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Fermion fields in η-ξ spacetime

Yuan-Xing Gui
*Chinese Center of Advanced Science and Technology, World Laboratory, P.O. Box 8730, Beijing 100080, China and Department of Physics, Dalian University of Technology, Dalian 116023, China**

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Fermion fields in η-ξ spacetime are discussed. By the path-integral formulation of quantum field theory, we show that the (zero-temperature) Green's functions for Dirac fields on the Euclidean section in η-ξ spacetime are equal to the imaginary-time thermal Green's functions in Minkowski spacetime, and that the (zero-temperature) Green's functions on the Lorentzian section in η-ξ spacetime correspond to the real-time thermal Green's functions in Minkowski spacetime. The antiperiodicity of fermion fields in η-ξ spacetime originates from Lorentz transformation properties of the fields.

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In our previous paper [1] (hereafter referred to as I), a new spacetime, called η-ξ spacetime, was constructed. The scalar quantum field in η-ξ spacetime was discussed. The main conclusion was that the quantum fields in η-ξ spacetime relate closely to quantum field theory at finite temperature in Minkowski spacetime and (zero-temperature) quantum field theory in η-ξ spacetime corresponds to field theory at finite temperature in Minkowski spacetime. The geometrical origin of this connection is that η-ξ spacetime can be regarded as, as pointed out by Wald [2], a maximal complex analytic extension of $S^1\times R^3$. In I it was shown that the vacuum state of scalar fields in η-ξ spacetime is a thermal state for an inertial observer in Minkowski spacetime, and the vacuum Green's functions in η-ξ spacetime are the thermal Green's functions in Minkowski spacetime. To complete our argument, we must generalize this discussion to fermion fields in η-ξ spacetime, and show how the antiperiodic boundary conditions on fermion fields in thermal equilibrium can be satisfied in Euclidean η-ξ spacetime and how to express the thermal Green's functions as the vacuum Green's functions in η-ξ spacetime.

Now we consider the fermion fields in η-ξ spacetime. Let us start from the covariant form of the action for the Dirac fields on the Lorentzian section in η-ξ spacetime,

$$I^\eta=\int d\eta\, d\xi\, dy\, dz\sqrt{-g}\,\bar\psi(i\gamma^a\nabla_a-m)\psi\ , \tag{1}$$

which is a scalar under local changes in the vierbein, as well as under general coordinate transformations, where γ^a ($a=0,1,2,3$) are the Dirac matrices, and the covariant derivative is defined by

$$\nabla_a=e_a{}^\mu(\partial_\mu+\Gamma_\mu),\quad \Gamma_\mu=\tfrac{1}{2}\Sigma^{ab}\omega_{ab\mu} \tag{2}$$

the spin connection $\omega_{ab\mu}$ is defined by [3]

$$\omega_{ab\mu}=\tfrac{1}{2}e_a{}^\nu(e_{b\nu,\mu}-e_{b\mu,\nu})-\tfrac{1}{2}e_b{}^\nu(e_{a\nu,\mu}-e_{a\mu,\nu})-\tfrac{1}{2}e_a{}^\nu e_b{}^\sigma(e^c{}_{\nu,\sigma}-e^c{}_{\sigma,\nu})e_{c\mu}\ , \tag{3}$$

here $e_a{}^\nu$ are the vierbeins, Σ^{ab} are the generators of Lorentz transformation on the spinor. (Greek indices are referred to η-ξ coordinates, and Latin indices are referred to the local Lorentz frame.) As usual, we have

$$e_a{}^\mu e_{b\mu}=\eta_{ab}=(-1,1,1,1)$$

and

$$g_{\mu\nu}=e_{a\mu}e_{b\nu}\eta^{ab},\quad \sqrt{-g}=|e_{a\mu}|\ .$$

It is convenient to choose the vierbein components $e_a{}^\mu$ and $e_{a\mu}$ in η-ξ coordinates as shown in Table I. The local frames are oriented so that the axes are parallel to the coordinate axes η, ξ, y, and z. The nonvanishing elements of the connection Γ_μ in all of regions I,II, and III,IV are

$$\Gamma_\eta=\Sigma^{01}\frac{\xi}{\xi^2-\eta^2},\quad \Gamma_\xi=-\Sigma^{01}\frac{\eta}{\xi^2-\eta^2}\ . \tag{4}$$

