δ expansion applied to quantum electrodynamics

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(Received 4 June 1991)

A recently proposed technique known as the δ expansion provides a nonperturbative treatment of a quantum field theory. The δ-expansion approach can be applied to electrodynamics in such a way that local gauge invariance is preserved. In this paper it is shown that for electrodynamic processes involving only external photon lines and no external electron lines the δ expansion is equivalent to a fermion loop expansion. That is, the coefficient of δ^n in the δ expansion is precisely the sum of all n-electron-loop Feynman diagrams in a conventional weak-coupling approximation. This equivalence does not extend to processes having external electron lines. When external electron lines are present, the δ expansion is truly nonperturbative and does not have a simple interpretation as a resummation of conventional Feynman diagrams. To illustrate the nonperturbative character of the δ expansion we perform a speculative calculation of the fermion condensate in the massive Schwinger model in the limit of large coupling constant.

PACS number(s): 11.15.Tk, 11.15.Bt, 12.20.Ds

I. INTRODUCTION

A recent series of papers [1–5] has explored a new technique for generating nonperturbative expansions of quantum field theories. This technique, known as the δ expansion, is implemented by inserting a small parameter δ in the exponent of the nonlinear interaction terms in the Lagrangian. For example, to treat a $\lambda\phi^4$ field theory we consider a $\lambda(\phi^2)^{1+\delta}$ theory, where the parameter δ is regarded as small: $\delta \ll 1$. Then, following well-defined rules explained in previous papers [1], we express the Green's functions of the theory as series in powers of δ. At the end of the calculation, we set $\delta=1$ to obtain the result for a $\lambda\phi^4$ theory. In models we have studied so far, the numerical results have been good; a small number of terms in the δ series gives a good numerical approximation [1–6]. We emphasize that the δ expansion is nonperturbative; the construction and convergence of the δ series does not depend on the value of λ.

To apply the principles of the δ expansion to quantum electrodynamics it is necessary to insert the small parameter δ in such a way that local gauge invariance is preserved. It appears that the only way to do this is to replace the minimal coupling term $i\partial - eA$ by $(i\partial\!\!\!/ - eA\!\!\!/)^\delta$ [7]. Thus, we consider the Lagrangian

$$\mathcal{L}_\delta = -\tfrac{1}{4}(F^{\mu\nu})^2 + M\bar{\psi}[(i\partial\!\!\!/ - eA\!\!\!/)/M]^\delta\psi , \qquad (1.1)$$

where M is a mass parameter that has been inserted to make the quantity raised to the power δ dimensionless. Note that when $\delta=1$ the Lagrangian in (1.1) reduces to the standard Lagrangian for massless electrodynamics and no longer depends on the parameter M. If the fermion mass m is nonzero we consider the Lagrangian

$$\mathcal{L}_\delta = -\tfrac{1}{4}(F^{\mu\nu})^2 + M\bar{\psi}[(m + i\partial\!\!\!/ - eA\!\!\!/)/M]^\delta\psi . \qquad (1.2)$$

It is easy to show [7] that for all values of the parameter δ the Lagrangians in (1.1) and (1.2) are invariant under the local gauge transformation

$$\psi(x)\to\psi(x)\exp[ie\Lambda(x)] , \quad A^\mu(x)\to A^\mu(x) - \partial^\mu\Lambda(x) . \qquad (1.3)$$

The general procedure for computing the Green's functions in d-dimensional space-time as series in powers of δ is straightforward. We describe the first-order calculation below. Using the fact that for small δ

$$a^\delta = 1 + \delta\ln a + O(\delta^2) ,$$

we replace $\mathcal{L}_\delta$ in (1.2) by

$$\mathcal{L}_\delta = -\tfrac{1}{4}(F^{\mu\nu}) + M\bar{\psi}\psi + \delta M\bar{\psi}\ln[(m + i\partial\!\!\!/ - eA\!\!\!/)/M]\psi + O(\delta^2) . \qquad (1.4)$$

It is not possible to give diagrammatic rules for the Lagrangian in (1.4) because the interaction term is nonpolynomial. Thus, we construct a provisional Lagrangian $\tilde{\mathcal{L}}$ of the form

$$\tilde{\mathcal{L}} = -\tfrac{1}{4}(F^{\mu\nu})^2 + M\bar{\psi}\psi + \delta M\bar{\psi}[(m + i\partial\!\!\!/ - eA\!\!\!/)/M]^N\psi . \qquad (1.5)$$

If we regard N as a non-negative integer, we can derive Feynman rules for computing the Green's functions of $\tilde{\mathcal{L}}$. Having computed a Green's function to order δ as a function of N, we now treat N as a continuous variable, differentiate with respect to N, and set $N=0$. This gives the Green's function for $\mathcal{L}_\delta$ to order δ.

The reader will note that (1.5) and the technique of differentiating with respect to N and setting $N=0$ bears a strong resemblance to the replica method well known in statistical mechanics [8]. (This analogy with the replica method does not persist beyond leading order [1].) Although the analytic continuation described here cannot be justified rigorously, we have investigated this procedure in many models, and have found it to be invariably correct. These examples include zero- and one-dimensional models [1], the large-N limit [1], supersymmetric quantum field theory [3], stochastic quantization [9], and the renormalization of scalar quantum field theory [2]. Although there are many obvious and crucial questions of analyticity awaiting further study, we feel that it is valid to proceed on the basis of our experience.

As we will see later on, the Feynman rules for $\tilde{\mathcal{L}}$ are rather unusual. The electron propagator is just the constant $1/M$. Furthermore, there are many different vertices, all of order δ. The number of vertices depends on the parameter N. Using these Feynman rules in conjunction with the computational procedure described above (differentiating with respect to N) we have been able to compute the anomaly in two-dimensional electrodynamics [10].

In this paper we will show that, for the special case of Green's functions having no external electron lines, the δ series at $\delta=1$ is an infinite resummation of ordinary weak-coupling Feynman graphs. Specifically, we will show that the coefficient of δ^n in the δ expansion of such a Green's function is the sum of all Feynman diagrams having n internal electron loops. For Green's functions having external electron lines, there is no known way to express the coefficients of the δ expansion as a resummation of Feynman diagrams. Thus, for such processes, the δ expansion appears to provide a new nonperturbative method of calculation that is not accessible to ordinary graphical perturbation theory. One such process is described by the Green's function having two external electron lines and one external photon line. This Green's function gives the value of the anomalous magnetic moment $g-2$ of the electron.

This paper is organized as follows. First, in Sec. II, we illustrate the equivalence of the δ expansion and the loop expansion of a zero-dimensional field theory for the case of Green's functions having no external electron lines. We also show that this equivalence no longer holds for the case of Green's functions having external electron lines. We conclude that for such Green's functions the δ expansion is truly nonperturbative in character. This conclusion suggests a simple speculative calculation of the fermion condensate in the massive Schwinger model. This calculation is described in Sec. III.

Next, in Sec. IV we derive and explain the diagrammatic rules for the provisional Lagrangian in (1.5). These rules may be used to calculate Green's functions to first order in δ. (Successively more elaborate, but well-defined and straightforward, sets of rules must be used to calculate the Green's functions to higher order in δ.) Using these graphical rules to compute a Green's function to leading order in δ is a lengthy and difficult procedure. To present the relevant computational ideas in a simple context we set ourselves the more limited task in Secs. V and VI of computing Green's functions to first order in δ and to second order in the electric charge. Specifically, we compute the photon Green's function in Sec. V and the electron Green's function in Sec. VI. We will see that the photon Green's function to first order in δ and to second order in the electric charge, e, is *identical* to the $O(e^2)$ one-fermion-loop contribution to this Green's function. However, we will see that the electron Green's function to first order in δ and to second order in e bears no resemblance to the $O(e^2)$ weak-coupling contribution to this Green's function.

In Sec. VII we generalize the results of Sec. V to all orders in e. That is, we show that to order δ a Green's function having no external electron lines is precisely the (infinite) sum of all one-fermion-loop weak-coupling Feynman graphs. Thus, for such Green's functions the δ expansion is an infinite resummation of ordinary weak-coupling Feynman graphs. Finally, in Sec. VIII we generalize the conclusions of Sec. VII to second order in δ and indicate how this equivalence persists to all orders in δ.

II. ZERO-DIMENSIONAL ILLUSTRATIVE MODEL

Consider a trivial zero-dimensional model of electrodynamics described by the Lagrangian

$$L=\tfrac{1}{2}m\psi^2+\tfrac{1}{2}\mu^2A^2+\tfrac{1}{2}g\psi^2A \ , \tag{2.1}$$

where ψ represents the electron field and A represents the photon field.

Our objective in this section will be to compare the loop expansion and the δ expansion of the Green's functions for the Lagrangian L in (2.1). For simplicity we will consider two Green's functions, the photon propagator and the electron propagator. We will show that for the case of the photon propagator the one-electron-loop expansion is identical with the δ expansion to first order. On the other hand, the leading term in the electron-loop expansion for the electron propagator bears little resemblance to the δ expansion of this Green's function.

