{24R})\rho^A M\rho^A\rho^A = (Q^{12I}+Q^{34I})\rho^{12I}M\rho^{12I}\rho^{12I},$$

$$(Q^{12R}-Q^{34R})\rho^A M\rho^A\rho^A = (-Q^{13I}+Q^{24I})\rho^{12I}M\rho^{12I}\rho^{12I},$$

and

$$Q^{\Psi_2}\rho^{\Psi}M\rho^{\Psi}\rho^{\Psi} = (-Q^{14R}-Q^{23R})\rho^{12R}M\rho^{12R}\rho^{12R},$$
$$Q^{\Psi_3}\rho^{\Psi}M\rho^{\Psi}\rho^{\Psi} = (-Q^{14R}+Q^{23R})\rho^{12R}M\rho^{12R}\rho^{12R}, \quad (3.33)$$

$$(Q^{13R}-Q^{24R})\rho^{\Psi}M\rho^{\Psi}\rho^{\Psi} = (Q^{12R}+Q^{34R})\rho^{12R}M\rho^{12R}\rho^{12R},$$

$$(Q^{12I}-Q^{34I})\rho^{\Psi}M\rho^{\Psi}\rho^{\Psi} = (-Q^{13I}-Q^{24I})\rho^{12R}M\rho^{12R}\rho^{12R}.$$

Note that if the nondiagonal Δ spin-space operators $Q^{12R(I)}$, $Q^{34R(I)}$, $Q^{13R(I)}$, and $Q^{24R(I)}$ have no determined parity character, it is not the case for combinations of them, which appear in Bohr's rules; some are scalar, some are pseudoscalar. More explicitly in Eqs. (3.32), the quantities $(Q^{12R}-Q^{34R})$, $(Q^{12I}+Q^{34I})$, $(Q^{13R}+Q^{24R})$, and $(-Q^{13I}+Q^{24I})$ are scalar; in Eqs. (3.33), $(Q^{12R}+Q^{34R})$, $(Q^{12I}-Q^{34I})$, $(Q^{13R}-Q^{24R})$, and $(-Q^{13I}-Q^{24I})$ are pseudoscalar. These Bohr's rules correspond to relationships between 3 or 4 helicity secondary observables. Therefore, by mixing Eqs. (3.32) and (3.33) with (3.29), relationships between 4 observables are simplified and one finally obtains

$$\mathcal{L}^h(\alpha,\beta;A,\delta) = \pm[\mathcal{L}^h(\alpha',\beta';14I,\delta') - \mathcal{L}^h(\alpha',\beta';23I,\delta')],$$

$$\mathcal{L}^h(\alpha,\beta;\Psi_1,\delta) = \pm[\mathcal{L}^h(\alpha',\beta';14I,\delta') + \mathcal{L}^h(\alpha',\beta';23I,\delta')],$$

$$\mathcal{L}^h(\alpha,\beta;\Psi_2,\delta) = \pm[\mathcal{L}^h(\alpha',\beta';14R,\delta') + \mathcal{L}^h(\alpha',\beta';23R,\delta')],$$

$$\mathcal{L}^h(\alpha,\beta;\Psi_3,\delta) = \pm[\mathcal{L}^h(\alpha',\beta';14R,\delta') - \mathcal{L}^h(\alpha',\beta';23R,\delta')], \quad (3.34)$$

and

$$\mathcal{L}^h(\alpha,\beta;12R,\delta) = \begin{cases} \pm\mathcal{L}^h(\alpha',\beta';13I,\delta'), \\ \text{if } [\alpha]+[\beta]+[\delta] \text{ even}, \\ \pm\mathcal{L}^h(\alpha',\beta';13R,\delta'), \\ \text{if } [\alpha]+[\beta]+[\delta] \text{ odd}, \end{cases} \quad (3.35)$$

and

$$\mathcal{L}^h(\alpha,\beta;12I,\delta) = \begin{cases} \pm\mathcal{L}^h(\alpha',\beta';13R,\delta'), \\ \text{if } [\alpha]+[\beta]+[\delta] \text{ even}, \\ \pm\mathcal{L}^h(\alpha',\beta';13I,\delta'), \\ \text{if } [\alpha]+[\beta]+[\delta] \text{ odd}, \end{cases} \quad (3.36)$$

where α,β,δ and α',β',δ' interchange as

$$A \leftrightarrow 12I, \quad \Psi \leftrightarrow 12R, \quad (3.37)$$

and where $[\alpha]$, $[\beta]$, $[\delta]$ are given by Eq. (3.30).

Then, we may choose 256 linearly independent observables, for example, among only 6 of the 16 initial Δ polarization states, namely, 32 observables for each of the four Δ argument A, Ψ_1, Ψ_2, Ψ_3, and 64 for $12R$ and for $12I$.

IV. AMPLITUDE AND OBSERVABLE TRANSFORMATIONS

The 16 complex spin amplitudes f_i and g_i defined in Eq. (2.1) correspond to the spin-space transition operators Q_i and $Q_i(\boldsymbol{\sigma}_2\cdot\hat{\boldsymbol{n}})$ for $i=1,\ldots,8$ and depend on the orthonormal $(\hat{\boldsymbol{l}}, \hat{\boldsymbol{m}}, \hat{\boldsymbol{n}})$ fixed basis which is the same for all particles involved in the reaction.