Now the action (1) can be written as

$$I^\eta=\int_{\rm I,II}d\eta\, d\xi\, dx_\perp\frac{1}{\alpha^2(\xi^2-\eta^2)}\bar\psi\left[i\varepsilon\alpha(\xi^2-\eta^2)^{1/2}\left[\gamma^0\frac{\partial}{\partial\eta}+\gamma^1\frac{\partial}{\partial\xi}\right]+i\gamma^\perp\cdot\nabla_\perp-m+\frac{i\varepsilon\alpha}{2}\frac{\gamma^0\eta-\gamma^1\xi}{(\xi^2-\eta^2)^{1/2}}\right]\psi$$
$$+\int_{\rm III,IV}d\eta\, d\xi\, dx_\perp\frac{1}{\alpha^2(\eta^2-\xi^2)}\bar\psi\left[i\varepsilon\alpha(\eta^2-\xi^2)^{1/2}\left[\gamma^0\frac{\partial}{\partial\xi}+\gamma^1\frac{\partial}{\partial\eta}\right]+i\gamma^\perp\cdot\nabla_\perp-m+\frac{i\varepsilon\alpha}{2}\frac{\gamma^0\xi-\gamma^1\eta}{(\eta^2-\xi^2)^{1/2}}\right]\psi\ , \tag{5}$$

*Mailing address.

where $\varepsilon=1(-1)$ for regions I and III (II and IV), $x_\perp$ denotes y,z coordinates. Introduce the transformation $(\eta,\xi)\to(\lambda,X)$, where λ and X defined by

$$\frac{\partial}{\partial\lambda}=\varepsilon\alpha\left[\xi\frac{\partial}{\partial\eta}+\eta\frac{\partial}{\partial\xi}\right],\quad \frac{\partial}{\partial X}=\alpha\left[\eta\frac{\partial}{\partial\eta}+\xi\frac{\partial}{\partial\xi}\right]. \tag{6}$$

The action (5) becomes

$$I^\eta=\int d^4X\,\bar\psi\left[i\Sigma^0\frac{\partial}{\partial(\varepsilon\lambda)}+i\Sigma^1\frac{\partial}{\partial X}+i\gamma^\perp\cdot\nabla_\perp -m-\frac{i\alpha}{2}\Sigma^1\right]\psi\,, \tag{7}$$

where

$$\Sigma^0=\gamma^0\cosh(\varepsilon\alpha\lambda)-\gamma^1\sinh(\varepsilon\alpha\lambda)\,,$$
$$\Sigma^1=-\gamma^0\sinh(\varepsilon\alpha\lambda)+\gamma^1\cosh(\varepsilon\alpha\lambda)\,,$$
$$d^4X=\varepsilon\,d\lambda\,dX\,dy\,dz$$

and the integral $\int d^4X$ runs over the whole η-ξ space-time, i.e., over all of regions I, II, III, and IV.

It is useful to make a change of spinor

$$\psi\to(\cosh\tfrac{1}{2}\varepsilon\alpha\lambda-\gamma^0\gamma^1\sinh\tfrac{1}{2}\varepsilon\alpha\lambda)\psi\,,$$
$$\bar\psi\to\bar\psi(\cosh\tfrac{1}{2}\varepsilon\alpha\lambda+\gamma^0\gamma^1\sinh\tfrac{1}{2}\varepsilon\alpha\lambda)\,. \tag{8}$$

Then Eq. (7) has the form

$$I^\eta=\int d^4X\,\bar\psi\left[i\gamma^0\frac{\partial}{\partial(\varepsilon\lambda)}+i\gamma^1\frac{\partial}{\partial X}+i\gamma^\perp\cdot\nabla_\perp-m\right]\psi$$
$$\equiv\int d^4X\,\bar\psi(i\gamma\cdot D-m)\psi\,. \tag{9}$$

If restricting the integral domain in (9) only to region I, we obtain the action for Dirac fields in Minkowski space-time:

$$I=\int d^4x\,\bar\psi(x)(i\gamma\cdot\partial-m)\psi(x)\,. \tag{10}$$

We shall see below that the transformation (8) plays a critical role for getting thermal properties. In order to make the sense of the transformation (8) clear, we now go through another way to get Eq. (9) starting from the vierbein form (1). Using the coordinate λ and X in (1) and choosing the new vierbeins $\bar e_{\bar a}{}^{\mu}$, which are parallel to the coordinate axes $\varepsilon\lambda,X,y,z$, we get Eq. (9) immediately. It shows that Eq. (9) is the expression of the action (1) on the vierbeins $\bar e_{\bar a}{}^{\mu}$.