A. Loop expansion for the photon propagator

It is easy to read off the Feynman rules for the weak-coupling expansion of the Lagrangian in (2.1): The amplitude for an electron propagator, represented by a thick line, is $1/m$, the amplitude for a photon propagator, represented by a thin line, is $1/\mu^2$, and the amplitude for a photon-electron-electron vertex is g. The graphs for the one-electron-loop expansion of the photon propagator are shown in Fig. 1. Note that there is one graph of order g^2, two graphs of order g^4, nine graphs of order g^6, and so on. The sum of the symmetry numbers for all one-loop graphs having $2n$ vertices is given by the simple formula $(2n-1)!!/2$. Let us consider the amplitude of all one-loop $2n$-vertex graphs contributing to the photon propagator. Such graphs have $2n$ electron lines and $n+1$ photon lines. Thus, the amplitude for the sum of all such graphs is

$$\frac{(2n-1)!!}{2}g^{2n}(\mu^2)^{-n-1}m^{-2n} \ . \tag{2.2}$$

We sum all these amplitudes with respect to n to obtain the exact one-electron-loop contribution to the electron propagator:

$$\Gamma^{(2)}_{1\text{ loop}}=\frac{1}{2\mu^2\sqrt{\pi}}\sum_{n=1}^{\infty}\Gamma\left[n+\frac{1}{2}\right]\left[\frac{2g^2}{\mu^2m^2}\right]^n , \tag{2.3}$$

where we have expressed the double factorial in terms of a Γ function. This series is formally divergent, but we can use the integral representation of the Γ function,

$$\Gamma\left[n+\frac{1}{2}\right]=\int_0^{\infty}dt\,\exp(-t)t^{n-1/2} \tag{2.4}$$

to perform a Borel sum of the series in (2.3):

$$\Gamma^{(2)}_{1\text{ loop}}=\frac{g^2}{\mu^4m^2\sqrt{\pi}}\int_0^{\infty}dt\frac{\exp(-t)\sqrt{t}}{1-2g^2t/(\mu^2m^2)} . \tag{2.5}$$

We point out that this integral only has a formal existence because the path of integration passes through a simple pole. The origin of this divergence is simply that the Hamiltonian for this theory is not bounded below and thus the vacuum functional for this theory is a divergent integral.

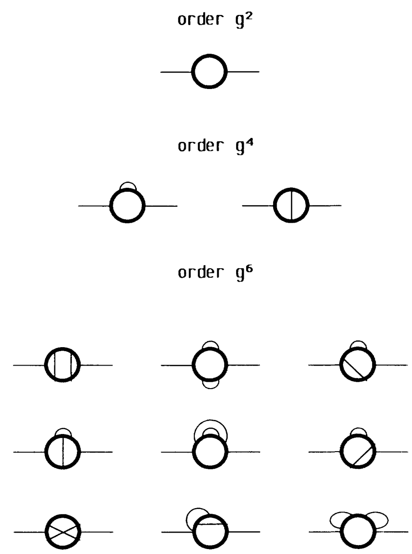


FIG. 1. One-electron-loop weak-coupling graphs contributing to the photon propagator. There is one graph of order g^2, two graphs of order g^4, and nine graphs of order g^6.

B. δ expansion for the photon propagator

To solve for the Green's functions of $\mathcal{L}$ in (2.1) we introduce the parameter δ in a manner similar to that used in (1.2):

$$L_\delta=\tfrac{1}{2}\mu^2A^2+\tfrac{1}{2}M\psi[(m+gA)/M]^{\delta}\psi . \tag{2.6}$$

The vacuum functional for this field theory in the presence of external sources J and η for the A and ψ field is

$$Z[J,\eta]=\int\int d\psi\,dA\,\exp(-L_\delta+JA+\eta\psi) . \tag{2.7}$$

Since we are only interested in the Green's functions to first order in δ we expand $Z[J,\eta]$ in (2.7) in powers of δ and neglect terms of order δ^2. The result is

$$Z[J,\eta]=\int\int d\psi\,dA\left[1-\frac{1}{2}M\delta\psi^2\ln\left[\frac{m+gA}{M}\right]\right]\exp\left[-\frac{1}{2}\mu^2A^2-\frac{1}{2}M\psi^2+JA+\eta\psi\right] . \tag{2.8}$$

To compute the connected two-point photon Green's function we differentiate $\ln Z[J,\eta]$ twice with respect to J and set the sources J and η to zero. The result is

$$\Gamma^{(2)}=\frac{Z_{JJ}[0,0]}{Z[0,0]}-\left[\frac{Z_J[0,0]}{Z[0,0]}\right]^2 . \tag{2.9}$$

We substitute (2.8) into (2.9), expand to first order in δ, and evaluate the Gaussian integrals to obtain

$$\Gamma^{(2)}_{\delta}=\frac{1}{\mu^2}\left[1-\frac{\delta}{\sqrt{8\pi}}\int_{-\infty}^{\infty}dt(t^2-1)\ln\left[\frac{m+gt/\mu}{M}\right]\exp(-t^2/2)\right] . \tag{2.10}$$

The integral in (2.10) can be rewritten by adding it to itself after making the change of integration variable $t\to -t$

$$\Gamma^{(2)}_{\delta}=\frac{1}{\mu^2}\left[1-\frac{\delta}{\sqrt{32\pi}}\int_{-\infty}^{\infty}dt(t^2-1)\ln\left[\frac{m^2-g^2t^2/\mu^2}{M^2}\right]\exp(-t^2/2)\right] . \tag{2.11}$$

Finally, we perform an integration by parts using the identity

$$(t^2-1)\exp(-t^2/2)=-\frac{d}{dt}[t\exp(-t^2/2)] .$$

The result, after the change of variable $x=\frac{1}{2}t^2$, is

$$\Gamma^{(2)}_{\delta}=\frac{1}{\mu^2}+\frac{\delta g^2}{\mu^4m^2\sqrt{\pi}}\int_0^{\infty}dx\frac{\sqrt{x}}{1-2g^2x/(\mu^2m^2)}e^{-x} . \tag{2.12}$$

Observe that the parameter M has completely disappeared from this integral representation. To leading order in δ the result in (2.12) is the zero-electron-loop contribution to the photon propagator that comes from the simple graph consisting of a single photon line. To first order in δ the result in (2.12) is the contribution from all one-electron-loop graphs as given in (2.5).

C. Loop expansion for the electron propagator

For simplicity, we compute only the zero-electron-loop contribution to the electron propagator. Using the same

Feynman rules that we followed in Sec. II A we can evaluate all graphs of the form shown in Fig. 2. The resulting Green's function, obtained by multiplying together the symmetry number and amplitude for each graph and summing over all graphs, is

$$\Delta^{(2)}=\frac{1}{m\sqrt{\pi}}\sum_{n=0}^{\infty}\Gamma\left[n+\frac{1}{2}\right]\left[\frac{2g^2}{m^2\mu^2}\right]^n . \tag{2.13}$$

We can obtain the Borel sum of this series by using the integral representation (2.4) for the Γ function. The result is

$$\Delta^{(2)}=\frac{1}{m\sqrt{\pi}}\int_0^{\infty}\frac{dt}{\sqrt{t}}e^{-t}\frac{1}{1-2g^2t/(\mu^2m^2)} . \tag{2.14}$$

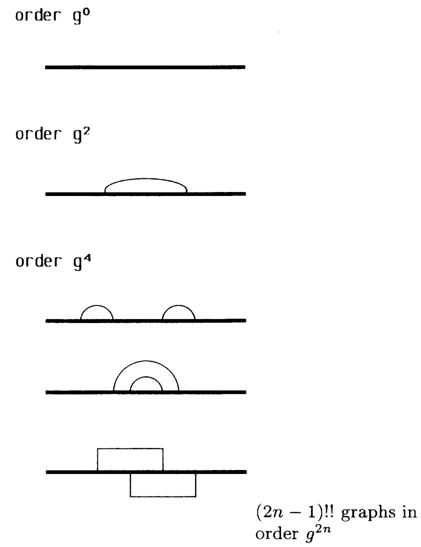


FIG. 2. Zero-electron-loop weak-coupling graphs contributing to the electron propagator. There are exactly $(2n-1)!!$ graphs of order g^{2n}.

D. δ expansion for the electron propagator

To compute the connected two-point electron Green's function we differentiate the generating function $\ln Z[J,\eta]$ in (2.8) twice with respect to η and set the sources J and η to zero. The result is

$$\Delta^{(2)}=\frac{Z_{\eta\eta}[0,0]}{Z[0,0]}-\left[\frac{Z_{\eta}[0,0]}{Z[0,0]}\right]^2 . \tag{2.15}$$

Substituting (2.8) into (2.15) and evaluating the Gaussian integrals we obtain to first order in δ

$$\Delta_{\delta}^{(2)}=\frac{1}{M}\left[1-\frac{\delta}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dt\,\ln\left[\frac{m+gt/\mu}{M}\right]\exp(-t^2/2)\right] . \tag{2.16}$$

The integral in (2.16) can be rewritten by adding it to itself after making the change of integration variable $t\to -t$

$$\Delta_{\delta}^{(2)}=\frac{1}{M}\left[1-\delta\ln(m/M)-\frac{\delta}{\sqrt{8\pi}}\int_{-\infty}^{\infty}dt\,\ln[1-g^2t^2/(\mu^2m^2)]\exp(-t^2/2)\right] . \tag{2.17}$$

Unlike the case of the photon propagator, this expression is distinctly different from that in (2.14). Moreover, after we set δ=1 the expression in (2.17) still depends on the mass parameter M. Recall that this parameter was not present in the final expression for the photon propagator in (2.12).

