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CP^{N-1} models in the $1/N$ expansion

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Two-dimensional CP^{N-1} models are analyzed by means of the $1/N$ expansion, performed up to the first nonleading order. The expectation values of gauge- and renormalization-group-invariant quantities are computed in a regulated continuum version of the theory: renormalizability and absence of infrared divergences are explicitly verified. Special attention is devoted to open and closed loops of gauge fields and to their correlations. No single-particle mass for the "elementary" gauge-dependent fields can be consistently defined. The observable mass parameter is related to the lowest-energy bound state and shows a nonanalytic dependence on $1/N$. The qualitative picture of the CP^{N-1} models resulting from the large-N approximation is quantitatively confirmed in the $1/N$ expansion.

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I. INTRODUCTION

Two-dimensional CP^{N-1} models take a special place in the realm of quantum-field-theory models because of the close resemblance between many of their dynamical features and the properties expected to hold (but often very difficult to prove) in QCD [1,2]. Among these properties we mention asymptotic freedom, spontaneous mass generation, unbroken gauge invariance, dynamical appearance of a linear confining potential between non-gauge-invariant states, and topological structure (instantons, anomalies, θ vacua). These models have however a number of computational advantages with respect to QCD: they involve spin (instead of gauge) fields and lower space dimensionality and, last but not least, they possess a viable $1/N$ expansion [1,2,3]. These considerations make CP^{N-1} models a quite natural candidate for a detailed exploration of dynamical mechanisms and a testing ground for both analytical and numerical methods in strongly interacting relativistic quantum field theory. However, despite the already extensive literature on the subject, it appears that most properties have been only qualitatively discussed, and quantitative predictions beyond the leading large-N limit have not been presented yet.

In view of the renewed interest around these models, especially in the context of the numerical simulation approach to quantum field theories [4,5], we decided to perform a systematic analysis of the first nonleading order in the $1/N$ expansion, with the ultimate goal of offering the possibility of quantitative tests of agreement between this theoretical approach and other analytical and numerical methods. The purpose of the present work is therefore twofold: (a) we want to identify the physical quantities that are "measurable" (i.e., gauge and renormalization-group invariant) and receive nontrivial contributions from the first order of the $1/N$ expansion; (b) we want to compute these contributions and show that they are consistent with the qualitative picture of the model. In order to do this, we decided to focus on the continuum version of the models and on the very simple sharp-momentum (SM) cutoff regularization scheme. Converting our results to different continuum or lattice schemes may not be technically trivial but is certainly conceptually straightforward, if we follow the lines of Refs. [6] and [7].

The present paper is organized as follows. In Sec. II we discuss the $1/N$ expansion of the CP^{N-1} models in their continuum version. In Sec. III we compute the $1/N$ contribution to the free energy density and extract the $1/N$ dependence of the renormalization-group β function in the SM scheme. In Sec. IV we study a class of gauge-invariant two-point functions obtained by introducing a path-dependent gauge field "string" between two fundamental field operators, and show that no single-particle mass for the fundamental field can be consistently defined. Renormalization of ultraviolet singularities appearing in open and closed loops is discussed and a completely finite, calculable self-energy function is presented. In Sec. V we extract quantitative predictions for the magnetic susceptibility and the second moment of the two-point correlation function. In Sec. VI we discuss the properties of closed loops of gauge fields ("Wilson loops"), and compute their $1/N$ expectation values for circular and "long" rectangular loops, checking in particular the asymptotic area law and fixing the coefficient of the perimeter term. In Sec. VII loop-loop correlation functions are discussed and evaluated. In Sec. VIII we consider the two-point correlation function of local gauge-invariant composite operators made out of two fundamental fields. We explicitly prove the relationship between the asymptotic large-distance behavior of this correlation and the mass of the two-particle bound state corresponding to the lowest-energy eigenvalue of the confining potential. We give an explicit representation of the $1/N$ contribution to the correlation and extract some qualitative and quantitative physical information.

Throughout all our analysis we find full agreement with the qualitative picture rising from the large-N approximation, while we put rigorous quantitative constraints on the speed of approach to large-N asymptopia.

II. THE $1/N$ EXPANSION OF THE CP^{N-1} MODELS

The bare continuum Lagrangian of the two-dimensional CP^{N-1} models is [1,2]

$$S=\frac{N}{2f}\int d^2x\,\overline{D_\mu z}\,D_\mu z\ , \tag{2.1}$$

where z is an N-component complex vector field subject to the constraint

$$\bar{z}z=1 \tag{2.2}$$

and a covariant derivative $D_\mu\equiv\partial_\mu+iA_\mu$ has been defined in terms of the composite gauge fields:

$$A_\mu=\tfrac{1}{2}i(\bar{z}\partial_\mu z-\partial_\mu\bar{z}z)\equiv i\bar{z}\partial_\mu z\ . \tag{2.3}$$

The $1/N$ expansion of the generating functional is obtained by introducing the external currents J coupled to the z fields and Lagrange multiplier fields introduced in order to implement the constraint. By standard manipulations one obtains

$$\begin{aligned}S&=\frac{N}{2f}\int d^2x[\partial_\mu\bar{z}\partial_\mu z+(\bar{z}\partial_\mu z)^2+i\alpha(\bar{z}z-1)\\&\qquad+(\lambda_\mu-i\bar{z}\partial_\mu z)^2+\bar{z}J+z\bar{J}] \qquad (2.4)\\&=\frac{N}{2f}\int d^2x[\overline{D_\mu z}D_\mu z+i\alpha(\bar{z}z-1)+\bar{z}J+z\bar{J}]\ ,\end{aligned}$$

where now $D_\mu\equiv\partial_\mu+i\lambda_\mu$. A Gaussian integration on the z variable leads to

$$\begin{aligned}S_{\rm eff}&=N\,{\rm Tr}\ln(-\overline{D_\mu}D_\mu+m_0^2+i\alpha_q)\\&\quad+\frac{N}{2f}\int d^2x\left[-i\alpha_q+\bar{J}\frac{1}{-\overline{D_\mu}D_\mu+m_0^2+i\alpha_q}J\right]\ ,\end{aligned} \tag{2.5}$$

where $-im_0^2$ is the large-N vacuum expectation value of the α field and it is determined as a function of f from the saddle-point condition imposed on $S_{\rm eff}$.

For the sake of simplicity in computations, throughout the present paper we shall make use of the SM regularization scheme, whose main features are discussed in Ref. [7]. Physical quantities depend on the cutoff parameter M^2; however, renormalization-group-invariant quantities can be shown to depend only on the physical mass parameter $m^2(M^2,f)$, related to the coupling constant by the standard renormalization-group relationship. The large-N limit of this relationship is exactly the saddle-point condition we mentioned before. In our notation it takes the form

$$\frac{\pi}{f}=\frac{1}{2}\ln\frac{M^2}{m_0^2}\ . \tag{2.6}$$

By taking the second functional derivative of the effective action around the saddle point we may obtain the propagators of the quantum fluctuations associated with the fields α_q and λ_μ; both are $O(1/N)$ quantities that can be expressed by the functions

$$\begin{aligned}\Delta_{(\alpha)}^{-1}&=\int\frac{d^2q}{(2\pi)^2}\frac{1}{q^2+m_0^2}\frac{1}{(p+q)^2+m_0^2}\\&=\frac{1}{2\pi p^2\xi}\ln\frac{\xi+1}{\xi-1}\ ,\end{aligned} \tag{2.7}$$

$$\begin{aligned}\Delta_{\mu\nu}^{(\lambda)-1}&=2\delta_{\mu\nu}\int\frac{d^2q}{(2\pi)^2}\frac{1}{q^2+m_0^2}\\&\quad-\int\frac{d^2q}{(2\pi)^2}\frac{(p_\mu+2q_\mu)(p_\nu+2q_\nu)}{(q^2+m_0^2)[(p+q)^2+m_0^2]}\\&=\left[\delta_{\mu\nu}\frac{p_\mu p_\nu}{p^2}\right]\frac{1}{2\pi}\left[\xi\ln\frac{\xi+1}{\xi-1}-2\right]\\&\equiv\left[\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right]\Delta_{(\lambda)}^{-1}\ ,\end{aligned} \tag{2.8}$$

where $\xi=\sqrt{1+4m_0^2/p^2}$.

A well-known result concerning the vector propagator is the appearance of a massless pole [1,2], whose Landau-gauge representation is

$$\Delta_{\mu\nu}^{(\lambda)}\underset{p^2\to 0}{\sim}\frac{12\pi m_0^2}{p^2}\left[\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right]\ . \tag{2.9}$$

As a consequence, λ_μ becomes a dynamical gauge field, giving rise to a linear confining potential between z and $\bar{z}$. A more subtle consequence is the systematic appearance of infrared divergences in all expectation values of operators that are not gauge invariant. In particular, as we shall see, this will prevent the possibility of giving a physical definition of mass for the z particle; we shall interpret this fact as a further signal of confinement.

The $1/N$ perturbative expansion can now be performed by expanding the effective action around the saddle point and then generating effective vertices (which are nothing but one-loop integrals over massive z-field propagators) for the interactions of the α and λ fields. The $1/N$ expansion is also a loop expansion in the loops generated by the vertices and propagators of the effective fields. An explicit construction of the first few relevant effective vertices is given in Appendix A; we are indebted to H. Panagopoulos for some of the techniques and results presented there.

The interaction of the effective fields with the "elementary" fields z can be expressed in terms of Feynman rules that can be easily extracted from Eq. (2.5). These Feynman rules are collected in Fig. 1. There is however some computational convenience in choosing a slightly different, but completely equivalent, form of the effective action, obtained by an explicit use of the constraint $\bar{z}z=1$ in an intermediate stage of the computation:

$$S_{\rm eff}=N\,{\rm Tr}\ln(-\partial_\mu\partial_\mu-i\{\partial_\mu,\lambda_\mu\}+m_0^2+i\alpha_q)+\frac{N}{2f}\int d^2x\left[-i\alpha_q+\lambda_\mu\lambda_\mu+\bar{J}\frac{1}{-\partial_\mu\partial_\mu-i\{\partial_\mu,\lambda_\mu\}+m_0^2+i\alpha_q}J\right]\ . \tag{2.10}$$

This choice allows for the complete elimination of the two-vector-field vertex ("seagull diagram") from the Feynman rules. One may explicitly verify that the sum of the corresponding contributions is identically zero.

A last comment concerns renormalization: we expect to be able to renormalize the theory just by renormalizing the coupling, the mass parameter, and the z-field wave function, order by order in the $1/N$ expansion [8]. We shall verify this statement to second nontrivial order.

