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Massive Schwinger model and four-dimensional QED: The connection

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We state the connection between the fermion determinant in four-dimensional QED (QED_4) and the massive Schwinger model, QED_2, for the case of smooth, polynomial-bounded, unidirectional magnetic fields. Using the diamagnetic bound on the fermion determinant in QED_2, we obtain an upper bound on the fermion determinant in QED_4 for this class of fields. Using Kato's inequality, we obtain an upper bound on the one-loop effective action in scalar QED_4 for smooth, polynomial-bounded but otherwise general fields with fast decrease at infinity.

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I. INTRODUCTION

The unrenormalized Euclidean Green's functions in QED can be defined by the functional integral

$$G_{\mu_1\cdots\mu_m}(x_1,\ldots,x_n;y_1,\ldots,y_n;z_1,\ldots,z_m) = \frac{1}{Z}\int d\mu(A)\det[S(x_i,y_j;eA)]|_{i,j=1}^{n} \times \prod_{k=1}^{m} A_{\mu_k}(z_k)\det{}_{\text{ren}}(1-eS_F A)\ ,$$

where S is the fermion two-point function in the external potential A_μ, S_F is the free-fermion propagator, $\det_{\text{ren}}(1-eS_F A)$ is the fermion determinant defined below, and

$$Z=\int d\mu(A)\det{}_{\text{ren}}(1-eS_F A)\ .$$

The Gaussian measure for A_μ is chosen to have mean zero and covariance:

$$\int d\mu\ A_\mu(x)A_\nu(y)=D_{\mu\nu}(x-y)\ ,$$

where $D_{\mu\nu}$ is the free-photon propagator in a gauge determined by the measure, $d\mu(A)$. The measure may be chosen to give $D_{\mu\nu}$ an infrared cutoff mass; a way of introducing an ultraviolet cutoff will be mentioned below.

An open question is the following: how does $\det_{\text{ren}}(1-eS_F A)$ behave for large values of A_μ? If the logarithm of the gauge-invariant fermion determinant grows more than quadratically in the field strength $F_{\mu\nu}$, then there is doubt that $\det_{\text{ren}}(1-eS_F A)$ can be integrated for any Gaussian measure: four-dimensional QED (QED_4) would be unstable. Our goal here is to obtain a gauge-invariant upper bound on the fermion determinant.

As we shall see in Sec. II, we will have to confine ourselves to smooth, polynomial-bounded, unidirectional magnetic fields with fast decrease at infinity. For such fields we are able to link the massive Schwinger model, QED_2, directly to $\det_{\text{ren}}$ via Eq. (8) below.

This result immediately places the massive Schwinger model on an entirely new level: it is no longer a model; it contains physical information about the four-dimensional world. For example, if the fermion determinant of the massive Schwinger model were known, one could, via Eq. (8) below, calculate the, at present unknown, effective action for QED_4 for unidirectional, smooth, polynomial-bounded magnetic fields with rapid decrease at infinity. We use the diamagnetic bound on the functional determinant in QED_2 to obtain an upper bound on $\det_{\text{ren}}$ for these special field configurations. Our results are summarized in Sec. III.

Let us state straightaway that our special field configurations are a set of measure zero, and, therefore, the long-standing problem of how the spinor determinant $\det_{\text{ren}}$ behaves for strong general fields remains unanswered. Nevertheless, our bound, Eq. (15), does represent progress. Hitherto, explicit strong-field results for $\det_{\text{ren}}$ were known only for constant field strength [1,2] and expansions around constant field strength [3].

An upper bound on $\det_{\text{ren}}([(P-A)^2+m^2]/(P^2+m^2))$ for the case of scalar QED_4 for smooth, polynomial-bounded but otherwise general fields $F_{\mu\nu}$ with fast decrease at infinity is obtained in the Appendix with the help of Kato's inequality.