On the other hand, transversity amplitudes $D^t(\lambda,l;\Lambda,L)$ are defined with coordinates for each particle as shown in Fig. 1, where the $\hat{\boldsymbol{z}}$ quantization direction of each particle is the normal to the scattering plane, while $\hat{\boldsymbol{x}}$ is along the momentum for particles N_1 and Δ_1 but opposite to the momentum for N_2 and N_2'. Then, for initial particles, $(\hat{\boldsymbol{l}}, \hat{\boldsymbol{m}}, \hat{\boldsymbol{n}})$ is simply tranformed to $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$. In addition, for final particles, a θ_Δ angle rotation with respect to $\hat{\boldsymbol{z}}$ axis is needed.

Finally, transversity amplitudes defined by Kotanski [18] (see Fig. 2) are equal to zero for $(\lambda+l+\Lambda+L)$ odd [see Eq. (3.11)] and are given in terms of f_i and g_i, for $(\lambda+l+\Lambda+L)$ even, by

$$D^t(\lambda,l;\Lambda,L) = e^{i(\lambda+\Lambda)\theta_\Delta}\sum_i c_i\,[f_i + (-)^{1/2-L}g_i], \quad (4.1)$$

where i runs from 1 to 4 for $\Lambda=-L$ [$(\lambda+l)$ being even], and from 5 to 8 for $\Lambda=L$ [$(\lambda+l)$ being odd]. The c_i coefficients are given in Table I in terms of λ and Λ.

Particular transversity amplitudes defined with the same $(\hat{\boldsymbol{l}},\hat{\boldsymbol{m}},\hat{\boldsymbol{n}})$ fixed basis for all particles are denoted $D^f(\lambda,l;\Lambda,L)$ and given by Eq. (4.1) putting the overall

TABLE I. Coefficients c_i defined in Eq. (4.1).

	$\lvert\lambda\rvert=3/2$	$\lvert\lambda\rvert=1/2$
$c_1=c_7$	$\frac{\sqrt{3}}{2}(-)^{1/2-\lambda}$	$-\frac{1}{2}(-)^{1/2-\lambda}$
$c_2=-c_8$	$\frac{1}{2}(-)^{1/2-\lambda}$	$\frac{\sqrt{3}}{2}(-)^{1/2-\lambda}$
c_3	$\frac{\sqrt{3}}{2}(-)^{1/2-\Lambda}$	$\frac{1}{2}(-)^{1/2-\Lambda}$
c_4	$\frac{1}{2}(-)^{1/2-\Lambda}$	$-\frac{\sqrt{3}}{2}(-)^{1/2-\Lambda}$
c_5	0	$(-)^{1/2-\Lambda}$
c_6	$(-)^{1/2+\Lambda}$	0

$e^{i(\lambda+\Lambda)\theta_\Delta}$ phase factor equal to unity.

The helicity amplitudes are defined with coordinates for each particle as shown in Fig. 1, where the $\hat{z}$ quantization direction of each particle is along its own momentum while $\hat{y}$ is normal to the scattering plane. Note that standard phases for the helicity states put the $\hat{y}$ axis parallel to $\hat{n}$ for N_1 and Δ_1 but antiparallel for N_2 and N_2'. Then, the rotation operator transforming the transversity state into an helicity state is $R(0,\pi/2,\pi/2)$ for N_1 and Δ_1 particles and $R(0,-\pi/2,\pi/2)$ for N_2 and N_2'. Therefore, helicity amplitudes are related to transversity ones via

$$D^h(\lambda',l';\Lambda',L') = e^{i(\lambda'+\Lambda'-l'-L')\pi/2}\sum_{\lambda,l,\Lambda,L} D^t(\lambda,l;\Lambda,L)r^{3/2}_{\lambda\lambda'}(\pi/2)r^{1/2}_{ll'}(\pi/2)r^{1/2}_{\Lambda\Lambda'}(-\pi/2)r^{1/2}_{LL'}(-\pi/2). \tag{4.2}$$

Expliciting the $\frac{3}{2}$- and $\frac{1}{2}$-spin rotation matrices, one gets

$$\begin{aligned} D^h(\lambda',l';\Lambda',L') = {}& \frac{1}{8}e^{i(\lambda'+\Lambda'-l'-L')\pi/2}\sum_{\lambda,\Lambda,l,L} D^t(\lambda,l;\Lambda,L) \\ & \{(\delta_{|\lambda|,3/2}+\sqrt{3}\delta_{|\lambda|,1/2})[\delta_{\lambda',3/2}+(-)^{1/2-\lambda}\delta_{\lambda',-3/2}]+(\sqrt{3}\delta_{|\lambda|,3/2}-\delta_{|\lambda|,1/2})[\delta_{\lambda',-1/2}+(-)^{1/2-\lambda}\delta_{\lambda',1/2}]\} \\ & [\delta_{l',1/2}+(-)^{1/2+l}\delta_{l',-1/2}][(-)^{1/2-\Lambda}\delta_{\Lambda',1/2}+\delta_{\Lambda',-1/2}][(-)^{1/2-L}\delta_{L',1/2}+\delta_{L',-1/2}]. \end{aligned} \tag{4.3}$$