Previous investigations have had to address the appearance of the mass parameter M in the coefficients of the δ expansion. In general, if one calculates to first order in δ, one finds that after setting δ=1 the mass parameter M remains. To eliminate the dependence on this mass parameter one can use the method of minimal sensitivity; to wit, one argues that since the exact theory is independent of M when δ=1 the optimal value of M for the approximate theory (the δ series truncated at some finite order) is that for which the series is least sensitive to variations in M [11]. Thus, our procedure is to set δ=1 in the δ series and find the value of M for which the derivative of this series with respect to M vanishes. Applying this procedure to the formula in (2.17) with δ=1 gives the following value for M:

$$M=m\exp\left[\frac{1}{\sqrt{8\pi}}\int_{-\infty}^{\infty}dt\,\ln[1-g^2t^2/(\mu^2m^2)]\exp(-t^2/2)\right] . \tag{2.18}$$

Now, substituting this value of M into (2.17) with δ=1 gives

$$\Delta_{\delta=1}^{(2)}=\frac{1}{m}\exp\left[-\frac{1}{\sqrt{8\pi}}\int_{-\infty}^{\infty}dt\,\ln[1-g^2t^2/(\mu^2m^2)]\exp(-t^2/2)\right] . \tag{2.19}$$

The form of this expression indicates that the δ expansion is nonperturbative in character. Furthermore, the result in (2.19) is totally different from the result of the loop expansion given in (2.14). This clearly demonstrates that the δ expansion and the loop expansion are inequivalent computational schemes. The large-g behavior of this result is discussed in the next section.

III. SPECULATIVE COMPUTATION OF THE FERMION CONDENSATE IN THE MASSIVE SCHWINGER MODEL

In the previous section we used the δ expansion to compute a leading-order approximation to the electron propagator. The result is given in (2.19). The result in (2.19) has a nonperturbative structure; it is not a series in powers of g. It is therefore interesting to examine this structure in the strong-coupling limit. In the limit of large g, the formula in (2.19) has a simple asymptotic behavior:

$$\Delta^{(2)}_{\delta=1} \sim \frac{\mu\sqrt{2}}{ig} e^{\gamma/2} \; (g \to \infty) , \tag{3.1}$$

where we have used the identity

$$\frac{1}{\sqrt{\pi}} \int_0^\infty \frac{dt}{\sqrt{t}} \ln t \, e^{-t} = \psi(1/2) = -\gamma - 2\ln 2 ,$$

where $\gamma = 0.577\,215\ldots$ is Euler's constant. The asymptotic behavior in (3.1) is nonperturbative because it contains the coupling constant g to the power -1. Such a result could never be obtained from a weak-coupling expansion to any finite order because every Feynman graph contributing to the fermion propagator is proportional to an even power of g.

The result in (3.1) bears a striking similarity to the formula for the fermion condensate in the limit of large g for the massive Schwinger model [12]. The exact result for this quantity is known to be

$$\langle \bar\psi\psi \rangle \sim \frac{g}{2\pi^{3/2}} e^{\gamma} \; (g \to \infty) . \tag{3.2}$$

The similarity of the two formulas in (3.1) and (3.2) encourages us to attempt a direct calculation of the strong-coupling behavior of the fermion condensate in the massive Schwinger model using δ expansion techniques. We present the calculation below.

The Lagrangian for the massive Schwinger model (two-dimensional quantum electrodynamics) is

$$\mathcal{L} = -\tfrac{1}{4}(F^{\mu\nu})^2 + \tfrac{1}{2}\mu^2 (A^\mu)^2 + \bar\psi(m + i\partial\!\!\!/ - gA\!\!\!/)\psi , \tag{3.3}$$

where we have given the photon a mass μ. We introduce the parameter δ as follows:

$$\mathcal{L}_\delta = -\tfrac{1}{4}(F^{\mu\nu})^2 + \tfrac{1}{2}\mu^2 (A^\mu)^2 + M\bar\psi \left[\frac{m + i\partial\!\!\!/ - gA\!\!\!/}{M} \right]^\delta \psi . \tag{3.4}$$

Treating δ as a small parameter, we expand (3.4) to first order in δ:

$$\mathcal{L}_\delta = -\tfrac{1}{4}(F^{\mu\nu})^2 + \tfrac{1}{2}\mu^2 (A^\mu)^2 + M\bar\psi\psi + \delta M \bar\psi \ln \left[\frac{m + i\partial\!\!\!/ - gA\!\!\!/}{M} \right] \psi + O(\delta^2) . \tag{3.5}$$

The fermion condensate can be expressed as a ratio of two functional integrals:

$$\langle \bar\psi\psi \rangle = \frac{\int \mathcal{D}A\,\mathcal{D}\bar\psi\,\mathcal{D}\psi\; \bar\psi(0)\psi(0) \exp\left[-\int dx\, \mathcal{L}_\delta \right]}{\int \mathcal{D}A\,\mathcal{D}\bar\psi\,\mathcal{D}\psi\; \exp\left[-\int dx\, \mathcal{L}_\delta \right]} . \tag{3.6}$$

Next, we substitute (3.5) into (3.6) and expand the functional integrals in the numerator and denominator to first order in δ. Also, because we are interested in the strong-coupling ($g \to \infty$) behavior of (3.6), we neglect the terms $m + i\partial\!\!\!/$ in comparison with $gA\!\!\!/$. In addition, we make a further intuitive approximation based on the fact that the limit $g \to \infty$ is equivalent to the massless limit $m \to 0$. In this limit the photon propagator has a pole at g^2/π. We will be working in the approximation where there are no fermion loops; thus, to maintain the essential content of the theory, we make the identification $\mu^2 = g^2/\pi$. Thus, we will treat μ^2 as large and neglect $\tfrac{1}{4}(F^{\mu\nu})^2$ in comparison with $\tfrac{1}{2}\mu^2 A^2$. Thus,

$$\begin{aligned}\langle \bar\psi\psi \rangle \sim & \left[\int \mathcal{D}A\,\mathcal{D}\bar\psi\,\mathcal{D}\psi\; \bar\psi(0)\psi(0) \exp\left[-\tfrac{1}{2}\mu^2 \int dx\, A^2 - M \int dx\, \bar\psi\psi \right] \right. \\ & \left. - \delta M \int \mathcal{D}A\,\mathcal{D}\bar\psi\,\mathcal{D}\psi\; \bar\psi(0)\psi(0) \exp\left[-\tfrac{1}{2}\mu^2 \int dx\, A^2 - M \int dx\, \bar\psi\psi \right] \int dy\, \bar\psi \ln(-gA\!\!\!//M)\psi \right] \\ & \times \left[\int \mathcal{D}A\,\mathcal{D}\bar\psi\,\mathcal{D}\psi\; \exp\left[-\tfrac{1}{2}\mu^2 \int dx\, A^2 - M \int dx\, \bar\psi\psi \right] \right. \\ & \left. - \delta M \int \mathcal{D}A\,\mathcal{D}\bar\psi\,\mathcal{D}\psi\; \exp\left[-\tfrac{1}{2}\mu^2 \int dx\, A^2 - M \int dx\, \bar\psi\psi \right] \int dy\, \bar\psi \ln(-gA\!\!\!//M)\psi \right]^{-1} \; (g \to \infty) .\end{aligned} \tag{3.7}$$

Before we begin the evaluation of this ratio of functional integrals, we perform an important simplification. We replace the term $\ln(-g\mathbf{A}/M)$ by $\frac{1}{2}\ln(g^2A^2/M^2)$. This simplification eliminates the γ matrices from the integrands in (3.7).

Our objective is now to evaluate the functional integrals in (3.7). To do so we must discretize space-time. This discretization reduces functional integrals to products of ordinary integrals, which we will be able to evaluate in closed form. However, this discretization involves the introduction of a lattice spacing a, which regulates the theory. We will choose the value of a to be consistent with the phase space (uncertainty) relation $dx\,dp/2\pi=1$:

$$ag=2\pi\ . \tag{3.8}$$

Here, a, the unit of distance on the lattice, plays the role of dx and g plays the role of the basic mass or momentum unit, dp.

We now perform each of the lattice fermion integrals in turn. The simplest such integral occurs in the first term in the denominator of (3.7):

$$\int\mathcal{D}\bar{\psi}\mathcal{D}\psi\exp\left[-M\int dx\,\bar{\psi}\psi\right]=\prod_i\int d\bar{\psi}_i d\psi_i\exp(-Ma^2\bar{\psi}_i\psi_i)$$
$$=\prod_i\det(-Ma^2)=(M^2a^4)^N\ , \tag{3.9}$$

where N is the number of lattice sites.

Next we evaluate the fermion integral in the first term in the numerator of (3.7):

$$\int\mathcal{D}\bar{\psi}\mathcal{D}\psi\,\bar{\psi}(0)\psi(0)\exp\left[-M\int dx\,\bar{\psi}\psi\right]=\prod_i\int d\bar{\psi}_i d\psi_i\bar{\psi}_0\psi_0\exp(-Ma^2\bar{\psi}_i\psi_i)$$
$$=\prod_{i\neq 0}\det(-Ma^2)(-2Ma^2)=(M^2a^4)^N\left[-\frac{2}{Ma^2}\right]. \tag{3.10}$$

We continue by evaluating the fermion integral in the second term in the denominator of (3.7):

$$\int\mathcal{D}\bar{\psi}\mathcal{D}\psi\int dy\,\bar{\psi}(y)\tfrac{1}{2}\ln[g^2A^2(y)/M^2]\psi(y)\exp\left[-M\int dx\,\bar{\psi}\psi\right]=\sum_j a^2\tfrac{1}{2}\ln[g^2A_j^2/M^2]\prod_i\int d\bar{\psi}_i d\psi_i\bar{\psi}_j\psi_j\exp(-Ma^2\bar{\psi}_i\psi_i)$$
$$=\sum_j a^2\tfrac{1}{2}\ln[g^2A_j^2/M^2]\prod_{i\neq j}\det(-Ma^2)(-2Ma^2)$$
$$=(M^2a^4)^N\left[-\frac{1}{M}\sum_j\ln[g^2A_j^2/M^2]\right]. \tag{3.11}$$