Note that the vierbeins $\bar e_{\bar a}{}^{\mu}$ relate $e_a{}^{\mu}$ by Lorentz boost

$$L_{\bar a}{}^{a}=\begin{bmatrix}\cosh\varepsilon\alpha\lambda & \sinh\varepsilon\alpha\lambda\\ \sinh\varepsilon\alpha\lambda & \cosh\varepsilon\alpha\lambda\end{bmatrix} \tag{11}$$

so that

$$\bar e_{\bar a}{}^{\mu}=L_{\bar a}{}^{a}e_a{}^{\mu}\,, \tag{12}$$

where we have omitted the y,z coordinates, which are trivial here. The corresponding matrix of the transformation of spinor is given by

$$M=\exp(\tfrac{1}{2}\varepsilon\alpha\lambda\gamma^0\gamma^1)$$
$$=\cosh\left[\frac{\varepsilon\alpha\lambda}{2}\right]-\gamma^0\gamma^1\sinh\left[\frac{\varepsilon\alpha\lambda}{2}\right], \tag{13}$$

which leads to the transformation (8). That is, the transformation (8) is necessary if we transform vierbeins from

TABLE I. (a) Vierbein components $e_a{}^{\mu}$ for η-ξ coordinates. The local frames are oriented so that the axes are parallel to the coordinate axes. The upper signs refer to regions I,III while the lower signs refer to regions II,IV. (b) Vierbein components $e_{a\mu}$ for η-ξ coordinates. The sign conventions are the same as for (a).

Lorentz index \ η-ξ coordinate index	η	ξ	y	z
$e_a{}^{\mu}$		(a)		
0	$\pm\alpha(\xi^2-\eta^2)^{1/2}$ for regions I,II	$\pm\alpha(\eta^2-\xi^2)^{1/2}$ for regions III,IV	0	0
1	$\pm\alpha(\eta^2-\xi^2)^{1/2}$ III,IV	$\pm\alpha(\xi^2-\eta^2)^{1/2}$ I,II	0	0
2	0	0	1	0
3	0	0	0	1
$e_{a\mu}$		(b)		
0	$\mp\alpha^{-1}(\xi^2-\eta^2)^{-1/2}$ I,II	$\mp\alpha^{-1}(\eta^2-\xi^2)^{-1/2}$ III,IV	0	0
1	$\pm\alpha^{-1}(\eta^2-\xi^2)^{-1/2}$ III,IV	$\pm\alpha^{-1}(\xi^2-\eta^2)^{-1/2}$ I,II	0	0
2	0	0	1	0
3	0	0	0	1

$e_a{}^\mu$ to $\bar{e}_{\bar{a}}{}^\mu$ (in fact, the vierbeins $\bar{e}_{\bar{a}}{}^\mu$ are equivalent to Minkowski coordinates t,x).

Now we shall show that the Euclidean generating functional in η-ξ spacetime is proportional to the partition function at temperature $T=\alpha/2\pi$ in Minkowski spacetime. The Euclidean generating functional for Dirac fields in η-ξ spacetime has the form (the shorthand where the normalization factor is suppressed will often be used in the following)

$$Z=\int D\psi D\bar{\psi}\exp\left\{-\int d\sigma\, d\xi\, dx_\perp \frac{1}{\alpha^2(\xi^2+\sigma^2)}\bar{\psi}\left[\alpha(\xi^2+\sigma^2)^{1/2}\left[\gamma^0\frac{\partial}{\partial\sigma}+i\gamma^1\frac{\partial}{\partial\xi}\right]+i\gamma^\perp\cdot\nabla_\perp-m+\frac{i\alpha}{2}\frac{i\gamma^0\sigma-\gamma^1\xi}{(\xi^2+\sigma^2)^{1/2}}\right]\psi\right\}. \tag{14}$$