Then, helicity amplitudes are given in terms of f_i and g_i by

$$\begin{aligned} D^h(\tfrac{3}{2},l;\Lambda,\Lambda) = \frac{1}{4}e^{i(3/2-l)\pi/2}\Bigg[& \left(\cos\frac{3\theta_\Delta}{2}E_{3/2}(l,\Lambda)+i\sin\frac{3\theta_\Delta}{2}E_{3/2}(-l,-\Lambda)\right) \\ & +\sqrt{3}\left(\cos\frac{\theta_\Delta}{2}E_{1/2}(l,\Lambda)-i\sin\frac{\theta_\Delta}{2}E_{1/2}(-l,-\Lambda)\right)\Bigg], \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} D^h(-\tfrac{1}{2},l;\Lambda,\Lambda) = \frac{1}{4}e^{i(-1/2-l)\pi/2}\Bigg[& \sqrt{3}\left(\cos\frac{3\theta_\Delta}{2}E_{3/2}(l,\Lambda)+i\sin\frac{3\theta_\Delta}{2}E_{3/2}(-l,-\Lambda)\right) \\ & -\left(\cos\frac{\theta_\Delta}{2}E_{1/2}(l,\Lambda)-i\sin\frac{\theta_\Delta}{2}E_{1/2}(-l,-\Lambda)\right)\Bigg], \end{aligned} \tag{4.5}$$

where $E_{3/2}(l,\Lambda)$ and $E_{1/2}(l,\Lambda)$ are listed as functions of f_i and g_i amplitudes in Table II. Both of the last two equations define eight helicity amplitudes. The eight other independent amplitudes $D^h(\frac{3}{2},l;\Lambda,-\Lambda)$ and $D^h(-\frac{1}{2},l;\Lambda,-\Lambda)$ are obtained from Eqs. (4.4) and (4.5), respectively, by exchanging f_i and g_i in $E_{3/2}$ and $E_{1/2}$ and multiplying the right-hand side of the equations by the phase factor $e^{i\Lambda\pi}$. The 16 nonindependent helicity amplitudes are related by Eq. (3.10) to the preceding defined amplitudes. Note that the normalization of these various amplitudes yields

$$\sum_{\lambda,l,\Lambda,L}|D^t(\lambda,l;\Lambda,L)|^2 = \sum_{\lambda,l,\Lambda,L}|D^h(\lambda,l;\Lambda,L)|^2 = 4\sum_{i=1}^{8}(|f_i|^2+|g_i|^2). \tag{4.6}$$

To explicit optimal observables in terms of f_i and g_i amplitudes, Eq. (4.1) for transversity, Eqs. (4.4) and

TABLE II. Coefficients $E_{3/2}(l,\Lambda)$ and $E_{1/2}(l,\Lambda)$ defined in Eqs. (4.4) and (4.5).

	$E_{3/2}(l,\Lambda)$
$l=\frac{1}{2},\Lambda=\frac{1}{2}$	$\cos\frac{\theta_\Delta}{2}(\sqrt{3}g_3+g_4-2g_6)+i\sin\frac{\theta_\Delta}{2}(-\sqrt{3}f_3-f_4-2f_6)$
$l=\frac{1}{2},\Lambda=-\frac{1}{2}$	$\cos\frac{\theta_\Delta}{2}(-\sqrt{3}g_3-g_4-2g_6)+i\sin\frac{\theta_\Delta}{2}(\sqrt{3}f_3+f_4-2f_6)$
$l=-\frac{1}{2},\Lambda=\frac{1}{2}$	$\cos\frac{\theta_\Delta}{2}(-\sqrt{3}f_1-f_2-\sqrt{3}f_7+f_8)+i\sin\frac{\theta_\Delta}{2}(\sqrt{3}g_1+g_2-\sqrt{3}g_7+g_8)$
$l=-\frac{1}{2},\Lambda=-\frac{1}{2}$	$\cos\frac{\theta_\Delta}{2}(\sqrt{3}f_1+f_2-\sqrt{3}f_7+f_8)+i\sin\frac{\theta_\Delta}{2}(-\sqrt{3}g_1-g_2-\sqrt{3}g_7+g_8)$
	$E_{1/2}(l,\Lambda)$
$l=\frac{1}{2},\Lambda=\frac{1}{2}$	$\cos\frac{\theta_\Delta}{2}(g_3-\sqrt{3}g_4+2g_5)+i\sin\frac{\theta_\Delta}{2}(-f_3+\sqrt{3}f_4+2f_5)$
$l=\frac{1}{2},\Lambda=-\frac{1}{2}$	$\cos\frac{\theta_\Delta}{2}(-g_3+\sqrt{3}g_4+2g_5)+i\sin\frac{\theta_\Delta}{2}(f_3-\sqrt{3}f_4+2f_5)$
$l=-\frac{1}{2},\Lambda=\frac{1}{2}$	$\cos\frac{\theta_\Delta}{2}(f_1-\sqrt{3}f_2+f_7+\sqrt{3}f_8)+i\sin\frac{\theta_\Delta}{2}(-g_1+\sqrt{3}g_2+g_7+\sqrt{3}g_8)$
$l=-\frac{1}{2},\Lambda=-\frac{1}{2}$	$\cos\frac{\theta_\Delta}{2}(-f_1+\sqrt{3}f_2+f_7+\sqrt{3}f_8)+i\sin\frac{\theta_\Delta}{2}(g_1-\sqrt{3}g_2+g_7+\sqrt{3}g_8)$

TABLE III. Relations between frame unit vectors.