Finally, we evaluate the fermion integral in the second term in the numerator of (3.7):

$$\int\mathcal{D}\bar{\psi}\mathcal{D}\psi\,\bar{\psi}(0)\psi(0)\int dy\,\bar{\psi}(y)\tfrac{1}{2}\ln[g^2A^2(y)/M^2]\psi(y)\exp\left(-M\int dx\,\bar{\psi}\psi\right)$$
$$=\sum_j a^2\tfrac{1}{2}\ln[g^2A_j^2/M^2]\prod_i\int d\bar{\psi}_i d\psi_i\bar{\psi}_0\psi_0\bar{\psi}_j\psi_j\exp(-Ma^2\bar{\psi}_i\psi_i)$$
$$=(M^2a^4)^N\left[\frac{2}{M^2a^2}\sum_{j\neq 0}\ln[g^2A_j^2/M^2]+\frac{1}{M^2a^2}\ln[g^2A_0^2/M^2]\right]. \tag{3.12}$$

We insert the results of the above four fermion integrations into (3.7) to obtain

$$\langle\bar{\psi}\psi\rangle\sim\left[\int\mathcal{D}A\left[\frac{-2}{Ma^2}\right]\exp\left[-\frac{1}{2}\mu^2\int dx\,A^2\right]\right.$$
$$\left.-\delta M\int\mathcal{D}A\frac{1}{M^2a^2}\exp\left[-\frac{1}{2}\mu^2\int dx\,A^2\right]\left[2\sum_{j\neq 0}\ln(g^2A_j^2/M^2)+\ln(g^2A_0^2/M^2)\right]\right]$$
$$\times\left[\int\mathcal{D}A\exp\left[-\frac{1}{2}\mu^2\int dx\,A^2\right]+\delta M\int\mathcal{D}A\exp\left[-\frac{1}{2}\mu^2\int dx\,A^2\right]\frac{1}{M}\sum_j\ln[g^2A_j^2/M^2]\right]^{-1}\quad(g\to\infty)\ . \tag{3.13}$$

Next, we proceed to the evaluation of the integrals over the photon field. On the lattice we replace

$$\int \mathcal{D}A \exp\left[-\frac{1}{2}\mu^2 \int dx\, A^2\right]$$

by

$$\prod_i \int dx_i dy_i \exp\left[-\tfrac{1}{2}a^2\mu^2(x_i^2+y_i^2)\right],$$

where we have written the components of the vector A^μ as x and y. We then make analogous replacements for each of the functional integrals in (3.13). Before evaluating the resulting products of ordinary integrals it is useful to expand the resulting ratio to first order in δ. When this is done, we see that an enormous simplification occurs; all of the integrals but one cancel from the numerator and denominator leaving a simple ratio of double integrals:

$$\langle \bar{\psi}\psi \rangle \sim -\frac{2}{Ma^2}\left[1-\frac{\delta}{2}\frac{\int dx\, dy \exp[-\frac{1}{2}a^2\mu^2(x^2+y^2)]\ln[g^2(x^2+y^2)/M^2]}{\int dx\, dy \exp[-\frac{1}{2}a^2\mu^2(x^2+y^2)]}\right] \quad (g\to\infty)\,. \tag{3.14}$$

We evaluate the elementary integrals in (3.14) by introducing polar coordinates:

$$\langle \bar{\psi}\psi \rangle \sim -\frac{2}{Ma^2}\{1-\delta\ln[g\sqrt{2}/(Ma\mu)]+\delta\gamma/2\} \quad (g\to\infty)\,. \tag{3.15}$$

To simplify this expression we replace μ by $g/\sqrt{\pi}$ as discussed above after (3.6) and we eliminate the lattice spacing a in favor of $2\pi/g$ according to (3.8):

$$\langle \bar{\psi}\psi \rangle \sim -\frac{g^2}{2M\pi^2}\{1-\delta\ln[g/(M\sqrt{2\pi})]+\delta\gamma/2\} \quad (g\to\infty)\,. \tag{3.16}$$

Finally, we impose the principle of minimal sensitivity as we did at the beginning of this section for the case of zero dimensions. We set $\delta=1$ and differentiate (3.16) with respect to M to determine that value of M for which (3.16) is stationary:

$$M=\frac{g}{\sqrt{2\pi}}e^{-\gamma/2}\,. \tag{3.17}$$

Substituting M in (3.17) back into (3.16) gives our final result for the first order in δ calculation of the electron condensate in the strong-coupling limit:

$$\langle \bar{\psi}\psi \rangle \sim -\frac{g}{\sqrt{2}\pi^{3/2}}e^{\gamma/2} \quad (g\to\infty)\,. \tag{3.18}$$

This result is similar in structure to the exact answer in (3.2) and is larger numerically by about 6%. Of course, the coefficient of g in (3.18) is sensitive to the manner in which the theory is regulated. However, the accuracy of the result and the simplicity of the calculation encourages us to apply δ expansion techniques to other nonperturbative problems in the context of gauge theories.

IV. DIAGRAMMATIC RULES FOR THE δ EXPANSION IN ABELIAN GAUGE THEORIES

The speculative calculation presented in Sec. III lends support to the use of the δ expansion for nonperturbative calculations in gauge theories. However, we must emphasize that the above calculation was particularly easy because it was carried out in the limit of strong coupling. In this region we were able to make powerful asymptotic approximations that enabled us to evaluate the functional integrals directly. In general, these sorts of asymptotic approximations are not possible and it is necessary to determine the δ expansion systematically by using diagrammatic rules. We discuss these rules for quantum electrodynamics in this section.

It is easiest to express the graphical rules for the δ expansion in momentum space. In this section we will state the rules for performing calculations to first order in δ. We will then use these rules in Sec. V to calculate the two-point photon Green's function to first order in δ. We will see in Sec. V that this approximation to the two-point photon Green's function is identical to the sum of all one-fermion-loop weak-coupling Feynman graphs contributing to this Green's function. (There are an infinite number of such graphs.)

We read off the graphical rules of the δ expansion to first order in δ from the provisional Lagrangian in (1.5) written in d-dimensional momentum space. The amplitude for an electron propagator, represented by a thick line, is $1/M$. The amplitude, $D^{\mu\nu}(k)$, for a photon propagator, represented by a thin line, is identical to that in the weak-coupling expansion; for example, in the Feynman gauge it has the form $g^{\mu\nu}/k^2$. There are many vertices, all proportional to the small parameter δ. Each vertex has two fermion lines and any number of photon lines ranging from 0 to N. We find the amplitudes for all of the vertices by expanding the expression

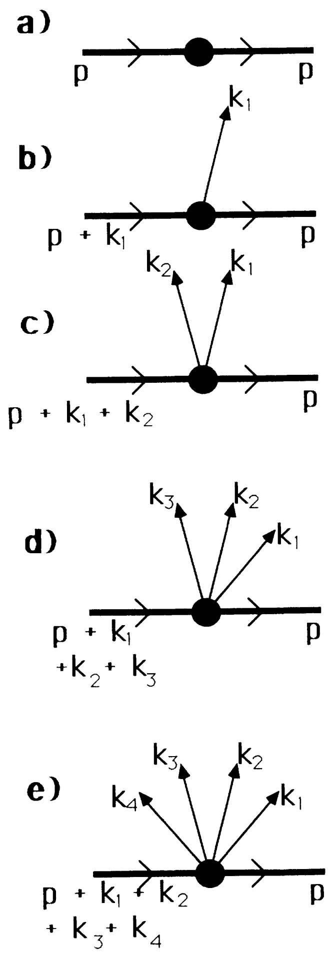
$$\delta M^{1-N}\bar{\psi}(m-\not{p}-e\not{A})^N\psi$$

as a polynomial in A. Thus, the amplitude for a vertex having n photon lines is proportional to e^n. Some of the vertex amplitudes are shown on Fig. 3. The simplest vertex [Fig. 3(a)] has no photon lines. Its amplitude is

$$-\delta M^{1-N}(m-\not{p})^N\,. \tag{4.1}$$

There are N vertices having one photon line [see Fig. 3(b)]. For each of these vertices momentum $p+k_1$ enters on one electron line, momentum p exits on the other electron line, and momentum k_1 exits on the photon line. A representative vertex amplitude is

$$e\delta M^{1-N}(m-\not{p}-\not{k}_1)^{N-a-1}\gamma^\lambda(m-\not{p})^a\,, \tag{4.2}$$



a) $-\delta M^{1-N}(m-\not{p})^N$

b) $\delta e M^{1-N}(m-\not{p}-\not{k}_1)^{N-a-1}\gamma^\lambda(m-\not{p})^a$
$(a:\ 0\to N-1)$

c) $-\delta e^2 M^{1-N}(m-\not{p}-\not{k}_1-\not{k}_2)^{N-b-a-2}\gamma^\lambda \times(m-\not{p}-\not{k}_1)^b\gamma^\mu(m-\not{p})^a$
$(a:\ 0\to N-b-2,\ \ b:\ 0\to N-2)$

d) $\delta e^3 M^{1-N}(m-\not{p}-\not{k}_1-\not{k}_2-\not{k}_3)^{N-a-b-c-3}\gamma^\lambda \times(m-\not{p}-\not{k}_1-\not{k}_2)^c\gamma^\mu(m-\not{p}-\not{k}_1)^b \times\gamma^\nu(m-\not{p})^a$
$(a:\ 0\to N-b-c-3,\ \ b:\ 0\to N-c-3,\ \ c:\ 0\to N-3)$

e) $-\delta e^4 M^{1-N}(m-\not{p}-\not{k}_1-\not{k}_2-\not{k}_3-\not{k}_4)^{N-a-b-c-d-3} \times\gamma^\lambda(m-\not{p}-\not{k}_1-\not{k}_2-\not{k}_3)^d\gamma^\mu \times(m-\not{p}-\not{k}_1-\not{k}_2)^c\gamma^\nu(m-\not{p}-\not{k}_1)^b \times\gamma^\sigma(m-\not{p})^a$
$(a:\ 0\to N-b-c-d-4,\ \ b:\ 0\to N-c-d-4,\ \ c:\ 0\to N-d-4\ \ d:\ 0\to N-4)$

FIG. 3. Vertices for the δ expansion in order δ. Shown are (a) the vertex having no photon lines, (b) vertices having one photon line, (c) vertices having two photon lines, (d) vertices having three photon lines, and (e) vertices having four photon lines.

where a is an integer that ranges from 0 to $N-1$.