Under the transformation

$$\frac{\partial}{\partial\tau}=\alpha\left[\xi\frac{\partial}{\partial\sigma}-\sigma\frac{\partial}{\partial\xi}\right],\quad \frac{\partial}{\partial x}=\alpha\left[\sigma\frac{\partial}{\partial\sigma}+\xi\frac{\partial}{\partial\xi}\right] \tag{15}$$

with the change of spinor corresponding to (8),

$$\psi\rightarrow\left[\cos\frac{\alpha\tau}{2}-i\gamma^0\gamma^1\sin\frac{\alpha\tau}{2}\right]\psi,\quad \bar{\psi}\rightarrow\bar{\psi}\left[\cos\frac{\alpha\tau}{2}+i\gamma^0\gamma^1\sin\frac{\alpha\tau}{2}\right], \tag{16}$$

Eq. (14) becomes

$$Z=\int_{\psi(\tau=0)=-\psi(\tau=\beta)}D\psi D\bar{\psi}\exp\left[-\int_0^\beta d\tau\int d^3x\,\bar{\psi}\left[\gamma^0\frac{\partial}{\partial\tau}+i\gamma^1\frac{\partial}{\partial x}+i\gamma^\perp\cdot\nabla_\perp-m\right]\psi\right], \tag{17}$$

where we have used the following facts.

(1) If we want the transformation (15) to be single valued, we must have $0\le\alpha\tau\le 2\pi$, or $0\le\tau\le 2\pi/\alpha\equiv\beta$. It determines the integral domain for τ in (17).

(2) The transformation (16) leads to the antiperiodic boundary condition for Dirac fields in (17):

$$\psi(\tau=0)=-\psi(\tau=\beta)\ . \tag{18}$$

Equation (17) is the path-integral expression of the partition function $\mathrm{Tr}(e^{-\beta H})$ in Minkowski spacetime. The thermal properties of fermion fields in (17) are characterized by the antiperiodic boundary condition (18), which comes from the transformation property (8) of spinor under Lorentz transformation (11). Note that the vierbeins $\bar{e}_{\bar{a}}{}^\mu$ are, in fact, Minkowski coordinates $\mathbf{x},t$, and the spinors ψ in (9) and (17) are defined with respect to the vierbeins $\bar{e}_{\bar{a}}{}^\mu$, so the thermal properties in (17) are measured only by a static observer in Minkowski spacetime.

We are now in the position to calculate the Green's functions. It is well known [4] that for thermal Green's functions, its imaginary-time form is characterized by the imaginary-time periodicity (antiperiodicity) for boson (fermion) fields, and its real-time form is characterized by the doubling of the degrees of freedom. In the η-ξ formulation, the structure of the Euclidean section in η-ξ spacetime will automatically provide periodicity for imaginary time τ, the imaginary-time periodicity (antiperiodicity) of Green's functions for boson (fermion) fields on the Euclidean section in η-ξ spacetime will be given by Lorentz transformation properties of the fields. On the Lorentzian section in η-ξ spacetime, the existence of "horizons" will lead to the doubling of the degrees of freedom, corresponding to the fields in regions I and II. It is the character of real-time thermal Green's functions. Now we shall show that the (zero-temperature) Green's functions for Dirac fields on the Euclidean section in η-ξ spacetime are equal to the imaginary-time thermal Green's functions in Minkowski spacetime, and the (zero-temperature) Green's functions on the Lorentzian section in η-ξ spacetime correspond to the real-time thermal Green's functions in Minkowski spacetime.

References [5] and [6] have told us how to express a temperature Green's function for Dirac fields by the path-integral formulation

$$G_\beta(\mathbf{x}_1,\tau_1;\mathbf{x}_2,\tau_2)=Z^{-1}\frac{\delta^2}{\delta\bar{J}(\mathbf{x}_2,\tau_2)\delta J(\mathbf{x}_1,\tau_1)}\times\int_{\psi(\tau=0)=-\psi(\tau=\beta)}D\bar{\psi}D\psi\exp\left[-\int_0^\beta d\tau\int d^3x\,\bar{\psi}\left[\gamma^0\frac{\partial}{\partial\tau}+i\gamma^1\frac{\partial}{\partial x}+i\gamma^\perp\cdot\nabla_\perp-m\right]\psi+\bar{J}\psi+\bar{\psi}J\right]_{J=\bar{J}=0}. \tag{19}$$