	Fixed frame	Helicity frame	Transversity frame
Particle N_1	$\hat{l}$	$\hat{z}$	$\hat{x}$
	$\hat{m}$	$\hat{x}$	$\hat{y}$
	$\hat{n}$	$\hat{y}$	$\hat{z}$
Particle N_2	$\hat{l}$	$-\hat{z}$	$\hat{x}$
	$\hat{m}$	$\hat{x}$	$\hat{y}$
	$\hat{n}$	$-\hat{y}$	$\hat{z}$
Particle Δ_1	$\hat{l}$	$\hat{z}\cos\theta_\Delta - \hat{x}\sin\theta_\Delta$	$\hat{x}\cos\theta_\Delta - \hat{y}\sin\theta_\Delta$
	$\hat{m}$	$\hat{x}\cos\theta_\Delta + \hat{z}\sin\theta_\Delta$	$\hat{y}\cos\theta_\Delta + \hat{x}\sin\theta_\Delta$
	$\hat{n}$	$\hat{y}$	$\hat{z}$
Particle N_2'	$\hat{l}$	$-\hat{z}\cos\theta_\Delta - \hat{x}\sin\theta_\Delta$	$\hat{x}\cos\theta_\Delta - \hat{y}\sin\theta_\Delta$
	$\hat{m}$	$\hat{x}\cos\theta_\Delta - \hat{z}\sin\theta_\Delta$	$\hat{y}\cos\theta_\Delta + \hat{x}\sin\theta_\Delta$
	$\hat{n}$	$-\hat{y}$	$\hat{z}$

(4.5) for helicity may be used, this procedure leading to straightforward but tedious calculations. A simpler procedure is to connect directly various observables among themselves, taking advantage of dyadic notation for Δ spin-space operators. For this, relations between $\hat{l}$, $\hat{m}$, $\hat{n}$ unit vectors of the fixed basis and helicity or transversity frame unit vectors are presented in Table III for each particle. For example, assuming a Δ polarization state with (I,p) symbol equal to $(3,0)$, the corresponding spin-space operator $\Omega_3(0)$ of Eq. (2.13) becomes

$$\frac{1}{4\sqrt{3}}\{\cos 2\theta_\Delta[(\hat{x}\cdot\overleftrightarrow{\mathrm{T}}_\Delta\cdot\hat{x})-(\hat{y}\cdot\overleftrightarrow{\mathrm{T}}_\Delta\cdot\hat{y})] -2\sin 2\theta_\Delta(\hat{x}\cdot\overleftrightarrow{\mathrm{T}}_\Delta\cdot\hat{y})\}, \quad (4.7)$$

in the transversity frame by using Table III. Finally, from the Appendix, one gets

$$\Omega_3(0) = \cos 2\theta_\Delta(Q^{13R}+Q^{24R})-\sin 2\theta_\Delta(Q^{13I}+Q^{24I}). \quad (4.8)$$

A transcription of this equality in terms of arguments characterizing observables may be written

$$(3,0) \leftrightarrow \cos 2\theta_\Delta(13R+24R)-\sin 2\theta_\Delta(13I+24I). \quad (4.9)$$

The same procedure for $\Omega_3(0)$ gives, in the helicity frame,

$$\frac{1}{4\sqrt{3}}\{\cos 2\theta_\Delta[(\hat{z}\cdot\overleftrightarrow{\mathrm{T}}_\Delta\cdot\hat{z})-(\hat{x}\cdot\overleftrightarrow{\mathrm{T}}_\Delta\cdot\hat{x})] -2\sin 2\theta_\Delta(\hat{x}\cdot\overleftrightarrow{\mathrm{T}}_\Delta\cdot\hat{z})\}, \quad (4.10)$$

TABLE IV. Argument transcriptions from Sec. II to optimal transversity notations.

Particle	Section II	Transversity
N_1, N_2	0	A
	l	$12R$
	m	$12I$
	n	Ψ
N_2'	0	A
	l	$\cos\theta_\Delta(12R)-\sin\theta_\Delta(12I)$
	m	$\cos\theta_\Delta(12I)+\sin\theta_\Delta(12R)$
	n	Ψ
Δ_1	$(1,0)$	A
	$(2,0)$	Ψ_1
	$(3,0)$	$\cos 2\theta_\Delta(13R+24R)-\sin 2\theta_\Delta(13I+24I)$
	$(4,0)$	$-\cos 2\theta_\Delta(13I-24I)-\sin 2\theta_\Delta(13R-24R)$
	$(1,n)$	Ψ_2
	$(2,n)$	Ψ_3
	$(3,n)$	$-\cos 2\theta_\Delta(13R-24R)+\sin 2\theta_\Delta(13I-24I)$
	$(4,n)$	$\cos 2\theta_\Delta(13I+24I)+\sin 2\theta_\Delta(13R+24R)$
	$(1,l)$	$-\cos 3\theta_\Delta(14R)+\sin 3\theta_\Delta(14I)-\cos\theta_\Delta(23R)+\sin\theta_\Delta(23I)$
	$(2,l)$	$-\cos 3\theta_\Delta(14R)+\sin 3\theta_\Delta(14I)+\cos\theta_\Delta(23R)-\sin\theta_\Delta(23I)$
	$(3,l)$	$-\cos\theta_\Delta(12R+34R)+\sin\theta_\Delta(12I+34I)$
	$(4,l)$	$\cos\theta_\Delta(12I-34I)+\sin\theta_\Delta(12R-34R)$
	$(1,m)$	$\cos 3\theta_\Delta(14I)+\sin 3\theta_\Delta(14R)-\cos\theta_\Delta(23I)-\sin\theta_\Delta(23R)$
	$(2,m)$	$\cos 3\theta_\Delta(14I)+\sin 3\theta_\Delta(14R)+\cos\theta_\Delta(23I)+\sin\theta_\Delta(23R)$
	$(3,m)$	$\cos\theta_\Delta(12I+34I)+\sin\theta_\Delta(12R+34R)$
	$(4,m)$	$\cos\theta_\Delta(12R-34R)-\sin\theta_\Delta(12I-34I)$

TABLE V. Argument transcriptions from transversity to helicity optimal notations.

