Interpolating Lagrangians and SU(2) gauge theory on the lattice

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We apply the linear δ expansion to non-Abelian gauge theory on the lattice, with SU(2) as the gauge group. We establish an appropriate parametrization and evaluate the average plaquette energy E_P to $O(\delta)$. As a check on our results, we recover the large-β expansion up to $O(1/\beta^2)$, which involves some $O(\delta^2)$ contributions. Using these contributions we construct a variant of the $1/\beta$ expansion which gives a good fit to the data down to the transition region.

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I. INTRODUCTION

The δ expansion as originally formulated [1] was an expansion in the degree of nonlinearity of the interaction: for example a φ^4 interaction was written as $\varphi^{2(1+\delta)}$ and δ treated as an (artificial) expansion parameter, to be set equal to 1 at the end of the calculation. Although Feynman rules were developed [1,2] for the resulting logarithmic interaction Lagrangians, the evaluation [3] of higher-order contributions rapidly becomes extremely complicated, and it has not proved possible, in field-theory calculations, to go beyond second order. Moreover, it is difficult to extend the range of the technique beyond scalar field theories, although some progress has been made recently in treating U(1) gauge theory [4].

In the light of these difficulties, a variant of the method, the linear δ expansion, was developed by Duncan and Moshe [5] and applied in the first instance to the Gross-Neveu model and U(1) gauge theory on the lattice in three dimensions, where they obtained an improvement on the usual strong-coupling expansion. The linear δ expansion shares some important features with its predecessor, although in other respects it differs significantly. Like the original δ expansion it introduces an action S_δ which interpolates, in this case linearly, between a soluble action S_0 and the action S for the theory we are trying to solve:

$$S_\delta = \delta S + (1-\delta) S_0 \ . \tag{1.1}$$

The advantage of this interpolation, apart from its range of applicability, is that the Feynman rules are very close to those of ordinary weak-coupling perturbation theory, but the price to pay is that one loses the improved convergence properties which were such an attractive feature of the original δ expansion [1].

However, it is at this point that an aspect of that expansion which was a useful but not indispensable additional tool, now becomes of vital importance. We are referring here to the fact that there is a great deal of arbitrariness in S_0, which should nonetheless be chosen to embrace as much of the relevant physics as possible. In particular, the overall scale μ is at our disposal, and will have no effect on calculated quantities if these are evaluated exactly using the infinite expansion. However, in practice we will always be dealing with a truncated expansion, and that approximate answer will indeed depend on μ. How then should we choose that scale? Faced with such an ambiguity, we adopt the principle of minimal sensitivity (PMS), developed in another context [6], namely that at any order K we should choose μ such that the truncated result $R^{(K)}$ does not depend on μ, at least locally. In other words we demand that

$$\frac{\partial R^{(K)}}{\partial \mu} = 0 \ . \tag{1.2}$$

This fixing of μ via the PMS criterion gives the δ expansion some features of a variational principle. It is this which can turn [7] a divergent series (for fixed μ) into a convergent sequence (where μ is recalculated at each K) and which makes the linear δ expansion genuinely nonperturbative.

Among the many problems to which the linear δ expansion has subsequently been applied, perhaps one of the most testing was U(1) gauge theory on the lattice [8] (in four dimensions). There the Monte Carlo results [9] showed a clear nonperturbative transition in the plaquette energy E_P between the weak-coupling and strong-coupling regimes which neither strong- nor weak-coupling expansions could reproduce. The δ expansion, based on an S_0 appropriate to the weak-coupling (large β) regime, gave a distinct improvement on the usual $1/\beta$ expansion and contained within itself a signal for the crossover, in the disappearance of a solution to the PMS condition analogous to (1.2).

The purpose of the present paper is to attempt to extend these calculations to a non-Abelian gauge group, specifically SU(2). As might be expected, the calculations are considerably more involved: the following Sec. II is devoted to the technicalities—choice of parametrization, evaluation of traces, etc.—and the calculation of the lowest-order $[O(\delta^0)]$ contribution to E_P. At this stage there is no dependence on the scale parameter β' and so it is not determined by the PMS condition. This only comes into play in $O(\delta)$, which is the subject of Sec. III. In this section we are obliged to take account of the Haar measure, which can be done in a variety of ways. The various $O(\delta)$ contributions to E_P are calculated and col-

lected together. The results are disappointing: although the expansion again senses the transition by the disappearance of a stationary point, the actual values for E_P lie slightly above the weak-coupling curve. In order to understand the reasons for this, in Sec. IV we rederive the $1/\beta$ expansion from the δ expansion up to $O(1/\beta^2)$. This serves as a check on our calculation and brings out some important differences from the U(1) case. It also suggests a variant of the $1/\beta$ expansion which incorporates some of the features of the δ expansion and gives a rather good fit to the data right down into the transition region. The overall situation and the prospects of an $O(\delta^2)$ calculation are discussed in Sec. V.

II. BASIC FRAMEWORK AND LOWEST-ORDER CALCULATION

A. The lattice action

For a non-Abelian gauge group SU(N) the Wilson action [10] is based on "plaquettes," elementary squares on the lattice labeled by a position $\mathbf{m}$ and two directions, $(\mu\nu)$ say. One each plaquette we define

$$U_P=\frac{1}{N}\mathrm{Re\,Tr}(U_1U_2U_3^{-1}U_4^{-1})\ , \tag{2.1}$$

where $U_1,\ldots,U_4$ are the SU(N) group elements associated with each link around the plaquette (see Fig. 1). This construction preserves local gauge invariance, but of course breaks Lorentz invariance for finite lattice spacing a. The Wilson action is then defined as

$$S=\beta\sum_P U_P \tag{2.2}$$

where by considering the continuum limit it is easy to establish that β is related to the coupling constant g by $\beta=2N/g^2$.

Such systems have been extensively studied by Monte Carlo methods. In the case of SU(2), Lautrup and Nauenberg [11] have established the form of the average plaquette energy E_P as a function of β. As shown in Fig. 2, this exhibits a fairly smooth transition between the regions of small and large β, which are fitted by the strong- and weak-coupling expansions, respectively. As far as the weak-coupling ($1/\beta$) expansion is concerned, the $1/\beta$

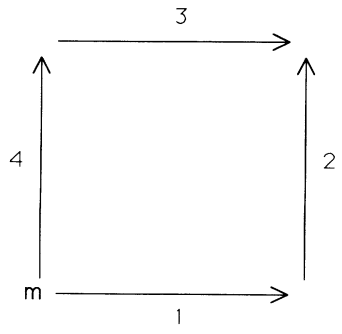


FIG. 1. Notation for the standard (12) plaquette positioned at $\mathbf{m}$.