III. THE FREE ENERGY

From a conceptual and computational point of view the free energy density, being gauge and renormalization-group invariant and carrying no dependence on external momenta, is the simplest physical quantity we may evaluate in the $1/N$ expansion. The first two nontrivial contributions are contained in the expression

$$\begin{aligned} F &\cong N\,\mathrm{Tr}\ln\frac{p^2+m_0^2}{p^2}-\frac{N}{2f}m_0^2+\frac{1}{2}\mathrm{Tr}\ln\frac{\Delta^{(0)}_{(\alpha)}(p)}{\Delta_{(\alpha)}(p)}+\frac{1}{2}\mathrm{Tr}\ln\frac{\Delta^{(0)}_{(\lambda)}(p)}{\Delta_{(\lambda)}(p)} \\ &=N\int\frac{d^2p}{(2\pi)^2}\ln\left[1+\frac{m_0^2}{p^2}\right]-\frac{N}{2f}m_0^2+\frac{1}{2}\int\frac{d^2p}{(2\pi)^2}\left[\ln\frac{\frac{1}{\xi}\ln\frac{\xi+1}{\xi-1}}{\ln\frac{p^2}{m_0^2}}+\ln\frac{\xi\ln\frac{\xi+1}{\xi-1}-2}{\ln\frac{p^2}{m_0^2}-2}\right], \end{aligned} \tag{3.1}$$

where

$$\Delta_{(\alpha)}(p)\underset{p\to\infty}{\sim}\frac{2\pi p^2}{\ln p^2/m_0^2}\equiv\Delta^{(0)}_{(\alpha)}(p)\,, \tag{3.2}$$

$$\Delta_{(\lambda)}(p)\underset{p\to\infty}{\sim}\frac{2\pi}{\ln p^2/m_0^2-2}\equiv\Delta^{(0)}_{(\lambda)}(p)\,. \tag{3.3}$$

We have normalized F to its zero-mass value, which corresponds to subtracting the perturbative tail and keeping only the scaling part of the free energy density [6]. This is a necessary prerequisite to our computation, but it is not yet a regularization; to make Eq. (3.1) ultraviolet finite we still have to subtract the leading term of the expansion in powers of $1/p^2$. The resulting regularized expression is

$$\begin{aligned} F &\cong\frac{Nm_0^2}{4\pi}+\frac{1}{2}\int\frac{d^2p}{(2\pi)^2}\left[\ln\ln\frac{\xi+1}{\xi-1}\ln\ln\frac{p^2}{m_0^2}\right] \\ &\quad+\frac{1}{2}\int\frac{d^2p}{(2\pi)^2}\left[\ln\left[\ln\frac{\xi+1}{\xi-1}-\frac{2}{\xi}\right]-\ln\left[\ln\frac{p^2}{m_0^2}-2\right]\right] \\ &\quad-\frac{1}{2}\int_{M^2}^{\infty}\frac{d^2p}{(2\pi)^2}\left[\frac{2m_0^2}{p^2}\frac{1}{\ln p^2/m_0^2}+\frac{6m_0^2}{p^2}\frac{1}{\ln p^2/m_0^2-2}\right] \\ &=\frac{Nm_0^2}{4\pi}+\frac{m_0^2}{4\pi}\left[\ln\ln\frac{M^2}{m_0^2}+\gamma_E+3\ln\left[\ln\frac{M^2}{m_0^2}-2\right]+c_\lambda\right], \end{aligned} \tag{3.4}$$

where $c_\lambda=0.611\,671\,457\ldots$ c_λ is a numerical constant related to the finite part of the vector propagator loop. It can be computed at all desired numerical accuracy but, in contrast with the scalar propagator loop it does not seem to be amenable to the properties of any known special function. We suspect this phenomenon to be related to the lack of complete integrability of the quantum CP^{N-1} models [9].

As a straightforward consequence of our result (3.4), we are ready to extract the $O(1/N)$ contributions to the renormalization-group β function in the SM scheme, via the renormalization-group equation

$$\left[M\frac{\partial}{\partial M}+\beta(f)\frac{\partial}{\partial f}\right]F=0\,. \tag{3.5}$$

Applying the saddle-point condition (2.6) we easily obtain

$$\begin{aligned} \frac{1}{\beta(f)}&\cong-\frac{1}{2}\left\{\frac{2\pi}{f^2}+\frac{1}{N}\frac{\partial}{\partial f}\left[\ln\frac{2\pi}{f}+3\ln\left[\frac{2\pi}{f}-2\right]\right]\right\}, \\ \beta(f)&\cong-\frac{f^2}{\pi}\left[1+\frac{1}{N}\frac{f}{2\pi}\left[1+\frac{3}{1-f/\pi}\right]\right]+O\left[\frac{1}{N^2}\right]. \end{aligned} \tag{3.6}$$

The first two nontrivial powers of f are associated with universal coefficients; these may be compared and found to agree with previous results appearing in the literature [10]. In the next section we shall also verify the self-consistency of our determination of β.

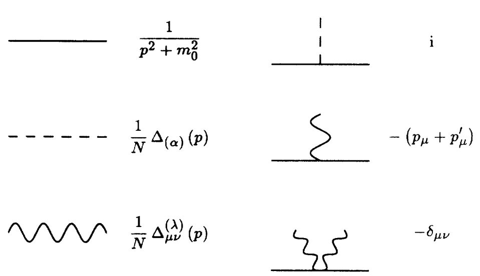

FIG. 1. The Feynman rules of the models.

IV. THE GAUGE-INVARIANT TWO-POINT FUNCTION

As we anticipated in Sec. II, non-gauge-invariant Green's functions are plagued by the infrared divergencies originated by the massless propagator $\Delta^{(\lambda)}_{\mu\nu}$. Let us indeed consider the $O(1/N)$ contributions to the two-point function $2fG(x,y)=\langle\bar{z}(y)z(x)\rangle$. The corresponding Feynman diagrams are drawn in Fig. 2, and their momentum-space representation is

$$-\frac{1}{N}\frac{1}{p^2+m_0^2}\Sigma(p)\frac{1}{p^2+m_0^2}\,, \tag{4.1}$$

where

$$\Sigma(p)=\int\frac{d^2k}{(2\pi)^2}\frac{\Delta_{(\alpha)}(k)}{(p+k)^2+m_0^2}-\Delta_{(\alpha)}(0)\int\frac{d^2k}{(2\pi)^2}\frac{d^2q}{(2\pi)^2}\frac{1}{(q^2+m_0^2)^2}\frac{\Delta_{(\alpha)}(k)}{(q+k)^2+m_0^2}$$
$$-\int\frac{d^2k}{(2\pi)^2}\Delta_{(\lambda)}(k)\left[\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right]\frac{(2p_\mu+k_\mu)(2p_\nu+k_\nu)}{(p+k)^2+m_0^2}$$
$$+\Delta_{(\alpha)}(0)\int\frac{d^2k}{(2\pi)^2}\frac{d^2q}{(2\pi)^2}\frac{1}{(q^2+m_0^2)^2}\Delta_{(\lambda)}(k)\left[\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right]\frac{(2q_\mu+k_\mu)(2q_\nu+k_\nu)}{(q+k)^2+m_0^2}\ . \tag{4.2}$$

It is easy to obtain from Eq. (4.2) the representation

$$\Sigma(p)=\int\frac{d^2k}{(2\pi)^2}\frac{\Delta_{(\alpha)}(k)}{(p+k)^2+m_0^2}+\frac{1}{2}\Delta_{(\alpha)}(0)\int\frac{d^2k}{(2\pi)^2}\Delta_{(\alpha)}(k)\frac{\partial}{\partial m_0^2}\Delta_{(\alpha)}^{-1}(k)+\int\frac{d^2k}{(2\pi)^2}\Delta_{(\lambda)}(k)\left[1-\frac{4p^2k^2-4(pk)^2}{k^2[(p+k)^2+m_0^2]}\right]+\frac{1}{2}\Delta_{(\alpha)}(0)\int\frac{d^2k}{(2\pi)^2}\Delta_{(\lambda)}(k)\frac{\partial}{\partial m_0^2}\Delta_{(\alpha)}^{-1}(k)\ , \tag{4.3}$$

and by making use of the explicit expressions of the propagators we also obtain

$$\Sigma(p)=\int\frac{d^2k}{(2\pi)^2}\frac{\Delta_{(\alpha)}(k)}{(p+k)^2+m_0^2}-\int\frac{d^2k}{(2\pi)^2}\frac{\Delta_{(\alpha)}(k)}{k^2+4m_0^2}+\int\frac{d^2k}{(2\pi)^2}\Delta_{(\lambda)}(k)\left[1-\frac{4p^2k^2-4(pk)^2}{k^2[(p+k)^2+m_0^2]}\right]-\int\frac{d^2k}{(2\pi)^2}\Delta_{(\lambda)}(k)\frac{1}{k^2+4m_0^2}\ . \tag{4.4}$$

In order to extract the corrections to the mass parameter we must evaluate $\Sigma(p^2=-m_0^2)$ [6], that is

$$\Sigma(-m_0^2)=\int\frac{d^2k}{(2\pi)^2}\frac{\Delta_{(\alpha)}(k)}{k^2}\left[\frac{1}{\xi}-\frac{1}{\xi^2}\right]+\int\frac{d^2k}{(2\pi)^2}\frac{\Delta_{(\lambda)}(k)}{k^2}\left[\xi-\frac{1}{\xi^2}\right]\ . \tag{4.5}$$

Ultraviolet regularization by SM cutoff subtraction leads to the following parameterization of the ultraviolet divergence:

$$m_0^2\left[\ln\ln\frac{M^2}{m_0^2}+\gamma_E\right]+3m_0^2\left[\ln\left[\ln\frac{M^2}{m_0^2}-2\right]+\text{const}\right]\ , \tag{4.6}$$

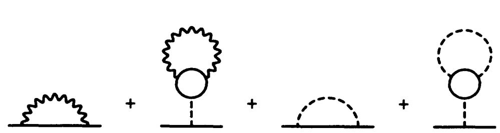

FIG. 2. $O(1/N)$ contributions to the two-point function.

showing the expected consistency with the renormalization-group β function obtained in the previous section. However, when trying to evaluate the constant in the last term of Eq. (4.6) one easily recognizes that this quantity is divergent when the infrared regulator is sent to zero. The obvious origin of this infrared catastrophe is the lack of gauge invariance of the correlation functions we are studying.

