Spinor field theory at finite temperature in the early Universe

N. Banerjee and S. Mallik
Saha Institute of Nuclear Physics, Block-AF, Bidhannagar, Calcutta-700 064, India
(Received 19 July 1991)

We consider the Dirac field on a spatially flat Robertson-Walker space-time. We find the exact expression for the Dirac propagator for an arbitrary scale factor in the real-time formulation of finite-temperature field theory. The mode functions used in the construction satisfy uncoupled ordinary differential equations.

PACS number(s): 98.80.Cq, 11.90.+t

I. INTRODUCTION

For times larger than the Planck time, the Universe can be described as an appropriate system of quantized matter fields on the curved space-time of general relativity. Such a semiclassical system has, of course, been extensively studied in the conventional (zero-temperature) framework of quantum field theory [1]. But the appropriate setting to describe the temporal evolution of physical quantities characterizing the early Universe requires a real-time formulation of finite-temperature field theory.

The authors of the real-time theory [2,3] base their formulations on the example of a (Higgs) scalar field on a spatially flat Robertson-Walker metric. Of these, the one by Semenoff and Weiss [3] has the advantage that it does not require the continuation of the scale factor in the metric to imaginary time. Their original formulation can be greatly simplified [4] with respect to both the technical problem of renormalizability and the expression for the finite-temperature propagator of the scalar field. We then extended the construction of the propagator to the (Abelian) gauge field [5].

In the present Brief Report, we find the propagator of a Dirac field in real-time finite-temperature field theory. This construction follows in outline our earlier work on the gauge-field propagator. While for the gauge propagator we could find its form in the radiation-dominated era, the Dirac propagator is obtained here in terms of mode functions for any form of the scale factor. In Sec. II we write ordinary differential equations for the Fourier-transformed O(3)-invariant functions into which the spinor Green's function can be decomposed. In Sec. III we solve them with thermal boundary conditions in parallel with that of the scalar field case. Our concluding remarks are contained in Sec. IV.

II. PROPAGATOR EQUATIONS

The action for the Dirac field in the presence of an external spinor source on curved space-time is

$$S=\int d^4x\sqrt{-g}\,[i\bar\psi\gamma^\mu(\partial_\mu+\omega_\mu)\psi-M\bar\psi\psi+\bar\psi j+\bar j\psi]\ . \quad (1)$$

We use Greek and Latin letters to denote coordinate and Lorentz indices, respectively. The vierbein $e^a_\mu(x)$ relates the general metric $g_{\mu\nu}$ to the flat (Lorentz) metric η_{ab}:

$$g_{\mu\nu}(x)=e^a_\mu(x)e^b_\nu(x)\eta_{ab},\quad \eta_{ab}=\mathrm{diag}(1,-1,-1,-1)\ .$$

Then the matrices γ^μ are given by $\gamma^\mu(x)=\gamma^a e^\mu_a(x)$, where γ^a are the usual Dirac matrices satisfying $\{\gamma^a,\gamma^b\}=2\eta^{ab}$. The spin connection is

$$\omega_\mu=-\frac{i}{4}\sigma^{ab}e^\nu_a\nabla_\mu e_{b\nu},\quad \sigma^{ab}=\frac{i}{2}[\gamma^a,\gamma^b]\ ,$$

∇_μ being the covariant derivative. M in (1) is a mass parameter.

Here we point out that the action describing our early Universe is actually that of an interacting (nonabelian gauge) field theory. The reason for not including interactions (with the gauge and Higgs fields) in (1) is due to the fact that we shall, in this Brief Report, confine ourselves to determining the free Dirac field propagator in the background gravitational field. But the presence of these interactions is needed even to realize the assumptions in this formulation. They are responsible for collisions leading to the initial thermal equilibrium of the matter in the early Universe. Also, these interactions generate additional temperature-dependent masses for the fields, which for the spinor field is included in M, implying the presence of a compensating quadratic counterterm in the (omitted) interaction terms. This modification in the mass term is needed to restore the breakdown of naive perturbation theory at sufficiently high temperature [6]. It also plays a crucial role in defining the positive- and negative-frequency mode functions around the initial time in our formulation [4].