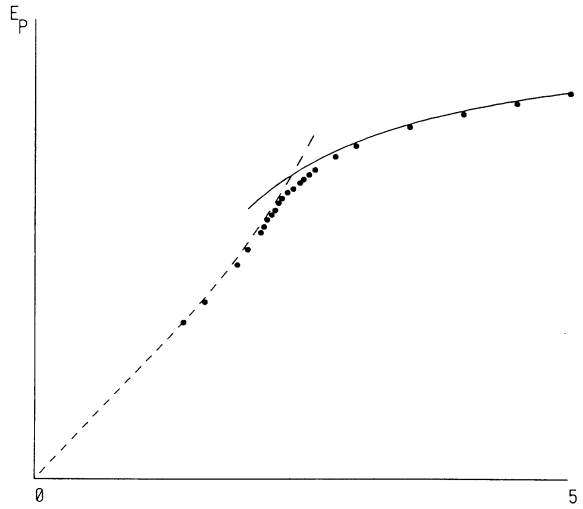


FIG. 2. Monte Carlo data on the plaquette energy compared with the weak-coupling expansion (solid line) and the strong-coupling expansion (dashed line).

coefficient is easily seen to be $-\frac{3}{4}$, but the $1/\beta^2$ and $1/\beta^3$ coefficients were obtained by a least-squares fit to the Monte Carlo data over the region $4\le\beta<10.5$, giving

$$E_P=1-\frac{3}{4\beta}-\frac{0.13}{\beta^2}-\frac{0.29}{\beta^3}+\cdots\ . \tag{2.3}$$

The coefficient c_2 of $-1/\beta^2$ was subsequently calculated by Heller and Karsch [12] and others [13] as $c_2=0.1514$.

In applying the δ expansion to (1.2) we will approach the problem from the weak-coupling side, aiming at obtaining an improvement over the large-β expansion. Accordingly we take S_0 to be the weak-coupling limit of S, up to a scale factor β'. The large-β limit forces the link variables U_l to unity and the angles parametrizing U_l to zero. For our parametrization of U_l, to be introduced shortly, S_0 is rather a simple quadratic S_Q in angles. Thus our interpolating action S_0 for SU(2) is

$$S_\delta=\sum_P\{\beta\delta[\tfrac{1}{2}\mathrm{Re\,Tr}(U_1U_2U_3^{-1}U_4^{-1})-1]\}$$
$$-\beta'(1-\delta)(S_Q+S_{\mathrm{gf}}) \tag{2.4}$$

where S_{gf} is the gauge-fixing term, to be specified later. β' is the variational parameter called μ in the Introduction, to be fixed by the PMS condition (1.2) on E_P. In common with the U(1) case it will turn out that the optimal value of β' is always less than β.

B. Parametrization of SU(2)

According to (2.4) the evaluation of successive terms in the δ expansion involves Gaussian integrals over the various angles. For this purpose the usual parametrization $U_l=\exp(\frac{1}{2}i\boldsymbol{\sigma}\cdot\boldsymbol{\alpha}_l)$ proves unsuitable, because the plaquette action (2.1) is a very complicated function of the individual angles α_i. The Euler parametrization $U_l=\exp(i\sigma_3\psi_l)\exp(i\sigma_2\vartheta_l)\exp(i\sigma_3\varphi_l)$ is more promising, but suffers from the problem that ψ_l and φ_l are not

forced to be small by the large-β limit (rather $\psi_l=-\varphi_l$), and this is reflected in the singular nature of S_Q.

A satisfactory variant of the Euler parametrization is found to be

$$U_l=e^{i\sigma_1\varphi_l}e^{i\sigma_2\vartheta_l}e^{i\sigma_3\psi_l} \tag{2.5}$$

which corresponds to shifting the origin of ϑ by $\pi/4$ and premultiplying by the constant element $e^{i\sigma_2\pi/4}$. With this parametrization we have a sensible quadratic limit, and U_P can be evaluated in terms of projection operators multiplied by ordinary exponentials which are amenable to Gaussian integration [see Eq. (2.9) below]. The Haar measure approximate to (2.5) is

$$H=\prod_l^{N_l}\cos(2\vartheta_l)\ , \tag{2.6}$$

where N_l is the total number of links.

The lattice action U_P involves the trace of the product of twelve elementary group elements, three for each of the four links. Each such element can be written in terms of two projection operators; thus

$$e^{i\sigma_1\varphi_1}\equiv\sum_{s=\pm1}e^{is\varphi_1}P_s^{(1)} \tag{2.7}$$

etc., where

$$P_s^{(i)}=\tfrac{1}{2}(1+s\sigma_i)\ . \tag{2.8}$$

Thus, U_P can be written as

$$U_P=\mathrm{Re}\sum_{s_i=\pm1}\zeta_{\mathbf{s}}e^{i\mathbf{s}\cdot\boldsymbol{\theta}} \tag{2.9}$$

where $\mathbf{s}$ is a twelve-component vector whose elements are all plus or minus one, and $\boldsymbol{\theta}$ comprises the twelve angles of the plaquette: $\boldsymbol{\theta}_l\equiv(\varphi_l,\vartheta_l,\psi_l)$. The complex quantity $\zeta_{\mathbf{s}}$ is given by the trace of all the projection operators:

$$\begin{aligned}\zeta_s=\tfrac{1}{2}\mathrm{Tr}[&(P_{s_1}^{(1)}P_{s_2}^{(2)}P_{s_3}^{(3)})(P_{s_4}^{(1)}P_{s_5}^{(2)}P_{s_6}^{(3)})\\&\times(P_{-s_7}^{(3)}P_{-s_8}^{(2)}P_{-s_9}^{(1)})(P_{-s_{10}}^{(3)}P_{-s_{11}}^{(2)}P_{-s_{12}}^{(1)})]\ ,\end{aligned} \tag{2.10}$$

from which it is clear that the 2^{12} possible values for $\mathbf{s}$ can immediately be reduced to 2^{10}, since $\zeta_{\mathbf{s}}$ vanishes unless $s_1=-s_{12}$ and $s_6=-s_7$.

The real and imaginary parts of $\zeta_{\mathbf{s}}$, which we call $\xi_{\mathbf{s}}$ and $\eta_{\mathbf{s}}$, respectively, have various important properties:

$$\begin{aligned}&\sum_{\mathbf{s}}\xi_{\mathbf{s}}=1,\quad\sum_{\mathbf{s}}\eta_{\mathbf{s}}=0\ ,\\&\sum_{\mathbf{s}}\xi_{\mathbf{s}}s_l^\mu=0,\quad\sum_{\mathbf{s}}\eta_{\mathbf{s}}s_l^\mu=0\ ,\\&\sum_{\mathbf{s}}\xi_{\mathbf{s}}s_l^\mu s_j^\nu=v_iv_j\delta^{\mu\nu},\quad\sum_{\mathbf{s}}\eta_{\mathbf{s}}s_l^\mu s_j^\nu=0\ ,\\&\sum_{\mathbf{s}}\xi_{\mathbf{s}}s_l^\mu s_j^\nu s_k^\tau=0,\quad\sum_{\mathbf{s}}\eta_{\mathbf{s}}s_i^\mu s_j^\nu s_k^\tau=T_{ijk}^{\mu\nu\tau}\ ,\\&\sum_{\mathbf{s}}\xi_{\mathbf{s}}s_i^\mu s_j^\nu s_k^\tau s_l^\lambda=F_{ijkl}^{\mu\nu\tau\lambda}\ ,\end{aligned} \tag{2.11}$$

where $i,j,k,l=1,2,3,4$ denote links within a plaquette and μ,ν,τ,λ denote angles: φ,ϑ,ψ. The quantities $v_i=\pm1$ take account of the inverses on links 3 and 4: $v_1=v_2=+1$, $v_3=v_4=-1$. The three-point and four-point non-Abelian vertices are encoded in T and F, respectively. $T^{\mu\nu\tau}$ is zero unless all three indices $(\mu\nu\tau)$ are different, while $F^{\mu\nu\tau\lambda}$ is zero unless $\mu=\nu=\tau=\lambda$ or they are equal in pairs.