As we shall see, it is possible to restore gauge invariance and regulate the infrared behavior without spoiling the nice ultraviolet structure we have found. This is achieved in a quite natural way by introducing a gauge "string" between the two gauge "charges." Since the gauge symmetry is Abelian, no path ordering is needed and we may simply consider the class of (path-dependent) correlations

$$2fG_{\mathcal{C}}(x,y)=\left\langle\bar z(y)\exp\left[i\int_y^x dt_\mu A_\mu(t)\right]z(x)\right\rangle\ , \tag{4.7}$$

where t_μ are the points of the path $\mathcal{C}$ connecting x and y. In the first nontrivial order of $1/N$ perturbation theory we therefore obtain the contributions

$$2fG_{\mathcal{C}}(x,y)\cong\langle\bar z(y)z(x)\rangle_{(0)}+\frac{1}{N}\langle\bar z(y)z(x)\rangle_{(1)}+\frac{i}{N}\int_y^x dt_\mu\langle\bar z(y)A_\mu(t)z(x)\rangle_{(0)}-\frac{1}{2N}\int_y^x dt_\mu\int_y^x dt'_\nu\langle\bar z(y)z(x)\rangle_{(0)}\times\langle A_\mu(t)A_\nu(t')\rangle_{(0)}\ , \tag{4.8}$$

where the second term is the infrared-divergent quantity we have just computed.

In what follows, we shall focus on a special subclass of these correlation functions, which we call "straight open strings" and corresponds to the choice of connecting the points x and y through the minimal straight path parametrized by

$$t_\mu=y_\mu+\lambda(x_\mu-y_\mu)\ ,\quad 0\le\lambda\le 1\ . \tag{4.9}$$

When the path belongs to this class, the correlation function depends only on the distance between the two points x and y, and it is therefore possible to consider its Fourier transform, with obvious advantages from a computational point of view. Most intermediate integrations can be performed analytically and one can with a reasonable effort show that

$$G_{\mathcal{C}}(p)=\int e^{-ip(y-x)}d(y-x)G_{\mathcal{C}}(x,y)$$
$$\cong\frac{1}{p^2+m_0^2}-\frac{1}{N}\frac{1}{p^2+m_0^2}\Sigma(p)\frac{1}{p^2+m_0^2}$$
$$+\frac{1}{N}\int\frac{d^2k}{(2\pi)^2}\Delta^{(\lambda)}_{\mu\nu}(k)\frac{\partial}{\partial p_\mu}\left[2p_\nu\int_0^1 d\lambda\frac{1}{[p+(1-\lambda)k]^2+m_0^2}\frac{1}{(p-\lambda k)^2+m_0^2}\right]$$
$$+\frac{1}{2N}\int\frac{d^2k}{(2\pi)^2}\Delta^{(\lambda)}_{\mu\nu}(k)\frac{\partial}{\partial p_\mu}\frac{\partial}{\partial p_\nu}\left[\int_0^1 d\lambda\int_0^1 d\lambda'\frac{1}{[p+(\lambda-\lambda')k]^2+m_0^2}\right]. \quad (4.10)$$

The infrared behavior of this expression is obtained by simply setting $k=0$ in the terms multiplying $\Delta^{(\lambda)}_{\mu\nu}(k)$:

$$-\frac{1}{(p^2+m_0^2)^2}\left[\delta_{\mu\nu}-\frac{4p_\mu p_\nu}{p^2+m_0^2}\right]+\frac{\partial}{\partial p_\mu}\frac{2p_\nu}{(p^2+m_0^2)^2}+\frac{1}{2}\frac{\partial}{\partial p_\mu}\frac{\partial}{\partial p_\nu}\frac{1}{p^2+m_0^2}$$
$$=-\frac{1}{(p^2+m_0^2)^2}\left[\delta_{\mu\nu}-\frac{4p_\mu p_\nu}{p^2+m_0^2}\right]+\frac{\partial}{\partial p_\mu}\frac{p_\nu}{(p^2+m_0^2)^2}=0\ . \quad (4.11)$$

Therefore there is no infrared divergence in the dk integration and the resulting Green's function has no infrared pathology. Let us now introduce the auxiliary functions

$$A(p,k)=\int_0^1 d\lambda\int_0^1 d\lambda'\frac{1}{[p+(\lambda-\lambda')k]^2+m_0^2}\ , \quad (4.12)$$

$$B(p,k)=\int_0^1 d\lambda\frac{1}{[p+(1-\lambda)k]^2+m_0^2}\frac{1}{[p-\lambda k]^2+m_0^2}\ , \quad (4.13)$$

and notice that, thanks to the transversality property $k_\mu\Delta^{(\lambda)}_{\mu\nu}(k)=0$, we may derive the relationships

$$\Delta^{(\lambda)}_{\mu\nu}(k)\frac{\partial}{\partial p_\nu}A=\Delta^{(\lambda)}_{\mu\nu}(k)2p_\nu\frac{\partial A}{\partial m_0^2}\ ,$$
$$\Delta^{(\lambda)}_{\mu\nu}(k)\frac{\partial}{\partial p_\mu}\frac{\partial}{\partial p_\nu}A=\Delta^{(\lambda)}_{\mu\nu}(k)\left[4p_\mu p_\nu\frac{\partial^2 A}{\partial(m_0^2)^2}+2\delta_{\mu\nu}\frac{\partial A}{\partial m_0^2}\right], \quad (4.14)$$
$$\Delta^{(\lambda)}_{\mu\nu}(k)\frac{\partial}{\partial p_\nu}B=\Delta^{(\lambda)}_{\mu\nu}(k)2p_\nu\frac{\partial B}{\partial m_0^2}\ .$$

We can therefore write our results in the compact form

$$G_{\mathcal{C}}(p)=\frac{1}{p^2+m_0^2}-\frac{1}{N}\frac{1}{p^2+m_0^2}\Sigma_{\mathcal{C}}(p)\frac{1}{p^2+m_0^2}\ , \quad (4.15)$$

$$\Sigma_{\mathcal{C}}(p)=\Sigma(p)-(p^2+m_0^2)^2\times\int\frac{d^2k}{(2\pi)^2}\Delta^{(\lambda)}_{\mu\nu}(k)\left[2\delta_{\mu\nu}+4p_\mu p_\nu\frac{\partial}{\partial m_0^2}\right]\times\left[\frac{1}{2}\frac{\partial A}{\partial m_0^2}+B\right]. \quad (4.16)$$

In addition, it is possible to perform analytically the λ integrations, thus obtaining

$$A(p,k)=\frac{1}{2k^2}\ln\frac{(p^2+m_0^2)^2}{(p^2+m_0^2+k^2)^2-4(pk)^2}+\frac{k^2+(pk)}{k^2\delta}\arctan\frac{k^2+(pk)}{\delta}+\frac{k^2-(pk)}{k^2\delta}\arctan\frac{k^2-(pk)}{\delta}-\frac{2(pk)}{k^2\delta}\arctan\frac{(pk)}{\delta}\ , \quad (4.17)$$

$$B(p,k)=\frac{1}{(k^2)^2+4k^2(p^2+m_0^2)-4(pk)^2}\times\left[\ln\frac{(p^2+m_0^2+k^2)^2-4(pk)^2}{(p^2+m_0^2)^2}+\frac{k^2}{\delta}\arctan\frac{k^2+(pk)}{\delta}+\frac{k^2}{\delta}\arctan\frac{k^2-(pk)}{\delta}\right], \quad (4.18)$$

where $\delta=\sqrt{k^2(p^2+m_0^2)-(pk)^2}$.

However this is not the end of the story. In fact, when we consider the ultraviolet behavior of the integrand in Eq. (4.16), we recognize that for large values of k a new divergence may be originated by the leading contribution to $\partial A/\partial m_0^2$:

$$\frac{\partial A}{\partial m_0^2}\underset{k\to\infty}{\sim}-\frac{k^2}{\delta^3}\frac{\pi}{2}+\frac{1}{\delta^2}+\frac{(pk)}{\delta^3}\arctan\frac{(pk)}{\delta}\ , \quad (4.19)$$

where we can write $\delta=k\sqrt{p^2\sin^2\theta+m_0^2}$ with $(pk)=|p||k|\cos\theta$. An explicit evaluation of this divergence is given by

$$\int \frac{d^2k}{(2\pi)^2} \Delta^{(\lambda)(0)}_{\mu\nu}(k) \left[\delta_{\mu\nu} + 2p_\mu p_\nu \frac{\partial}{\partial m_0^2} \right] \left[-\frac{k^2}{\delta^3}\frac{\pi}{2} + \frac{1}{\delta^2} + \frac{pk}{\delta^3} \arctan\frac{pk}{\delta} \right]$$

$$= \int \frac{d^2k}{(2\pi)^2} \Delta^{(0)}_{(\lambda)}(k) \left[1 + 2p^2 \sin^2\theta \frac{\partial}{\partial m_0^2} \right] \left[-\frac{k^2}{\delta^3}\frac{\pi}{2} + \frac{1}{\delta^2} + \frac{pk}{\delta^3} \arctan\frac{pk}{\delta} \right] . \quad (4.20)$$

Now noticing that

$$\int \frac{d\theta}{2\pi} \left[1 + 2p^2 \sin^2\theta \frac{\partial}{\partial m_0^2} \right] \left[\frac{1}{p^2 \sin^2\theta + m_0^2} + \frac{p\cos\theta}{(p^2 \sin^2\theta + m_0^2)^{3/2}} \arctan \frac{p\cos\theta}{(p^2\sin^2\theta + m_0^2)^{1/2}} \right] = \frac{1}{p^2 + m_0^2} , \quad (4.21)$$

$$\int \frac{d\theta}{2\pi} \left[1 + 2p^2 \sin^2\theta \frac{\partial}{\partial m_0^2} \right] \frac{1}{(p^2\sin^2\theta + m_0^2)^{3/2}} = \frac{2}{\pi} \frac{1}{(p^2+m_0^2)^{3/2}} \left[2E\left[\frac{p}{\sqrt{p^2+m_0^2}} \right] - K\left[\frac{p}{\sqrt{p^2+m_0^2}} \right] \right] , \quad (4.22)$$

where E and K are the standard elliptic functions, we can parametrize the divergence by

$$\int \frac{d^2k}{(2\pi)^2} \frac{\Delta^{(0)}_{(\lambda)}(k)}{k^2} \frac{1}{p^2+m_0^2} - \int \frac{d^2k}{(2\pi)^2} \frac{\Delta^{(0)}_{(\lambda)}(k)}{k} \left[2E\left[\frac{p}{\sqrt{p^2+m_0^2}} \right] - K\left[\frac{p}{\sqrt{p^2+m_0^2}} \right] \right] \frac{1}{(p^2+m_0^2)^{3/2}} . \quad (4.23)$$