Particle	Transversity	Helicity
N_1	A	A
	$12R$	Ψ
	$12I$	$12R$
	Ψ	$12I$
N_2, N_2'	A	A
	$12R$	$-\Psi$
	$12I$	$12R$
	Ψ	$-12I$
Δ_1	A	A
	Ψ_1	$-\frac{1}{2}\Psi_1 - \frac{\sqrt{3}}{2}(13R + 24R)$
	Ψ_2	$14I - 23I$
	Ψ_3	$-\frac{1}{2}(14I + 23I) - \frac{\sqrt{3}}{2}(12I + 34I)$
	$13R + 24R$	$\frac{\sqrt{3}}{2}\Psi_1 - \frac{1}{2}(13R + 24R)$
	$13I + 24I$	$12R - 34R$
	$13R - 24R$	$-\frac{\sqrt{3}}{2}(14I + 23I) + \frac{1}{2}(12I + 34I)$
	$13I - 24I$	$13I - 24I$
	$12R + 34R$	$-\frac{\sqrt{3}}{2}\Psi_3 - \frac{1}{2}(13R - 24R)$
	$12I + 34I$	$-\frac{\sqrt{3}}{2}(14R - 23R) + \frac{1}{2}(12R + 34R)$
	$12R - 34R$	$12I - 34I$
	$12I - 34I$	$13I + 24I$
	$14R$	$-\frac{1}{2}\Psi_2 + \frac{1}{4}\Psi_3 - \frac{\sqrt{3}}{4}(13R - 24R)$
	$14I$	$-\frac{1}{4}(14R) - \frac{3}{4}(23R) + \frac{\sqrt{3}}{4}(12R + 34R)$
	$23R$	$-\frac{1}{2}\Psi_2 - \frac{1}{4}\Psi_3 + \frac{\sqrt{3}}{4}(13R - 24R)$
	$23I$	$\frac{3}{4}(14R) + \frac{1}{4}(23R) + \frac{\sqrt{3}}{4}(12R + 34R)$

TABLE VI. Argument transcriptions from optimal transversity to Sec. II notations.

Particle	Transversity	Section II
N_1, N_2	A	0
	$12R$	l
	$12I$	m
	Ψ	n
N_2'	A	0
	$12R$	$\cos\theta_\Delta(l) + \sin\theta_\Delta(m)$
	$12I$	$\cos\theta_\Delta(m) + \sin\theta_\Delta(l)$
	Ψ	n
Δ_1	A	$(1,0)$
	Ψ_1	$(2,0)$
	Ψ_2	$(1,n)$
	Ψ_3	$(2,n)$
	$12R + 34R$	$-\cos\theta_\Delta(3,l) + \sin\theta_\Delta(3,m)$
	$12R - 34R$	$\cos\theta_\Delta(4,m) + \sin\theta_\Delta(4,l)$
	$12I + 34I$	$\cos\theta_\Delta(3,m) + \sin\theta_\Delta(3,l)$
	$12I - 34I$	$\cos\theta_\Delta(4,l) - \sin\theta_\Delta(4,m)$
	$13R + 24R$	$\cos 2\theta_\Delta(3,0) + \sin 2\theta_\Delta(4,n)$
	$13R - 24R$	$-\cos 2\theta_\Delta(3,n) - \sin 2\theta_\Delta(4,0)$
	$13I + 24I$	$\cos 2\theta_\Delta(4,n) - \sin 2\theta_\Delta(3,0)$
	$13I - 24I$	$-\cos 2\theta_\Delta(4,0) + \sin 2\theta_\Delta(3,n)$
	$14R$	$-\frac{1}{2}\cos 3\theta_\Delta[(1,l) + (2,l)] + \frac{1}{2}\sin 3\theta_\Delta[(1,m) + (2,m)]$
	$14I$	$\frac{1}{2}\cos 3\theta_\Delta[(1,m) + (2,m)] + \frac{1}{2}\sin 3\theta_\Delta[(1,l) + (2,l)]$
	$23R$	$-\frac{1}{2}\cos\theta_\Delta[(1,l) - (2,l)] - \frac{1}{2}\sin\theta_\Delta[(1,m) - (2,m)]$
	$23I$	$-\frac{1}{2}\cos\theta_\Delta[(1,m) - (2,m)] + \frac{1}{2}\sin\theta_\Delta[(1,l) - (2,l)]$

which leads to the argument transcription

$$(3,0) \leftrightarrow \frac{1}{2}\cos 2\theta_\Delta[\sqrt{3}\Psi_1 - (13R+24R)] - \sin 2\theta_\Delta(12R-34R). \quad (4.11)$$

The systematic exploitation of this method is shown in Tables IV and V for the four particles. Argument transcriptions are presented in Table IV from Sec. II notation to optimal transversity notation and in Table V from transversity to helicity frame notation. By combining Tables IV and V, a transcription from Sec. II to the optimal helicity formalism is obtained.