C. The quadratic action and the lattice propagator

Returning to Eq. (2.4) we are now in a position to be more explicit about S_Q and the gauge-fixing term S_{gf}. In our parametrization S_Q turns out to be simply three copies of the quadratic action for U(1), one for each angle, namely

$$S_Q\equiv\tfrac{1}{2}\sum_P(\varphi_P^2+\vartheta_P^2+\psi_P^2)\ , \tag{2.12}$$

where $\varphi_P=\sum_i v_i\varphi_i$, etc. Accordingly, we shall also take S_{gf} to be three copies of the U(1) gauge-fixing term, viz.

$$S_{\mathrm{gf}}=\tfrac{1}{2}\sum_{\mathbf{m},\mu}(\boldsymbol{\theta}_{\mathbf{m},\mu}-\boldsymbol{\theta}_{\mathbf{m}-\hat{\mathbf{x}}_\mu,\mu})^2\ , \tag{2.13}$$

Provided that S_{gf} is accompanied by the factor $(1-\delta)$, as in Eq. (2.4), we will be entitled to calculate in the absence of ghosts, even for the non-Abelian case. With this in mind, for the first part of the paper we use a gauge parameter $\alpha=1-\delta$, in close analogy with the U(1) case [8].

As already intimated, we adopt the notation that ordinary angles indicate vectors that span all the links of the lattice, but only for *one* angle, e.g., φ, whereas bold angle vectors span all links for *all* angles.

$$\boldsymbol{\theta}_{\mathbf{m}\mu}=(\varphi_{\mathbf{m},\mu},\vartheta_{\mathbf{m}\mu},\psi_{\mathbf{m}\mu})\ . \tag{2.14}$$

For the matrices M and $D=M^{-1}$, the lattice propagator, we use bold-face when all three angles are involved. In this notation we write $S_{\mathrm{quad}}=\beta'[(1-\delta)S_Q+\alpha S_{\mathrm{gf}})]$, the quadratic part of S, as

$$\begin{aligned}S_{\mathrm{quad}}&=\tfrac{1}{2}\beta'\boldsymbol{\theta}\mathbf{M}\boldsymbol{\theta}\\&=\tfrac{1}{2}\beta'\sum_{\mathbf{m},\mu,\mathbf{n},\nu}\boldsymbol{\theta}_{\mathbf{m}\mu}M_{\mathbf{m}\mu,\mathbf{n}\nu}\boldsymbol{\theta}_{\mathbf{n}\nu}\ .\end{aligned} \tag{2.15}$$

$\mathbf{M}$ is the diagonal matrix

$$\mathbf{M}=\begin{bmatrix}M_\varphi&0&0\\0&M_\vartheta&0\\0&0&M_\psi\end{bmatrix}\ , \tag{2.16}$$

where M is as it was [8] for U(1):

$$M_{\mathbf{m}\mu,\mathbf{n}\nu}=\frac{1}{L^d}\sum_{\mathbf{p}}e^{2\pi i\mathbf{p}\cdot(\mathbf{m}-\mathbf{n})/L}H_{\mu\nu}(\mathbf{p})\ , \tag{2.17}$$

where

$$\begin{aligned}H_{\mu\nu}(\mathbf{p})=&(1-\delta)[\delta_{\mu\nu}\Delta(\mathbf{p})-c_\mu^*(\mathbf{p})c_\nu(\mathbf{p})]\\&+\alpha c_\mu^*(\mathbf{p})c_\nu(\mathbf{p})+\rho\delta_{\mathbf{p}0}\delta_{\mu\nu}\end{aligned} \tag{2.18}$$

with

$$c_\mu(\mathbf{p})\equiv e^{2\pi ip_\mu/L}-1,\quad\Delta(\mathbf{p})\equiv\sum_\mu|\mathbf{c}_\mu(\mathbf{p})|^2\ .$$

So $D=M^{-1}$ is

$$D_{\mathbf{m}\mu,\mathbf{n}\nu}=\frac{1}{L^d}\sum_{\mathbf{p}\neq 0} e^{2\pi i\mathbf{p}\cdot(\mathbf{m}-\mathbf{n})/L}\frac{\delta_{\mu\nu}-[1-(1-\delta)/\alpha]c_\mu^*(\mathbf{p})c_\nu(\mathbf{p})/\Delta(\mathbf{p})}{(1-\delta)\Delta(\mathbf{p})}+\frac{1}{L^d}\frac{\delta_{\mu\nu}}{\rho}\ . \tag{2.19}$$

For a sufficiently large lattice, the last term, which cancels zero modes, is damped out by the $1/L^d$ factor, and in the Feynman gauge, $\alpha=1-\delta$, adopted in (2.4), the propagator reduces to a simpler expression, which only couples parallel links:

$$D_{\mathbf{m}\mu,\mathbf{n}\nu}=\frac{1}{L^d}\sum_{\mathbf{p}\neq 0} e^{2\pi i\mathbf{p}\cdot(m-n)/L}\frac{\delta_{\mu\nu}}{(1-\delta)\Delta(\mathbf{p})}\ . \tag{2.20}$$

As before, values for elements of the matrix D can be found by numerically integrating products of modified Bessel functions:

$$D_{\mathbf{m}\mu,\mathbf{n}\nu}=\frac{1}{2}\frac{1}{(1-\delta)}\delta_{\mu\nu}\int_0^\infty d\lambda e^{-d\lambda}\prod_\mu I_{|m_\mu-n_\mu|}(\lambda)\ . \tag{2.21}$$

The gauge choice $\alpha=1-\delta$ has the advantage of computational simplicity, but when we come to consider the $1/\beta$ expansion we shall be obliged to use the $\alpha=1$ gauge, as it is the only gauge that will yield correct $1/\beta$ expansion coefficients at finite order in δ.

We have several sum rules regarding the D and $\mathbf{D}=\mathbf{M}^{-1}$ propagators:

$$\left[vDv=\frac{1}{2(1-\delta)}\right]\Longrightarrow\left[\mathbf{vDv}=\frac{3}{2(1-\delta)}\right], \tag{2.22}$$

where $v=(+1,+1,-1,-1)$ reflects the direction conventions of links around a plaquette and $\mathbf{v}$ is the triple angle extension of this. Also for three plaquettes P_0, P_1, and P_2, we have the sum rules:

$$(1-\delta)\sum_{P_1}\mathbf{v}_{P_0}\mathbf{D}\mathbf{v}_{P_1}\mathbf{v}_{P_1}\mathbf{D}\mathbf{v}_{P_2}=\mathbf{v}_{P_0}\mathbf{D}\mathbf{v}_{P_2}\ . \tag{2.23}$$

Note, however, that

$$(1-\delta)(DvvD)_{\mathbf{m}\mu,\mathbf{n}\nu}=(D)_{\mathbf{m}\mu,\mathbf{n}\nu}-\frac{1}{L^d}\sum_{\mathbf{p}\neq 0} e^{2\pi i\mathbf{p}\cdot(\mathbf{m}-\mathbf{n})/L}\frac{c_\mu^*(\mathbf{p})c_\nu(\mathbf{p})}{(1-\delta)\Delta(\mathbf{p})^2}\ . \tag{2.24}$$

We can recover (2.23) from (2.24) by contracting at both ends with v and including all three angles, since contracting (2.24) at either end with v causes the second term to vanish.

