When can identical particles collide?

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It is customary, when discussing configuration spaces of identical particles in two or more dimensions, to discard the configurations where two or more particles overlap, the justification being that the configuration space ceases to be a manifold at those points, and also to allow for nonbosonic statistics. We show that there is in general a loss of physical information in discarding these points by studying the simple system of two free particles moving in the plane and requiring that the Hamiltonian be self-adjoint. We find that the Hamiltonian for fermions is unique, but that in all other cases (i.e., for particles obeying properly fractional or Bose statistics) there is a one-parameter family of possible self-adjoint extensions. We show how a plausible limiting procedure selects a unique extension from each family, the favored extension being the one for which the wave function remains finite at the points of overlap. We also test our procedure by applying it to the known case of the hydrogen atom.

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I. INTRODUCTION

The question posed in the title of this paper arises in connection with particle statistics, and specifically in connection with what might be called "intrinsic" or "topological" formulations of this fundamental trait of quantum particles. In formulations of this sort, the mutual indistinguishability of the particles is coded into the topology of configuration space itself, and the different statistical types then correspond to different choices of boundary conditions on the wave function Ψ, or—more generally and more invariantly—to different choices of the underlying vector bundle of which Ψ is a section. (In a path-integral formulation the different bundles or boundary conditions would correspond to different possible global phase factors, or their generalization.)

In comparison with older formulations in which the configuration space is topologically trivial and statistics is defined in terms of the behavior of Ψ under "exchange of labels," the intrinsic approach is more natural in that it avoids the introduction of "gauge variables" (the labels) which only have to be eliminated at a later stage by imposing symmetry conditions on Ψ. It is also more general, in that it allows particle statistics to be understood as a kind of "force" in essence similar to other interactions with a topological character, like the interaction between an electric and magnetic charge in three spatial dimensions, or the type of interaction in two dimensions which is responsible for the Bohm-Aharonov effect and fractional statistics ("anyons") [1,2]. Indeed all these examples appear as part of a much broader range of situations in which the topology of an appropriate configuration space manifests itself in the possibility of nontrivial quantum bundles over that space, whose structure captures qualitatively important aspects of the dynamics. Such situations include, for example, the "θ-vacuum" phenomenon of non-Abelian gauge theories and the possibility of spin $\frac{1}{2}$ in pure quantum gravity [3] or in SU(N)-chiral models (Skyrmions) [4], as well as a whole range of other phenomena of "Berry-phase" type, such as the fact that the nonrelativistic wave function cannot transform as a scalar under Galilean transformations [5].

Although the "intrinsic" or "topological" approach to particle statistics has clarified many issues and unified many phenomena, it has also introduced a technical complication which is absent in the older, more limited approach, namely, the introduction of configuration spaces which are not (smooth) manifolds. In the older approach, the configuration space for two identical spinless particles in three dimensions, for example, would be $\mathbb{R}^3\times\mathbb{R}^3$, on which a free Hamiltonian $\nabla_1^2+\nabla_2^2$ can unambiguously be defined. The intrinsic attitude, however, identifies $(x,y)\in\mathbb{R}^3\times\mathbb{R}^3$ with (y,x), leading to a configuration space Q which has a (n orbifold-type) singularity at each "diagonal" configuration (x,x). At such singular points of Q it is not obvious how to introduce the mathematical objects in terms of which a Hamiltonian is normally defined (gradient operator). What is worse, the presence of these singular points might even seem to destroy the possibility of nontrivial statistics altogether, because, for example, the configuration space for two identical particles no longer possesses nontrivial (i.e., noncontractible) loops if the diagonal points are included in Q (cf. [6]).

Now this last difficulty is only apparent (only an inconvenience) because the physical meaning of statistics resides in processes of *exchange* of identical partners (more precisely in the correlative interference effects), and such

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exchange necessarily refers to paths in Q which *avoid* the diagonal, it being no longer possible to say which particle is which, once they have coincided. Hence a specification of the dynamics *away* from the points of overlap already defines the statistics, independently of what may occur when two or more particles coincide.[1] Therefore, if including the diagonal in Q renders an exchange-loop trivial, this can only mean that the most straightforward topological definition of the statistical phase factors breaks down at points of coincidence; it cannot of itself imply that nontrivial statistics is no longer possible. Indeed, it is not at all obvious just what the inclusion of the diagonal points entails for the dynamics, because the singularity of Q there prevents the Schrödinger equation from being unambiguously meaningful there, as we have already emphasized.

It is this ambiguity in the dynamics at the points of coincidence that represents the genuine incompleteness in what we have called the intrinsic formulations of the quantum mechanics of identical particles. Until appropriate dynamical rules have been specified at these singular points the effects of actual collisions remain undetermined, even if the particles' statistical type has already been defined. A similar incompleteness pertains to the (nonrelativistic) hydrogen atom, where the overlap of a point proton and a point electron is a singular configuration, not for reasons of particle identity, but because the electrostatic potential becomes infinite there.

The appropriate mathematical setting for studying these dynamical ambiguities would seem to depend on the framework adopted for quantum mechanics in general. Here we will take the Hamiltonian operator as basic (as opposed to the path amplitudes of the sum over histories) in order to take advantage of the extensive body of results available concerning the mathematics of linear operators in Hilbert space. In this setting our question about the boundary conditions for the Schrödinger equation appears as a question about self-adjoint extensions: we know how to define a Hamiltonian H_0 away from the diagonal of Q and we wish to extend it to a self-adjoint[2] operator H_{sa} for all of $L^2(Q)$.

In what follows we examine this problem in the relatively simple case of a pair of identical, structureless, free particles moving in two spatial dimensions. We will see that the answer to our title question depends on the value of the phase angle θ defining the "fractional statistics" of the particles.[3] In the fermionic case ($\theta=\pi$) collision is strictly forbidden, the self-adjoint extension being unique; but in all other cases a U(1)'s worth of Hamiltonians is conceivable, most of which allow collisions (in the sense that Ψ need not vanish on the diagonal of Q), but some of which do not. However, we can try to remove this ambiguity in the dynamics by means of a suitable regularization at the points of overlap. Doing so in a natural way, and taking the limit as the regularization is removed, will lead us to a unique choice of Hamiltonian for each value of θ in the range of ambiguity $-\pi<\theta<\pi$. With respect to this unique choice we will find that collisions are allowed *only* in the bosonic case ($\theta=0$); in all other cases the particles avoid each other. In every case, moreover, the self-adjoint extension picked out is the unique one for which Ψ remains finite at the points of overlap. In this way, we will have remedied the incompleteness in the intrinsic formulation of statistics.

As a check on our procedure we examine in the Appendix the known example of the nonrelativistic hydrogen atom, where there is an analogous problem of extending the Hamiltonian to the singular configuration where proton and electron overlap. Again it turns out that a unique extension is selected, and in this case it is the one known by experiment to be correct.