The Dirac equation is

$$[i\gamma^\mu(\partial_\mu+\omega_\mu)-M]\psi(x)=j(x)\ , \quad (2)$$

and the Green's function or propagator satisfies

$$[i\gamma^\mu(\partial_\mu+\omega_\mu)-M]S(x,x')=\frac{\delta(x-x')}{\sqrt{-g}}\ . \quad (3)$$

As in the case of gauge fields [5], the construction of the propagator $S(x,x')$ is greatly simplified by analyzing Eq. (2) for the mode functions $\psi(x)$ and relating its components to $S(x,x')$ by

$$\psi(x)=\int d^4x'\sqrt{-g'}S(x,x')j(x')\ . \quad (4)$$

In Robertson-Walker space-time

$$ds^2=dt^2-a^2(t)d\mathbf{x}^2 , \tag{5}$$

with which we shall work henceforth, (2) becomes

$$\left[\frac{\partial}{\partial t}+\frac{3}{2}\frac{\dot a}{a}+\frac{1}{a}\gamma^0\gamma\cdot\nabla+iM\gamma^0\right]\psi(\mathbf{x},t)=-i\gamma^0 j(\mathbf{x},t) , \tag{6}$$

an overdot indicating the derivative with respect to time t. Because the metric is spatially flat, it is convenient to work with three-dimensional Fourier transforms of the above quantities, denoted by a tilde; for example,

$$\psi(\mathbf{x},t)=\int\frac{d^3\mathbf{k}}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{x}}\tilde\psi(\mathbf{k},t) . \tag{7}$$

We also decompose all four-component spinors into two-component ones,

$$\psi=\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix},\quad j=\begin{bmatrix}j_1\\ j_2\end{bmatrix} ,$$

so that (6) breaks up, in the Dirac representation of the gamma matrices, into the two coupled equations

$$\left[\frac{d}{dt}+\frac{3}{2}\frac{\dot a}{a}+iM\right]\tilde\psi_1(t)+i\frac{\sigma\cdot\mathbf{k}}{a(t)}\tilde\psi_2(t)=-i\tilde j_1(t) , \tag{8a}$$

$$\left[\frac{d}{dt}+\frac{3}{2}\frac{\dot a}{a}-iM\right]\tilde\psi_2(t)+i\frac{\sigma\cdot\mathbf{k}}{a(t)}\tilde\psi_1(t)=i\tilde j_2(t) . \tag{8b}$$

Here and in the following, we omit the k dependence of all Fourier transforms.

It is easy to obtain uncoupled, second-order equations for ψ_1 and ψ_2 from (8a) and (8b). Apply $(d/dt+\frac{5}{2}\dot a/a-iM)$ on (8a) and use (8b) to eliminate $\tilde\psi_2$ and get

$$\left[\frac{d^2}{dt^2}+4\frac{\dot a}{a}\frac{d}{dt}+iM\frac{\dot a}{a}+c+\omega^2\right]\tilde\psi_1(t) =\frac{\sigma\cdot\mathbf{k}}{a(t)}\tilde j_2(t)-i\left[\frac{d}{dt}+\frac{5}{2}\frac{\dot a}{a}-iM\right]\tilde j_1(t) , \tag{9a}$$

where $c=\frac{3}{2}\{\ddot a/a+\frac{3}{2}(\dot a/a)^2\}$ and $\omega^2=\omega^2(t)=M^2(t)+k^2/a^2(t)$. Similarly, we eliminate $\tilde\psi_1$ to get

$$\left[\frac{d^2}{dt^2}+4\frac{\dot a}{a}\frac{d}{dt}-iM\frac{\dot a}{a}+c+\omega^2\right]\tilde\psi_2(t) =-\frac{\sigma\cdot\mathbf{k}}{a(t)}\tilde j_1(t)+i\left[\frac{d}{dt}+\frac{5}{2}\frac{\dot a}{a}+iM\right]\tilde j_2(t) . \tag{9b}$$

Next, it simplifies things to get rid of the first derivative on the right-hand side of (9a) and (9b) by a redefinition of the amplitudes and sources:

$$\tilde\chi_{1,2}=a^2\tilde\psi_{1,2},\quad \tilde J_{1,2}=a^2\tilde j_{1,2} . \tag{10}$$