There are $N(N-1)/2$ vertices having two photon lines [see Fig. 3(c)]. For each of these vertices momentum $p+k_1+k_2$ enters on one electron line, momentum p exits on the other electron line, and momenta k_1 and k_2 exit on the photon lines. A representative vertex amplitude is

$$-e^2\delta M^{1-N}(m-\not{p}-\not{k}_1-\not{k}_2)^{N-a-b-2} \times\gamma^\lambda(m-\not{p}-\not{k}_1)^b\gamma^\mu(m-\not{p})^a\ , \qquad (4.3)$$

where a is an integer that ranges from 0 to $N-2-b$ and b is an integer that ranges from 0 to $N-2$.

There are $N(N-1)(N-2)/6$ vertices having three photon lines [see Fig. 3(d)]. For each of these vertices momentum $p+k_1+k_2+k_3$ enters on one electron line, momentum p exits on the other electron line, and momenta k_1, k_2, and k_3 exit on the photon lines. A representative vertex amplitude is

$$e^3\delta M^{1-N}(m-\not{p}-\not{k}_1-\not{k}_2-\not{k}_3)^{N-a-b-c-3}\gamma^\lambda(m-\not{p}-\not{k}_1-\not{k}_2)^c\gamma^\mu(m-\not{p}-\not{k}_1)^b\gamma^\nu(m-\not{p})^a\ , \qquad (4.4)$$

where a is an integer that ranges from 0 to $N-3-b-c$, b is an integer that ranges from 0 to $N-3-c$, and c is an integer that ranges from 0 to $N-3$.

In Fig. 3(e) a vertex having four photon lines is shown. The amplitude for this vertex is

$$-e^4\delta M^{1-N}(m-\not{p}-\not{k}_1-\not{k}_2-\not{k}_3-\not{k}_4)^{N-a-b-c-d-4}\gamma^\lambda(m-\not{p}-\not{k}_1-\not{k}_2-\not{k}_3)^d\gamma^\mu(m-\not{p}-\not{k}_1-\not{k}_2)^c \times\gamma^\nu(m-\not{p}-\not{k}_1)^b\gamma^\sigma(m-\not{p})^a\ . \qquad (4.5)$$

V. PHOTON PROPAGATOR TO FIRST ORDER IN δ

The graphical rules described in Sec. IV allow us to calculate straightforwardly any Green's function to first order in δ. However, the calculation is necessarily long and tedious because the δ expansion is nonperturbative; it contains an enormous amount of information (equivalent to an infinite number of ordinary weak-coupling Feynman diagrams) in every order in δ. The purpose of this section is to illustrate the use of the graphical rules described in Sec. IV by beginning the calculation of the two-point photon Green's function to first order in δ; we will confine our attention to just the e^2 contribution to the photon propagator. Specifically, we will consider only the vertices of order e^2 shown in Fig. 3(c). There are two contributions to the Green's function arising from these vertices. The first is

$$\delta e^2 M^{1-N} \sum_{b=0}^{N-2} \sum_{a=0}^{N-b-2} \int \frac{d^d p}{(2\pi)^d} \frac{1}{M} \mathrm{Tr}(m - \not p)^{N-a-b-2} \gamma^\lambda (m - \not p - \not k)^b \gamma^\mu (m - \not p)^a . \tag{5.1}$$

The second contribution comes from the crossed graph; that is, the amplitude in (5.1) with μ and λ interchanged and with k replaced by $-k$. To obtain (5.1) we have used the vertex amplitude given in (4.3) and proceed as follows. First, we take $k = k_1 = -k_2$ to be the momentum flowing through the graph. Second, we connect the electron lines together with the electron propagator $1/M$ and integrate over the electron momentum p. This electron loop is associated with a trace and a factor of -1. Note that the momentum integral over p appears to be strongly divergent because the power of p in the integrand is positive. This amplitude can be represented by a graph having one vertex to which one fermion loop and two external photon lines are attached (see Fig. 4).

The expression in (5.1) can be simplified dramatically using the cyclic property of the trace to combine the first and last factors in the trace. The resulting integrand no longer depends on the summation variable a. This allows us to perform the sum on a:

$$\delta e^2 M^{1-N} \sum_{b=0}^{N-2} (N-b-1) \int \frac{d^d p}{(2\pi)^d} \frac{1}{M} \mathrm{Tr}(m - \not p)^{N-b-2} \gamma^\lambda (m - \not p - \not k)^b \gamma^\mu . \tag{5.2}$$

The result in (5.2) simplifies further if we combine it with that for the crossed graph. To obtain the simplification we make the change of summation variable $b \to N-b-2$ and the change of integration variable $p \to p+k$ in the amplitude for the crossed graph. The result, after using the cyclic property of the trace, is

$$\delta e^2 M^{1-N} \sum_{b=0}^{N-2} (b+1) \int \frac{d^d p}{(2\pi)^d} \frac{1}{M} \mathrm{Tr}(m - \not p)^{N-b-2} \gamma^\lambda (m - \not p - \not k)^b \gamma^\mu . \tag{5.3}$$

This expression differs only slightly from that in (5.2); namely, the factor $(N-b-1)$ is replaced by $(b+1)$. Now, when we combine (5.2) and (5.3) we obtain a result proportional to N:

$$\delta e^2 N M^{-N} \sum_{b=0}^{N-2} \int \frac{d^d p}{(2\pi)^d} \mathrm{Tr}(m - \not p)^{N-b-2} \times \gamma^\lambda (m - \not p - \not k)^b \gamma^\mu . \tag{5.4}$$

The fact that (5.4) is proportional to N is crucial because we must differentiate with respect to N and set $N=0$. This procedure removes the factor of N and evaluates the sum at $N=0$. Thus, we must understand what it means to evaluate a sum whose limits range from $b=0$ to $b=N-2$ at $N=0$. To do so we consider the following generic sum $S(N)$:

$$S(N) = \sum_{b=0}^{N-2} \alpha_b . \tag{5.5}$$

We rewrite $S(N)$ by adding and subtracting the $b=N-1$ and $b=N$ terms:

$$S(N) = -\alpha_N - \alpha_{N-1} + \sum_{b=0}^{N} \alpha_b .$$

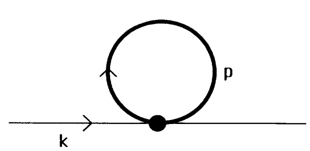


FIG. 4. Graph contributing to the photon propagator in order δe^2.

Now, if we evaluate $S(N)$ at $N=0$, we obtain

$$S(0) = -\alpha_{-1} . \tag{5.6}$$

Using this result, we reduce the expression in (5.4) to

$$-\delta e^2 \int \frac{d^d p}{(2\pi)^d} \mathrm{Tr}(m - \not p)^{-1} \gamma^\lambda (m - \not p - \not k)^{-1} \gamma^\mu . \tag{5.7}$$

Apart from the factor of δ, this amplitude is precisely the lowest-order one-fermion-loop contribution to the photon propagator in the conventional weak-coupling expansion. The graph representing this amplitude is shown in Fig. 5. We make several observations. First, the graph in Fig. 5 has two vertices; this is in distinct contrast with the one-vertex graph in Fig. 4. The usual weak-coupling amplitude for the electron propagator seems to have materialized from the elaborate structure of the vertex amplitude in the δ expansion. Second, while the momentum integral in (5.1) appeared to be horribly divergent, we now see that it is no more divergent than the one-loop graph in Fig. 5. We control the divergence of the integral by differentiating with respect to N before attempting to perform the momentum integration. Third, while (5.1) appears to depend on the mass parameter M, it is actually independent of M. The disappearance of this parameter

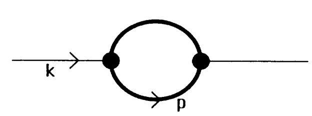


FIG. 5. Lowest-order one-fermion-loop contribution to the photon propagator in the conventional weak-coupling expansion.

is the signal that the δ expansion reduces to an ordinary weak-coupling expansion. The parameter M does not drop out if the Green's function has external electron lines, as we will see in Sec. VI when we calculate the electron propagator to order δ. For such a case the δ expansion is not equivalent to another approximation scheme; it provides a new nonperturbative window on the theory.

VI. ELECTRON PROPAGATOR TO FIRST ORDER IN δ

In this section we continue with the approach of the previous section. Our objective here will be to perform a calculation of the electron propagator to leading order in δ, but only to order e^2. The vertices shown in Figs. 3(a) and 3(c) contribute to the amplitude.