II. CONFIGURATION SPACE FOR TWO IDENTICAL PARTICLES

The configuration space of two identical structureless particles in two space dimensions is, in the center-of-mass coordinate system, a plane with diametrically opposite points with respect to the origin identified. This is equivalent to a cone of half-angle 30° (defect angle 180°), the vertex of the cone corresponding to the overlap of the two particles [1]. A trajectory in configuration space which exchanges the two particles corresponds to a closed loop on the cone encircling the vertex. Exchanging the particles twice gives a curve encircling the vertex twice and this curve also cannot be continuously deformed to a point if we exclude the vertex as a possible configuration. The "punctured" configuration space $\mathring{Q}$ is thus infinitely connected. This means that, unlike in three or higher space dimensions, the wave function of the system may acquire a phase after the exchange of the two particles which does not square to one. The particles are said to obey fractional statistics characterized by the angle θ acquired in the phase after (counter-clockwise) exchange. This angle can take any value and one recovers Fermi ($\theta=\pi$) or Bose ($\theta=0$) statistics for particular values of θ.

A more abstract description of the meaning of θ is in terms of the fundamental group $\pi_1(\mathring{Q})$ of $\mathring{Q}$. It is known

[1] A possible exception to these remarks might occur, in the path-integral framework, if one were dealing with differentiable paths; for them it would sometimes be possible to still recognize which particle was which after a collision occurred. Another possible exception might be for particles moving on a line, where exchange without collision is impossible (unlike on the circle, or in more highly ramified one-dimensional networks [7]). In that situation the only remnant of statistics which can be defined refers to boundary conditions at the diagonal points, which now constitute a boundary of Q.

[2] Self-adjointness is needed in order that one can exponentiate H to get a unitary operator of time evolution. In certain situations this "conservation of probability" might not be appropriate (e.g., if the particles could annihilate on contact), but we will restrict ourselves here to situations for which it is appropriate.

[3] Notice, incidentally, that for $\theta\neq 0,\pi$ the points of overlap are singular even in the traditional setting of labeled particles.

[6,8,9] that there is an inequivalent quantum theory on a given configuration space Q for each distinct unitary irreducible representation (UIR) of $\pi_1(Q)$. Since $\pi_1(\mathring{Q})=\mathbb{Z}$ here, its UIR's are given by $\rho_\theta: n \to e^{in\theta}$ and possible quantum theories are labeled by the angle $\theta(\mathrm{mod}2\pi)$ which parametrizes the UIR.

A still more general description of the same system can be had in terms of a bundle [8,9]. If the particles obeying fractional statistics do not overlap, and moreover if they are "free," then their dynamics can be globally described in terms of a U(1) bundle over $\mathring{Q}$ with a locally flat connection. Now suppose that we remove the restriction on overlapping by restoring the origin to $\mathring{Q}$, obtaining thereby an enlarged configuration space $\bar{Q}$ with the topology of a complete cone. There is no topological obstruction to extending our bundle to all of $\bar{Q}$ (unlike for fermions in three spatial dimensions or higher), but there is a problem with smoothness, and therefore with the notion of connection. In the special cases $\theta=0$, π we might evade this difficulty by passing to the twofold cover of $\bar{Q}$ (i.e., by labeling the particles), but we know of no similar trick which would apply for other values of θ.

In the absence of such guidance, we will fall back on the general theory of self-adjoint extensions, only noting on one hand, the question whether different such extensions can be related to any generalization of the notion of bundle with connection (called "gauge space" in [10]), and on the other hand, the possible answer that the question is moot, because the physically favored extensions will turn out to involve only wave functions which vanish at the origin (except for $\theta=0$, where the covering-space is regular).

III. SELF-ADJOINT EXTENSIONS AND THE HAMILTONIAN

Away from the origin, the Hamiltonian for a system of two identical particles moving freely in the plane is given, in c.m. coordinates, by the differential operator,

$$H\tilde{\Psi}=-\frac{\hbar^2}{2\mu}\nabla^2\tilde{\Psi}=-\frac{\hbar^2}{2\mu}\left[\frac{1}{r}\frac{\partial\tilde{\Psi}}{\partial r}+\frac{\partial^2\tilde{\Psi}}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2\tilde{\Psi}}{\partial\varphi^2}\right], \tag{1}$$

where $\mu=m/2$ is the reduced mass and (r,φ) are the relative radius and angle coordinates. However, for nonzero statistics parameter θ, $\tilde{\Psi}$ satisfies discontinuous boundary conditions: the exchange of the two particles corresponds to the coordinate shift $\varphi\to\varphi+\pi$ in configuration space, and the wave function acquires a phase according to

$$\tilde{\Psi}(r,\varphi+\pi)=e^{i\theta}\tilde{\Psi}(r,\varphi)\ , \tag{2}$$

the points (r,φ) and $(r,\varphi+\pi)$ being identified.

Equations (1) and (2) describe the system in terms of multivalued wave functions with no (nontrivial) connection or potential. As referred to earlier, we can equivalently work with single valued wave functions Ψ by introducing an explicit interaction term (or bundle connection). The two different descriptions just represent different choices of gauge with respect to a single U(1) bundle with a (locally flat) connection. The first choice is such as to trivialize the connection (and is therefore necessarily discontinuous), the second is the simplest continuous gauge choice. In the second description different values of θ are explicitly associated with different interactions (different values of the connection):

$$H\Psi=-\frac{\hbar^2}{2\mu}\left[\frac{1}{r}\frac{\partial\Psi}{\partial r}+\frac{\partial^2\Psi}{\partial r^2}+\frac{1}{r^2}\left[\frac{\partial}{\partial\varphi}+\frac{i\theta}{\pi}\right]^2\Psi\right], \tag{3}$$

$$\Psi(r,\varphi+\pi)=\Psi(r,\varphi)\ . \tag{4}$$

The two Hamiltonian operators are related by

$$H=\exp\left[-\frac{i\theta}{\pi}\varphi\right]\tilde{H}\exp\left[\frac{i\theta}{\pi}\varphi\right].$$

As is made explicit in the description (3), (4), our boundary condition on Ψ can be expressed by saying that it is a continuous cross section of a trivial U(1) bundle over the configuration space Q. (Notice that the bundles corresponding to different θ *must* all be topologically equivalent to the trivial bundle, since one can continuously change the parameter θ to take any value.) Thus the only distinction among different statistics resides in the connection; if it were not locally flat, it could not be characterized by a single parameter θ, and statistics would have no gauge independent meaning. This differs strikingly from the situation in three (or higher) dimensions, where the Bose and Fermi bundles differ topologically, and the meaning of statistics is altogether independent of any connection.

Now, the operator in (3) is singular at the origin for two reasons: As mentioned earlier, the particle is moving on a cone (because of the identification of x with $-x$ in $\mathbb{R}^2$ that expresses the identity of the two particles), and the cone is not smooth at the origin. Moreover, the connection term $(\partial/\partial\varphi+i\theta/\pi)$ is itself singular at $r=0$. We are now interested in checking whether the Hamiltonian admits any self-adjoint extensions at the singular point where the two particles overlap, and if so, how many. We will use Von Neumann's theory of deficiency indices [11] to answer these questions.

Let us review some definitions.

A densely defined linear operator T on a separable Hilbert space is called symmetric if and only if $T\subset T^\dagger$ ($T^\dagger$ being the adjoint operator), that is, if and only if $D(T)\subset D(T^\dagger)$ and $T\Phi=T^\dagger\Phi$ for all $\Phi\in D(T)$. (D is the domain of T.) This is equivalent to saying that T is symmetric if and only if $\langle T\Phi|\Psi\rangle=\langle\Phi|T\Psi\rangle$ for all Φ, $\Psi\in D(T)$. An operator T is called self-adjoint if and only if $T=T^\dagger$, that is, if and only if T is symmetric and $D(T)=D(T^\dagger)$. This last statement implies that if the functions in $D(T)$ obey certain boundary conditions in order that T be symmetric, then the functions in $D(T^\dagger)$ must obey the same boundary conditions.