This apparently discouraging result is nevertheless quite easily interpreted in coordinate space. Actually it has been known for some time [11,12,13] that the introduction of a gauge "string" may lead to the appearance of a new class of ultraviolet divergences, related to self-energy effects (self-mass of the string) and to wave-function renormalization of the end points in the case of an open string. These divergences can be parametrized in terms of a multiplicative renormalization factor $\exp(\gamma_1 L + \gamma_0)$, where L is the physical length of the string and γ_1, γ_0 are divergent factors growing linearly and logarithmically, respectively, with the cutoff parameter M. We therefore expect to be able to renormalize our "straight open string" by a multiplicative renormalization defined by

$$G^{\rm ren}_{\mathcal{C}}(x,y) = \exp(-\gamma_1 |x-y| - \gamma_0) G_{\mathcal{C}}(x,y) . \quad (4.24)$$

The momentum-space counterpart of Eq. (4.24) is

$$G^{\rm ren}_{\mathcal{C}}(p) = \int \frac{d^2q}{(2\pi)^2} G_{\mathcal{C}}(p-q) \int d^2z \, e^{-\gamma_1 z - \gamma_0 + iqz}$$

$$= \int \frac{d^2q}{(2\pi)^2} G_{\mathcal{C}}(p-q) \frac{2\pi\gamma_1}{(2^2+\gamma_1^2)^{3/2}} e^{-\gamma_0} , \quad (4.25)$$

where we notice that

$$\lim_{\gamma_1 \to 0} \frac{2\pi\gamma_1}{(q^2+\gamma_1^2)^{3/2}} = (2\pi)^2 \delta^{(2)}(q) . \quad (4.26)$$

Within the $1/N$ expansion γ_0 and γ_1 are $O(1/N)$ quantities, and we may therefore keep the leading orders in γ after performing the q integration. As a consequence we consistently obtain

$$G^{\rm ren}_{\mathcal{C}}(p) = G_{\mathcal{C}}(p) + \gamma_1 \lim_{\gamma_1\to 0} \int \frac{d^2q}{(2\pi)^2} G_0(p-q) \left[\frac{2\pi}{(q^2+\gamma_1^2)^{3/2}} - \frac{(2\pi)^2}{\gamma_1} \delta^{(2)}(q) \right]$$

$$- \gamma_0 \lim_{\gamma_1\to 0} \int \frac{d^2q}{(2\pi)^2} G_0(p-q) \frac{2\pi\gamma_1}{(q^2+\gamma_1^2)^{3/2}}$$

$$= G_{\mathcal{C}}(p) + \gamma_1 \lim_{\gamma_1 \to 0} \left[\int \frac{d^2q}{(2\pi)^2} \left[\frac{2\pi}{(q^2+\gamma_1^2)^{3/2}} \frac{1}{(p-q)^2+m_0^2} \right] - \frac{1}{\gamma_1 (p^2+m_0^2)} \right] - \frac{\gamma_0}{p^2+m_0^2} . \quad (4.27)$$

The q integration can be performed by standard Feynman parameter techniques, the $\gamma_1 \to 0$ limit can be easily taken and the final result is

$$G^{\rm ren}_{\mathcal{C}}(p) = G_{\mathcal{C}}(p) - \frac{\gamma_1}{(p^2+m_0^2)^{3/2}} \left[2E\left[\frac{p}{\sqrt{p^2+m_0^2}} \right] - K\left[\frac{p}{\sqrt{p^2+m_0^2}} \right] \right] - \frac{\gamma_0}{p^2+m_0^2} . \quad (4.28)$$

By comparing Eq. (4.28) with Eq. (4.23) we immediately check that the gauge-invariant two-point function is made ultraviolet finite by the choice

$$\gamma_1 = -\int \frac{d^2k}{(2\pi)^2} \frac{\Delta^{(0)}_{(\lambda)}(k)}{k} , \quad \gamma_0 = \int_{M^2} \frac{d^2k}{(2\pi)^2} \frac{\Delta^{(0)}_{(\lambda)}(k)}{k^2} . \quad (4.29)$$

It may be interesting to rederive from strict coordinate-space considerations the origin of the divergent terms in the open-string Green's function. Appendix B is devoted to this analysis.

We can now summarize all the results of this section by writing down a completely ultraviolet and infrared-finite expression for the expectation value of the "straight open-string" effective self-energy function. This expression is suitable for numerical evaluation for all values of the momentum p such that $p^2 > -m_0^2$:

$$\begin{aligned}
\Sigma_{\mathcal{C}}^{\rm ren}(p) = & \int \frac{d^2k}{(2\pi)^2} \frac{\Delta_{(\alpha)}(k)}{(p+k)^2+m_0^2} - \int \frac{d^2k}{(2\pi)^2} \frac{\Delta_{(\alpha)}(k)}{k^2+4m_0^2} \\
& - \int_{M^2} \frac{d^2k}{(2\pi)^2} \frac{2\pi}{k^2 \ln k^2/m_0^2}(p^2+3m_0^2) \\
& + \int \frac{d^2k}{(2\pi)^2} \Delta_{(\lambda)}(k) \left[1 - \frac{4p^2\sin^2\theta}{(p+k)^2+m_0^2} \right] - \int \frac{d^2k}{(2\pi)^2} \Delta_{(\lambda)}(k) \frac{k^2}{k^2+4m_0^2} \\
& - (p^2+m_0^2)^2 \int \frac{d^2k}{(2\pi)^2} \Delta_{(\lambda)}(k) \left[1 + 2p^2\sin^2\theta \frac{\partial}{\partial m_0^2} \right] \left[\frac{\partial A}{\partial m_0^2} + 2B \right] \\
& - \int_{M^2} \frac{d^2k}{(2\pi)^2} \frac{2\pi}{k^2(\ln k^2/m_0^2 - 2)}(3m_0^2 - 3p^2) \\
& - \sqrt{p^2+m_0^2} \fint \frac{d^2k}{(2\pi)^2} \frac{2\pi}{\ln k^2/m_0^2 - 2} \frac{1}{k} \left[2E \left[\frac{p}{\sqrt{p^2+m_0^2}} \right] - K \left[\frac{p}{\sqrt{p^2+m_0^2}} \right] \right] .
\end{aligned} \tag{4.30}$$

The principal-part prescription [6] and the choice of zero as the lower limit of integration in the last integral (string self-mass counterterm) are dictated by the interpretation of this effect as purely perturbative and by the consequent request that it be removed by all physically relevant amplitudes, exactly as we proceeded in the evaluation of the free energy density. An alternative representation of $\Sigma_{\mathcal{C}}^{\rm ren}(p)$ suitable for analytic continuation in the region $p^2 < -m_0^2$ is discussed in Appendix C.

We can now extract the finite (M-independent) part of $\Sigma_{\mathcal{C}}^{\rm ren}$:

$$\begin{aligned}
\Sigma_{\mathcal{C}}^{\rm ren}(p) = & \Sigma_{\mathcal{C}}^{\rm fin}(p) + (p^2+3m_0^2)\frac{1}{2}\ln\ln\frac{M^2}{m_0^2} \\
& + (3m_0^2 - 3p^2)\frac{1}{2}\ln\left[\ln\frac{M^2}{m_0^2} - 2\right] .
\end{aligned} \tag{4.31}$$

$\Sigma_{\mathcal{C}}^{\rm fin}(p)$ is plotted as a function of p^2 in Fig. 3.

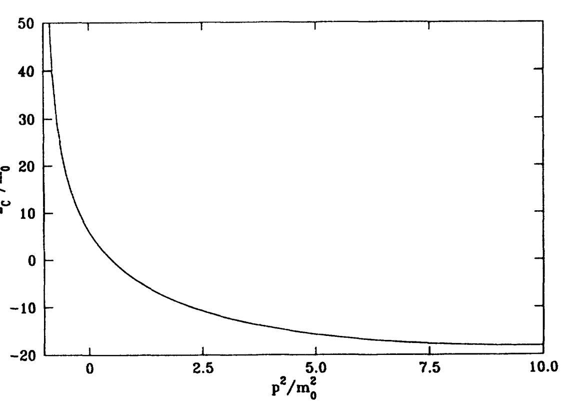

FIG. 3. The finite part of $\Sigma_{\mathcal{C}}$, $\Sigma_{\mathcal{C}}^{\rm fin}$, plotted as a function of p^2.

V. EXPECTATION VALUES OF RENORMALIZATION-GROUP-INVARIANT QUANTITIES

In our analysis of the CP^{N-1} models we are constantly keeping in mind the possibility of performing numerical lattice simulations for specific values of N and comparing the results to our theoretical predictions. In this perspective it is quite important to focus on those special objects of quantum field theory that correspond to renormalization-group-invariant (or covariant) functions. These are also the typical quantities one might try to evaluate in a numerical simulation.