Finally, using the argument transcription of Tables IV and V for the four particles involved in the Δ production reaction, an observable transformation is achieved. Taking into account normalization, an example of the $X_{(3,0)0lm}$ observable is developed, which gives, using Table IV,

$$X_{(3,0)0lm} = [1/\mathcal{L}^t(A,A;A,A)]\{\cos 2\theta_\Delta[\mathcal{L}^t(12R,12I;13R,A)+\mathcal{L}^t(12R,12I;24R,A)] - \sin 2\theta_\Delta[\mathcal{L}^t(12R,12I;13I,A)+\mathcal{L}^t(12R,12I;24I,A)]\}, \quad (4.12)$$

and, using Table V in addition and Eq. (3.29),

TABLE VII. Argument transcriptions from optimal helicity to Sec. II notations.

Particle	Helicity	Section II
N_1	A	0
	Ψ	l
	$12R$	m
	$12I$	n
N_2	A	0
	Ψ	$-(l)$
	$12R$	m
	$12I$	$-(n)$
N_2'	A	0
	Ψ	$-\cos\theta_\Delta(l) - \sin\theta_\Delta(m)$
	$12R$	$\cos\theta_\Delta(m) - \sin\theta_\Delta(l)$
	$12I$	$-(n)$
Δ_1	A	$(1,0)$
	Ψ_1	$\frac{1}{2}[-(2,0)+\sqrt{3}\cos 2\theta_\Delta(3,0)+\sqrt{3}\sin 2\theta_\Delta(4,n)]$
	Ψ_2	$\frac{1}{2}[(\cos 3\theta_\Delta+\cos\theta_\Delta)(1,l)+(\cos 3\theta_\Delta-\cos\theta_\Delta)(2,l)$ $-(\sin 3\theta_\Delta-\sin\theta_\Delta)(1,m)-(\sin 3\theta_\Delta+\sin\theta_\Delta)(2,m)]$
	Ψ_3	$-\frac{1}{4}[(\cos 3\theta_\Delta-\cos\theta_\Delta)(1,l)+(\cos 3\theta_\Delta+\cos\theta_\Delta)(2,l)$ $-(\sin 3\theta_\Delta+\sin\theta_\Delta)(1,m)-(\sin 3\theta_\Delta-\sin\theta_\Delta)(2,m)]$ $+\frac{\sqrt{3}}{2}[\cos\theta_\Delta(3,l)-\sin\theta_\Delta(3,m)]$
	$12R+34R$	$\frac{\sqrt{3}}{4}[(\cos 3\theta_\Delta-\cos\theta_\Delta)(1,m)+(\cos 3\theta_\Delta+\cos\theta_\Delta)(2,m)$ $+(\sin 3\theta_\Delta+\sin\theta_\Delta)(1,l)+(\sin 3\theta_\Delta-\sin\theta_\Delta)(2,l)]$ $+\frac{1}{2}[\cos\theta_\Delta(3,m)+\sin\theta_\Delta(3,l)]$
	$12R-34R$	$\cos 2\theta_\Delta(4,n)-\sin 2\theta_\Delta(3,0)$
	$12I+34I$	$-\frac{\sqrt{3}}{2}(2,n)-\frac{1}{2}[\cos 2\theta_\Delta(3,n)+\sin 2\theta_\Delta(4,0)]$
	$12I-34I$	$\cos\theta_\Delta(4,m)+\sin\theta_\Delta(4,l)$
	$13R+24R$	$-\frac{1}{2}[\sqrt{3}(2,0)+\cos 2\theta_\Delta(3,0)+\sin 2\theta_\Delta(4,n)]$
	$13R-24R$	$\frac{\sqrt{3}}{4}[(\cos 3\theta_\Delta-\cos\theta_\Delta)(1,l)+(\cos 3\theta_\Delta+\cos\theta_\Delta)(2,l)$ $-(\sin 3\theta_\Delta+\sin\theta_\Delta)(1,m)-(\sin 3\theta_\Delta-\sin\theta_\Delta)(2,m)]$ $+\frac{1}{2}[\cos\theta_\Delta(3,l)-\sin\theta_\Delta(3,m)]$
	$13I+24I$	$\cos\theta_\Delta(4,l)-\sin\theta_\Delta(4,m)$
	$13I-24I$	$-\cos 2\theta_\Delta(4,0)+\sin 2\theta_\Delta(3,n)$
	$14R$	$-\frac{1}{8}[(\cos 3\theta_\Delta+3\cos\theta_\Delta)(1,m)+(\cos 3\theta_\Delta-3\cos\theta_\Delta)(2,m)$ $+(\sin 3\theta_\Delta-3\sin\theta_\Delta)(1,l)+(\sin 3\theta_\Delta+3\sin\theta_\Delta)(2,l)]$ $-\frac{\sqrt{3}}{4}[\cos\theta_\Delta(3,m)+\sin\theta_\Delta(3,l)]$
	$23R$	$-\frac{1}{8}[(3\cos 3\theta_\Delta+\cos\theta_\Delta)(1,m)+(3\cos 3\theta_\Delta-\cos\theta_\Delta)(2,m)$ $+(3\sin 3\theta_\Delta-\sin\theta_\Delta)(1,l)+(3\sin 3\theta_\Delta+\sin\theta_\Delta)(2,l)]$ $+\frac{\sqrt{3}}{4}[\cos\theta_\Delta(3,m)+\sin\theta_\Delta(3,l)]$
	$14I$	$\frac{1}{2}(1,n)-\frac{1}{4}(2,n)+\frac{\sqrt{3}}{4}[\cos 2\theta_\Delta(3,n)+\sin 2\theta_\Delta(4,0)]$
	$23I$	$-\frac{1}{2}(1,n)-\frac{1}{4}(2,n)+\frac{\sqrt{3}}{4}[\cos 2\theta_\Delta(3,n)+\sin 2\theta_\Delta(4,0)]$