Now suppose that T is a self-adjoint operator and that there is a $\Phi\in D(T^\dagger)=D(T)$ for which $T^\dagger\Phi=i\Phi$. Then $T\Phi=i\Phi$ and $-i\langle\Phi|\Phi\rangle=\langle i\Phi|\Phi\rangle=\langle T\Phi|\Phi\rangle=\langle\Phi|T^\dagger\Phi\rangle=\langle\Phi|T\Phi\rangle=i\langle\Phi|\Phi\rangle$, whence $\Phi=0$. A similar proof shows that $T^\dagger\Phi=-i\Phi$ can have no solutions (as is well known, the spectrum must be real). Let

us look at the converse statement: if T is a symmetric operator and $T^\dagger\Phi=\pm i\Phi$ has no solutions, then T is essentially self-adjoint, in the sense that its *closure* $T^{\dagger\dagger}$ is self-adjoint. This is the basic criterion for self-adjointness [11].

Now, let

$$\mathcal{H}_+=\mathrm{Ker}(i-T^\dagger)\ ,\quad n_+(T)=\dim[\mathcal{H}_+]\ ,$$

$$\mathcal{H}_-=\mathrm{Ker}(i+T^\dagger)\ ,\quad n_-(T)=\dim[\mathcal{H}_-]\ ,$$

$\mathcal{H}_+$ and $\mathcal{H}_-$ are called the deficiency subspaces of T, and $n_\pm$ are the deficiency indices. The theory then states the following.

Let T by a symmetric operator with deficiency indices n_+ and n_-. Then (i) T is essentially self-adjoint if and only if $n_+=0=n_-$. (ii) If $n_+=n_-=n$ then T possesses an n^2-dimensional family of self-adjoint extensions; and there is[4] a one-one correspondence between self-adjoint extensions of T and unitary maps from $\mathcal{H}_+$ onto $\mathcal{H}_-$. (iii) If $n_+\neq n_-$ then T is not self-adjoint and has no self-adjoint extensions.

Let us go back to the form (3) of our Hamiltonian, with the domain taken to be, say, the C^∞ functions of compact support on the punctured version of Q. We want to find the normalizable solutions to $H^\dagger\Psi=\pm i\Psi$ and thereby determine the possible self-adjoint extensions at the origin. (Note that $H^\dagger$ is given by the same differential operator as H itself.)

Let us set $2\mu/\hbar^2$ equal to 1. Then we are solving the equation

$$\frac{\partial^2\Psi}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\Psi+\frac{1}{r^2}\left[\frac{\partial}{\partial\varphi}+\frac{i\theta}{\pi}\right]^2\Psi=-\lambda\Psi\ , \tag{5}$$

where $\lambda=\pm i$ is the eigenvalue of $H^\dagger$. (Strictly speaking λ has dimensions of length^{-2}, but we can imagine that some particular choice of units has been made.)

Separating out the different angular fourier modes (and recalling that the period of φ is π rather than 2π), we can set

$$\Psi(r,\varphi)=\Phi(r)e^{2in\varphi}\ .$$

Then (5) becomes

$$\frac{1}{r}\frac{\Phi'(r)}{\Phi(r)}+\frac{\Phi''(r)}{\Phi(r)}-\frac{1}{r^2}(2n+\theta/\pi)^2=-\lambda\ .$$

Setting

$$\nu=|2n+\theta/\pi|\ , \tag{6}$$

the radial equation can be put in the standard Bessel form:

$$\frac{d^2\Phi(z)}{dz^2}+\frac{1}{z}\frac{d\Phi(z)}{dz}+\left[1-\frac{\nu^2}{z^2}\right]\Phi(z)=0\ , \tag{7}$$

where $z=\sqrt{\lambda}r$.

The general solution of (7) is

$$\Phi(z)=aJ_\nu(z)+bN_\nu(z)\ ,$$

where $a,b\in\mathbb{C}$ and J_ν is the ordinary Bessel function and N_ν is the Bessel function of the second kind $\{N_\nu(z)=[J_\nu(z)\cos\nu\pi-J_{-\nu}(z)]/(\sin\nu\pi)$ for ν noninteger, and $N_m(z)=\lim_{\nu\to m}N_\nu(z)$ for m integer$\}$.

We will need the asymptotic expansions of J_ν and N_ν for complex argument z. We give them here for immediate and future reference [12].

For $z\to 0$:

$$J_\nu(z)\sim\frac{1}{\Gamma(\nu+1)}\left[\frac{z}{2}\right]^\nu\ ,$$

$$N_\nu(z)\sim-\frac{\Gamma(\nu)}{\pi}\left[\frac{2}{z}\right]^\nu\ ,\quad \nu>0\ ,$$

$$N_0(z)\sim\frac{2}{\pi}\ln\left[\frac{z}{2}\right]\ .$$

For $z\to\infty$ and z away from the negative real axis:

$$J_\nu(z)\sim\left[\frac{2}{\pi z}\right]^{1/2}\cos(z-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi)\left[\sum_{k=0}^{n}(-1)^k(\nu,2k)(2z)^{-2k}+O(|z|^{-2n-2})\right]\ ,$$

$$N_\nu(z)\sim\left[\frac{2}{\pi z}\right]^{1/2}\sin(z-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi)\left[\sum_{k=0}^{n}(-1)^k(\nu,2k)(2z)^{-2k}+O(|z|^{-2n-2})\right]\ ,$$

$$(\nu,k)=\{(4\nu^2-1)(4\nu^2-3^2)\cdots[4\nu^2-(2k-1)^2]\}/(2^{2k}k!)\ ,$$

$$(\nu,0)=1\ .$$

To find the deficiency indices, we set $\lambda=\pm i$ and demand that $\Phi(z)$ be $\mathcal{L}^2[0,\infty;r\,dr]$: $\int|\Phi(z)|^2r\,dr<\infty$. Looking at the behavior of this integral for large z, we find a set of terms in $\exp(\sqrt{2}r)$. In order that these divergent terms all cancel out, one finds that there is a unique linear combination of J_ν and N_ν possible: $b=ia$ for $\lambda=i$ and $b=-ia$ for $\lambda=-i$. Let us adopt this combination. Then, by looking at the behavior of the integral near the origin, we see that we have a term in $r^{1-2\nu}$ under the integral sign. In order that the integral be finite, we need $\nu<1$, or in terms of θ, $|\theta/\pi+2n|<1$. Now let us normalize θ by the condition

$$-\pi<\theta\leq\pi\ .$$

[4]More generally there is a correspondence between closed symmetric extensions and partial isometries of $\mathcal{H}_+$ into $\mathcal{H}_-$.

Then $|\theta/\pi| \leq 1$ for all θ, and ν can be <1 only for $n=0$. Conversely, for $n=0$, we *will* have $\nu<1$, except in the fermionic case $\theta=\pi$, where there is no solution for any value of n. We thus conclude that the deficiency indices are always equal, and take the values $n_+=n_-=1$ except for $\theta=\pi$, where both vanish. The fermionic Hamiltonian is therefore unique, with the functions in its domain vanishing at $r=0$. In all other cases there is a U(1)'s worth of physically inequivalent boundary conditions available at $r=0$, the deficiency subspaces $\mathcal{H}_\pm$ being one-dimensional and spanned, respectively, by

$$\Psi_+(r,\varphi)=J_\nu(\sqrt{i}r)+iN_\nu(\sqrt{i}r)$$
$$=H_\nu^{(1)}(\sqrt{i}r)\ ,$$
$$\Psi_-(r,\varphi)=J_\nu(\sqrt{-i}r)-iN_\nu(\sqrt{-i}r)$$
$$=H_\nu^{(2)}(\sqrt{-i}r)\ ,$$

where $H_\nu^{(1,2)}(z)$ are Bessel functions of the third kind.