The graph arising from the vertex in Fig. 3(a) is shown in Fig. 6(a). The amplitude for this graph is

$$-\delta M^{1-N}(m-\not p)^N . \tag{6.1}$$

We must differentiate the expression in (6.1) with respect to N and set $N=0$. The result is simply

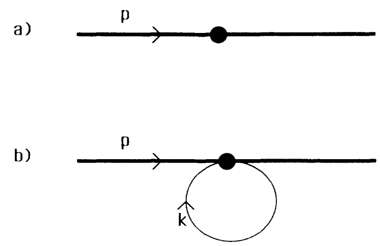


FIG. 6. Graphs contributing to the electron propagator in order δ. Shown are (a) the graph of order e^0 and (b) the graph of order e^2.

$$-\delta M \ln[(m-\not p)/M] . \tag{6.2}$$

The graphs arising from the vertices in Fig. 3(c) are shown in Fig. 6(b). There are actually two graphs to consider. The first has the amplitude

$$-e^2\delta M^{1-N}\sum_{b=0}^{N-2}\sum_{a=0}^{N-b-2}\int\frac{d^dk}{(2\pi)^d}D_{\lambda\mu}(k)(m-\not p)^{N-a-b-2}\gamma^\lambda(m-\not p-\not k)^b\gamma^\mu(m-\not p)^a , \tag{6.3}$$

where $D_{\lambda\mu}$ is the free photon propagator. The second graph is the crossed graph, whose amplitude is obtained from (6.3) by replacing p by $-p$.

We must now differentiate the results in (6.3) and the crossed amplitude with respect to N and set $N=0$. This procedure is more complicated than it was for the photon propagator because there are eight cases to consider, depending on whether N, a, and b are even or odd. Here, we just consider one of these cases to illustrate what is involved in the calculation. Let us take N, a, and b to be even:

$$N=2n,\quad a=2\alpha,\quad b=2\beta .$$

Also, we will make a further simplification by setting $m=0$. With these replacements, (6.3) becomes

$$-e^2\delta M^{1-2n}\sum_{\beta=0}^{n-1}\sum_{\alpha=0}^{n-\beta-1}\int\frac{d^dk}{(2\pi)^d}D_{\lambda\mu}(k)\gamma^\lambda\gamma^\mu(p^2)^{n-\beta-1}[(p+k)^2]^\beta . \tag{6.4}$$

Summing on α gives

$$-e^2\delta M^{1-2n}(p^2)^{n-1}\int\frac{d^dk}{(2\pi)^d}D_{\lambda\mu}(k)\gamma^\lambda\gamma^\mu\sum_{\beta=0}^{n-1}(n-\beta)[(p+k)^2/p^2]^\beta . \tag{6.5}$$

Next, we perform the sum over β using the identity

$$\sum_{\beta=0}^{n-1}(n-\beta)z^\beta=\frac{1}{(z-1)^2}[n(1-z)-z(1-z^n)] . \tag{6.6}$$

Note that this quantity vanishes when $N=n=0$. Thus, the only contribution comes when we differentiate (6.6) with respect to n and we may set $n=0$ elsewhere in (6.5). Our final result for the contribution to the electron propagator is

$$-e^2\delta\frac{M}{2p^2}\int\frac{d^dk}{(2\pi)^d}D_{\lambda\mu}(k)\gamma^\lambda\gamma^\mu\frac{p^2}{[(p+k)^2-p^2]^2}\left[p^2-(p+k)^2-(p+k)^2\ln\left[\frac{(p+k)^2}{p^2}\right]\right] . \tag{6.7}$$

It is apparent that the $O(\delta)$ contributions to the electron propagator in (6.2) and (6.7) bear no resemblance to the weak-coupling graphical expansion of the electron propagator to order e^2. Furthermore, these δ expansion amplitudes have a nontrivial dependence on the mass parameter M. When the value of M is determined by the principle of minimal sensitivity, as in Secs. II and III, the resulting expressions involve exponentials of the coupling constant and are truly nonperturbative in character.

VII. EQUIVALENCE OF THE δ AND FERMION-LOOP EXPANSIONS IN THE PHOTON SECTOR TO FIRST ORDER IN δ

In this section we generalize the calculation in Sec. V to higher orders in e. We show that the order-δ term in the δ expansion in the photon sector (by the photon sector we mean those Green's functions whose external lines are photon lines) is equivalent to the sum of all one-fermion-loop weak-coupling Feynman diagrams for that Green's function.

The approach used in this section is based on considering only graphs in the δ expansion having no photon loops. To illustrate our procedure we begin by examining a one-fermion-loop graph in the weak-coupling expansion having four external photon lines. This graph is shown in Fig. 7. The amplitude for this graph from the conventional Feynman rules is

$$e^4\int\frac{d^dp}{(2\pi)^d}\mathrm{Tr}(m-\not p)^{-1}\gamma^\lambda(m-\not p-\not k_1)^{-1}\gamma^\mu(m-\not p-\not k_1-\not k_2)^{-1}\gamma^\nu(m-\not p-\not k_1-\not k_2-\not k_3)^{-1}\gamma^\sigma\ . \tag{7.1}$$

Let us reconstruct this amplitude using the δ expansion. To do so use the vertex shown in Fig. 3(e), whose amplitude is given in (4.5). Recall that in Sec. V it was necessary to combine two graphs, (5.1) and its crossed graph, to establish the equivalence in order e^2 of the δ expansion and the conventional weak-coupling expansion. To higher order in e we will see that in the δ expansion a set of n distinct graphs (related by cyclic permutation of the labeling of the photon lines) is needed to obtain each weak-coupling graph in the loop expansion of the n-point function. For the graph in Fig. 7 the appropriate four δ-expansion graphs are depicted in Figs. 8(a)–8(d). For example, the amplitudes for the graphs in Figs. 8(a) and 8(b) are

$$\delta e^4M^{1-N}\int\frac{d^dp}{(2\pi)^d}\frac{1}{M}\sum_{0\le a+b+c+d\le N-4}\mathrm{Tr}(m-\not p)^{N-4-a-b-c-d}\gamma^\lambda(m-\not p-\not k_1)^d$$
$$\times\gamma^\mu(m-\not p-\not k_1-\not k_2)^c\gamma^\nu(m-\not p-\not k_1-\not k_2-\not k_3)^b\gamma^\sigma(m-\not p)^a \tag{7.2}$$

and

$$\delta e^4M^{1-N}\int\frac{d^dp}{(2\pi)^d}\frac{1}{M}\sum_{0\le a+b+c+d\le N-4}\mathrm{Tr}(m-\not p)^{N-4-a-b-c-d}\gamma^\sigma(m-\not p-\not k_4)^d$$
$$\times\gamma^\lambda(m-\not p-\not k_4-\not k_1)^c\gamma^\mu(\not p-\not k_4-\not k_1-\not k_2)^b\gamma^\nu(m-\not p)^a\ . \tag{7.3}$$

First, we use the cyclic property of the trace to combine the first and last factors therein and carry out the now trivial summation over a in each of these amplitudes. This introduces into each amplitude a factor of $N-3-b-c-d$ and changes the range of the summation to $0\le b+c+d\le N-4$. Then we again use the cyclic property of the trace to align the γ matrices in the four amplitudes of the δ expansion according to the order in (7.1). To make each factor in between the γ matrices coincident with those in (7.1) we shift the integration variable of the momentum integral in each amplitude accordingly and use conservation of momentum $k_1+k_2+k_3+k_4=0$. That is, we leave the integration over p in (7.2) unchanged, shift $p\to p-k_4$ in (7.3), and so on. Now the four amplitudes are

$$\delta e^4M^{-N}\int\frac{d^dp}{(2\pi)^d}\sum_{0\le b+c+d\le N-4}(N-3-b-c-d)\mathrm{Tr}(m-\not p)^{N-4-b-c-d}\gamma^\lambda$$
$$\times(m-\not p-\not k_1)^d\gamma^\mu(m-\not p-\not k_1-\not k_2)^c\gamma^\nu(m-\not p-\not k_1-\not k_2-\not k_3)^b\gamma^\sigma\ , \tag{7.4}$$

$$\delta e^4M^{-N}\int\frac{d^dp}{(2\pi)^d}\sum_{0\le b+c+d\le N-4}(N-3-b-c-d)\mathrm{Tr}(m-\not p)^d\gamma^\lambda(m-\not p-\not k_1)^c\gamma^\mu$$
$$\times(m-\not p-\not k_1-\not k_2)^b\gamma^\nu(m-\not p-\not k_1-\not k_2-\not k_3)^{N-4-b-c-d}\gamma^\sigma\ , \tag{7.5}$$

$$\delta e^4M^{-N}\int\frac{d^dp}{(2\pi)^d}\sum_{0\le b+c+d\le N-4}(N-3-b-c-d)\mathrm{Tr}(m-\not p)^c\gamma^\lambda(m-\not p-\not k_1)^b\gamma^\mu$$
$$\times(m-\not p-\not k_1-\not k_2)^{N-4-b-c-d}\gamma^\nu(m-\not p-\not k_1-\not k_2-\not k_3)^d\gamma^\sigma\ , \tag{7.6}$$

and

$$\delta e^4M^{-N}\int\frac{d^dp}{(2\pi)^d}\sum_{0\le b+c+d\le N-4}(N-3-b-c-d)\mathrm{Tr}(m-\not p)^b\gamma^\lambda(m-\not p-\not k_1)^{N-4-b-c-d}\gamma^\mu$$
$$\times(m-\not p-\not k_1-\not k_2)^d\gamma^\nu(m-\not p-\not k_1-\not k_2-\not k_3)^c\gamma^\sigma\ . \tag{7.7}$$