The literature on two-dimensional spin models usually focuses on two such quantities: the magnetic susceptibility, whose definition would be in our case

$$\chi \equiv \int d^2x \langle \bar{z}(x) z(0) \rangle \tag{5.1}$$

and the mass gap, whose standard definition refers to the asymptotic long-distance behavior of the two-point function; for practical purposes one may sometimes consider, as an alternative definition of the correlation length, the second moment of the two-point correlation function:

$$\langle x^2 \rangle = \frac{\int d^2x (x^2/4) \langle \bar{z}(x) z(0) \rangle}{\int d^2x \langle \bar{z}(x) z(0) \rangle} . \tag{5.2}$$

However in the CP^{N-1} models many problems are to be faced. First of all, in order to get finite results we must replace the two-point function with a gauge-invariant expression, which we did in the previous section by introducing the straight open strings. Now, applying our results and going to momentum space, we can replace Eqs. (5.1) and (5.2) by the "string susceptibility"

$$\chi_{\mathcal{C}}=2fG_{\mathcal{C}}^{\rm ren}(0)\cong\frac{2f}{m_0^2}\left[1-\frac{\Sigma_{\mathcal{C}}^{\rm ren}(0)}{Nm_0^2}\right] \tag{5.3}$$

and the "string second correlation moment"

$$\langle x^2\rangle_{\mathcal{C}}\cong\frac{1+\frac{1}{N}\frac{\partial\Sigma_{\mathcal{C}}^{\rm ren}(0)}{\partial p^2}}{m_0^2+\frac{1}{N}\Sigma_{\mathcal{C}}^{\rm ren}(0)}\cong\frac{1}{m_0^2}\left[1+\frac{1}{N}\frac{\partial\Sigma_{\mathcal{C}}^{\rm ren}(0)}{\partial p^2}-\frac{1}{N}\frac{\Sigma_{\mathcal{C}}^{\rm ren}(0)}{m_0^2}\right]. \tag{5.4}$$

There is no conceptual problem in the numerical evaluation of the quantities entering Eqs. (5.3) and (5.3). Our results, extracted from Eq. (4.30), are

$$\frac{\Sigma_{\mathcal{C}}^{\rm ren}(0)}{m_0^2}=\frac{3}{2}\ln\ln\frac{M^2}{m_0^2}+\frac{3}{2}\ln\left[\ln\frac{M^2}{m_0^2}-2\right]+c_1\,, \quad c_1=5.915\,75\ldots\,; \tag{5.5}$$

$$\frac{\partial\Sigma_{\mathcal{C}}^{\rm ren}(0)}{\partial p^2}=\frac{1}{2}\ln\ln\frac{M^2}{m_0^2}-\frac{3}{2}\ln\left[\ln\frac{M^2}{m_0^2}-2\right]+c_1'\,, \quad c_1'=115.725\ldots \tag{5.6}$$

implying, in particular,

$$\langle x^2\rangle_{\mathcal{C}}\cong\frac{1}{m_0^2}\left\{1-\frac{1}{N}\left[\ln\ln\frac{M^2}{m_0^2}+3\ln\left[\ln\frac{M^2}{m_0^2}-2\right]+c_1-c_1'\right]\right\}. \tag{5.7}$$

Let us only mention a few observations about the physical interpretation of these results. Equation (5.4) implies an anomalous dimension for the open-string operator:

$$\gamma_{\mathcal{C}}=\frac{1}{N}\frac{f}{2\pi}\left[1-\frac{3}{1-f/\pi}\right]. \tag{5.8}$$

If we extract an anomalous dimension of the z field from the wave-function renormalization of $\Sigma(p)$, we in turn find

$$2\gamma=\frac{1}{N}\frac{f}{2\pi}\left[1-\frac{2}{1-f/\pi}\right]. \tag{5.9}$$

The difference between $\gamma_{\mathcal{C}}$ and 2γ is obviously due to the string end-point wave-function renormalization we have already discussed in Sec. IV.

The correlation length $\langle x^2\rangle_{\mathcal{C}}$ is a renormalization-group-invariant quantity, whose renormalization-group structure is easily seen to be consistent with that found in Eq. (3.4) for the scaling part of the free energy density. However, while $\langle x^2\rangle_{\mathcal{C}}$ is certainly a well defined, scaling physical quantity, we cannot mimic the analysis of the $O(N)$ models and interpret it as an alternative, approximate definition of the mass gap. In the $O(N)$ case this interpretation rested on the essentially linear dependence of $\Sigma(p)$ on p^2 in the region $-m^2<p^2<0$, implying

$$\Sigma(0)-m^2\frac{\partial\Sigma(0)}{\partial p^2}\approx\Sigma(-m^2)\,. \tag{5.10}$$

What about CP^{N-1} models? We can certainly take the formal limit $p^2\to -m_0^2$ in Eq. (4.30) and, recalling Eq. (4.5), write

$$\Sigma_{\mathcal{C}}^{\rm ren}(-m_0^2)=\Sigma^{\rm ren}(-m_0^2)+\int\frac{d^2k}{(2\pi)^2}[\Delta_{(\lambda)}(k)-\Delta_{(\lambda)}^{(0)}(k)]\frac{2m}{k}\,. \tag{5.11}$$

Equation (5.11) is ultraviolet finite, but the limits $p^2+m_0^2\to 0$ and $k^2\to 0$ do not commute; as a consequence the infrared singularity cancellation holding for all nonzero values of $p^2+m_0^2$ is no longer present and $\Sigma_{\mathcal{C}}^{\rm ren}(-m_0^2)$ is a divergent quantity. Therefore no mild extrapolation from $p^2=0$ can reflect the real asymptotic behavior of the Green's functions at large distance; only a careful analysis of the approach to singularity might lead us to a determination of the large-distance behavior.

It is certainly worth observing that the relatively big numerical factor $c_1-c_1'\approx 21.641$ appearing in Eq. (5.7) is already a signal of the existence of the nearby singularity, and at the same time it implies a rather weak convergence of the $1/N$ expansion for the quantity under examination.

As we already discussed, in CP^{N-1} models it is possible to define a $1/N$ expandable renormalization-group-invariant mass scale $m(M,f)$ (the so-called Λ parameter of perturbative asymptotically free quantum field theory). However, Eq. (5.11) shows that it is not possible to identify unambiguously a $1/N$ expandable single-particle mass. This phenomenon is obviously related to the existence of a linearly rising confining potential.

In fact, due to confinement, the mass scale appearing in the large-distance behavior of the Green's functions can only be related to the two-particle $\bar{z}z$ bound state corresponding to the lowest-energy level of the linear potential. In the large-N approximation the leading contributions to this mass can be found by evaluating the lowest eigenvalue of the corresponding Schrödinger equation [2,14]. Since $m=m_0+O(1/N)$ one obtains

$$\frac{m_B}{m}=2+\varepsilon+O\left[\frac{1}{N}\right], \tag{5.12}$$

where ε is the first eigenvalue of the operator

$$-\frac{d}{dx^2}+\frac{6\pi}{N}x \tag{5.13}$$

subject to the condition of even-parity eigenfunction, and therefore

$$\varepsilon=-a_1'\left[\frac{6\pi}{N}\right]^{2/3}, \tag{5.14}$$

where $a_1'=-1.018\,792\,97\ldots$ is the first zero of the derivative of the Airy function Ai (cf. [15]) [in the existing literature, the coefficient of the linear potential in Eq. (5.13) is sometimes erroneously written as $12\pi/N$].

The main features of the quantity m_B are its nonanalytic dependence on $1/N$, apparent in Eq. (5.14), and the condition $\varepsilon>0$ implying that the bound-state mass lies above the large-N two-particle threshold $2m_0$. As a consequence one gets precise predictions on the structure of the singularity in $\Sigma_{\mathscr{C}}^{\rm fin}(p^2)$: Eqs. (5.12) and (5.14) can be shown to imply that, in the region around $p^2+m_0^2=0$, $\Sigma_{\mathscr{C}}^{\rm fin}(p^2)\sim 1/\sqrt{p^2+m_0^2}$, and moreover, in order not to find bound states below the two-particle threshold, $\Sigma_{\mathscr{C}}^{\rm fin}(p^2)>0$ when $0<p^2+m_0^2<m_0^2$. We performed a numerical analysis of Eq. (4.30) in the region $0<p^2+m_0^2<m_0^2$ and found that the parametrization

$$\frac{\Sigma_{\mathscr{C}}^{\rm fin}(p^2)}{m_0^2}\cong\frac{c_+}{\omega}+c'_+ +O(\omega)\quad\text{for } p^2+m_0^2>0\ ,$$

$$\frac{\Sigma_{\mathscr{C}}^{\rm fin}(p^2)}{m_0^2}\cong\frac{c_-}{\omega}+c'_- +O(\omega)\quad\text{for } p^2+m_0^2<0\ ,\tag{5.15}$$

$$\omega\equiv\sqrt{|1+p^2/m_0^2|}\ ,\quad c_+\cong 22.20\ ,\quad c'_+\cong -4.90\ ,$$

is quite satisfactory, in full agreement with the above discussion. Technical difficulties prevented us from numerically analyzing the region $p^2+m_0^2<0$; we do not however expect any major surprise and we believe that the physical picture that has been drawn is completely plausible. The function $\omega\Sigma_{\mathscr{C}}^{\rm fin}(p)$ is compared in Fig. 4 with Eq. (5.15), for small values of $p^2+m_0^2$.

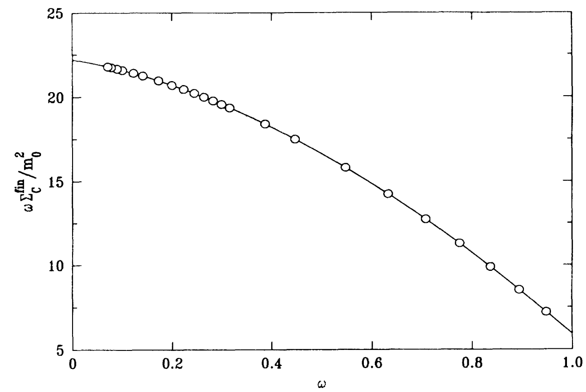

FIG. 4. $\omega\Sigma_{\mathscr{C}}^{\rm fin}$, plotted as a function of ω. The circles are the results of numerical integration; the solid line is a third degree polynomial fit.

VI. CLOSED LOOPS

In our analysis of gauge-invariant operators developing nontrivial vacuum expectation values within the $1/N$ expansion, we are now naturally led to considering closed gauge field loops:

$$L(\mathscr{C})=\left\langle \exp\left[i\oint_{\mathscr{C}}dt_\mu A_\mu(t)\right]\right\rangle\ ,\tag{6.1}$$

where $\mathscr{C}$ is an arbitrary closed path. Truncating the $1/N$ expansion to $O(1/N)$ we obtain

$$\begin{aligned}L(\mathscr{C})&\cong 1-\frac{1}{2}\left\langle\oint_{\mathscr{C}}dt_\mu\oint_{\mathscr{C}}dt'_\nu A_\mu(t)A_\nu(t')\right\rangle\\&=1-\frac{1}{2N}\oint_{\mathscr{C}}dt_\mu\oint_{\mathscr{C}}dt'_\nu\int\frac{d^2k}{(2\pi)^2}e^{ik(t-t')}\left[\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right]\Delta_{(\lambda)}(k)\ .\end{aligned}\tag{6.2}$$

We shall for simplicity focus on two special classes of loops: circular loops and "long" rectangular loops. As we shall see, this analysis is sufficient to allow for some general physical conclusions.