D. The plaquette energy in lowest order

We start by defining a generating functional for connected Green's functions, with $\sum_P$ representing a sum over positions and orientations of the plaquette P:

$$W\equiv\ln\int\prod_l d\theta_l\cos(2\vartheta_l)\exp\left[\delta\beta\sum_P U_P-\tfrac{1}{2}\beta'\boldsymbol{\theta}\mathbf{M}\boldsymbol{\theta}\right]\ . \tag{2.25}$$

The plaquette energy is obtained as before [8]:

$$E_P(\beta,\beta')\equiv 1+\frac{1}{N_P\delta}\frac{\partial W}{\partial\beta}=\langle U_P\rangle+\frac{\delta}{2!}\sum_{P_1}\langle U_P U_{P_1}\rangle_C+\cdots \tag{2.26}$$

where the angular brackets mean

$$\langle f\rangle\equiv\frac{1}{Z_0(\delta)}\int\prod_l d\theta_l\cos(2\vartheta_l)f\exp(-\tfrac{1}{2}\beta'\boldsymbol{\theta}\mathbf{M}\boldsymbol{\theta}) \tag{2.27}$$

and $Z_0(\delta)$ is a normalization such that $\langle 1\rangle=1$.

To evaluate the $O(\delta^0)$ contribution $\langle U_P\rangle$ we use the expression (2.9). Ignoring the Haar measure for the moment, this gives

$$\langle U_P\rangle=\frac{1}{Z_0}\sum_{\mathbf{s}}\int d\boldsymbol{\theta}\xi_{\mathbf{s}}\exp(i\mathbf{s}\cdot\boldsymbol{\theta}-\tfrac{1}{2}\beta'\boldsymbol{\theta}\mathbf{M}\boldsymbol{\theta}) \tag{2.28}$$

with no contribution from the imaginary parts $\eta_{\mathbf{s}}$. The notation here is that $\mathbf{s}$ and $\boldsymbol{\theta}$ run over all angles in the lattice, but all elements of $\mathbf{s}$ are zero apart from the twelve that relate to the plaquette P. Performing the Gaussian integrations[1] we obtain

$$\langle U_P\rangle=\sum_{\mathbf{s}}\xi_{\mathbf{s}}\exp\left[-\frac{1}{2\beta'}\mathbf{s}\mathbf{D}\mathbf{s}\right]\ . \tag{2.29}$$

$\langle U_P\rangle$ is thus the sum of at most 2^{10} exponential terms, but all nonzero terms are of three kinds only and the expression reduces to

$$\langle U_P\rangle=\tfrac{1}{4}e^{-6a/\beta'}(e^{6b/\beta'}-3e^{-2b/\beta'}+6e^{2b/\beta'}) \tag{2.30}$$

where

$$a=D^{\mu=\nu}_{\mathbf{m}-\mathbf{n}=0}|_{\delta=0}=0.1549334\ ,\qquad b=D^{\mu=\nu}_{\mathbf{m}-\mathbf{n}=(1,0,0,0)}|_{\delta=0}=0.0299334\ , \tag{2.31}$$

with $a-b=\frac{1}{8}$.

This lowest-order calculation does not represent a realistic evaluation of E_P since it is independent of β'. We will have to go to $O(\delta)$ to obtain an explicit β dependence, which is then modified by the PMS condition, which makes β' a nontrivial function of β. However, at

[1]Note that the integrals initially run over finite ranges. The error incurred in extending the range to infinity has been studied in Ref. [8], and can be neglected for the values of β' encountered in Sec. III.

this stage we can perform a check of (2.30) by setting $\beta'=\beta$ and expanding in $1/\beta$, in which case, following the arguments of Ref. [8], we would expect to obtain the correct $1/\beta$ coefficient. Indeed, we then obtain

$$E_P=1-6(a-b)/\beta+\cdots=1-3/4\beta+\cdots \tag{2.32}$$

which can also be derived directly from Eq. (2.29) with the help of (2.11) and (2.22).

The Haar measure does not contribute at order $1/\beta$, since there is no linear term in the expansion of $\cos(2\vartheta_l)$. Although it is reassuring to achieve the correct $1/\beta$ coefficient at order δ^0 we must check that the result is not spoilt by higher-order corrections. We shall show later that we get the same $1/\beta$ coefficient at $O(\delta)$, in any gauge, and that the result is stable at all higher orders in δ.

III. FIRST-ORDER CALCULATION

A. The Haar measure

It is now time to come to grips with the Haar measure [Eq. (2.6)], which we have so far neglected, with no justification other than the large-β expansion. As it stands, the Haar measure comprises a $\cos(2\vartheta_l)$ factor for each of the N_l links of the lattice, not just those in a few plaquettes. It is therefore quite impractical to deal with it without approximation, since its exact inclusion into the Gaussian integral would increase the size of the calculation by the prohibitive factor of 2^{N_l}. Among the many possible ways of dealing with the problem in the context of the δ expansion, we have settled on two.

The first of these, which we call the "cosine-Haar" or "CH" expansion, is simply to insert a factor of $\delta^{1/2}$ into the argument and then expand the cosine. In that case $\langle U_P\rangle$ becomes

$$\begin{aligned}\langle U_P\rangle&=\frac{1}{Z_0(\delta)}\int\prod_l d\theta_l\cos(2\delta^{1/2}\vartheta_l)\sum_{\mathbf{s}}\xi_{\mathbf{s}}\exp(i\mathbf{s}\cdot\boldsymbol{\theta}-\tfrac{1}{2}\beta'\boldsymbol{\theta}\mathbf{M}\boldsymbol{\theta})\\&=\sum_{\mathbf{s}}\xi_{\mathbf{s}}(1-\delta s D_\vartheta^2 s)\exp(-\mathbf{s}\mathbf{D}\mathbf{s}/2\beta')+O(\delta^2)\ .\end{aligned} \tag{3.1}$$