Equations (9a) and (9b) become

$$\left[\frac{d^2}{dt^2}\pm iM\frac{\dot a}{a}+d+\omega^2\right]\tilde\chi_{1,2}(t) =\pm\frac{\sigma\cdot\mathbf{k}}{a(t)}\tilde J_{2,1}(t)\mp i\left[\frac{d}{dt}+\frac{1}{2}\frac{\dot a}{a}\mp iM\right]\tilde J_{1,2}(t) , \tag{11}$$

with $d=-\frac{1}{2}\ddot a/a+\frac{1}{4}(\dot a/a)^2$. Here and below the upper (lower) sign corresponds to the first (second) subscript on the amplitudes and sources. We also need the original coupled equations (8a) and (8b) in the following. With the new amplitudes, they become

$$\left[\frac{d}{dt}-\frac{1}{2}\frac{\dot a}{a}\pm iM\right]\tilde\chi_{1,2}(t)+i\frac{\sigma\cdot\mathbf{k}}{a(t)}\tilde\chi_{2,1}(t)=\mp i\tilde J_{1,2}(t) . \tag{12}$$

We next turn to the propagator and Eq. (4) relating it to ψ. the propagator may be decomposed,

$$\tilde S(k,t,t')=\tilde A+\tilde B\gamma^0+\tilde C\gamma\cdot\mathbf{k}+\tilde D\gamma^0\gamma\cdot\mathbf{k} , \tag{13}$$

into amplitudes $\tilde A$, $\tilde B$, $\tilde C$, and $\tilde D$, invariant under parity and O(3) transformations, each of which is a function of t,t' as well as of k. In two-component notation, (4) becomes

$$\tilde\psi_{1,2}=\int dt'\,a^3(t')[(\tilde A\pm\tilde B)\tilde j_{1,2}(t')\pm(\tilde C\pm\tilde D)\sigma\cdot\mathbf{k}\tilde j_{2,1}(t')] . \tag{14}$$

We express it in terms of $\tilde\chi_{1,2}$ and $\tilde J_{1,2}$ and also redefine the invariant Green's functions to get rid of the scale factor, obtaining

$$\tilde\chi_{1,2}=\int dt'\left[\tilde U_{1,2}(t,t')\tilde J_{1,2}(t')\pm\tilde V_{1,2}(t,t')\frac{\sigma\cdot\mathbf{k}}{a(t')}\tilde J_{2,1}(t')\right] , \tag{15}$$

with

$$\tilde U_{1,2}(t,t')=a^2(t)a(t')(\tilde A\pm\tilde B)(t,t') , \tag{16a}$$

$$\tilde V_{1,2}(t,t')=a^2(t)a^2(t')(\tilde C\pm\tilde D)(t,t') . \tag{16b}$$

We are now in a position to obtain uncoupled equations for the invariant Green's functions. First, set

$$\tilde J_1(t)=0,\quad \tilde J_2(t)=\delta(t-t')\zeta(t') , \tag{17}$$

$\zeta(t)$ being an arbitrary two-component spinor. Equation (15) then yields simple algebraic relations among the $\tilde\chi$ amplitudes and invariant Green's functions,

$$\begin{aligned}\tilde\chi_1(t)&=\tilde V_1(t,t')\frac{\sigma\cdot\mathbf{k}}{a(t')}\zeta(t') ,\\ \tilde\chi_2(t)&=\tilde U_2(t,t')\zeta(t') ,\end{aligned} \tag{18}$$

and Eqs. (11) and (12) with the upper sign immediately give the desired equations

$$\left[\frac{d^2}{dt^2}+iM\frac{\dot a}{a}+d+\omega^2\right]\tilde V_1(t,t')=\delta(t-t') , \tag{19a}$$

$$\tilde{U}_2(t,t')=i\frac{a(t)}{a(t')}\left[\frac{d}{dt}-\frac{1}{2}\frac{\dot{a}}{a}+iM\right]\tilde{V}_1(t,t') . \quad (19b)$$

Note that the other two equations [(11) and (12) with the lower sign] are not convenient to determine the remaining amplitudes $\tilde{V}_2$ and $\tilde{U}_1$. To determine the latter ones, we put

$$\tilde{J}_2(t)=0, \quad \tilde{J}_1(t)=\delta(t-t')\xi(t') , \quad (20)$$

$\xi(t)$ being again an arbitrary spinor. As before, we get

$$\left[\frac{d^2}{dt^2}-iM\frac{\dot{a}}{a}+d+\omega^2\right]\tilde{V}_2(t,t')=\delta(t-t') , \quad (21a)$$

$$\tilde{U}_1(t,t')=-i\frac{a(t)}{a(t')}\left[\frac{d}{dt}-\frac{1}{2}\frac{\dot{a}}{a}-iM\right]\tilde{V}_2(t,t') . \quad (21b)$$

Thus the construction of the propagator for the Dirac field reduces to solving the ordinary differential equations for $\tilde{V}_1, \tilde{V}_2$, the invariant Green's functions, subject to appropriate boundary conditions.