Using the change (15) of integration variables on the exponential in (19) and the inverse transformation of (16), we obtain

$$G_\beta(\mathbf{x}_1,\tau_1,\mathbf{x}_2,\tau_2)=Z^{-1}\frac{\delta^2 Z[j,\bar{j}]}{\delta\bar{J}(\xi_2,\sigma_2)\delta J(\xi_1,\sigma_1)}\Bigg|_{J=\bar{J}=0}\equiv G_E^\eta(\xi_1,\sigma_1,\xi_2,\sigma_2)\ , \tag{20}$$

where

$$Z[j,\bar{j}]=\int D\psi D\bar{\psi}\exp\left\{-\int d\sigma\, d\xi\, dx_\perp \frac{1}{\alpha^2(\xi^2+\sigma^2)}\bar{\psi}\left[\alpha(\xi^2+\sigma^2)^{1/2}\left[\gamma^0\frac{\partial}{\partial\sigma}+i\gamma^1\frac{\partial}{\partial\xi}\right]+i\gamma^\perp\cdot\nabla_\perp -m+\frac{i\alpha}{2}\frac{i\gamma^0\sigma-\gamma^1\xi}{(\xi^2+\sigma^2)^{1/2}}\right]\psi+\bar{j}\psi+\bar{\psi}j\right\}\ , \tag{21}$$

with

$$\bar{J}(\xi,\sigma)=e^{-2\alpha x}\bar{j}(\xi,\sigma),\quad J(\xi,\sigma)=e^{-2\alpha x}j(\xi,\sigma)\ ,$$
$$\bar{j}(\xi,\sigma)=e^{2\alpha x}\bar{J}(\mathbf{x},\tau),\quad j(\xi,\sigma)=e^{2\alpha x}J(\mathbf{x},\tau)\ . \tag{22}$$

The right-hand side G_E^η of Eq. (20) is just the Euclidean (zero-temperature) Green's function for Dirac fields in η-ξ spacetime.

The generating functional on Lorentzian section in η-ξ spacetime has the form

$$Z[\bar{J},J]=\int D\psi D\bar{\psi}\exp\left[i\left[I^\eta+\int d^4X(\bar{J}\psi+\bar{\psi}J)\right]\right]\ , \tag{23}$$

where I^η is given by (9). By taking the transformation

$$\psi(X)\to\psi(X)-\int d^4Y\, S_\lambda(X-Y)J(Y)\ ,$$
$$\bar{\psi}(X)\to\bar{\psi}(X)-\int d^4Y\,\bar{J}(Y)S_\lambda(Y-X)\ ,$$

where

$$(i\gamma\cdot D-m)S_\lambda(X-Y)=\delta^4(X-Y) \tag{24}$$

we can write Eq. (23) as

$$Z[\bar{J},J]=\exp\left[i\int d^4X\, d^4Y[\bar{J}(X)S_\lambda(X,Y)J(Y)]\right]\ . \tag{25}$$

The integral $\int d^4X\, d^4Y$ runs over all of regions I, II, III, and IV, in which the regions III and IV are spacelike for time λ. We can drop these spacelike regions by letting $J(X)=\bar{J}(Y)=0$ if $X,Y\in$ III or IV, while noting that the regions I and II are spacelike disjoint, Eq. (25) then becomes

$$Z[\bar{J},J]=\exp\left[-i\int d^4x\, d^4y\,\bar{J}_a(x)S^{ab}(x-y)J_b(y)\right]\ , \tag{26}$$

where $a,b=1,2$ and

$$\begin{aligned}
&J_1(x)=J(X),\quad X\in\text{I}\ ,\\
&J_2(x)=-J(X),\quad X\in\text{II}\ ,\\
&S_\lambda^{11}(x-y)=S_\lambda(X-Y),\quad X,Y\in\text{I}\ ,\\
&S_\lambda^{22}(x-y)=S_\lambda(X-Y),\quad X,Y\in\text{II}\ ,\\
&S_\lambda^{12>}(x-y)=-S_\lambda(X-Y),\quad X\in\text{I},\ Y\in\text{II}\ ,\\
&S_\lambda^{21<}(x-y)=-S_\lambda(X-Y),\quad X\in\text{II},\ Y\in\text{I}\ .
\end{aligned} \tag{27}$$

Equation (26) is the generating functional for the real-time thermal Green's functions [7], in which the thermal propagator is a 2×2 matrix. In order to get Eq. (26), we let $J(X)$ and $\bar{J}(Y)$ in regions III and IV be zero here. If we keep $J(X)$ and $\bar{J}(Y)$ in regions III and IV not zero, we will get, similar to the Rindler case [8], the real-time thermal Green's functions with 4×4 matrix.

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