Let us remark that the equality of n_+ and n_- is no accident. It follows from a generalized reality-condition that H obeys: H is preserved by complex conjugation *followed by* the reflection $\varphi\to-\varphi$. For the same reason, $\Psi_\pm$ are conjugates, and therefore have the same $\mathcal{L}^2$ norm.

We may now construct the general self-adjoint extension of (3). In the present case, since the spaces $\mathcal{H}_+$ and $\mathcal{H}_-$ are one-dimensional, the only unitary maps of $\mathcal{H}_+$ into $\mathcal{H}_-$ are the maps $\Psi_+\to u\Psi_-$ where $|u|=1$. Therefore [11] the self-adjoint extensions $H_{\rm sa}$ are defined by the following domains:

$$D(H_{\rm sa})=\{\Psi+\beta(\Psi_++e^{i\alpha}\Psi_-)|\Psi\in D(H^{\dagger\dagger})\ ,\ \beta\in\mathbb{C}\}\ , \tag{8}$$

where α characterizes the self-adjoint extension. (In effect α selects that linear combination of $\Psi_\pm$ which epitomizes the boundary condition determining $D(H_{\rm sa})$. Notice also that, as pointed out in [13], α is not really dimensionless because λ is really not dimensionless. Rather, it should be regarded as equivalent to a sort of "scattering length" with which the extension is associated.)

IV. FAVORED BOUNDARY CONDITIONS ON THE WAVE FUNCTION

The differential operator (3) is singular at the origin because the particle is moving on a cone, and also because the connection is singular at $r=0$. *A priori* therefore, the Hamiltonian is not globally defined, and we have found that (3) is compatible with more than one unitary evolution, except in the fermionic case $\theta=\pi$. We will now see, however, that a plausible regularization-scheme will allow us to pick a unique self-adjoint extension (and therefore unitary evolution) for each θ.

It proves convenient to introduce two independent regularizing parameters, dealing separately with the "connection-singularity" and the "geometrical" one. To smooth out the former we can allow θ to become a function of r which goes continuously to 0 at the origin, for example,

$$\tilde{\theta}(r)=\theta(1-e^{-r^2/a^2})\ , \tag{9}$$

which reverts to the constant value θ as $a\to0$. This will define our regularized connection. We also need to smooth out the conical singularity of our configuration space. To that end we will replace its metric by the following metric of a rounded cone:

$$ds^2=dr^2+f(r)^2r^2d\varphi^2\ ,$$

where φ runs from 0 to π and $f(r)$ is some smooth function such that $f(0)=2$ and $f(\infty)=1$, for example,

$$f(r)=\mathrm{sech}(r/b)+1=\frac{1}{\cosh(r/b)}+1\ . \tag{10}$$

We then recover the pointed vertex as $b\to0$. This defines our geometric regularization.

Now let us imagine that we let the regularization-parameters a and b revert continuously to zero. If, as $a,b\to0$, our regularized Hamiltonian $\tilde{H}$ has a limit in some appropriate sense, then we can designate this limit as the "favored self-adjoint extension of H." Since we are dealing with unbounded operators, the notion of limit is not so straightforward,[5] so for convenience we will employ an easy-to-use criterion related to the concept of weak topology for bounded operators. Specifically, we will require that certain matrix elements of $\tilde{H}$ behave continuously as $a,b\to0$. In fact, we will really only need to require that these matrix elements be bounded, and will see that this very weak condition suffices to select the favored extension unambiguously. Recall now that the possible extensions $H_{\rm sa}$ of H are parametrized by the angle α which appears in (8), and which determines which of the vectors $\Psi_{\rm sa}=\Psi_++e^{i\alpha}\Psi_-$ belongs to $D(H_{\rm sa})$. Our rule will be to choose the α for which $\langle\Psi_{\rm sa}|\tilde{H}|\Psi_{\rm sa}\rangle$ is bounded as $a,b\to0$. In the Appendix we verify that this rule leads to the right answer in the case of the hydrogen atom.

With our regulated connection and metric, the (gauge-and-geometrically covariant) Laplacian becomes

$$\tilde{\nabla}^2=\left[\left(\frac{1}{r}+\frac{f'(r)}{f(r)}\right)\frac{\partial}{\partial r}+\frac{\partial^2}{\partial r^2}\right.$$
$$\left.+\frac{1}{r^2f(r)^2}\left(\frac{\partial}{\partial\phi}+\frac{i\tilde{\theta}(r)}{\pi}\right)^2\right]\ , \tag{11}$$

and the regularized Hamiltonian is $\tilde{H}=-\tilde{\nabla}^2$. We will now examine $\lim_{a,b\to0}\langle\Psi_{\rm sa}|\tilde{H}|\Psi_{\rm sa}\rangle$, where

$$\Psi_{\rm sa}=\Psi_++e^{i\alpha}\Psi_-\ . \tag{12}$$

In order to check for square integrability of the expectation value, we need to look at the behavior as $r\to0$ and make sure that the divergence at the origin cancels out.

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[5] See [14] for one possible definition. Other possibilities might be to work with the bounded operators $\exp(i\tilde{H}t)$, or with the spectral projection operators corresponding to $\tilde{H}$ (i.e., to say that a sequence of operators converges if the corresponding eigenvalues and eigensubspaces converge).

We will need the asymptotic forms of $H_\nu^{(1,2)}$ for small z:

$$H_\nu^{(q)}(z)\sim \mp \frac{i\Gamma(\nu)}{\pi}\left[\frac{2}{z}\right]^\nu, \quad q=1,2 \text{ and } \nu\neq 0,$$
$$H_0^{(q)}(z)\sim \mp \left[i\frac{2}{\pi}\ln\frac{2}{z}\right], \quad q=1,2. \tag{13}$$

Also, we have

$$\Psi_{sa}=H_\nu^{(1)}(\sqrt{i}\,r)+e^{i\alpha}H_\nu^{(2)}(\sqrt{-i}\,r).$$

Now, we can write from (11) (and noting that Ψ_{sa} is independent of φ):

$$\langle\Psi_{sa}|\tilde{H}|\Psi_{sa}\rangle=\langle\Psi_{sa}|H_{sa}|\Psi_{sa}\rangle + \left\langle\Psi_{sa}\left|-\frac{f'}{f}\frac{\partial}{\partial r}+\frac{\tilde{\theta}^2 f^{-2}-\theta^2}{\pi^2 r^2}\right|\Psi_{sa}\right\rangle. \tag{14}$$

We know that $\langle\Psi_{sa}|H_{sa}|\Psi_{sa}\rangle$ is bounded since Ψ_{sa} is in the domain of H_{sa}, so we need only consider the last two terms in (14).