Alternatively we may promote the Haar measure to the exponent, writing

$$\prod_l\cos(2\vartheta_l)=\exp\left[\delta\sum_l\ln\cos(2\vartheta_l)\right]\Bigg|_{\delta=1} \tag{3.2}$$

and setting $\delta=1$ *after* expansion of the exponential. We will call this the "log-cos-Haar," or "LCH" expansion. For a one-dimensional integral

$$\begin{aligned}\int\ln|\cos(2\vartheta)|e^{-m\vartheta^2/2}d\vartheta&=\frac{1}{2}\frac{\partial}{\partial n}\int\cos^{2n}(2\vartheta)e^{-m\vartheta^2/2}\Bigg|_{n=0}\\&=\left[\frac{\pi}{2m}\right]^{1/2}\frac{1}{2}\frac{\partial}{\partial n}\frac{\Gamma(2n+1)}{2^{2n}}\left[\frac{1}{[\Gamma(n+1)]^2}+2\sum_{r=1}^{\infty}\frac{e^{-8r^2d}}{\Gamma(n+r+1)\Gamma(n-r+1)}\right]\Bigg|_{n=0}\\&=\left[\frac{2\pi}{m}\right]^{1/2}\sum_{r=1}^{\infty}\frac{(-1)^r}{r}(1-e^{-8r^2d})\end{aligned} \tag{3.3}$$

and if we introduce a source

$$\int\ln|\cos(2\vartheta)|e^{is\vartheta}e^{-m\vartheta^2/2}d\vartheta=\left[\frac{2\pi}{m}\right]^{1/2}e^{-s^2d/2}\left[-\ln 2+\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{r}e^{-8r^2d}\cosh(4rsd)\right], \tag{3.4}$$

where $d=1/m$. Generalizing to matrices we find that

$$\frac{1}{Z_0}\int\sum_l\ln|\cos(2\vartheta_l)|\exp(i\mathbf{s}\boldsymbol{\theta}-\tfrac{1}{2}\beta'\boldsymbol{\theta}\mathbf{M}\boldsymbol{\theta})d\theta=e^{-\mathbf{s}\mathbf{D}\mathbf{s}/2\beta'}\left\{\sum_{r=1}^{R}\frac{(-1)^{r-1}}{r}e^{-8r^2a/\beta'}\sum_l\left[\cosh\left[\frac{4rs_iD_{il}}{\beta'}\right]-1\right]\right\} \tag{3.5}$$

where D_{il} is a ϑ-angle propagator. R should be ∞, but in practice a value in the range 10–20 is adequate in the numerical sums; l is summed over the whole lattice and there is an implied sum over $i=1,2,3,4$.

Unfortunately, both these methods of expanding the Haar measure give rise to infrared divergences. This is most clearly seen in Eq. (3.1), which involves matrix elements $(D_\vartheta^2)_{ij}=\sum_l D_{il}D_{lj}$, which grow logarithmically with the size of the lattice. This divergence can be controlled by modifying the ϑ-angle gauge propagator, by the inclusion of the quadratic part of the Haar measure in the exponent.

Thus in the CH expansion we replace H by

$$H_{\mathrm{CH}}=\prod_l e^{-2(1-\delta)\vartheta_l^2}\cos(2\delta^{1/2}\vartheta_l)\ . \tag{3.6}$$

Applying the same modification to the LCH expansion results in

$$H_{\rm LCH}=\prod_l \exp[-2(1-\delta)\vartheta_l^2+\delta\ln\cos(2\vartheta_l)]\ , \tag{3.7}$$

which is in line with the general philosophy of the δ expansion (1.1) if the Haar measure is absorbed into the action S.

The additional factor of $e^{2(1-\delta)\vartheta_l^2}$ amounts to a modification of the ϑ-angle propagator, causing it to fall off exponentially with distance and eliminating infrared divergences. The effect on the weak-coupling matrix M for the ϑ angle is

$$M\to M'=M+\frac{4}{\beta'}1 \tag{3.8}$$

which causes D to become a function of β':

$$D'_{\mathbf{m}\mu,\mathbf{n}\nu}(\beta')=\frac{1}{2}\frac{1}{(1-\delta)}\delta_{\mu\nu}\int_0^\infty d\lambda e^{-(4+2/\beta')\lambda}\times\prod_\mu I_{|m_\mu-n_\mu|}(\lambda)\ , \tag{3.9}$$

to be compared with Eq. (2.21). We will use a prime on D to denote a damped propagator.

These are the minimal modifications we can make. Equation (3.7) has a firmer justification in the general framework of the δ expansion, but is more difficult to implement than the simpler *ad hoc* modification of Eq. (3.6). But there is still more freedom: for example, we could multiply by a damping factor $\exp[-2(1-\delta)\vartheta_i^2\beta'/\beta]$, which would lead to the modified propagators being a function of β rather than β'. Further, although there is no need to modify the φ and ψ propagators from the point of view of infrared divergences, subsequent calculations are greatly simplified if they are also damped in exactly the same way as the ϑ propagators. In the calculation of E_P to order δ we shall use all these variants, which are compared in Table II below.

B. E_P to $O(\delta)$

There are three parts to the $O(\delta)$ contribution: $\langle U_P\rangle$ with the $(1-\delta)^{-1}$ factor in each propagator expanded, and the connected correlators $\delta\langle UH\rangle_C$ and $\delta\sum_{P_1}\langle U_PU_{P_1}\rangle_C$.

In view of the modifications we have introduced in order to deal with the Haar measure, $\langle U_P\rangle$ will now be a more complicated expression than (2.30), involving a' and b' as well as a and b, in the case where only the ϑ propagator is damped. There are now five different types of nonvanishing terms, and

$$\langle U_P\rangle=\tfrac{1}{4}e^{-(4a+2a')/\beta'(1-\delta)}(e^{-(4b-2b')/\beta'(1-\delta)}+e^{(4b+2b')/\beta'(1-\delta)}+2e^{2b'/\beta'(1-\delta)}-4e^{-2b/\beta'(1-\delta)}+4e^{2b(\beta')(1-\delta)}) \tag{3.10}$$

which simplifies to an expression analogous to (2.30) if all three propagators are damped, when $a\to a', b\to b'$. To find the $O(\delta)$ contribution from (3.10) we expand

$$e^{A/(1-\delta)}=e^{A(1+\delta+\cdots)}=e^A(1+\delta A+\cdots)\ . \tag{3.11}$$

Turning now to $\delta\langle U_PH\rangle$, in the CH expansion we have to evaluate

$$\frac{1}{Z_0}\left[2\delta\sum_{\mathbf{s}}\int\prod d\theta\,\vartheta_i^2\exp\left[i\mathbf{s}\cdot\theta-\frac{1}{2}\beta'\theta\mathbf{M}'\theta\right]\right]_C=-\frac{2\delta}{\beta'^2}\sum_{\mathbf{s}}\xi_s sD'^2_\vartheta s\exp(-\mathbf{sD's}/2\beta')\ . \tag{3.12}$$

This sum can be performed algebraically, with the result:

$$\delta\langle U_PH\rangle=\frac{4\delta}{\beta'^2}e^{-(2a+4a')/\beta'}\{(a'_2-b'_2)e^{2b'/\beta'}[1+\cosh(4b/\beta')]+4a'_2\sinh(2b/\beta')\} \tag{3.13}$$

where

$$a'_2=(D'^2_\vartheta)_{\mathbf{m}-\mathbf{n}=0}|_{\delta=0},\quad b'_2=(D'^2_\vartheta)_{\mathbf{m}-\mathbf{n}=(1,0,0,0)}|_{\delta=0}\ . \tag{3.14}$$

We calculate a'_2 and b'_2 by summing our damped propagators over sites. They converge to sufficient accuracy within an 11^4 lattice.

In the LCH expansion method we can again perform the sum over permutations of s algebraically. The resulting expression, which is rather more complicated than (3.13), is used to find values of $\langle U_PH\rangle_C$ for different values of β and β'.