Let us start with circular loops of radius R. In order to get rid of the angular degrees of freedom we compute

$$\begin{aligned}&\oint_{\mathscr{C}}dt_\mu\oint_{\mathscr{C}}dt'_\nu\left[\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right]e^{ik(t-t')}\\&\quad=R^2\left|\int_0^{2\pi}d\theta\cos\theta e^{ikR\cos\theta}\right|^2=(2\pi)^2R^2J_1^2(kR)\ ,\end{aligned}\tag{6.3}$$

where J_1 is the standard Bessel function. Therefore

$$L(R)=1-\frac{1}{2N}\int d^2k\ \Delta_{(\lambda)}(k)R^2J_1^2(kR)\ .\tag{6.4}$$

To this stage, we have not yet analyzed the problem of ultraviolet divergences arising from the string "self-mass" (closed loops have no wave-function renormalization problems). We can extract the large-k behavior of the integrand from the asymptotic expansion of J_1:

$$J_1^2(z)\sim\frac{1}{\pi z}\left[1-\sin 2z+\frac{3}{4z}\cos 2z\right]+O\left[\frac{1}{z^3}\right]\ .\tag{6.5}$$

Therefore the only divergent term is originated from the large-k part of

$$-\frac{1}{2\pi N}\frac{R^2}{kR}\Delta_{(\lambda)}(k)$$

and can be easily removed by subtraction, leading us with the renormalized expression

$$\begin{aligned}L_{\rm ren}(R)=1-\frac{1}{N}\pi R^2\Bigg[&\int d^2k\ J_1^2(kR)\frac{\Delta_{(\lambda)}(k)}{2\pi}\\&-⨍ d^2k\frac{1}{\pi kR}\frac{\Delta_{(\lambda)}^{(0)}(k)}{2\pi}\Bigg]\ .\end{aligned}\tag{6.6}$$

As one may easily appreciate, the subtraction we have performed is exactly the same we might have guessed from our discussion of open-string renormalization presented in Sec. IV, and Eq. (6.6) is just the $1/N$ expansion of

$$L_{\rm ren}(R)=\exp(-\gamma_1 2\pi R)L(R)\ , \tag{6.7}$$

where γ_1 is the same as in Eq. (4.29). It is also easy to recognize that Eq. (6.6) is free of infrared divergences, as we might have expected from the gauge-invariant character of quantity whose expectation value we are evaluating. Equation (6.6) is therefore ready for numerical evaluation. Moreover, its large-R behavior can be extracted without much effort: we must only recognize that in this case the integration is dominated by the small-k region, where

$$\Delta_{(\lambda)}(k)\underset{k\to 0}{\sim}\frac{12\pi m_0^2}{k^2}\ . \tag{6.8}$$

Since

$$\int \frac{d^2k}{k^2}J_1^2(kR)=\pi\ , \tag{6.9}$$

we obtain

$$L_{\rm ren}(R)\underset{R\to\infty}{\sim}1-\frac{6\pi^2}{N}m_0^2R^2-\frac{2\pi c_L}{N}m_0R\ . \tag{6.10}$$

The constant c_L can be determined by use of the asymptotic expansion (6.5) and turns out to be

$$c_L=\frac{1}{m_0}\int_0^\infty\left[\Delta_{(\lambda)}(k)-\Delta_{(\lambda)}^{(0)}(k)-12\pi\frac{m_0^2}{k^2}\right]\frac{dk}{2\pi}$$
$$=-0.116\,375\ldots\ . \tag{6.11}$$

We have numerically verified that once the asymptotic behavior (6.10) has been subtracted one is left with a bounded and rapidly oscillating function of R.

Let us now consider "long" rectangular loops, characterized by the lengths of the two sides R and T, and consider the limit $T\to\infty$. According to standard lore, the quantity

$$V(R)=-\lim_{T\to\infty}\frac{1}{T}\ln L_{\rm ren}(R,T) \tag{6.12}$$

can be interpreted as the interaction potential generated by gauge fields between two static sources. All computations are straightforward. We start from

$$L(R,T)=1-\frac{1}{2N}\int\frac{d^2k}{(2\pi)^2}\left|\int_0^R dx\int_0^T dt\, e^{ik_0t+ik_1x}\right|^2$$
$$\times k^2\Delta_{(\lambda)}(k) \tag{6.13}$$

and obtain

$$\lim_{T\to\infty}\ln L(R,T)=-\frac{1}{2N}\int\frac{d^2k}{(2\pi)^2}\left|\frac{e^{ik_1R}-1}{ik_1}\right|^2$$
$$\times 2\pi\delta(k_0)Tk^2\Delta_{(\lambda)}(k)$$
$$=-\frac{T}{N}\int_0^\infty 4\sin^2\frac{k_1R}{2}\Delta_{(\lambda)}(k_1)\frac{dk_1}{2\pi}\ . \tag{6.14}$$

Ultraviolet regularization is in agreement with previous cases and leads to

$$NV(R)=-2\int_0^\infty \cos k_1R\,\Delta_{(\lambda)}(k_1)\frac{dk_1}{2\pi}$$
$$+2\int_0^\infty[\Delta_{(\lambda)}(k_1)-\Delta_{(\lambda)}^{(0)}(k_1)]\frac{dk_1}{2\pi}\ . \tag{6.15}$$

This is essentially the Fourier transform of the gauge field propagator; the only physically relevant observation concerns the fact that this procedure automatically removes both the infrared ambiguity resulting from the $1/R^2$ behavior of the propagator at small k and the ultraviolet singularity resulting from self-mass effects and thus fixes the zero-point value of the potential in an apparently unambiguous way. By very simple manipulations Eq. (6.15) can be rearranged in the form

$$NV(R)=6\pi m_0^2R+2c_Lm_0$$
$$-2\int_0^\infty \cos k_1R\left[\Delta_{(\lambda)}(k_1)-\frac{12\pi m_0^2}{k_1^2}\right]\frac{dk_1}{2\pi}\ , \tag{6.16}$$

where the last integral is infrared and ultraviolet regular and its large-distance contribution is exponentially depressed. $NV(R)$ is plotted in Fig. 5, together with the first two terms of the right-hand side (RHS) of Eq. (6.16).

The comparison between Eqs. (6.6) and (6.16) leads to the very plausible conjecture that the most general closed gauge loop may be parametrized, at least to $O(1/N)$ in the $1/N$ expansion, by the following "geometrical" large-distance representation:

$$L(\mathcal{C})=1-\frac{6\pi}{N}m_0^2(\text{area})-\frac{c_L}{N}m_0(\text{perimeter})+\ldots\ , \tag{6.17}$$

where neglected terms are rapidly vanishing when the space dimensions of the loop become larger than $1/m_0$. This picture is consistent with the description of the large-distance physics dominated by the "Coulombic" potential emerging form Eq. (6.8) plus a "small" zero-

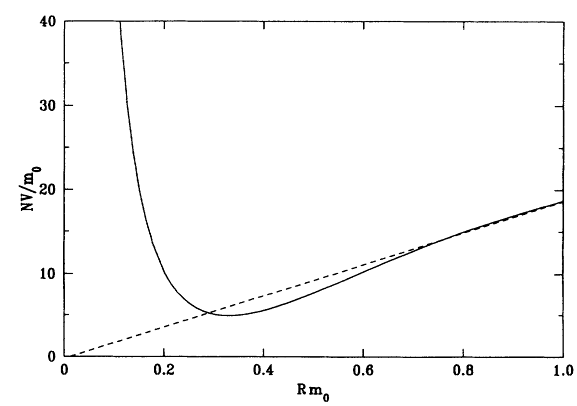

FIG. 5. The $\bar{z}z$ potential $NV(R)$ (solid line), compared to the area + perimeter law (dashed line).

point energy renormalization effect; we remind that the exact, all-order prediction for a purely Coulombic potential would be

$$L(\mathcal{C})=\exp\left[-\frac{6\pi}{N}m_0^2(\text{area})\right] \tag{6.18}$$

for all non-self-intersecting loops.

VII. LOOP-LOOP CORRELATIONS

The correlation between two loops is another gauge-invariant object that can easily be evaluated to $O(1/N)$. From the general definition

$$G(\mathcal{C}_1,\mathcal{C}_2)=\left\langle \exp\left[i\oint_{\mathcal{C}_1}A_\mu dt_\mu\right]\exp\left[i\oint_{\mathcal{C}_2}A_\nu dt'_\nu\right]\right\rangle_{\text{conn}} \tag{7.1}$$

we immediately obtain

$$G(\mathcal{C}_1,\mathcal{C}_2) \cong -\frac{1}{N}\left\langle \oint_{\mathcal{C}_1}dt_\mu \oint_{\mathcal{C}_2}dt'_\nu \int \frac{d^2k}{(2\pi)^2}e^{ik\cdot(t-t')}\Delta^{(\lambda)}_{\mu\nu}(k)\right\rangle . \tag{7.2}$$

If we consider for simplicity and definiteness the case of two circular loops of radius r and distance between centers equal to R, we can apply Eq. (6.3) and find

$$G(r,R)\cong -\frac{1}{N}\int r^2J_1^2(kr)e^{ik\cdot R}\Delta_{(\lambda)}(k)d^2k = -\frac{2\pi}{N}r^2\int J_1^2(kr)J_0(kR)\Delta_{(\lambda)}(k)k\,dk . \tag{7.3}$$

Equation (7.3) is both infrared and ultraviolet regular, and therefore it requires no subtraction.

When the distance R between the two loops is large compared to both the loop radius r and to the correlation length $1/m_0$, the integration is dominated by the small-k region. However an orthogonality condition holds:

$$\int \frac{dk}{k}J_1^2(kr)J_0(kR)=0 \quad \text{if } R>2r , \tag{7.4}$$

and therefore the large-distance loop-loop correlation is rapidly vanishing. Actually an analysis of the singularities of $\Delta_{(\lambda)}(k)$ in the complex k plane shows, as proven in Appendix D, that the asymptotic behavior is

$$\chi(r,R)\equiv N\frac{G(r,R)}{(\pi r^2m_0^2)^2} \underset{R\to\infty}{\sim} \frac{I_1^2(2m_0r)}{(m_0r)^2}\frac{\pi}{(m_0R)^2}e^{-2m_0R} \underset{r\to 0}{\sim} \frac{\pi}{(m_0R)^2}e^{-2m_0R} , \tag{7.5}$$

consistently with our physical insight. The relationship $G(\mathcal{C}_1,\mathcal{C}_2)=0$ is exact in the case of a purely Coulombic potential.