We are interested in the integrand for small r, so we will need the asymptotic expansion of the function $f(r)$ and related terms for $r\to 0$:

$$f(r)\sim 2-\tfrac{1}{2}(r/b)^2,$$
$$\frac{1}{f^2}\sim\tfrac{1}{4}[1+\tfrac{1}{2}(r/b)^2],$$
$$f'/f\sim -\frac{1}{2b}r/b.$$

Notice that the general forms $f^{-2}\sim\frac{1}{4}+O(r^2)$ and $f'/f\sim O(r)$ are independent of the specific ansatz (10). Since our conclusions will rest only on these general forms, they will be independent of the particular f we choose.

We first consider the case $\nu=|\theta/\pi|\neq 0$. For this case, $\langle\Psi_{sa}|\tilde{H}|\Psi_{sa}\rangle-\langle\Psi_{sa}|H_{sa}|\Psi_{sa}\rangle$ will have two terms, one coming from the derivative term on the right-hand-side of (14), and the other from the expression involving the connection. Let us begin by asking whether these terms are even finite when $a,b\neq 0$. Inasmuch as $(f'/f)\partial/\partial r\sim r\partial/\partial r$ near $r=0$, it is plain that the first of the two terms just mentioned will not diverge faster than $|\Psi|^2$ at $r=0$; hence it must be integrable because Ψ itself is $\mathcal{L}^2$ by construction. However the second term

$$\left\langle\Psi_{sa}\left|\frac{\tilde{\theta}^2 f^{-2}-\theta^2}{\pi^2 r^2}\right|\Psi_{sa}\right\rangle \tag{15}$$

is more troublesome. Since $\tilde{\theta}(r=0)=0$, this term behaves near $r=0$ like

$$\left[\frac{\theta}{\pi}\right]^2\int \pi r\,dr\,r^{-2}|\Psi_{sa}|^2\sim\nu^2\int\frac{dr}{r}|\Psi_{sa}|^2.$$

From (13) we will have $\Psi_{sa}\sim r^{-\nu}$ for a generic choice of $e^{i\alpha}$, and then the integral will diverge for all positive values of ν. This means that Ψ_{sa} will not even be in the domain of $\tilde{H}$ for such extensions, and our "weak boundedness" criterion will be badly violated. The only self-adjoint extension which escapes this infinity is that for which $e^{i\alpha}$ is chosen so that the $r^{-\nu}$ divergences in Ψ_+ and Ψ_- cancel each other. From (13) it is easy to compute that the coefficient of $r^{-\nu}$ in Ψ_{sa} is proportional to $e^{i\alpha}e^{i\pi\nu/4}-e^{-i\pi\nu/4}$, so the desired cancellation occurs for the unique choice of self-adjoint extension,

$$e^{i\alpha}=e^{-i\pi\nu/2}, \tag{16}$$

that is for $\alpha=-\pi\nu/2$. For this special extension, Ψ_{sa} is finite at the origin, behaving there like r^ν rather than $r^{-\nu}$, and the integral giving $\langle\Psi_{sa}|\tilde{H}|\Psi_{sa}\rangle$ is now convergent, taking the form near $r=0$ of $\int(dr/r)r^{2\nu}$, which is finite. Even in this case, Ψ_{sa} is not strictly speaking in $D(\tilde{H})$, since $\tilde{H}\Psi_{sa}\sim r^{\nu-2}$, which has infinite norm, but our concern is with the expectation value $\langle\Psi_{sa}|\tilde{H}|\Psi_{sa}\rangle$, which *is* finite. Thus, for $\theta\neq 0$, our regularization has led us to a favored extension, even without our examining what happens as the regularization is removed. We will take up this question in a moment, but first let us examine the finiteness of $\langle\Psi_{sa}|\tilde{H}|\Psi_{sa}\rangle$ for the case of $\theta=0$ (bosons).

For $\theta=0$, the expression (15) vanishes and only the derivative term in (14) contributes to $\langle\Psi_{sa}|\tilde{H}-H_{sa}|\Psi_{sa}\rangle$. Near $r=0$ this derivative term behaves, in virtue of (13), like

$$\int r\,dr(\ln r)(r\partial/\partial r)\ln r\sim\int r\ln r\,dr,$$

which is a very convergent integral. Thus, for bosons $\langle\Psi_{sa}|\tilde{H}|\Psi_{sa}\rangle$ is finite for all α (if $b>0$); and only a closer examination of the $a,b\to 0$ limit can select the favored extension.

In order, then, to estimate $\langle\Psi_{sa}|\tilde{H}-H_{sa}|\Psi_{sa}\rangle$ for small a and b (or really just for small b, since a is irrelevant when $\theta=0$), we may notice that, as $b\downarrow 0$, $f(r)$ is designed to approach a step function. It follows from this (at least heuristically) that f'/f will approach a δ function whose strength is of order unity and whose location is near $r=b$:

$$-f'/f\sim\delta(r-b). \tag{17}$$

For the generic ($\alpha\neq 0$) behavior of $\Psi_{sa}\sim\ln r$, (17) leads to

$$\langle\Psi_{sa}|(-\ln f)'\partial/\partial r|\Psi_{sa}\rangle\sim\int r\,dr\ln r\delta(r-b)(\ln r)' \sim\int dr\,\delta(r-b)\ln r\sim\ln b,$$

which blows up as $b\to 0$. Hence our criterion of weak boundedness has again selected the extension for which $\Psi_{sa}(0)$ is finite, although the derivation required more care in this case. Continuing a bit further we can see that, for the favored extension, $\langle\Psi_{sa}|\tilde{H}-H_{sa}|\Psi_{sa}\rangle$ is not only bounded, but actually vanishes as $b\to 0+$, as follows from the estimate

$$\langle\Psi|(-\ln f)'\partial/\partial r|\Psi\rangle\sim\langle c_1+c_2r^2|\delta(r-b)\partial/\partial r|c_1+c_2r^2\rangle\sim\int r\,dr\delta(r-b)r\sim b^2$$

and the observation that no difficulty can occur away from $r=0$. Thus in the bosonic case, and for the favored extension, $\alpha=0$,

$$\langle\Psi_{sa}|\tilde{H}|\Psi_{sa}\rangle\to\langle\Psi_{sa}|H_{sa}|\Psi_{sa}\rangle$$

as the regularization is removed, the limit being ∞ for all other choices of extension.

Finally, let us return to the "properly anyonic" cases $0<|\theta|<\pi$ and ask whether the favored extension yields weak convergence (and *a fortiori* also weak boundedness) in this case also. For the favored extension, the leading term in Ψ_{sa} comes from J_ν rather than N_ν, and $\Psi_{sa}\sim r^\nu$. Hence $\langle\Psi|(-\ln f)'\partial/\partial r|\Psi\rangle\sim\int r\,dr\,r^\nu\delta(r-b)r^{\nu-1}\sim b^{2\nu}\to 0$. This was the first term in (14). For the second term, namely (15), we have the convergent behavior,

$$\left\langle\Psi\left|\frac{1}{r^2}\right|\Psi\right\rangle\sim\int r\,dr\,r^\nu r^{-2}r^\nu\sim\int dr\,r^{2\nu-1}\sim r^{2\nu},$$

whence (15) itself also goes to zero, its integrand vanishing pointwise as $a,b\to 0$. We conclude that, for $\theta\neq 0$ also, $\langle\Psi_{sa}|\tilde{H}-H_{sa}|\Psi_{sa}\rangle\to 0$ as the regularization is removed.