II. BOUND ON $\det_{\text{ren}}$ IN SPINOR QED_4

The functional measure $d\mu(A)$ for the free Maxwell field A_μ can be realized on $\mathcal{S}'$, the space of tempered distributions. Our procedure for smoothing these rough fields will also serve to regulate QED_4. Specifically, we smooth A_μ by convoluting it with an ultraviolet-cutoff function $h_\Lambda \in \mathcal{S}$, the functions of rapid decrease; that is, let $A_\mu^\Lambda = A_\mu \star h_\Lambda$. A choice for h_Λ might be a function whose Fourier transform $\hat{h}_\Lambda \in C_0^\infty$, such as $\hat{h}_\Lambda(p)=1$ for $p^2\le\Lambda^2$, $\hat{h}_\Lambda(p)=0$ for $p^2\ge(\Lambda+m_R)^2$, where m_R is the renormalized fermion mass. Then A_μ^Λ is a polynomial-bounded C^∞ function. To deal with volume divergences, multiply A_μ by a volume cutoff $g\in C_0^\infty$ and replace the potential in $\det_{\text{ren}}(1-eS_F A)$ with gA^Λ; the potentials in S can be simply replaced with A_μ^Λ, as no volume cutoff is needed here. Note that the photon propagator is now regulated:

$$\int d\mu\ A_\mu^\Lambda(x)A_\nu^\Lambda(y)=D_{\mu\nu}^\Lambda(x-y)\ ,$$

where $D_{\mu\nu}^\Lambda$'s Fourier transform $\hat{D}_{\mu\nu}^\Lambda \propto |\hat{h}_\Lambda|^2$. Since the fermion determinant has lowest-order charge renormalization built into it (see below), the unrenormalized theory is now free of divergences. Hereafter, the product $gA_\mu^\wedge$ will be denoted by A_μ, where A_μ is now a smooth, polynomial-bounded potential with fast decrease at infinity.

We define the fermion determinant, based on Schwinger's proper-time definition [1] as

$$\ln \det_{\rm ren}(1-eS_F A)=\frac{1}{2}\int_0^\infty \frac{dt}{t}\int d^4x\left\{\operatorname{tr}\left\langle x\left|e^{-P^2t}-\exp\left[-\left[(P-eA)^2+\frac{e}{2}\sigma_{\mu\nu}F_{\mu\nu}\right]t\right]\right|x\right\rangle+\frac{e^2F^2}{24\pi^2}\right\}e^{-tm^2}\,, \tag{1}$$

where $\sigma_{\mu\nu}=(1/2i)[\gamma_\mu,\gamma_\nu],[\gamma_\mu,\gamma_\nu]_+=-2\delta_{\mu\nu}$, $\gamma_\nu^\dagger=-\gamma_\mu$, and m is the unrenormalized fermion mass. This definition makes sense out of the formal expressions

$$\det{}^2(1-eS_F A)=\det\left[\frac{(P-eA)^2+(e/2)\sigma_{\mu\nu}F_{\mu\nu}+m^2}{P^2+m^2}\right],$$

and provides a gauge-invariant representation of the fermion determinant. It includes a second-order charge renormalization subtraction to make the proper time integral well defined for small t. This definition yields the conventional power-series expressions for the one-loop fermion graphs. For example, the $O(e^2)$ expansion of Eq. (1) gives

$$\ln \det_{\rm ren}(1-eS_F A)=-\frac{1}{2}\int d^4x\,d^4y\,A_\mu(x)\times\Pi_{\mu\nu}(x-y)A_\nu(y)\,,$$

with the Fourier transform of $\Pi_{\mu\nu}$ given by the second-order vacuum-polarization tensor

$$\hat\Pi_{\mu\nu}(k)=-\frac{e^2}{2\pi^2}(k^2\delta_{\mu\nu}-k_\mu k_\nu)\int_0^1 dz\,z(1-z)\times\ln\left[\frac{z(1-z)k^2+m^2}{m^2}\right].$$

Definition (1) also renders the box diagram gauge invariant and respects Furry's theorem. The point of these elementary remarks is to emphasize that we are dealing with QED in its entirety and not some abbreviated version of it.

We do not know how $\det_{\rm ren}(1-eS_F A)$ behaves for strong general fields $F_{\mu\nu}$. But it is a legitimate and well-posed question to ask how it behaves for strong magnetic fields. Since we are in Euclidean space, **E** and **B** are on the same footing, and it may be the case that the behavior of the determinant for large **B** fields will remain true for general fields $F_{\mu\nu}$. This is the case for *constant* fields. To reduce notation we will absorb e into A_μ and replace eA_μ by A_μ. Then [1,2]

$$\ln \det_{\rm ren}(1-S_F A)=\frac{V}{24\pi^2}(\mathbf{B}^2+\mathbf{E}^2-3|\mathbf{E}\cdot\mathbf{B}|)\times\ln\left[\frac{\mathbf{B}^2+\mathbf{E}^2}{m^2}\right]+O(F^2)\,,$$

where V is the volume of the space-time box. The determinant is seen to grow more than quadratically for $|\mathbf{E}\cdot\mathbf{B}|/(\mathbf{B}^2+\mathbf{E}^2)<\frac{1}{3}$.