The major part of the calculation in terms of computing time is the evaluation of $\sum_{P_1}\langle U_PU_{P_1}\rangle_C=\sum_{P_1}(\langle U_PU_{P_1}\rangle-\langle U_P\rangle\langle U_{P_1}\rangle)$. Here

$$\sum_{P_1}\langle U_PU_{P_1}\rangle=\frac{1}{Z_0}\sum_{P_1}\sum_{\mathbf{s},\mathbf{t}}\int\prod d\theta\,\mathrm{Re}(\zeta_{\mathbf{s}}e^{i\mathbf{s}\cdot\theta})\mathrm{Re}(\zeta_{\mathbf{t}}e^{i\mathbf{t}\cdot\theta})e^{-\beta'\theta\mathbf{M}\theta/2}$$
$$=\sum_{P_1}\sum_{\mathbf{s},\mathbf{t}}(\xi_{\mathbf{s}}\xi_{\mathbf{t}}-\eta_{\mathbf{s}}\eta_{\mathbf{t}})e^{-(\mathbf{s}+\mathbf{t})\mathbf{D}(\mathbf{s}+\mathbf{t})/2\beta'}\ . \tag{3.15}$$

TABLE I. The number of different types of exponential term in Eq. (3.15). When the two plaquettes have no directions in common, the exponent is always zero. There is less degeneracy between the terms if the ϑ-angle propagator is different from those for the other two angles.

	$\xi_s\xi_t$		$\eta_s\eta_t$	
Directions in common	1	2	1	2
$D'_\vartheta=D'_\varphi=D'_\psi$	38	221	38	212
$D'_\vartheta\neq D'_\varphi=D'_\psi$	140	808	140	791

The double sum over permutations of s and t is large, generating in the first instance $2^{10}\times 2^{10}\approx 1$ million terms, each to be summed over the lattice, which typically has 11^4 sites. However, due to symmetries of the exponential, and of ξ_s and η_s, we can reduce the number of terms to $2^8\times 2^6$, and of these there is much degeneracy between the arguments of the exponentials, as shown in Table I. We treat the $\xi_s\xi_t$ and $\eta_s\eta_t$ contributions in isolation, and within these treat separately the two cases where P and P_1 have one direction or two directions in common.

Thus by storing the details of each type of term and its total weight, we can cut the computing time by a factor of the order of a thousand. So, although on the face of it the $O(\delta)$ calculation for SU(2) involves an increase in computing time by a factor of a million over the corresponding U(1) calculation, this factor is reduced to some 2000 if $D'_\vartheta\neq D'_\varphi=D'_\psi$ and down to 500 if $D'_\vartheta=D'_\varphi=D'_\psi$.

The total $O(\delta)$ contribution to E_P is obtained by summing Eqs. (3.10), (3.13), and (3.15). Depending on the type of infrared damping chosen, some or all of the lattice propagators are functions of β or β'. Because of constraints on long-term memory these propagator matrices have to be recalculated each time a new value of $\beta(\beta')$ is required in the evaluation of (3.15). We opted to calculate the relevant quantities a', b', a'_2, b'_2, and $\sum_{P_1}\langle U_P U_{P_1}\rangle$ on a coarse β' grid, and interpolated between these points using Fortran library routines, to produce smooth curves of E_P against β', for different values of β.

In Table II we give a comparison between the various damping methods introduced in Sec. 3.1 at the value $\beta=2.8$, which corresponds roughly to the onset of the transition in E_P. At this value the weak-coupling expansion to $O(1/\beta)$ gives $E_P=1-3/4\beta=0.732$, which lies somewhat above the Monte Carlo point. All of these methods give rather similar results, with the simple CH expansion with full damping being marginally favored. Disappointingly, however, the $O(\delta)$ expansion is marginally further away from the experimental value than is the large-β expansion, in contrast to the situation for U(1). The reason for this difference will be explored in the next section.

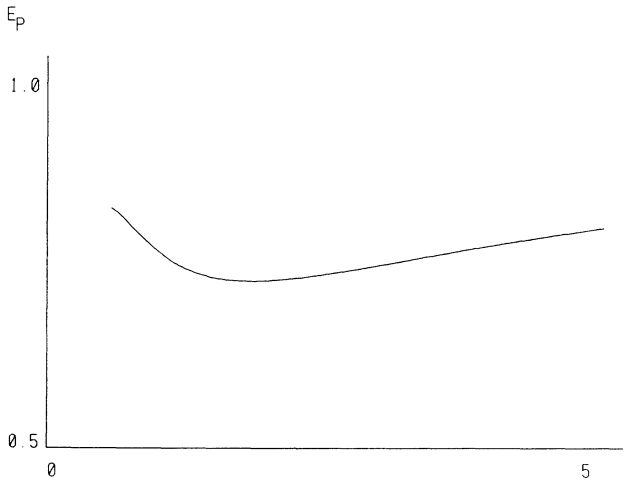


FIG. 3. The minimum of E_P as a function of β' for $\beta=2.8$.

Nonetheless, the δ expansion contains a signal of the transition which prevents it being used beyond a certain point, in contrast to the $1/\beta$ expansion, which contains no internal measure of its range of validity. This signal arises from the implementation of the PMS criterion, which in this case reads $\partial E_P(\beta,\beta')/\partial\beta'=0$. Above the transition region there is a broad flat minimum in β', illustrated in Fig. 3, but as the transition region is approached, this becomes less pronounced and eventually disappears.

In Fig. 4 we show the $O(\delta)$ curve for E_P using the simple CH prescription with partial damping which was favored at $\beta=2.8$. For reasons just discussed, the curve terminates at $\beta=2.1$.

TABLE II. $O(\delta)$ results for eight possibilities of infrared divergence damping scheme and Haar expansion, for the specific case $\beta=2.8$. Note that the Monte Carlo studies yielded $E_P=0.70$, and $1-3/4\beta=0.732$.

Type of damping	$D'_\vartheta, D_\varphi=D_\psi$		$D'_\vartheta=D'_\varphi=D'_\psi$	
	$D'(\beta')$	$D'(\beta)$	$D'(\beta')$	$D'(\beta)$
CH	0.73442	0.73554	0.73890	0.73878
LCH	0.74069	0.74245	0.74797	0.74822

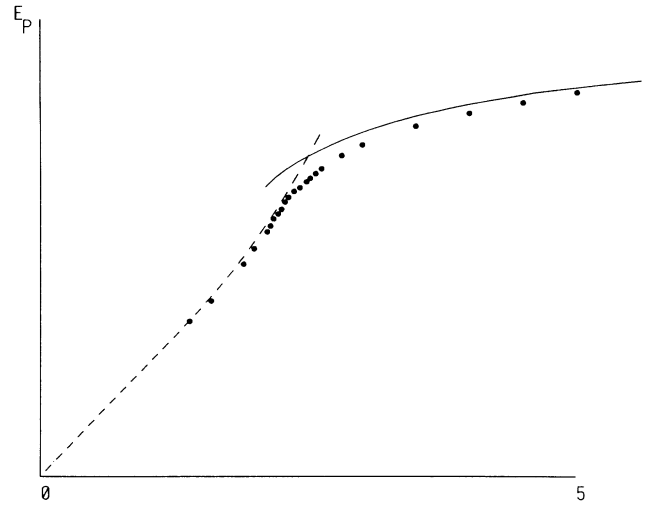


FIG. 4. The $O(\delta)$ expansion for E_P. The curve terminates at $\beta=2.1$ due to the disappearance of the minimum in β'.