To summarize then: our criterion of weak boundedness selects a unique self-adjoint extension for all values of θ in the range $-\pi<\theta<\pi$, the favored extension being that defined by (8) and (16), or equivalently by the requirement that the functions in $D(H_{sa})$ be finite at $r=0$. For the favored extensions we have not only weak boundedness, but weak convergence in the sense that

$$\langle\Psi_{sa}|\tilde{H}|\Psi_{sa}\rangle\to\langle\Psi_{sa}|H_{sa}|\Psi_{sa}\rangle$$

as the regularization is removed. In the case of fermions ($\theta=\pi$), this analysis is not needed, but we may note that there is consistency with the limit $\theta\to\pi$ in the sense that for $\nu=|\theta/\pi|=1$ and $e^{i\alpha}=e^{-i\pi\nu/2}=-i$, the corresponding Ψ_{sa} vanishes at $r=0$, being in fact $O(r)$ there. For properly fractional statistics ($0<|\theta|<\pi$) the favored Ψ_{sa} also vanishes at $r=0$, though now only like $r^\nu,\nu=|\theta/\pi|$.

V. CONCLUSIONS AND COMMENTS

So what shall we say about whether identical point particles can collide, and about the closely related question whether the points of coincidence or "diagonal" Δ must be excluded from the configuration space Q or not? In the situation we have examined (two particles in $\mathbb{R}^2$) the answer seems to depend very much on the "statistical parameter" θ.

Things are most clear cut in the fermionic case $\theta=\pi$. There the "centrifugal potential/Fermi-repulsion" is strong enough to keep the particles apart, this repulsion being expressed mathematically by the essential self-adjointness of H [with domain such that $\Psi(0)=0$] and the corresponding lack of ambiguity in the dynamics. In this case then, the correct answer seems to be that the particles *cannot* collide and that it therefore does not matter whether we include Δ in Q.

In the bosonic case, on the other hand, there is a one-parameter family of possible self-adjoint extensions, and all of their domains include wave functions for which $\Psi(0)\neq 0$. Thus the correct answer for $\theta=0$ seems to be that the particles *can* collide, and therefore Δ must be included in Q. In fact, by including it and then regularizing H at $r=0$, we have seen how to select a unique H_{sa}, which turns out to be the one for which Ψ remains finite on Δ (and is also the "standard quantization" for bosons, of course). By excluding Δ we would preclude such a regularization, and this is another reason for keeping the points of coincidence.

Finally the case $0<|\theta|<\pi$ seems to be logically intermediate between $\theta=0$ and $\theta=\pi$, as might perhaps have been anticipated. On one hand, it resembles $\theta=0$ in that the centrifugal barrier (cf. [15]) is too weak to drive the particles apart, and there is again a circle's worth of inequivalent choices for H_{sa}. On the other hand, although the regularization procedure again selects a unique[6] H_{sa}, this time the corresponding Ψ_{sa} vanishes at $r=0$. The answer in this case thus seems to be—at least for this favored H_{sa}—that the particles cannot collide, but that Δ must nonetheless still be included in Q, its presence being felt indirectly in the fact that Ψ vanishes there.

(Of course, one can also consider situations where the particles are spatially extended, and then the replacement of H_{sa} by a regularized $\tilde{H}$ might have direct physical meaning. In such a case we would not subject $\tilde{H}$ to a limiting process, and some other self-adjoint extension might be more appropriate than the ones we are calling "favored" (especially since there would then be a distinguished length scale in the physics). This sort of criterion for selecting an extension is discussed in [13], whose authors point out that such "finite-size" effects can be particularly important in two dimensions, where one often has logarithmic fall off at large r instead of power-law decay.)

By means of the analysis carried out above, we hope to have completed the intrinsic formulation of identical-particle quantum mechanics, by providing a derivation, in a certain sense, of what must occur at points of overlap. There are several comments to be made about possible directions to look at from here.

To extend our analysis to higher dimensions would seem to be straightforward. In three dimensions, presumably, the free fermionic Hamiltonian would again turn out to be essentially self-adjoint, while the free bosonic Hamiltonian would admit a circle's worth of self-adjoint extensions, with the standard bosonic extension once again emerging as the favored one via our limiting process. One should also study systems of three or more identical particles (in both two and three dimensions) to see whether new ambiguities can arise at points of multiple coincidence.

As we mentioned before, when particles are able to overlap, their dynamics cannot globally be described in terms of a U(1) bundle over Q with locally flat connec-

[6]It is interesting that the mere fact that *some* unique extension can be selected tells us which one it must be: since the non-favored extensions are transformed into each other by scaling, no scale-invariant selection procedure can pick any one of them.

tion. It would be interesting to find out if it is possible to generalize (where appropriate) the description in terms of a bundle to handle configuration spaces where the diagonal is included.

Let us observe again that topology is a poor guide in the case where particles are allowed to overlap, since $\pi_1(Q)=0$ when the diagonal is included in Q. Statistics, on the other hand, is well defined whether or not the overlap is allowed, since it physically concerns what the particles are doing when they change places without meeting. The potential for conflict between this viewpoint on statistics and the purely topological one relying on $\pi_1(Q)$ seems to have been removed by our conclusion that Ψ vanishes at $r=0$ for all nontrivial (i.e., nonbosonic) statistics, which frees us from having to extend our bundle with connection to the points of overlap. Still one can wonder whether this outcome was just a matter of luck.

However this may be, it raises the important general issue of the dynamical implications of kinematical structures such as configuration spaces. A statement such as "the configuration space is $Q\simeq\mathbb{R}^3$" (where Q is thought of as specified, say, by its topology and differentiable structure) would seem to acquire its meaning from an implicit completion like "and the Hamiltonian is a differential operator on Q with continuous coefficients." We know from many examples that this particular completion is too restrictive, but we do not know what the most suitable replacement for it is. Perhaps it involves vector bundles over Q, or perhaps some formulation in terms of a sum over paths should be preferred. Any definite choice would have implications for attempts to place the spin-statistics theorem on a topological footing [16], and for related attempts to understand how novel topological phenomena arising in the quantum-gravity of globally hyperbolic spacetimes (emergent spin $\frac{1}{2}$, emergent Fermi and more general statistics, quantum multiplicity, etc.) are influenced by the exigencies of topology-change [9].

We would also like to be able to describe the meaning of the different possible self-adjoint extensions found in Sec. III in path-integral terms (cf. [17]), which in particular might help us interpret the choice of extension made in Sec. IV.[7]

Finally, it would be very interesting to try to generalize our methods to non-Abelian UIR's, as with parastatistics, and also to bundles with curvature, such as the case of a charge-monopole system in three dimensions. It also might be rewarding to consider the case of a particle and its antiparticle, where the extension might not be self-adjoint due to annihilation. Analogous but more complicated cases would be Skyrmions in an s wave and topological geons [9], both of these being extended objects which might be expected to decay in certain situations.

As we were writing-up the results contained in this paper, we received Ref. [18], which also considers self-adjoint extensions of the Hamiltonian (3). For the case $\theta\neq\pi$ they appear to find the same extensions as we do by a somewhat different method. They also study a specific regularization of the connection (but not of the conical geometry) reaching by an examination of the energy eigenfunctions, the conclusion that the extensions which survive as their regularization is removed are the same ones which our criterion of weak boundedness has selected.