We still cannot estimate the large-**B** behavior of $\det_{\rm ren}(1-S_F A)$ unless **B** is further restricted to be unidirectional, thereby reducing the problem to an effective two-dimensional one. Specifically, we set $\mathbf{B}=B(x,y)\hat{\mathbf{k}}$, where $B(x,y)$ is a smooth function with fast decrease at infinity. Then in the chiral representation of the γ matrices $\sigma_{12}=\begin{pmatrix}-\sigma_3 & 0\\ 0 & -\sigma_3\end{pmatrix}$, and Eq. (1) reduces to

$$\ln \det_{\rm ren}(1-S_F A)=\frac{V_\parallel}{2}\int_0^\infty\frac{dt}{t}\int d^2x_\perp\left[\frac{2}{4\pi t}\operatorname{tr}\langle x_\perp|e^{-P_\perp^2 t}-\exp-[(\mathbf{P}_\perp-\mathbf{A})^2-\sigma_3 B]t|x_\perp\rangle+\frac{B^2}{12\pi^2}\right]e^{-tm^2}\,, \tag{2}$$

where $\perp$ ($\|$) refers to the x and y (z and t) coordinates, and $V_\parallel$ is the volume of the zt box. We used

$$\operatorname{tr} e^{-P_\parallel^2 t}=\frac{V_\parallel}{4\pi t}\,;$$

the factor 2 in Eq. (2) comes from a partial spin sum. Then

$$\begin{aligned}-2\pi\frac{\partial}{\partial m^2}\ln\det_{\rm ren}&=\frac{V_\parallel}{2}\int_0^\infty\frac{dt}{t}\int d^2x_\perp\operatorname{tr}\langle x_\perp|e^{-P_\perp^2 t}-\exp\{-[(\mathbf{P}_\perp-\mathbf{A})^2-\sigma_3 B]t\}|x_\perp\rangle e^{-tm^2}+\frac{V_\parallel\|B\|^2}{12\pi m^2}\\&=V_\parallel\ln\det_{\rm Sch}(1-S_F A)+\frac{V_\parallel\|B\|^2}{12\pi m^2}\,,\end{aligned} \tag{3}$$

where $\|B\|^2=\int d^2x\,B^2(x,y)$. The determinant $\det_{\rm Sch}$ is defined by

$$\ln\det_{\rm Sch}(1-S_F A)=\frac{1}{2}\int_0^\infty\frac{dt}{t}\operatorname{tr}(e^{-P^2t}-\exp\{-[(\mathbf{P}-\mathbf{A})^2-\sigma_3 B]t\})e^{-tm^2}\,, \tag{4}$$

where the trace is over two-dimensional space-time and spin spaces. That is to say, $\det_{\rm Sch}$ is nothing but the fermion determinant of the massive Schwinger model [4] or, in other words, it is the fermion determinant of QED_2. The propagator S_F and the γ matrices in the argument of $\det_{\mathrm{Sch}}$ are now two dimensional. It might be objected that in four dimensions B (e) has dimension m^2 (1) while in two dimensions B and e both have dimension m. But recall that B stands for the product eB, which has the invariant dimension of m^2.

Expanding Eq. (4) to $O(B^2)$ and denoting the remainder by $\ln\det_3$, we get

$$\ln\det_{\mathrm{Sch}}(1-S_F A\!\!\!/)=-\frac{1}{2\pi}\int\frac{d^2k}{(2\pi)^2}|\hat{B}(k)|^2\int_0^1 dz\frac{z(1-z)}{k^2z(1-z)+m^2}+\ln\det_3(1-S_F A\!\!\!/)\ . \tag{5}$$