In the especial case when the loop radius is much smaller than the correlation length, i.e., $R\gg 1/m_0\gg r$, we may approximate in the integrand $J_1(kr)$ by $kr/2$ and obtain

$$G(r,R)\approx -\frac{(\pi r^2)^2}{N}\int \frac{d^2k}{(2\pi)^2}k^2\Delta_{(\lambda)}(k)e^{ik\cdot R} . \tag{7.6}$$

Equation (7.6) admits a very simple interpretation, since it corresponds to applying Stokes' theorem to the loop integral

$$\oint_{\mathcal{C}}dt_\mu A_\mu(t)=\int d\sigma\, \varepsilon_{\mu\nu}\partial_\mu A_\nu , \tag{7.7}$$

where σ is the surface bounded by the loop, and approximating the RHS in the case $r\ll 1/m_0$ with the area of the loop times the value of the field at its center. Indeed, we obtain

$$\langle \varepsilon_{\mu\nu}\partial_\mu A_\nu(R)\varepsilon_{\rho\sigma}\partial_\rho A_\sigma(0)\rangle=\frac{1}{N}\int\frac{d^2k}{(2\pi)^2}e^{ik\cdot R}k^2\Delta_{(\lambda)}(k) \tag{7.8}$$

exactly to $O(1/N)$.

A last result about the loop-loop correlation concerns the small-R regime, where the predictions of standard perturbation theory are expected to hold. We are now considering the regime $1/m_0\gg R\gg r$ and we must evaluate

$$G(r,R)\approx -\frac{(\pi r^2)^2}{N}\int\frac{d^2k}{(2\pi)^2}k^2\Delta^{(0)}_{(\lambda)}(k)e^{ik\cdot R} , \tag{7.9}$$

where

$$\Delta^{(0)}_{(\lambda)}(k)=\frac{2\pi}{\ln\frac{k^2}{m_0^2}-2}\equiv\frac{\pi}{\ln\frac{k}{\mu}+\frac{\pi}{f}-1} =\lim_{\varepsilon\to 0}\frac{\pi}{\frac{\pi}{f}-1+\frac{1}{\varepsilon}\left[\left[\frac{k}{\mu}\right]^\varepsilon-1\right]} , \tag{7.10}$$

where f is the renormalized perturbative coupling constant and μ is the renormalization scale. By applying the techniques developed in Ref. [7] we can show that the integral in Eq. (7.9) can be expressed in terms of the asymptotic non-Borel-summable series

$$-\frac{8}{R^4}\sum_{k=0}^{\infty}a_k\frac{(k+1)!}{[\ln m_0^2R^2e^2/4]^{k+2}} , \tag{7.11}$$

where a_k are defined by

$$(1+x)^2\frac{\Gamma(1+x)}{\Gamma(1-x)}=\sum_{k=0}^{\infty}a_kx^k ; \tag{7.12}$$

the leading terms are

$$a_0=1 , \quad a_1=2(1-\gamma_E) , \quad a_2=2(1-\gamma_E)^2-1 .$$

Equations (7.5), (7.6), and (7.11) have been numerically verified and describe accurately the leading behaviors of the correlation functions, as shown in Figs. 6 and 7.

VIII. COMPOSITE OPERATORS AND THEIR CORRELATIONS

The introduction of strings is not the only way of constructing gauge-invariant operators in the CP^{N-1} models.

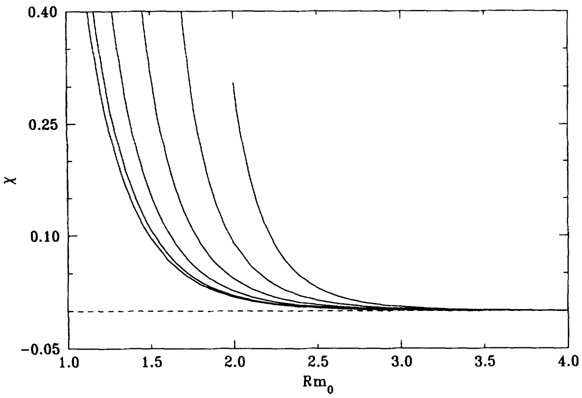

FIG. 6. The normalized $O(1/N)$ contribution to the loop-loop correlation $\chi(r,R)$, plotted as a function of R for $r=0.0,0.2,0.4,0.6,0.8,1.0$ (from bottom to top).

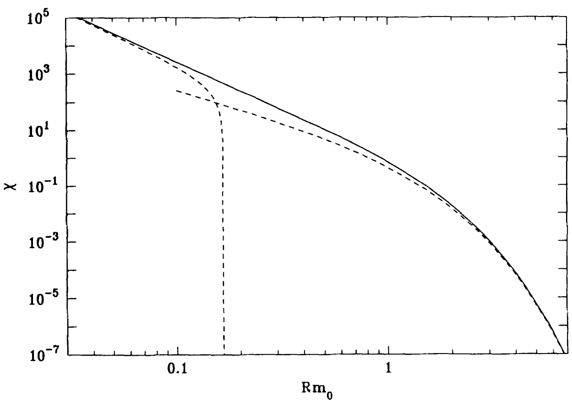

FIG. 7. The loop-loop correlation $\chi(0,R)$ (solid line) compared to the last term of Eq. (7.5) and to the first three orders of Eq. (7.11) (dashed lines).

Actually one may define the local composite operator

$$P_{ij}(x)=\bar{z}_i(x)z_j(x) \tag{8.1}$$

and easily verify its local gauge invariance. P_{ij} is a projection operator, enjoying the properties

$$P^2=P\ , \quad \mathrm{tr}P=1\ , \tag{8.2}$$

and the Lagrangian itself can be reformulated in terms of P, via the relationship

$$\overline{D_\mu z}D_\mu z=\tfrac{1}{2}\mathrm{Tr}(\partial_\mu P\partial_\mu P)\ . \tag{8.3}$$

It is worth noticing that in the case $N=2$ we may parametrize P by

$$P=\tfrac{1}{2}(+\sigma\cdot\mathbf{S})\ , \tag{8.4}$$

where σ are the Pauli matrices $\mathbf{S}$ are the constituent fields of the O(3) nonlinear σ model, equivalent to CP^1.

Correlation functions of the composite field P at different locations are automatically gauge invariant and therefore we expect them to be free of infrared pathologies. SU(N) invariance implies that

$$\langle P_{ij}(x)\rangle=\frac{1}{N}\delta_{ij}\ . \tag{8.5}$$

We can therefore focus on the connected part of the two-point correlation function of the field P, which is the same as the Green's function of its traceless (multiplet) component:

$$G_{ij,kl}(x-y)=\langle P_{ij}(x)P_{kl}(y)\rangle_{\mathrm{conn}}=\langle P_{ij}(x)P_{kl}(y)\rangle-\frac{1}{N^2}\delta_{ij}\delta_{kl}$$
$$=\left\langle\left[P_{ij}(x)-\frac{1}{N}\delta_{ij}\right]\left[P_{kl}(y)-\frac{1}{N}\delta_{kl}\right]\right\rangle\ . \tag{8.6}$$

SU(N) invariance of the amplitude immediately implies that

$$G_{ij,kl}(x-y)=\frac{B(x-y)}{N(N+1)}\left[\delta_{il}\delta_{jk}-\frac{1}{N}\delta_{ij}\delta_{kl}\right]\ , \tag{8.7}$$

where we have imposed the tracelessness condition and the normalization condition $B(0)=1$. The problem is therefore reduced to the evaluation of the scalar quantity $B(x-y)$, that may in turn be represented by

$$B(x-y)=\frac{N\langle\bar{z}_i(x)z_j(x)z_i(y)\bar{z}_j(y)\rangle-1}{N-1}\ . \tag{8.8}$$

When evaluating $B(x-y)$ in the context of the $1/N$ expansion, it is convenient to introduce the (nonlocal) operator

$$D(x,y)=\left\langle x\left|\frac{2f}{-\partial_\mu\partial_\mu-i\{\partial_\mu,\lambda_\mu\}+m_0^2-i\alpha_q}\right|y\right\rangle \tag{8.9}$$

and notice that

$$\langle\bar{z}_i(x)z_j(x)z_i(y)\bar{z}_j(y)\rangle$$
$$=\langle D^2(x,y)\rangle+\frac{1}{N}\langle D(x,x)D(y,y)\rangle\ . \tag{8.10}$$

Since, by the gap equation, we have

$$\langle D(x,x)\rangle=1\ , \tag{8.11}$$

it is easy to obtain

$$B(x-y)=\frac{1}{N-1}[N\langle D^2(x,y)\rangle+\langle D(x,x)D(y,y)\rangle_{\mathrm{conn}}]\ . \tag{8.12}$$

We can now classify the diagrams of the $1/N$ expansion contributing to the full inverse propagator of the quantum field α_q within the following three classes: Δ_0^{-1} is just the one-loop contribution: $\Delta_0^{-1}=N\Delta_{(\alpha)}^{-1}$; Δ_1^{-1} is the sum of all the diagrams such that the (removed) external α legs emerge from the same effective vertex ("tadpoles"); Δ_2^{-1} is the sum of all the diagrams such that the external α legs emerge from two distinct irreducibly connected vertices. It is then rather easy to recognize that

$$\langle D^2(x,y)\rangle=\frac{4f^2}{N}(\Delta_0^{-1}+\Delta_1^{-1}) , \tag{8.13}$$

while it takes some combinatorial effort to prove that

$$\langle D(x,x)D(y,y)\rangle_{\rm conn}=\frac{4f^2}{N^2}\left[\Delta_2^{-1}-(\Delta_0^{-1}+\Delta_1^{-1}+\Delta_2^{-1})\Delta_0\frac{1}{1+\Delta_0(\Delta_1^{-1}+\Delta_2^{-1})}(\Delta_0^{-1}+\Delta_1^{-1}+\Delta_2^{-1})\right]$$
$$=-\frac{4f^2}{N^2}(\Delta_0^{-1}+\Delta_1^{-1}) . \tag{8.14}$$

In conclusion we managed to obtain the relationship

$$B(x-y)=\frac{4f^2}{N}\left[1+\frac{1}{N}\right](\Delta_0^{-1}+\Delta_1^{-1}) , \tag{8.15}$$

relating the two-point function of the composite operator P to the "tadpole" contributions in the α field propagator. We stress that the "tadpole" structure of Δ_1^{-1} is such only at a pictorial level: due to the nonlocal character of the effective vertices, Δ_1^{-1} cannot be seen as a momentum-independent insertion along the α line.