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APPENDIX

When solving for the energy eigenfunctions of the hydrogen atom, one encounters a candidate ground-state solution whose radial wave function is singular at the origin. Dirac [19] rejects this solution on the grounds that it fails to satisfy the Schrödinger equation (construed in a generalized-function manner) at the origin,[8] $r=0$. In this sense, one is *including* $r=0$ in Q, and using information available there to select the physically appropriate self-adjoint extension of H. In this appendix, we revisit the problem using the language of self-adjoint extensions, and show that our regularization criterion also selects the correct extension. It is, of course, very natural to introduce a smoothed Hamiltonian in this case, since the proton really does have nonzero size. Nonetheless the success of our procedure here tends to show that the criteria adopted in the main text are reasonable, and could be used in other situations as well.

Except in the $l=0$ sector, the Hamiltonian,

$$H=-\frac{\hbar^2}{2m}\nabla^2-\frac{e^2}{r}\ ,$$

is essentially self-adjoint. In that sector the solutions $\Psi_\pm$ of the equation $H\Psi_\pm=\pm i\Psi_\pm$ have the form $\Psi=y(r)/r$, where $y(r)$ (in effect the $\frac{1}{2}$-density version of Ψ) obeys the following differential equation:

[7]In particular one can ask whether the zero boundary condition we have favored for $\theta\neq0$ corresponds to two-dimensional "avoiding boundary conditions" in the path integral. If true, this would conflict with our above interpretation that in some sense the diagonal *should* be included in Q when $\theta\neq\pi$.

[8]There is no difficulty in extending the Laplacian to the origin in this case because there is no conical singularity there, the proton and electron being distinguishable particles. This, together with the fact that the Coulomb potential is integrable, makes it possible to give a generalized meaning to the Hamiltonian differential operator without resorting to actual regularization.

$$\frac{d^2y}{d\rho^2}+\left[\frac{\lambda}{\rho}-\frac{1}{4}\right]y=0\ . \tag{A1}$$

Here $\rho=\alpha r,\alpha^2=-8m(\pm i)/\hbar^2$, $\lambda=e^2 2m/\hbar^2\alpha$, and the sign of α is chosen so that its real part is positive.

For real λ, one ordinarily writes the solution in terms of Laguerre polynomials: $y(x)=e^{-(x/2)}L_n^{\alpha}(x)$, but they are defined for integral n, whereas for us here, λ is complex, whence n would have to be complex too. However the general solution of the complex differential equation

$$\frac{d^2u}{dz^2}+\left[\frac{\lambda}{z}-\frac{1}{4}+\frac{\frac{1}{4}-\mu^2}{z^2}\right]u=0$$

is known [13], being given (for $|\arg z|<\pi$) by the pair of linearly independent Whittaker functions

$$M_{\lambda,\mu}(z)=z^{\mu+1/2}e^{-z/2}\Phi(\tfrac{1}{2}-\lambda+\mu,2\mu+1;z)\ ,$$

$$W_{\lambda,\mu}(z)=z^{\mu+1/2}e^{-z/2}\Upsilon(\tfrac{1}{2}-\lambda+\mu,2\mu+1;z)\ ,$$

where $2\mu\neq -1,-2,-3,\ldots$. Our solutions are then given by

$$M_{\lambda,\frac{1}{2}}(z)=ze^{-z/2}\Phi(1-\lambda,2;z)\ ,$$

$$W_{\lambda,\frac{1}{2}}(z)=ze^{-z/2}\Upsilon(1-\lambda,2;z)\ ,$$

where

$$\Phi(\beta,\gamma;z)=\sum_{k=0}^{\infty}\frac{(\beta)_k}{(\gamma)_k}\frac{z^k}{k!}\ ,$$

$$|z|<\infty\ ,\quad \gamma\neq 0,-1,-2,\ldots$$

is the confluent hypergeometric function with

$$(\eta)_0=1,(\eta)_k=\frac{\Gamma(\eta+k)}{\Gamma(\eta)}=\eta(\eta+1)\cdots(\eta+k-1)\ ,\qquad k=1,2,\ldots$$

and

$$\Upsilon(\beta,n+1;z)=\frac{(-1)^{n+1}}{\Gamma(\beta-n)}\sum_{k=0}^{\infty}\frac{(\beta)_k z^k}{(n+k)!k!}[\psi(\beta+k)-\psi(1+k)-\psi(n+1+k)+\ln z]$$
$$+\frac{1}{\Gamma(\beta)}\sum_{k=0}^{n-1}\frac{(-1)^k(n-k-1)!(\beta-n)_k}{k!}z^{k-n}$$

with $|\arg z|<\pi$, $n=0,1,2,\ldots$, $\beta\neq 0,-1,-2,\ldots$. Here $\psi(z)=\Gamma'(z)/\Gamma(z)$ is the logarithmic derivative of the gamma function. Υ is called the confluent hypergeometric function of the second kind. (We have given here the expression of Υ for integer $\gamma,\gamma=n+1$.) Note that in our case (A1), $\gamma=2,n=1,\beta=1-\lambda$ and $\mu=\frac{1}{2}$.

Therefore the general solution for $\Psi=y(r)/r$ is given by

$$\Psi(\rho)=ae^{-\rho/2}\Upsilon(1-\lambda,2;\rho)+be^{-\rho/2}\Phi(1-\lambda,2;\rho)\ . \tag{A2}$$

For λ corresponding to the $+i$ eigenvalue, that is $\lambda\sim(1+i)/\sqrt{2}$, there will be one linear combination in (A2) which will give a square integrable eigenfunction, Ψ_+. To find it, we need the asymptotic behaviors of Υ and Φ.

For large $|z|$,

$$\Phi(\beta,\gamma;z)\sim\frac{\Gamma(\gamma)}{\Gamma(\gamma-\beta)}e^{\pm\beta\pi i}z^{-\beta}\left[\sum_{k=0}^{n}\frac{(-1)^k(\beta)_k(1+\beta-\gamma)_k}{k!}z^{-k}+O(|z|^{-n-1})\right]$$
$$+\frac{\Gamma(\gamma)}{\Gamma(\beta)}e^{z}z^{-(\gamma-\beta)}\left[\sum_{k=0}^{n}\frac{(\gamma-\beta)_k(1-\beta)_k}{k!}z^{-k}+O(|z|^{-n-1})\right],$$

where $|\arg z|\leq\pi-\delta$, where $\delta>0$ is arbitrarily small, $\gamma\neq 0,-1,-2,\ldots$, and the plus sign corresponds to $\mathrm{Im}(z)>0$ and the minus sign to $\mathrm{Im}(z)<0$. For $|\arg z|\leq\frac{1}{2}\pi-\delta$, the second term is the one which contributes, and we have

$$\Phi(\beta,\gamma;z)\sim\frac{\Gamma(\gamma)}{\Gamma(\beta)}e^{z}z^{-(\gamma-\beta)}\left[\sum_{k=0}^{n}\frac{(\gamma-\beta)_k(1-\beta)_k}{k!}z^{-k}+O(|z|^{-n-1})\right],$$

where $|\arg z|\leq\pi/2-\delta$, $\beta,\gamma\neq 0,-1,-2,\ldots$.

For large $|z|$,

$$\Upsilon(\beta,\gamma;z)\sim z^{-\beta}\left[\sum_{k=0}^{n}\frac{(-1)^k(\beta)_k(1+\beta-\gamma)_k}{k!}z^{-k}+O(|z|^{-n-1})\right]$$

with $|\arg z|\leq\pi-\delta$. It is easy to see that the exponential $e^{\rho/2}=e^{\alpha r/2}$ in Ψ coming from Φ will dominate for large $|z|$ and therefore we must take $b=0$ in (A2).