Restoring $F_{\mu\nu}(F_{12}=B,\sigma_{12}=-\sigma_3)$, by definition

$$\ln\det_3(1-S_F A\!\!\!/)=\frac{1}{2}\int_0^\infty\frac{dt}{t}\left[\mathrm{tr}(e^{-P^2t}-e^{-[(\mathbf{P}-\mathbf{A})^2+\sigma\cdot F/2]t})+\frac{t}{2\pi}\int_0^1 dz\,z(1-z)\int\frac{d^2k}{(2\pi)^2}|\hat{F}_{\mu\nu}(k)|^2e^{-k^2z(1-z)t}\right]e^{-tm^2} \tag{6}$$

$$=-\sum_{n=3}^{\infty}\frac{1}{n}\mathrm{tr}(S_F A\!\!\!/)^n\ . \tag{7}$$

We recognize $\ln\det_3$ as the sum of all one-loop fermion graphs in two dimensions, beginning with the box graph, since definition (6) respects Furry's theorem. Recall that these graphs vanish when $m=0$ [5,6]. By making a similarity transformation [7], $\ln\det_3$ can be interpreted as $\det_3(1-K)$, where

$$K=(-\Delta+m^2)^{-3/4}(i\partial\!\!\!/+m)A\!\!\!/(-\Delta+m^2)^{1/4}\ .$$

Then K is a compact operator on $L^2(\mathbb{R}^2)\oplus L^2(\mathbb{R}^2)$; it also belongs to the trace ideal C_q for $q>2$ [7–9], i.e.,

$$\mathrm{tr}(K^*K)^{q/2}<\infty\ .$$

Substituting Eq. (5) in Eq. (3) gives

$$-2\pi\frac{\partial}{\partial m^2}\ln\det_{\mathrm{ren}}=\frac{V_\parallel}{2\pi}\int\frac{d^2k}{(2\pi)^2}|\hat{B}(k)|^2\int_0^1 dz\,z(1-z)\left[\frac{1}{m^2}-\frac{1}{k^2z(1-z)+m^2}\right]+V_\parallel\ln\det_3\ .$$

Integrating with respect to m^2 gives

$$\ln\det_{\mathrm{ren}}=\frac{V_\parallel}{4\pi^2}\int\frac{d^2k}{(2\pi)^2}|\hat{B}(k)|^2\times\int_0^1 dz\,z(1-z)\ln\left[\frac{k^2z(1-z)+m^2}{m^2}\right]+\frac{V_\parallel}{2\pi}\int_{m^2}^\infty dM^2\ln\det_3(M^2)\ , \tag{8}$$

where we have set $\lim_{m^2=\infty}\det_{\mathrm{ren}}=1$. This is true graph by graph; it is true nonperturbatively for the constant field case, and it is physically reasonable that an infinite-mass fermion cannot respond to an external magnetic field. Referring to Eq. (8), we find it somewhat remarkable that the massive Schwinger model, through $\det_3$, has such a direct bearing on QED_4.

Unlike the original Schwinger model with $m^2=0$, the massive model has not yet been solved. But there are some important results. One of these is the "diamagnetic bound" [7,9,10], which states that

$$\det_{\mathrm{Sch}}\le 1\ , \tag{9}$$

for $m^2\ge 0$. Referring back to the definition of $\det_{\mathrm{Sch}}$, Eq. (4), it is evident that Eq. (9) is rather an expression of the paramagnetic property of fermions; it is also a statement of the positivity of the effective interaction Lagrangian of QED_2. From Eq. (5), Eq. (9) implies

$$\ln\det_3\le\frac{1}{2\pi}\int\frac{d^2k}{(2\pi)^2}|\hat{B}(k)|^2\int_0^1 dz\frac{z(1-z)}{k^2z(1-z)+m^2}\ . \tag{10}$$

Inserting (10) in (8) and choosing $\|B\|^2\ge m^2$, we get

$$\ln\det_{\mathrm{ren}}\le\frac{V_\parallel}{4\pi^2}\int\frac{d^2k}{(2\pi)^2}|\hat{B}(k)|^2\int_0^1 dz\,z(1-z)\times\ln\left[\frac{k^2z(1-z)+\|B\|^2)}{m^2}\right]+\frac{V_\parallel}{2\pi}\int_{\|B\|^2}^\infty dM^2\ln\det_3(M^2)\ . \tag{11}$$

Equation (11) gives a bound on $\det_{\mathrm{ren}}$ for strong magnetic fields. Indeed, letting $A_\mu\to\lambda A_\mu$ we get

$$\ln\det_{\mathrm{ren}}\underset{\lambda\gg 1}{\le}\frac{\lambda^2V_\parallel\|B\|^2}{24\pi^2}\ln\left[\frac{\lambda^2\|B\|^2}{m^2}\right]+\frac{V_\parallel}{2\pi}\times\int_{\lambda^2\|B\|^2}^\infty dM^2\ln\det_3(\lambda A\!\!\!/,M^2)+O(\lambda^0)\ . \tag{12}$$