III. FINITE TEMPERATURE

Equations (19a) and (21a) as well as the physics of the problem are similar to that of the scalar field case, treated in detail in Ref. [4]. It thus suffices to describe the finite-temperature formalism for the spinor case in outline only.

Consider a sufficiently small region of the early Universe so as to be well within the causal horizon. Assume that around time t_0 the collision rates among particles far exceed the expansion rate, so as to obtain an effective thermal equilibrium. Let the temperature be $T_0=1/\beta_0$ at time t_0. Now consider the ensemble average of any operator or product of operators, for example, the time-ordered product of two ψ fields,

$$\mathrm{Tr} e^{-\beta_0 H(t_0)} T\psi(\mathbf{x},t)\bar{\psi}(\mathbf{x}',t') , \quad (22)$$

which is of interest to us here, $H(t_0)$ being the Hamiltonian of the system at time t_0.

We can set up a path-integral representation for a quantity such as (22) using the complex time contour of Fig. 1 [7]. The two segments C_1 and C_2 of this contour are along the real-time axis [8], C_1 running forward and C_2 backward in time. The last segment C_3 runs parallel to the imaginary-time axis, where the scale factor $a(t)$ is fixed at time $t=t_0$. The field ψ and other functions are thus extended to the complex time variable τ. On the segments C_1 and C_2, $\tau=t$, while on C_3, we set $\tau=t_0-it$. Clearly, there is no discontinuity in the mode functions or any of its derivatives at the junction of C_1 and C_2. The mode functions turn out to be continuous also at the junction of C_2 and C_3, by virtue of the requirement that the geometry be stationary at $t=t_0$ $[\dot{a}(t_0)=\ddot{a}(t_0)=0]$, which was originally invoked to remove a difficulty with the renormalizability of the theory.

As is well known [9], the trace in (22) implies antiperiodic boundary conditions on the classical anticommuting (fermionic) ψ fields at the two ends of the contour of Fig. 1, in contrast with periodic boundary conditions

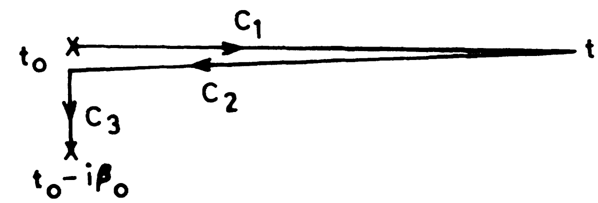

FIG. 1. Complex time contour.

for the classical commuting bosonic fields. Equations (12) then imply antiperiodic boundary conditions for the first derivative of the fields as well. It is then easy to see that the expression (22) is equal to the Green's function defined by (3) or equivalently by (13), (19), and (21) together with antiperiodic boundary conditions on these functions and their first derivatives.

Consider the construction of $\tilde{V}_1(t,t')$ obeying (19a), subject to the antiperiodic boundary conditions stated above. As in the scalar field case, we define real-time (on C_1 and C_2) mode functions $f^{\pm}(t,t')$ by the differential equation

$$\left[\frac{d^2}{dt^2}+iM\frac{\dot{a}}{a}+d+\omega^2\right]f^{\pm}(t)=0 , \quad (23)$$

and the boundary conditions $f^{\pm}(t_0)=1$, $\dot{f}^{\pm}(t_0)=\mp i\omega_0$, where $\omega_0=[M(t_0)+k^2/a^2(t_0)]^{1/2}$. These modes are then extended to functions defined on the entire complex time contour C by setting

$$f^{\pm}(\tau)=\begin{cases} f^{\pm}(t) & \text{on } C_1 \text{ and } C_2 , \\ e^{\mp\omega_0 t} & \text{on } C_3 . \end{cases} \quad (24)$$