We redefine the summation variables and add the four amplitudes in (7.4)–(7.7) to obtain

$$\delta e^4NM^{-N}\int\frac{d^dp}{(2\pi)^d}\sum_{0\le b+c+d\le N-4}\mathrm{Tr}(m-\not p)^{N-4-b-c-d}\gamma^\lambda(m-\not p-\not k_1)^d\gamma^\mu$$
$$\times(m-\not p-\not k_1-\not k_2)^c\gamma^\nu(m-\not p-\not k_1-\not k_2-\not k_3)^b\gamma^\sigma\ . \tag{7.8}$$

Again, we observe that, as in (5.4), the sum of the amplitudes of the cyclically permuted graphs in the δ expansion is proportional to N. Differentiating (7.8) with respect to N at $N=0$ removes the factor of N and requires us to evaluate the sum at $N=0$. Using the generalization (A1) of (5.6), which we have proved in the Appendix, yields

$$\delta e^4\int\frac{d^dp}{(2\pi)^d}\mathrm{Tr}(m-\not p)^{-1}\gamma^\lambda(m-\not p-\not k_1)^{-1}\gamma^\mu(m-\not p-\not k_1-\not k_2)^{-1}\gamma^\nu(m-\not p-\not k_1-\not k_2-\not k_3)^{-1}\gamma^\sigma\ . \tag{7.9}$$

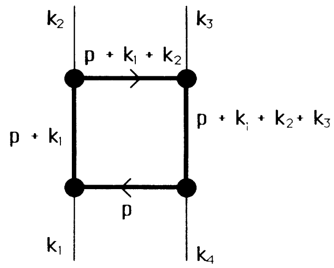


FIG. 7. Lowest-order one-fermion-loop contribution to the four-point photon Green's function in the weak-coupling expansion.

Setting $\delta=1$ we reproduce the corresponding one-loop weak-coupling amplitude in (7.1) and the observations following (5.7) apply accordingly. It is now easy to see how the above procedure generalizes to the n-point Green's function in the photon sector for any n. Thus, we will forego here the general proof of the equivalence.

It remains to show how the equivalence between the δ and the loop expansion to first order in the photon sector can be extended for the case of internal photon lines. Because the amplitude of the photon propagator in the δ expansion is identical to that in the weak-coupling expansion, if we link external photon lines to make internal photon lines the equivalence between the two expansions continues to hold. For example, if we link two external photon lines of the weak-coupling graph in Fig. 7 by operating with

$$\int\int \frac{d^dk_2}{(2\pi)^d}\frac{d^dk_4}{(2\pi)^d}(2\pi)^d\delta^d(k_2+k_4)D^{\mu\sigma}(k_2) \qquad (7.10)$$

on (7.1), we obtain the amplitude for the graph in Fig. 9. It is immediately clear that this amplitude is equal to the sum of the amplitudes for the graphs in Figs. 10(a)–10(d), which are obtained by operating with (7.10) on the amplitudes of the graphs in Figs. 8(a)–8(d). It is now obvious how the equivalence that we proved above for graphs with only external photon lines extends, through repeated application of (7.10), to graphs that include an arbitrary number of internal photon lines.

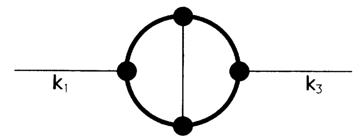


FIG. 9. One-fermion-loop contribution to the two-point photon Green's function having one internal photon line.

VIII. EQUIVALENCE OF THE δ AND FERMION-LOOP EXPANSIONS IN THE PHOTON SECTOR TO SECOND ORDER IN δ

In Ref. [1] a derivation of diagrammatic rules for higher orders in the δ expansion for Lagrangians such as those in (1.1) and (1.2) was given. For example, expanding (1.2) to all orders in δ gives

$$\mathcal{L}_\delta=-\tfrac{1}{4}(F^{\mu\nu})^2+M\bar\psi\psi+\mathcal{L}_I\ , \qquad (8.1)$$

where we define the interaction part of the Lagrangian $\mathcal{L}_I$ to be

$$\mathcal{L}_I=M\bar\psi\sum_{k=1}^{\infty}\frac{\delta^k}{k!}\{\ln[(m+i\partial\!\!\!/-eA\!\!\!/)/M]\}^k\psi\ . \qquad (8.2)$$

To a given order K in the δ expansion the interaction part of the provisional Lagrangian is

$$\tilde{\mathcal{L}}_I=M\bar\psi\sum_{k=1}^{K}[(m+i\partial\!\!\!/-eA\!\!\!/)/M]^{\alpha_k}P_k^{(K)}\psi\ . \qquad (8.3)$$

The Kth-order provisional Lagrangian has a diagrammatic interpretation if all α_k are regarded as integers. The Green's functions of the original theory in (8.1) to order δ^K can be obtained from those of the provisional Lagrangian, whose interaction term is given in (8.3), by applying the derivative operator

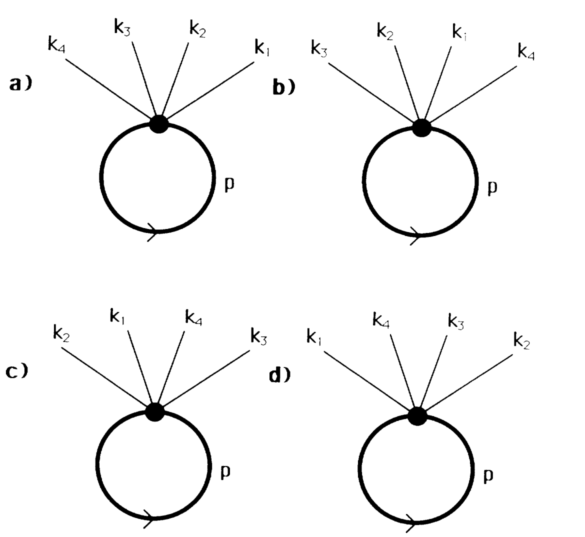


FIG. 8. The four graphs, related by cyclic permutation of the labeling of the photon lines, in the δ expansion corresponding to the weak-coupling graph in Fig. 7.

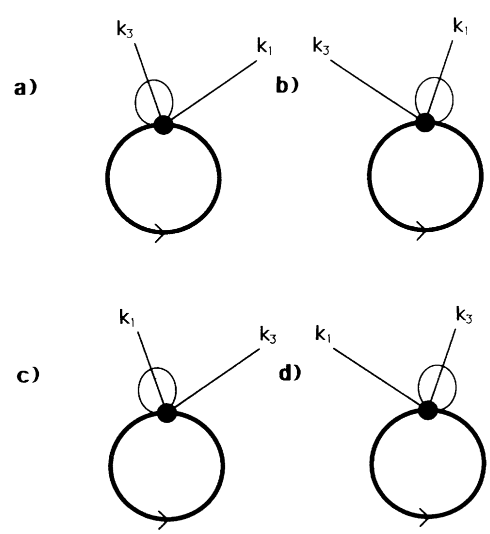


FIG. 10. The four graphs in the δ expansion corresponding to the weak-coupling graph in Fig. 9.

$$D^{(K)}=\frac{1}{K}\sum_{j=1}^{K}\sum_{k=1}^{K}\frac{\exp[2\pi ij(1-k)/K]}{j!}\left[\frac{\partial}{\partial\alpha_k}\right]^j \qquad (8.4)$$

at $\alpha_1=\alpha_2=\cdots=\alpha_K=0$. For $K=1$, we find $P_1^{(1)}=\delta$, and the procedure reduces to the one given in Sec. I. For $K=2$, $P_1^{(2)}=\delta+\delta^2$ and $P_2^{(2)}=-\delta+\delta^2$. In this case we get from (8.3)

$$\tilde{\mathcal{L}}_I=M(\delta+\delta^2)\bar{\psi}[(m+i\partial\!\!\!/-eA\!\!\!/)/M]^\alpha\psi$$
$$+M(-\delta+\delta^2)\bar{\psi}[(m+i\partial\!\!\!/-eA\!\!\!/)M]^\beta\psi\ , \qquad (8.5)$$

and from (8.4)

$$D^{(2)}=\frac{1}{2}\left[\frac{\partial}{\partial\alpha}-\frac{\partial}{\partial\beta}\right]+\frac{1}{4}\left[\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}\right]. \qquad (8.6)$$

A diagrammatic interpretation of (8.5) is straightforward. Two different types of vertices appear, one proportional to $\delta+\delta^2$, and the other proportional to $-\delta+\delta^2$. In the photon sector of the theory we consider four types of graphs, as depicted in Fig. 11. The two graphs in Figs. 11(a) and 11(b) have one vertex each and yield amplitudes that contribute to first as well as to second order in δ. The two graphs in Figs. 11(c) and 11(d) contain both types of vertices and contribute only to second order in δ. Note that in Fig. 11(c) there is a fermion loop at each vertex whereas in Fig. 11(d) a single fermion loop connects the two vertices.