From Eq. (6), the dominant contribution to $\det_3$ in Eq. (12) for large mass will come from the small-t region of its proper time representation. In QED_2 we have the heat-kernel expansion [11]

$$\mathrm{tr}(\exp\{-[(\mathbf{P}-\mathbf{A})^2+\tfrac{1}{2}\sigma\cdot F]t\}-e^{-tP^2})=\frac{1}{4\pi t}\int d^2x\left[\frac{t^2}{3}F^2+\frac{t^3}{15}F_{\mu\nu}\nabla^2F_{\mu\nu}-\frac{t^4}{90}(F^2)^2+\frac{t^4}{140}F_{\mu\nu}\nabla^4F_{\mu\nu}+O(t^5)\right]. \quad (13)$$

The second term in Eq. (6) is specifically designed to cancel all terms of $O(F^2)$. Thus, when (13) is substituted in Eq. (6) we get

$$\begin{aligned}\ln\det{}_3&=\frac{1}{2}\int_0^\infty\frac{dt}{t}\int d^2x\left[\frac{t^3}{360\pi}(F^2)^2+O(t^4)\right]e^{-tm^2}\\&=\frac{\int d^2x(F^2)^2}{360\pi m^6}+O\left[\frac{\int d^2x\,F^2F_{\mu\nu}\nabla^2F_{\mu\nu}}{m^8},\frac{\int d^2x(F^2)^3}{m^{10}}\right]\\&=\frac{\int d^2x\,B^4}{90\pi m^6}+O\left[\frac{\int d^2x\,B^3\nabla^2B}{m^8},\frac{\int d^2x\,B^6}{m^{10}}\right].\end{aligned}$$

Then

$$\int_{\lambda^2\|B\|^2}^\infty dM^2\det{}_3(\lambda A,M^2)=\frac{\int d^2x\,B^4}{180\pi\|B\|^4}+O\left[\frac{1}{\lambda^2}\right], \quad (14)$$

and hence the strong magnetic field bound

$$\ln\det{}_{\mathrm{ren}}\underset{\lambda\gg1}{\le}\frac{\lambda^2V_\parallel\|B\|^2}{24\pi^2}\ln\left[\frac{\lambda^2\|B\|^2}{m^2}\right]+O(\lambda^0). \quad (15)$$

The bound in Eq. (15) is in accord with the expectations of the authors in Ref. [12].

There remains the question of whether the logarithm term in Eq. (15) can be removed by a better estimate. If one goes back to the definition of $\det_{\mathrm{ren}}$ for a unidirectional magnetic field, Eq. (2), and breaks the t integral up into $\int_0^{1/\|B\|^2}$ and $\int_{1/\|B\|^2}^\infty$, then the charge renormalization term in $\det_{\mathrm{ren}}$ gives the contribution

$$\frac{V_\parallel\|B\|^2}{24\pi^2}\int_{1/\|B\|^2}^\infty\frac{dt}{t}e^{-tm^2}=\frac{V_\parallel\|B\|^2}{24\pi^2}\left[\ln\left[\frac{\|B\|^2}{m^2}\right]-\gamma\right]+O(\|B\|^0),$$

where γ is Euler's constant. This is the logarithm term in Eq. (15). Any hope of canceling it would have to come from the term

$$-\frac{V_\parallel}{4\pi}\int_{1/\|B\|^2}^\infty\frac{dt}{t^2}\mathrm{tr}(\exp\{-[(\mathbf{P}_\perp-\mathbf{A})^2-\sigma_3B]t\})e^{-tm^2}.$$

This integral is sensitive to the eigenvectors of $H=(\mathbf{P}_\perp-\mathbf{A})^2-\sigma_3B$ with eigenvalues at and near zero. Defining the flux ϕ of B by $\int d^2x\,B(\mathbf{x})$, the Aharonov-Casher theorem [13] states that H has exactly $[\phi/2\pi]$ eigenvectors with eigenvalue 0, all with $\sigma_3=1$ ($\sigma_3=-1$) if $\phi\ge0$ ($\phi\le0$). Here $[x]$ denotes the largest integer *strictly* less than x, and $[0]=0$. Thus the zero modes of H for large B fields with rapid decrease at infinity are highly degenerate. Further progress along these lines might shed some light on the question we have raised [14].