Then the behavior of Ψ_+ near the origin is

$$\Psi_+\sim\frac{1}{\Gamma(-\lambda)}\ln(\alpha r)+\frac{1}{\Gamma(1-\lambda)}\left[\frac{1}{\alpha r}\right]\quad\text{for } z\to 0\ .$$

Therefore Ψ_+ is $\mathcal{L}^2[0,\infty;r^2dr]$.

Clearly, Ψ_- is just the complex conjugate of Ψ_+ (the Hamiltonian being a real differential operator). The

deficiency indices are thus $n_+ = n_- = 1$. The wave function defining the self-adjoint extension is then written in the usual way by taking a linear combination

$$\Psi_{sa} = \Psi_+ + e^{i\eta}\Psi_- ,$$

where η is the phase which determines the self-adjoint extension H_{sa}.

Now, the singular term in the Hamiltonian is the potential in $1/r$, so we will pick our regularized Hamiltonian to have a potential smeared over radius r_0, and take the limit of the expectation value $\langle \Psi_{sa} | H_{r_0} | \Psi_{sa} \rangle$ as $r_0 \to 0$. We have

$$\langle \Psi_{sa} | H_{r_0} - H_{sa} | \Psi_{sa} \rangle = \int_0^{r_0} \Psi_{sa}^* \left[\frac{1}{r_0} - \frac{1}{r} \right] \Psi_{sa} r^2 dr .$$

Only the terms in $1/r$ of Ψ_{sa} can lead to divergences (logarithmic ones) of this integral at $r=0$. When such terms are present, the expectation value $\langle \Psi | H_{r_0} | \Psi \rangle$ will be infinite, similarly to what happened for properly fractional statistics in Sec. IV. In order to avoid this divergence, we must choose η so that the $1/r$ term in Ψ_{sa} disappears, which requires

$$\frac{2}{\Gamma(1-\lambda^*)\Gamma(1-\lambda)|\alpha|^2} + e^{i\eta} \frac{1}{\Gamma(1-\lambda^*)^2 \alpha^{*2}} + e^{-i\eta} \frac{1}{\Gamma(1-\lambda)^2 \alpha^2} = 0$$

or

$$e^{i\eta} = -\frac{\Gamma(1-\lambda^*)}{\Gamma(1-\lambda)} \frac{\alpha^*}{\alpha} . \tag{A3}$$

We now notice that this choice of η has the effect of eliminating the logarithmic divergence in Ψ_{sa} as well.

Indeed, the latter is given by

$$\frac{1}{\Gamma(-\lambda)} \ln r - \frac{\Gamma(1-\lambda^*)}{\Gamma(1-\lambda)} \frac{\alpha^*}{\alpha} \frac{1}{\Gamma(-\lambda^*)} \ln r$$

using (A3). But this vanishes, owing to the property

$$\Gamma(z+1) = z\Gamma(z)$$

and the fact that $\alpha^*/\alpha = \lambda/\lambda^*$.

Thus, both the $1/r$ *and* logarithmic terms drop out of this linear combination, and we find—once again—that an extension with $\Psi(0)$ finite exists, and is the only acceptable one. For that extension, the integral not only converges but vanishes with r_0, meaning that we have weak convergence: $\langle \Psi_{sa} | H_{r_0} | \Psi_{sa} \rangle \to \langle \Psi_{sa} | H_{sa} | \Psi_{sa} \rangle$ as $r_0 \to 0$. Our criterion for selecting a favored self-adjoint extension is therefore amply satisfied.

[1] J. M. Leinaas and J. Myrheim, Nuovo Cimento B **37**, 1 (1977).

[2] R. D. Sorkin, Phys. Rev. D **27**, 1787 (1983).

[3] J. L. Friedman and R. D. Sorkin, Phys. Rev. Lett. **44**, 1100 (1980); **45**, 148 (1980); Gen. Relativ. Gravit. **14**, 615 (1982).

[4] R. D. Sorkin, Commun. Math. Phys. **115**, 421 (1988), and references therein.

[5] A. P. Balachandran, H. Gomm, and R. D. Sorkin, Nucl. Phys. **B281**, 573 (1987).

[6] L. S. Schulman, Phys. Rev. **176**, 1558 (1968); and *Techniques and Applications of Path Integration* (Wiley, New York, 1981); C. Morette-Dewitt, Phys. Rev. D **3**, 1375 (1971); J. S. Dowker, J. Phys. A **5**, 936 (1972); University of Texas Report No. 80-0133 (unpublished); J. Phys. A **18**, 3521 (1985); R. D. Sorkin, Phys. Rev. D **27**, 1787 (1983); E. C. G. Sudarshan, T. D. Imbo, and T. R. Govindarajan, Phys. Lett. B **213**, 471 (1988).

[7] C. Aneziris, A. P. Balachandran, and D. Sen, Int. J. Mod. Phys. A **6**, 4721 (1991).

[8] A. P. Balanchandran, in *Conformal Field Theory, Anomalies and Superstrings,* Proceedings of the First Asia-Pacific Workshop on High Energy Physics, edited by B. E. Baaquie, C. K. Chew, Ch. Oh, and K. K. Phua (World Scientific, Singapore, 1988); in *Geometric and Algebraic Aspects of Nonlinear Field Theories,* Proceedings of the International Meeting Amalfi, Italy, 1988, edited by S. DeFillippo *et al.* (North-Holland, Amsterdam, 1989); C. J. Isham, Phys. Lett. **106B**, 188 (1981).

[9] R. D. Sorkin, in *Geometric and Algebraic Aspects of Nonlinear Field Theories* [8]; in *Topological Properties and Global Structure of Spacetime,* edited by P. G. Bergmann and V. de Sabbata (Plenum, New York, 1986).

[10] J. L. Friedman and R. D. Sorkin, Commun. Math. Phys. **73**, 161 (1980).

[11] M. Reed and B. Simon, *Methods of Modern Mathematical Physics* (Academic, New York, 1972), Vols. I and II.

[12] N. N. Lebedev, *Special Functions and Their Applications* (Dover, New York, 1972).

[13] B. S. Kay and U. M. Studer, Commun. Math. Phys. **139**, 103 (1991).

[14] R. D. Sorkin, Int. J. Theor. Phys. **18**, 309 (1979).

[15] R. MacKenzie and F. Wilczek, Int. J. Mod. Phys. A **3**, 2827 (1988), and references therein.

[16] A. P. Balachandran, A. Daughton, Z. C. Gu, G. Marmo, R. D. Sorkin, and A. M. Srivastava, Mod. Phys. Lett. A **5**, 1575 (1990).

[17] M. Carreau, E. Farhi, S. Gutmann, and P. F. Mende, Ann. Phys. (N.Y.) **204**, 186 (1990); E. Farhi and S. Gutmann, Int. J. Mod. Phys. A **5**, 3029 (1990).

[18] J. Grundberg, T. H. Hansson, A. Karlhede, and J. M. Leinaas, Mod. Phys. Lett. B **5**, 539 (1991).

[19] P. A. M. Dirac, *The Principles of Quantum Mechanics* (Oxford University Press, Oxford, England, 1958).