$$X_{(3,0)0lm} = [1/\mathcal{L}^h(A,A;A,A)]\left[\cos 2\theta_\Delta\left(\frac{\sqrt{3}}{2}\mathcal{L}^h(\Psi,12R;\Psi_1,A) - \mathcal{L}^h(\Psi,12R;13R,A)\right) - 2\sin 2\theta_\Delta \mathcal{L}^h(\Psi,12R;12R,A)\right]. \tag{4.13}$$

For sake of completeness, inversion of Tables IV and V is performed, which is presented in Tables VI and VII.

V. CONCLUSION

The present work is devoted to a detailed study of the spin observables in the $NN \to \Delta N$ transition. Two kinds of formalisms allowing one to express all observables by means of compact formulas are investigated. The first, using a spin-space decomposition of the transition matrix analogous to the Wolfenstein representation in $NN \to NN$ elastic scattering is convenient for studying nuclear reactions at intermediate-energy physics. The structure of the Δ spin-space operator basis is adapted to reflect relations analogous to the so-called "Bohr's rule" in $NN \to NN$ elastic scattering. The second formalism which optimally diagonalizes the matrix connecting observables and bilinear combinations of transition amplitudes (bicoms) is well adapted to the phenomenological amplitude determination.

In the optimal formalism, as far as "primary observables" are concerned, polarization structure analysis yields bicom-observable relations in a particularly simple form. Yet, it is much simpler to perform experiments in which some particles are unpolarized that correspond to averaged spin states, which lead to a redefinition of the observables in terms of "secondary observables." Unfortunately, the choice of "secondary observables," more adapted to experiment, increases the complexity of their relations with bicoms. The choice proposed in this work is adapted for taking into account Bohr's rules. The optimal formalism is explicitly developed in helicity and transversity frames.

The advantages of each formalism are clearly underlined and transformations between them are shown to be useful for a transition amplitude analysis. Tables are given which perform such transformations. A complete determination of all the transition amplitudes, namely, the determination of 16 magnitudes and 15 relative phases (the overall phase being irrelevant) is of importance in intermediate-energy physics. The discussion concerning the explicit choice of a set of experiments realizing a partial or complete phenomenological analysis is postponed to a future work.

ACKNOWLEDGMENT

Division de Physique Théorique is Unité de Recherche des Universités Paris XI et Paris VI associée au CNRS.

APPENDIX

In the optimal formalism, initial and final spin-space matrix elements are defined by Eq. (3.3). For spin-$\frac{1}{2}$ particles, one obtains two ρ^{uu} diagonal matrices,

$$\rho^{11} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \rho^{22} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \tag{A1}$$

and two ρ^{uvH_p} off-diagonal matrices,

$$\rho^{12R} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho^{12I} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{A2}$$

For spin-$\frac{3}{2}$ particles, one obtains four $Q^{\xi\xi}$ diagonal matrices,

$$Q^{11} = \begin{pmatrix} 2&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \quad Q^{22} = \begin{pmatrix} 0&0&0&0\\0&2&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix},$$
$$Q^{33} = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&2&0\\0&0&0&0 \end{pmatrix}, \quad Q^{44} = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&2 \end{pmatrix}, \tag{A3}$$

six $Q^{\xi\omega R}$ off-diagonal matrices,

$$Q^{12R} = \begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \quad Q^{13R} = \begin{pmatrix} 0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{pmatrix},$$
$$Q^{14R} = \begin{pmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{pmatrix}, \quad Q^{23R} = \begin{pmatrix} 0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0 \end{pmatrix},$$
$$Q^{24R} = \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0 \end{pmatrix}, \quad Q^{34R} = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0 \end{pmatrix}, \tag{A4}$$

and six $Q^{\xi\omega I}$ off-diagonal matrices,

$$Q^{12I} = \begin{pmatrix} 0&-i&0&0\\i&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \quad Q^{13I} = \begin{pmatrix} 0&0&-i&0\\0&0&0&0\\i&0&0&0\\0&0&0&0 \end{pmatrix},$$
$$Q^{14I} = \begin{pmatrix} 0&0&0&-i\\0&0&0&0\\0&0&0&0\\i&0&0&0 \end{pmatrix}, \quad Q^{23I} = \begin{pmatrix} 0&0&0&0\\0&0&-i&0\\0&i&0&0\\0&0&0&0 \end{pmatrix},$$
$$Q^{24I} = \begin{pmatrix} 0&0&0&0\\0&0&0&-i\\0&0&0&0\\0&i&0&0 \end{pmatrix}, \quad Q^{34I} = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&-i\\0&0&i&0 \end{pmatrix}. \tag{A5}$$