III. SUMMARY

The massive Schwinger model has been shown to be of direct physical relevance. Essentially, Eq. (8) says calculate $\det_{\mathrm{Sch}}$ and integrate over the fermion mass to get $\det_{\mathrm{ren}}$ and the effective action for spinor QED_4 for smooth, polynomial-bounded, unidirectional magnetic fields with fast decrease at infinity. If one wants to calculate classical trajectories, A_μ itself may be assumed smooth with rapid falloff at infinity so that the regulators g and h_Λ may be removed. This effective action would be relevant to the interior of pulsars, where trapped and more or less unidirectional magnetic fields near the electrodynamical critical field $B=m^2/e=(4.41)(10^{13})$ G are believed to exist.

Since $\det_{\mathrm{Sch}}$ is not yet known, we have resorted to the diamagnetic bound, Eq. (9), to bound $\det_{\mathrm{ren}}$ for a class of strong unidirectional magnetic fields. It is indeed rare when a two-dimensional result from constructive field theory has a direct bearing on physics in four dimensions.

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APPENDIX

For scalar QED_4 we can proceed as above and obtain a relation between the scalar determinant, $\det([(P-A)^2+m^2]/(P^2+m^2))$, and a determinant in scalar QED_2 analogous to Eq. (8) for the case of a unidirectional magnetic field. If one simply wishes to get an upper bound on the effective action for strong *general* fields $F_{\mu\nu}$, then there is a more direct way to proceed as we will now show.

We define the determinant of scalar QED_4 as

$$\ln\det{}_{\mathrm{ren}}\left[\frac{(P-A)^2+m^2}{P^2+m^2}\right]=\int_0^\infty\frac{dt}{t}\left[\mathrm{tr}(e^{-P^2}-e^{-(P-A)^2t})-\frac{\|F\|^2}{192\pi^2}\right]e^{-tm^2}, \quad (A1)$$

which contains a second-order charge renormalization subtraction. The dimensionless quantity $\|F\|^2 = \int d^4x\, F^2$. We are continuing to regard A_μ as a smooth, polynomial-bounded potential with fast decrease at infinity. Now break the t integral in Eq. (A1) into $\int_0^{1/m^2\|F\|^2}$ and $\int_{1/m^2\|F\|^2}^{\infty}$ and use Kato's inequality [15] in the form

$$\operatorname{tr} e^{-(P-A)^2 t} \le \operatorname{tr} e^{-P^2 t} , \tag{A2}$$

which expresses the universal diamagnetic tendency of spinless bosons in an external gauge field. Then

$$\ln \det_{\rm ren} \ge \int_0^{1/m^2\|F\|^2} \frac{dt}{t} \left[\operatorname{tr}(e^{-P^2 t} - e^{-(P-A)^2 t}) - \frac{\|F\|^2}{192\pi^2} \right] e^{-tm^2} - \frac{\|F\|^2}{192\pi^2} \int_{1/m^2\|F\|^2}^{\infty} \frac{dt}{t} e^{-tm^2} . \tag{A3}$$

The first integral in Eq. (A3) is dominated by its small-t behavior for $\|F\| \gg 1$. Using the expansion [16]

$$\operatorname{tr}(e^{-P^2 t} - e^{-(P-A)^2 t}) = \frac{1}{16\pi^2} \int d^4x \left[\frac{1}{12} F^2 + \frac{t}{120} F_{\mu\nu} \nabla^2 F_{\mu\nu} + \frac{37 t^2}{1680} F_{\mu\nu} \nabla^4 F_{\mu\nu} + \frac{t^2}{1440} [(F\tilde{F})^2 - 7 (F^2)^2] + O(t^3) \right] , \tag{A4}$$

we get

$$\ln \det_{\rm ren} \ge -\frac{\|F\|^2}{192\pi^2} (\ln^2 \|F\|^2 - \gamma) + \text{const} . \tag{A5}$$

The one-loop contribution to the Euclidean effective action is, for $\|F\| \gg 1$,

$$\Gamma^{(1)} = -\ln \det_{\rm ren} \le \frac{\|F\|^2}{192\pi^2} (\ln \|F\|^2 - \gamma) + \text{const} . \tag{A6}$$

Of course, this result may have no direct bearing on the stability of scalar QED_4 because the $\phi^{*2}\phi^2$ interaction, necessary to make the theory perturbatively renormalizable, has not been included.

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