The boundary conditions fix the coefficients of the homogenous pieces in $V_1(\tau,\tau')$, and we obtain

$$\tilde{V}_1(\tau,\tau')=\frac{i}{2\omega_0}\{[\theta(\tau-\tau')-F]f^{+}(\tau)f^{-}(\tau') + [\theta(\tau'-\tau)-F]f^{+}(\tau')f^{-}(\tau)\} , \quad (25)$$

where

$$F=\frac{1}{e^{\beta_0\omega_0}+1} . \quad (26)$$

An exactly similar construction holds for $V_2(\tau,\tau')$ with $f^{\pm}(t)$, in (24) replaced by $f^{\mp *}(t)$, the asterisk denoting complex conjugation. From (13), (16), (19), and (21), we get, for the expression for the propagator,

$$S(\mathbf{x}-\mathbf{x}';t,t')=\int\frac{d^3\mathbf{k}}{(2\pi)^3}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\tilde{S}(k,t,t') , \quad (27)$$

where

$$\tilde{S}(k,t,t')=-\frac{1}{2a(t)a^2(t')}\left\{\left[i\gamma^0\left[\frac{d}{dt}-\frac{1}{2}\frac{\dot a}{a}\right]-\frac{\gamma\cdot\mathbf{k}}{a}+M\right](\tilde V_1+\tilde V_2)-\gamma^0\left[i\gamma^0\left[\frac{d}{dt}-\frac{1}{2}\frac{\dot a}{a}\right]+\frac{\gamma\cdot\mathbf{k}}{a}+M\right](\tilde V_1-\tilde V_2)\right\}. \tag{28}$$

This completes our determination of the finite-temperature Dirac propagator in the spatially flat Robertson-Walker space-time geometry. For several forms of the scale factor $a(t)$, the mode functions can actually be solved in terms of known functions (see Ref. [10]).

In the radiation-dominated era, the requirement that thermal equilibrium be established around the initial time t_0 leads to the condition that the effective mass M be large compared to the curvature terms d in (19), (21), and (23) [4,5]. With such an adiabatic condition, the equations for $\tilde V_1$ and $\tilde V_2$ coincide and the Minkowski mode functions $f^{\pm}(t)$ become simply

$$f^{\pm}(t)=\left[\frac{\omega_0}{\omega(t)}\right]^{1/2}\exp\left[\mp i\int_{t_0}^{t}dt'\,\omega(t')\right]. \tag{29}$$

The propagator (28) then reduces to

$$\tilde S(k,t,t')==\frac{1}{a(t)a^2(t')}(i\gamma^\mu\partial_\mu+M)\tilde V_1(t,t'), \tag{30}$$

which is effectively the expression on flat space-time.

As the system leaves the radiation-dominated era, we have to reexamine the different conditions. In general, the mode functions in (28) have to be determined again by solving (23) with the new scale factor and requiring their continuity with the earlier ones in the transition region. If, however, the thermal equilibrium is maintained and the adiabatic condition also realized, then the form (30) for $\tilde S$ is still valid with t_0 referring to the initial time in the new era.

IV. CONCLUSION

Earlier we studied [4] a scalar Higgs field in an expanding Robertson-Walker geometry using a simplified form of the real-time formulation of the finite-temperature field theory originally proposed by Semenoff and Weiss. We obtained a much simpler expression for the thermal propagator for the scalar field. We also discussed, in particular, the physical notions associated with this formulation. We now discuss the construction of a thermal propagator for the Dirac field in the same framework. Thus the present work, together with that of Ref. [5], where we obtained the thermal gauge-field propagator, extends our earlier formulation for the scalar field to any gauge-field theory.

[1] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).

[2] I. T. Drummond, Nucl. Phys. **B190** [FS3], 93 (1984); B. L. Hu, Phys. Lett. **108B**, 19 (1982); A. Chowdhury and S. Mallik, Phys. Rev. D **36**, 1259 (1987).

[3] G. Semenoff and N. Weiss, Phys. Rev. D **31**, 689 (1985); **31**, 699 (1985).

[4] H. Leutwyler and S. Mallik, Ann. Phys. (N.Y.) **205**, 1 (1991).

[5] N. Banerjee and S. Mallik, Phys. Rev. D **44**, 3730 (1991).

[6] S. Weinberg, Phys. Rev. D **9**, 3357 (1974).

[7] The time contour usually chosen in the real-time formalism for flat space-time and its manipulation are beset with problems. See T. S. Evans, Z. Phys. C **41**, 333 (1988). The present contour of Fig. 1, where none of its parts are discarded, is free from such problems.

[8] The paths C_1 and C_2 are actually inclined infinitesimally downward to comply with the requirement of analyticity for the expectation value of the time-ordered field operators.

[9] F. A. Berezin, *The Method of Second Quantisation* (Academic, New York, 1966).

[10] A. O. Barut and I. H. Duru, Phys. Rev. D **36**, 3705 (1987); A. O. Barut and L. P. Singh, ICTP Report No. IC/90/205 (unpublished).