For the δ and loop expansion to be equivalent order by order, it is necessary that the one-fermion-loop contributions from the graphs in Figs. 11(a), 11(b), and 11(d) all cancel in second order in δ. This is most easily shown by examining the expansion of the exponential of the provisional action:

$$\tilde{S}=1-\int d^dx\tilde{\mathcal{L}}_I(x)+\tfrac{1}{2}\int\int d^dxd^dy\tilde{\mathcal{L}}_I(x)\tilde{\mathcal{L}}_I(y)+\cdots\ , \qquad (8.7)$$

where $\tilde{\mathcal{L}}_I$ is the interaction term in the provisional Lagrangian in (8.5). It is convenient to define

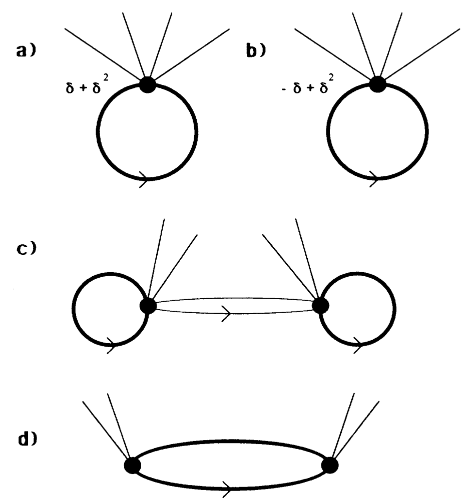


FIG. 11. Four graphs in the δ expansion that contribute to the four-point photon Green's function to second order in δ.

$$T(x,\alpha)=M\bar{\psi}(x)\{[m+i\partial\!\!\!/-eA\!\!\!/(x)]M\}^\alpha\psi(x)\ , \qquad (8.8)$$

and to rewrite (8.5) as

$$\tilde{\mathcal{L}}_I(x)=\delta[T(x,\alpha)-T(x,\beta)]+\delta^2[T(x,\alpha)+T(x,\beta)]\ . \qquad (8.9)$$

Utilizing the symmetries and antisymmetries in the variables α and β at $\alpha=\beta=0$ of (8.6) and (8.9), we derive

$$[D^{(2)}\tilde{\mathcal{L}}_I(x)]_{\alpha=\beta=0}=\left[\left[\delta\frac{\partial}{\partial\alpha}+\frac{1}{2}\delta^2\frac{\partial^2}{\partial\alpha^2}\right]T(x,\alpha)\right]_{\alpha=0} \qquad (8.10a)$$

and, to order δ^2,

$$[D^{(2)}\tilde{\mathcal{L}}_I(x)\tilde{\mathcal{L}}_I(y)]_{\alpha=\beta=0}=\delta^2\left[\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta}T(x,\alpha)T(y,\beta)\right]_{\alpha=\beta=0}. \qquad (8.10b)$$

We combine the results in (8.10) to obtain to order δ^2

$$S-1=(D^{(2)}\tilde{S})_{\alpha=\beta=0}=-\left[\left[\delta\frac{\partial}{\partial\alpha}+\frac{1}{2}\delta^2\frac{\partial^2}{\partial\alpha^2}\right]\tilde{S}_1(\alpha)\right]_{\alpha=0}+\frac{1}{2}\delta^2\left[\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta}\tilde{S}_2(\alpha,\beta)\right]_{\alpha=\beta=0} \qquad (8.11a)$$

with

$$\tilde{S}_1(\alpha)=M\int d^dx\bar{\psi}(x)\{[m+i\partial\!\!\!/-eA\!\!\!/(x)]/M\}^\alpha\psi(x) \qquad (8.11b)$$

and

$$\tilde{S}_2(\alpha,\beta)=M^2\int\int d^dxd^dy\bar{\psi}(x)\{[m+i\partial\!\!\!/-eA\!\!\!/(x)]/M\}^\alpha\psi(x)\bar{\psi}(y)\{[m+i\partial\!\!\!/-eA\!\!\!/(y)]/M\}^\beta\psi(y)\ . \qquad (8.11c)$$

In this form it is obvious that the contractions on $\tilde{S}_1$ give graphs of the types shown in Figs. 11(a) and 11(b). Note that the distinction between the two graphs disappears because of the action of $D^{(2)}$. Observe from (8.11a) that these graphs contribute to first as well as to second order in δ. The contractions on $\tilde{S}_2$ in the photon sector lead to the two types of graphs in Figs. 11(c) and 11(d):

$$\tilde{S}_2^a(\alpha,\beta)=M^2\int\int d^dx\,d^dy\,\overline{\psi}(x)\{[m+i\partial\!\!\!/-e A\!\!\!/(x)]/M\}^\alpha\psi(x)\overline{\psi}(y)\{[m+i\partial\!\!\!/-e A\!\!\!/(y)]/M\}^\beta\psi(y) \tag{8.12a}$$

and

$$\tilde{S}_2^b(\alpha,\beta)=M^2\int\int d^dx\,d^dy\,\overline{\psi}(x)\{[m+i\partial\!\!\!/-e A\!\!\!/(x)]/M\}^\alpha\psi(x)\overline{\psi}(y)\{[m+i\partial\!\!\!/-e A\!\!\!/(y)]/M\}^\beta\psi(y)\ . \tag{8.12b}$$

$\tilde{S}_2^a$ in (8.12a) contains the desired result that the second-order term in the δ expansion corresponds to the sum of all two-loop graphs. We will see that the contributions of $\tilde{S}_2^b$ in (8.12b) cancel. From the momentum-space Feynman amplitude for the electron propagator $1/M$ we have

$$\psi(x)\overline{\psi}(y)=\frac{1}{M}\delta^d(x-y)\ . \tag{8.13}$$

Using (8.13) we simplify $\tilde{S}_2^b$ by eliminating one integration and combining $\{[m+i\partial\!\!\!/-eA\!\!\!/(x)]/M\}^\alpha\{[m+i\partial\!\!\!/-eA\!\!\!/(x)]/M\}^\beta=\{[m+i\partial-eA\!\!\!/(x)]/M\}^{\alpha+\beta}$. Applying the derivative operator $D^{(2)}$ we find that

$$\left[\frac{\partial}{\partial\alpha}\frac{\partial}{\alpha\beta}\tilde{S}_2^b(\alpha,\beta)\right]_{\alpha=\beta=0}=\left[\frac{\partial^2}{\partial\alpha^2}\tilde{S}_1(\alpha)\right]_{\alpha=0}, \tag{8.14}$$

which cancels the first $O(\delta^2)$ term in (8.11a). Thus, in the photon sector of the original theory in (1.1) we obtain

$$S=1+\left[\left[-\delta\frac{\partial}{\partial\alpha}\right]\tilde{S}_1(\alpha)\right]_{\alpha=0}+\frac{1}{2}\left[\left[-\delta\frac{\partial}{\partial\alpha}\right]\left[-\delta\frac{\partial}{\partial\beta}\right]\tilde{S}_2^a(\alpha,\beta)\right]_{\alpha=\beta=0}, \tag{8.15}$$

which proves the equivalence to second order.

Several remarks appear to be in order here. First, it should be noted that nowhere in this section have we used the particular form of the minimal coupling term $(m+i\partial\!\!\!/-eA\!\!\!/)/M$. This indicates that the equivalence between the loop and the δ expansions extends to all topologically similar theories as long as the electron propagator is a constant in momentum space. This equivalence holds to all orders in δ. Finally, we note that no extra consideration was given to the case of internal photon lines because the observation at the end of Sec. VII regarding internal photon lines applies trivially to all orders. This result and (8.13) lead to the practical observation that, even if external electron lines are present, the Feynman rules will never be any more complicated than those given in Sec. IV despite the increasingly complex form of the interaction in (8.3). That is, no δ vertex will ever be connected to any other by an electron line. It thus is possible to assemble any n-vertex graph in the δ expansion from one-vertex graphs by linking photon lines according to the operation of (7.10).

ACKNOWLEDGMENTS

We thank H. Fried for a stimulating discussion. We thank the U.S. Department of Energy for financial support.

APPENDIX

For the proof in Sec. VII it is necessary to generalize the result of (5.6) to apply to multiple summations. Hence, we must show that for any number $n>1$ and for any function f of $n-1$ variables

$$\sum_{0\le x_2+\cdots+x_n\le N-n} f(x_2,\ldots,x_n)|_{N=0} =(-1)^{n-1}f(-1,\ldots,-1)\ . \tag{A1}$$

We establish (A1) by induction. First, note that (5.6) is precisely (A1) when $n=2$. Now suppose that we have verified (A1) for all numbers smaller than n. We rewrite

$$\sum_{0\le x_2+\cdots+x_n\le N-n} f(x_2,\ldots,x_n) \tag{A2}$$

by separating off the summation over x_n:

$$\sum_{0\le x_2+\cdots+x_{n-1}\le N-n}\ \sum_{x_n=0}^{N-n-x_2-\cdots-x_{n-1}} f(x_2,\ldots,x_n)\ . \tag{A3}$$

The expression in (A3) can now be split into two terms:

$$\sum_{0\le x_2+\cdots x_{n-1}\le N-(n-1)}\ \sum_{\alpha=0}^{N-n-x_2-\cdots-x_{n-1}} f(x_2,\ldots,x_{n-1},\alpha) - \sum_{x_2+\cdots+x_{n-1}=N-(n-1)}\ \sum_{\alpha=0}^{N-n-x_2-\cdots-x_{n-1}} f(x_2,\ldots,x_{n-1},\alpha)\ . \tag{A4}$$

The second term in (A4) vanishes. To show this we add a term to the sum for which $\alpha=N-(n-1)-x_2-\cdots-x_{n-1}$ and subtract it again. Because of the constraint $x_2+\cdots+x_{n-1}=N-(n-1)$ the sum reduces to

$$\sum_{x_2+\cdots+x_{n-1}=N-(n-1)}\left[\sum_{\alpha=0}^{0} f(x_2,\ldots,x_{n-1},\alpha)-f(x_2,\ldots,x_{n-1},0)\right]. \tag{A5}$$

The first term in (A4) at $N=0$ fulfills the induction assumption for $n-1$. Thus, (A4) at $N=0$ is

$$(-1)^{n-2}\sum_{\alpha=0}^{N-2} f(-1,\ldots,-1,\alpha)\big|_{N=0}\ . \tag{A6}$$

Again, we apply the induction assumption for $n=2$ in (A1) on (A6) and find that (A2) at $N=0$ is equal to

$$(-1)^{n-1}f(-1,\ldots,-1)\ . \tag{A7}$$

This completes the proof by induction.

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