IV. THE WEAK-COUPLING EXPANSION

A. The large-β limit and the δ expansion

The calculational framework we have set up for the δ expansion also allows us to calculate the $1/\beta^2$ term in the weak-coupling expansion. This provides a useful check of previous calculations [12,13], which used the conventional parametrization and were carried out in lattice momentum space, and brings out a very important difference from the U(1) case. Precisely because of the non-Abelian nature of SU(2), there exist three-point couplings between the gauge fields, encoded in the matrix T, which alter the relationship between the δ and $1/\beta$ expansions.

Let us briefly review the argument for U(1) to the effect that an order-δ^n calculation automatically gives the correct $1/\beta$ coefficients up to $O(1/\beta^{n+1})$. If we set $\beta'=\beta$, the δ action is

$$S_\delta=\beta\delta\sum_P(\cos\theta_P-1)-\tfrac{1}{2}\beta(1-\delta)\sum_P\theta_P^2$$
$$=-\tfrac{1}{2}\beta\sum_P\theta_P^2+\beta\delta(\cos\theta_P-1+\tfrac{1}{2}\theta_P^2)\ , \qquad (4.1)$$

ignoring the gauge-fixing term for the moment. The large-β limit in the functional integral forces the plaquette angles to zero as $\theta_P\sim 1/\beta^{1/2}$. Hence the factor which is brought down with δ is of order $\beta\theta_P^4=O(1/\beta)$. In calculating E_P we always have a factor of $U_P=O(1/\beta)$ by connectedness. So nth order in δ gives $(1/\beta)(\delta/\beta)^n$, i.e., $(n+1)$th order in $1/\beta$. In particular, the $O(\delta)$ calculation already gives the $1/\beta^2$ term. In retrospect this feature of U(1) may well have been an important element in the success of the δ expansion in that context. Another important point is that the gauge-fixing term does not affect the argument, because of the residual gauge invariance of the quadratic approximation. For example a change in gauge parameter from $\alpha=1-\delta$ to $\alpha=1$, which involves additional gauge insertions [see Eq. (4.2) below], has no additional effect because of the gauge invariance of the U(1) diagrams.

Turning now to SU(2), we note some crucial differences. The most important of these is that the difference between the action and its weak-coupling limit is now of order θ^3 rather than θ^4. Hence the factor brought down with δ is of order $\beta\theta^3=O(1/\beta^{1/2})$. [This occurs for contributions proportional to η, the imaginary part of ζ: those proportional to ξ follow the U(1) pattern.] Thus one must in general go to higher order in δ in order to obtain all the contributions of $O(1/\beta^n)$. In particular there is a second-order contribution $\sum\eta_s\eta_t\xi_u\mathbf{D}_{st}^2\mathbf{D}_{su}D_{tu}$ of order $1/\beta^2$ which is not included in the calculations of the previous section.

The second important difference is that the choice of gauge does now affect the argument, because the quadratic action on which the δ expansion is based lacks the residual gauge invariance of the U(1) case. In our discussion so far we have been using the gauge $\alpha=1-\delta$, which has the great simplification that the propagator only couples parallel links [cf. Eq. (2.20)]. However, from the point of view of the large β expansion it is not the optimal gauge to use, since the gauge-fixing insertion brought down with δ is of order $\beta\theta^2=O(1)$ and no longer vanishes. Thus, given any diagram of a particular order in δ and $1/\beta$, we can generate new diagrams of higher order in δ but the same order in $1/\beta$ by making such insertions on any of the propagators. In order to avoid this problem, we must choose a different gauge, the $\alpha=1$ gauge.

B. Calculating in the $\alpha=1$ gauge

In this gauge the propagator is modified by the addition of an extra term compared with (2.20)

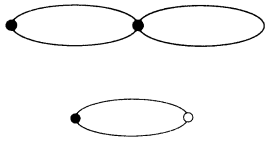
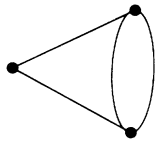
$$D^{\alpha=1}_{\mathbf{m}\mu,\mathbf{n}\nu}=D_{\mathbf{m}\mu,\mathbf{n}\nu}-\delta\frac{1}{L^d}\sum_{\mathbf{p}\neq 0}e^{2\pi i\mathbf{p}\cdot(\mathbf{m}-\mathbf{n})/L}\frac{c_\mu^*(\mathbf{p})c_\nu(\mathbf{p})}{(1-\delta)\Delta^2(\mathbf{p})}$$
$$=D_{\mathbf{m}\mu,\mathbf{n}\nu}+\delta[(1-\delta)DvvD-D]_{\mathbf{m}\mu,\mathbf{n}\nu} \qquad (4.2)$$

using Eq. (2.24). Thus to a given order n in δ the contributions are given by diagrams of the Feynman ($\alpha=1-\delta$) gauge up to order n, together with the appropriate number of insertions of the form (4.2) on propagators of the lower-order diagrams. In Table III below we show the relevant diagrams up to second order in δ and $1/\beta$ that remain *after* making the insertions and *after* expanding the $(1-\delta)^{-1}$ factor in each propagator. The notation is that the solid lines represent propagators $D_{\mathbf{m}\mu,\mathbf{n}\nu}$, solid blobs represent the interaction vertices arising from S, and the open circles represent the Haar measure.

The zeros are a consequence of the arguments given in the previous section. In the way we actually calculate, where the $(1-\delta)^{-1}$ factor is kept in the propagator and then expanded to the appropriate order, these zeros correspond to a guaranteed cancellation between diagrams. As expected, there are no higher-order corrections to the coefficient of $1/\beta$ given in Eq. (2.32).

The calculation of the coefficient of $1/\beta^2$ is free of in-

TABLE III. Connected diagrams up to $O(\delta^2)$ and $O(1/\beta^2)$ in the $\alpha=1$ gauge.

	δ^0	δ^1	δ^2
$1/\beta$		0	0
$1/\beta^2$ ξ-diagrams			0
$1/\beta^2$ η-diagrams			

frared divergences, and may thus be performed with the original undamped propagators. This is because the Haar diagram involves the *difference* $a_2-b_2=(D^2_{\mathbf{m}=\mathbf{n}}-D^2_{\mathbf{m}=\mathbf{n}+(1,0,0,0)})|_{\delta=1}$. Although each term is individually divergent, the difference converges to a value which is easily shown to be $a/8$. With undamped propagators it is possible, using sum rules like (2.23) and (2.24), to evaluate most of the diagrams semianalytically, i.e., in terms of a and b. Indeed it is only the two η diagrams that need to be computed numerically.

The $O(\delta^2)$ diagram, involving three plaquettes, each with its SU(2) trace factors, takes of the order of an hour of Cray XMP time.

C. Non-Abelian gauge fixing and ghosts

With the original choice of gauge parameter $\alpha=1-\delta$ there is no need to include ghosts. This is because when δ is set equal to one the gauge-fixing term vanishes and the action reverts to being gauge invariant. However, in the $\alpha=1$ gauge which we have been be obliged to use in this section in order to obtain the correct $1/\beta$ expansion at finite order in δ, this is no longer true. The gauge-fixing term survives at $\delta=1$, fixing the gauge of the full non-Abelian theory, and therefore gives rise to a functional determinant, which can be represented by ghost diagrams.