For nucleons, relations between Pauli operators and

ρ^{uvH_p} operators are written as

$$1=\frac{1}{2}(\rho^{11}+\rho^{22})=\rho^{A},\quad \text{scalar},$$
$$(\sigma\cdot\hat{z})=\frac{1}{2}(\rho^{11}-\rho^{22})=\rho^{\Psi},\quad \begin{array}{l}\text{scalar in transversity,}\\ \text{pseudoscalar in helicity,}\end{array}$$
$$(\sigma\cdot\hat{x})=\rho^{12R},\quad \begin{array}{l}\text{pseudoscalar in transversity,}\\ \text{pseudoscalar in helicity,}\end{array}$$
$$(\sigma\cdot\hat{y})=\rho^{12I},\quad \begin{array}{l}\text{pseudoscalar in transversity,}\\ \text{scalar in helicity.}\end{array}\qquad\text{(A6)}$$

For the Δ particle, relations are given which connect $Q^{\xi\omega H_q}$ operators and Cartesian tensors, defined in the $\hat{x},\hat{y},\hat{z}$ basis and written with dyadic notation, analogous to the $\Omega_I(p)$ operators presented in Sec. II. Four operators are scalar, which follow

$$1=\frac{1}{2}(Q^{11}+Q^{22}+Q^{33}+Q^{44})=Q^{A},$$
$$\frac{1}{4}(\hat{z}\cdot\overleftrightarrow{T}_\Delta\cdot\hat{z})=\frac{1}{2}(Q^{11}-Q^{22}-Q^{33}+Q^{44})=Q^{\Psi_1},$$
$$\frac{1}{4\sqrt{3}}[(\hat{x}\cdot\overleftrightarrow{T}_\Delta\cdot\hat{x})-(\hat{y}\cdot\overleftrightarrow{T}_\Delta\cdot\hat{y})]=Q^{13R}+Q^{24R},$$
$$\frac{1}{2\sqrt{3}}(T_3\hat{x}\hat{y}\hat{z})_0=-Q^{13I}+Q^{24I}.\qquad\text{(A7)}$$

The four following operators are scalar in transversity and pseudoscalar in helicity:

$$\Sigma(\hat{z})=\frac{1}{2}(-Q^{11}+Q^{22}-Q^{33}+Q^{44})=Q^{\Psi_2},$$
$$\frac{1}{4}(\hat{z}\cdot\overleftrightarrow{T}_\Delta\cdot\hat{z})\Sigma(\hat{z})=\frac{1}{2}(-Q^{11}-Q^{22}+Q^{33}+Q^{44})=Q^{\Psi_3},$$
$$\frac{1}{4\sqrt{3}}[(\hat{x}\cdot\overleftrightarrow{T}_\Delta\cdot\hat{x})-(\hat{y}\cdot\overleftrightarrow{T}_\Delta\cdot\hat{y})]\Sigma(\hat{z})=-Q^{13R}+Q^{24R},$$
$$\frac{1}{2\sqrt{3}}(T_3\hat{x}\hat{y}\hat{z})_0\Sigma(\hat{z})=Q^{13I}+Q^{24I}.\qquad\text{(A8)}$$

Four operators are pseudoscalar in transversity and helicity, which are written

$$\Sigma(\hat{x})=-Q^{14R}-Q^{23R},$$
$$\frac{1}{4}(\hat{z}\cdot\overleftrightarrow{T}_\Delta\cdot\hat{z})\Sigma(\hat{x})=-Q^{14R}+Q^{23R},$$
$$\frac{1}{4\sqrt{3}}[(\hat{x}\cdot\overleftrightarrow{T}_\Delta\cdot\hat{x})-(\hat{y}\cdot\overleftrightarrow{T}_\Delta\cdot\hat{y})]\Sigma(\hat{x})=-Q^{12R}-Q^{34R},$$
$$\frac{1}{2\sqrt{3}}(T_3\hat{x}\hat{y}\hat{z})_0\Sigma(\hat{x})=Q^{12I}-Q^{34I}.\qquad\text{(A9)}$$

Finally, the four following operators are pseudoscalar in transversity and scalar in helicity:

$$\Sigma(\hat{y})=Q^{14I}-Q^{23I},$$
$$\frac{1}{4}(\hat{z}\cdot\overleftrightarrow{T}_\Delta\cdot\hat{z})\Sigma(\hat{y})=Q^{14I}+Q^{23I},$$
$$\frac{1}{4\sqrt{3}}[(\hat{x}\cdot\overleftrightarrow{T}_\Delta\cdot\hat{x})-(\hat{y}\cdot\overleftrightarrow{T}_\Delta\cdot\hat{y})]\Sigma(\hat{y})=Q^{12I}+Q^{34I},$$
$$\frac{1}{2\sqrt{3}}(T_3\hat{x}\hat{y}\hat{z})_0\Sigma(\hat{y})=Q^{12R}-Q^{34R}.\qquad\text{(A10)}$$

We recall that the rank-2 and -3 tensorial operators satisfy the trace conditions

$$(\hat{x}\cdot\overleftrightarrow{T}_\Delta\cdot\hat{x})+(\hat{y}\cdot\overleftrightarrow{T}_\Delta\cdot\hat{y})+(\hat{z}\cdot\overleftrightarrow{T}_\Delta\cdot\hat{z})=0\qquad\text{(A11)}$$

and

$$(T_3\hat{a}\hat{x}\hat{x})_0+(T_3\hat{a}\hat{y}\hat{y})_0+(T_3\hat{a}\hat{z}\hat{z})_0=0,\qquad\text{(A12)}$$

for $\hat{a}=\hat{x},\hat{y}$ or $\hat{z}$. Useful tensorial relations can be found in Ref. [8], in particular in Appendix A.

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