The problem of gauge fixing on the lattice for a non-Abelian theory has been considered in detail by Baaquie [14]. It is more complicated than the continuum limit, where there is only a three-point interaction between a gauge meson and two ghosts. On the lattice there are higher-order vertices, of which we need only include the four-point vertex in evaluating the coefficient of $1/\beta^2$. There are thus two additional contributions, shown in Fig. 5, where dashed lines represent ghost propagators.

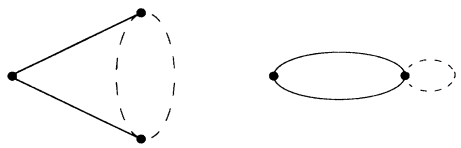
The second diagram, Fig. 5(b), is rather easy to evaluate, as it is simply related to the Haar diagram of Table III. In evaluating the second, and indeed in the entire formulation, we have to be careful to adapt the analysis of Baaquie, which was performed in the standard SU(2) parametrization, to our parametrization (2.5). The net result is that the three-point vertex is not simply proportional to the usual structure constants ϵ_{abc}, but also contains a term proportional to E_{abc}, the tensor relating our modified Euler angles to those of the standard parametrization near the origin, namely

$$\theta_a=\tfrac{1}{2}(\alpha_a+E_{abc}\alpha_b\alpha_c)+\cdots \tag{4.3}$$

[where we have written $(\varphi,\vartheta,\psi)=(\theta_1,\theta_2,\theta_3)$]. By virtue of their symmetry properties there is no interference between the two types of vertex in Fig. 5(a).

Combining the ghost diagrams with those of Table III we finally arrive at a value $c_2=0.15$, in essential agreement with Refs. [12,13]. This small coefficient is in fact the result of substantial cancellations between individual contributions.

The analysis we have just carried out suggests a way of improving on the $1/\beta$ expansion which does not involve



(a) (b)

FIG. 5. Ghost diagrams of $O(1/\beta^2)$.

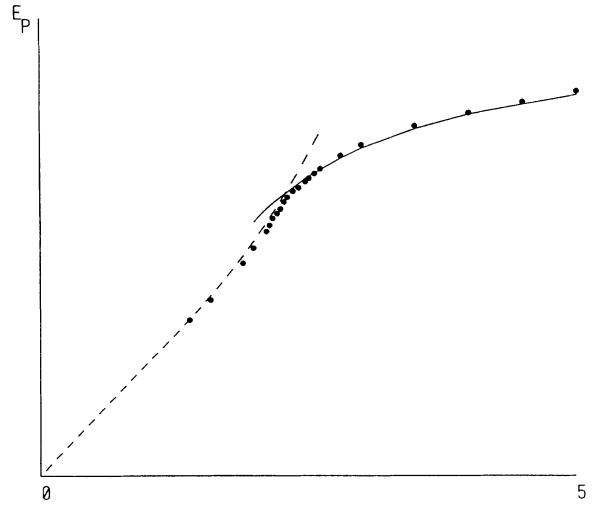


FIG. 6. The optimized $1/\beta$ expansion for E_P.

the full $O(\delta^2)$ calculation. Namely we can evaluate the diagrams of Table III and Fig. 5, but keeping β' distinct from β. In that case the expansion becomes

$$E_P(\beta,\beta')=1-(3/4\beta')(3-3x+x^2) \\ -(1/\beta'^2)(a_1+a_2x+a_3x^2) \quad (4.4)$$

where $x\equiv\beta/\beta'$ and the coefficients $a_1 \ldots a_3$ are $a_1=-1.8888$, $a_2=3.6326$, $a_3=-1.5940$.

In the limit $\beta'=\beta$ ($x=1$), this of course reduces to the $1/\beta$ expansion $E_P(\beta)=1-3/4\beta-c_2/\beta^2$, but in the spirit of the δ expansion one can regard β' as a free parameter, to be fixed as usual by the principle of minimal sensitivity.

We have carried out this procedure on Eq. (4.4), and the result is the curve shown in Fig. 6. In contrast to the full $O(\delta)$ calculation the curve does not naturally terminate: like the $1/\beta$ expansion itself it continues on into the small-β regime, where it is no longer appropriate. However, it does give a remarkably good fit to the data on the large-β side of the transition region, with only a small hiatus between this fit and the strong-coupling curve.

V. CONCLUSIONS

We have shown how it is possible to extend the applications of the linear δ expansion to include the non-Abelian group SU(2) on the lattice. In so doing we encountered several major obstacles which were not present in the U(1) case. A judicious choice of parametrization enabled us to reduce integrals of the group action to ordinary Gaussian integrals, albeit at the cost of dealing with a large number of permutations of the elementary traces ζ_s, which causes higher-order calculations to be rather demanding of computer time.

Another problem which did not occur in the U(1) case was the Haar measure. It was impractical to treat this exactly, and so various methods were proposed for incorporating it into the δ expansion, all with their own difficulties. The numerical results of the $O(\delta)$ calculation turned out to be not quite as accurate as the weak-coupling expansion to the same order, and in order to understand this we made a careful analysis of the relationship between the two expansions. This revealed that, unlike in the U(1) case, the $O(\delta)$ calculation is not guaranteed to include all contributions of $O(1/\beta^2)$. Indeed we identified the most important missing contribution, the $O(\delta^2)$ "cone" diagram of Table III. We also noted that in the gauge most appropriate for the $1/\beta$ expansion it was necessary to include ghost contributions, and the $1/\beta^2$ coefficient thus calculated agreed with previous results. The comparison with the $1/\beta$ expansion suggested a variational improvement whereby we kept just the relevant diagrams in the δ expansion but fixed β' by the PMS criterion rather than identifying it with β.

In view of the shortcomings of the full $O(\delta)$ calculation, what is the outlook for going to higher order? We believe it would just be possible, particularly in the method where all angles are treated equally, to extend the calculation to $O(\delta^2)$ in the Feynman gauge. The main problem is producing the table of nonvanishing coefficients for the different exponentials occurring in a three-plaquette correlator. Although we suspect that, as in Table I, the final result will show a dramatic reduction in the final number of surviving terms, a much larger number of terms have to be held in the computer in the intermediate stages. This problem is under active consideration.

However, a more promising way forward may be to choose an S_0 which is geared to the strong-coupling rather than the weak-coupling regime. Such a method, based on a maximal tree of plaquettes, was already investigated in the original paper [5] on the linear δ expansion, in the case of U(1) in three dimensions. We are currently exploring ways of extending these ideas to SU(2) and to four dimensions. The advantage here is that gauge invariance is maintained, and the angular integrals can be performed rather simply using the elegant machinery of characters.

Finally we should mention another method which has been used to deal with SU(2) on the lattice. This method, which was termed the "variational cumulant expansion" [15], can now be seen as a variant of the δ expansion, with an S_0 constructed from single links. The main difference is that, although it was E_P which was being calculated, the variational parameter was fixed by the PMS condition on the *first-order* expression for W. In our formulation of the δ expansion we would more naturally apply the PMS criterion to the nth *order* expression for E_P itself. We are currently investigating the changes that result when this criterion is applied to the model.

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