Equation (8.15) leads us to another, and physically more interesting, consideration about the interpretation of the function $B(x-y)$. Let us introduce the full ("dressed") propagator of the z fields in coordinate space, $G(x-y)$, and the t-channel four-point irreducible vertex $V(x_1,x_2;y_1,y_2)$, without worrying about their infrared convergence properties. Let us also define the inverse kernel of the product $G\otimes G$ by the condition

$$\int dy_1dy_2\Delta(x_1,x_2;y_1,y_2)G(y_1-z_1)G(y_2-z_2)=\delta(x_1-z_1)\delta(x_2-z_2) . \tag{8.16}$$

Finally let us notice that the quantity $\Delta_0^{-1}+\Delta_1^{-1}$ can be represented in coordinate space, in compact notation, by

$$\Delta_0^{-1}(x-y)+\Delta_1^{-1}(x-y)=\delta(x-x_1)\delta(x-x_2)\Delta^{-1}(x_1,x_2;t_1,t_2)\frac{1}{1-V\Delta^{-1}}(t_1,t_2;y_1,y_2)\delta(y-y_1)\delta(y-y_2)$$
$$=\delta(x-x_1)\delta(x-x_2)\frac{1}{\Delta-V}(x_1,x_2;y_1,y_2)\delta(y-y_1)\delta(y-y_2) , \tag{8.17}$$

where integration over repeated couples of coordinate indices is assumed. The graphical proof of this relationship is given in Fig. 8. Because of gauge invariance, the operator $\Delta-V$ will turn out to be free of infrared divergences.

Since we are interested in the large-distance behavior of the function $B(x-y)$, we may look for the lowest-energy zero-eigenvalue eigenfunction of the operator $\Delta-V$. However this is nothing but the Bethe-Salpeter equation of our model, and therefore the large-distance behavior of B is related to the lowest-mass bound state of the $\bar{z}z$ system. This is exactly the quantity m_B we have computed in the large-N limit in Eqs. (5.12) and (5.14) by solving the Schrödinger equation, which is nothing but the nonrelativistic limit of the Bethe-Salpeter equation.

The problem of finding the coefficient of the large-distance exponential decay is in principle solved by the previous considerations. However, we would like to get more insight in the detailed structure of the composite operator Green's function by a direct evaluation of the $O(1/N)$ contributions to B and possibly by a check of the above-mentioned asymptotic behavior. The diagrams contributing to Δ_1^{-1} to $O(1/N)$ are drawn in Fig. 9.

It is easy to write down an expression for these diagrams in terms of the effective four-point vertices defined in Appendix A:

$$\Delta_1^{-1}(p)=-\int\frac{d^2k}{(2\pi)^2}\Delta_{(\alpha)}(k)[V_4^{(\alpha)}(p,k)+V_4^{(\alpha)}(p,-k)+V_4^{(b)}(p,k)]$$
$$+\Delta_{(\alpha)}(0)\frac{\partial}{\partial m_0^2}\Delta_{(\alpha)}^{-1}(p)\int\frac{d^2k}{(2\pi)^2}\Delta_{(\alpha)}(k)\frac{1}{2}\frac{\partial}{\partial m_0^2}\Delta_{(\alpha)}^{-1}(k)$$
$$+\int\frac{d^2k}{(2\pi)^2}\Delta_{\mu\nu}^{(\lambda)}(k)\left[V_{\mu\nu}^{(\alpha)}(p,k)+V_{\mu\nu}^{(\alpha)}(p,-k)+V_{\mu\nu}^{(b)}(p,k)+\delta_{\mu\nu}\frac{\partial}{\partial m_0^2}\Delta_{(\alpha)}^{-1}(p)\right]$$
$$+\Delta_{(\alpha)}(0)\frac{\partial}{\partial m_0^2}\Delta_{(\alpha)}^{-1}(p)\int\frac{d^2k}{(2\pi)^2}\Delta_{(\lambda)}(k)\frac{1}{2}\frac{\partial}{\partial m_0^2}\Delta_{(\lambda)}^{-1}(k) . \tag{8.18}$$

By making use of the explicit form of the effective vertices it is also possible to show that

$$V^{(a)}_{\mu\nu}(p,k)+V^{(a)}_{\mu\nu}(p,-k)+V^{(b)}_{\mu\nu}(p,k)+\delta_{\mu\nu}\frac{\partial}{\partial m_0^2}\Delta^{-1}_{(\alpha)}(p)$$

$$=\left[\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right]\{-(k^2+4m_0^2)[V_4^{(a)}(p,k)+V_4^{(a)}(p,-k)]-(k^2+4m_0^2+2p^2)V_4^{(b)}(p,k)+4[V_3(p,k)+V_3(p,-k)]\}\ , \tag{8.19}$$

and by considering the $k\to 0$ limit of Eq. (8.19) one may check the infrared regularity of the only potentially dangerous contributions.

Collecting all contributions we get the form

$$\Delta_1^{-1}(p)=-\int\frac{d^2k}{(2\pi)^2}\Delta_{(\alpha)}(k)\left[V_4^{(a)}(p,k)+V_4^{(a)}(p,-k)+V_4^{(b)}(p,k)+\frac{1}{k^2+4m_0^2}\frac{\partial}{\partial m_0^2}\Delta^{-1}_{(\alpha)}(p)\right]$$

$$-\int\frac{d^2k}{(2\pi)^2}\Delta_{(\lambda)}(k)\left[(k^2+4m_0^2)[V_4^{(a)}(p,k)+V_4^{(a)}(p,-k)]\right.$$

$$\left.+(k^2+4m_0^2+2p^2)V_4^{(b)}(p,k)-4[V_3(p,k)+V_3(p,-k)]+\frac{k^2}{k^2+4m_0^2}\frac{\partial}{\partial m_0^2}\Delta^{-1}_{(\alpha)}(p)\right]. \tag{8.20}$$

Equation (8.20) is obviously still in bad need of ultraviolet regularization. Performing this regularization will also be a check of renormalizability, because we expect the divergences of Eq. (8.20) to be directly related to the mass, vertex, and wave-function renormalization of the model. The following large-k behaviors are easily derived:

$$\begin{aligned}V_4^{(a)}(p,k)+V_4^{(a)}(p,-k)&\to-\frac{1}{k^2}\frac{\partial}{\partial m_0^2}\Delta^{-1}_{(\alpha)}(p)\\&\quad+\frac{2}{k^4}\Delta^{-1}_{(\alpha)}(p)\\&\quad+\frac{2m_0^2}{k^4}\frac{\partial}{\partial m_0^2}\Delta^{-1}_{(\alpha)}(p)\ ,\\V_4^{(b)}(p,k)&\to\frac{2}{k^4}\Delta^{-1}_{(\alpha)}(p)\ ,\\V_3(p,k)+V_3(p,-k)&\to\frac{2}{k^2}\Delta^{-1}_{(\alpha)}(p)\ .\end{aligned} \tag{8.21}$$

Therefore the divergence of Eq. (8.20) is regularized by the addition of

$$\int_{M^2}\frac{d^2k}{(2\pi)^2}\frac{\Delta^{(0)}_{(\alpha)}(k)}{k^4}\left[4\Delta^{-1}_{(\alpha)}(p)-2m_0^2\frac{\partial}{\partial m_0^2}\Delta^{-1}_{(\alpha)}(p)\right]$$

$$-\int_{M^2}\frac{d^2k}{(2\pi)^2}\frac{\Delta^{(0)}_{(\lambda)}(k)}{k^2}\left[4\Delta^{-1}_{(\alpha)}(p)+6m_0^2\frac{\partial}{\partial m_0^2}\Delta^{-1}_{(\alpha)}(p)\right]. \tag{8.22}$$

Equation (8.22) is however nothing but the contribution of the diagrams originated by the insertion of the following counterterms in Δ_0^{-1}: mass counterterm

$$2m_0^2\int_{M^2}\frac{d^2k}{(2\pi)^2}\frac{\Delta^{(0)}_{(\alpha)}(k)}{k^4}+6m_0^2\int_{M^2}\frac{d^2k}{(2\pi)^2}\frac{\Delta^{(0)}_{(\lambda)}(k)}{k^2}\ ; \tag{8.23}$$

wave-function counterterm

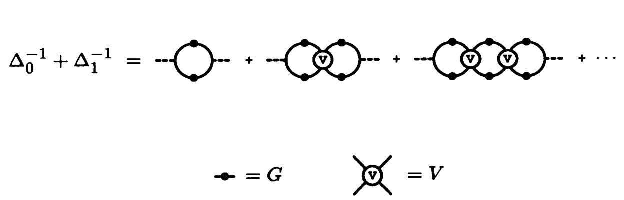

FIG. 8. Graphical proof of Eq. (8.16).

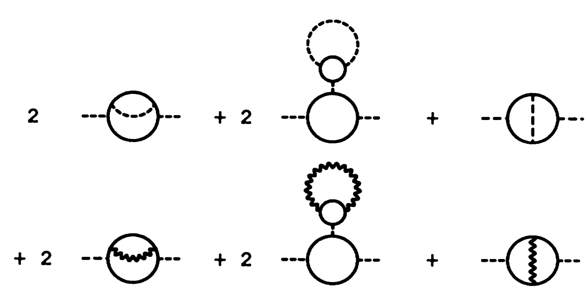

FIG. 9. $O(1/N)$ contributions to Δ_1^{-1}.

$$(p^2+m_0^2)\int_{M^2}\frac{d^2k}{(2\pi)^2}\frac{\Delta_{(\alpha)}^{(0)}(k)}{k^4} -2(p^2+m_0^2)\int_{M^2}\frac{d^2k}{(2\pi)^2}\frac{\Delta_{(\lambda)}^{(0)}(k)}{k^2} ; \quad (8.24)$$

scalar vertex counterterm

$$\int_{M^2}\frac{d^2k}{(2\pi)^2}\frac{\Delta_{(\alpha)}^{(0)}(k)}{k^4} . \quad (8.25)$$

The structure of Eqs. (8.23) and (8.24) is dictated by the request of ultraviolet finiteness for the two-point function; in particular compare Eq. (8.23) to Eq. (4.6) and Eq. (8.24) to Eq. (5.9). The check of the vertex renormalization is straightforward.

In conclusion we verified that the cancellation of ultraviolet divergences in the two-point function of the composite operator P is ensured by the renormalizability of the $1/N$ expansion. Equations (8.20) and (8.22) open the road to numerical evaluation of $\Delta_1^{-1}(p)$, at least in the region $p^2+4m_0^2>0$. The half-line $p^2+4m_0^2$ real and negative is a branch cut of the function $\Delta_1^{-1}(p)$ in the complex p^2 plane. This is a property shared by the function $\Delta_0^{-1}(p)$ and is nothing but the standard behavior above the two-particle threshold.

It is possible to define $\Delta_1^{-1\,\mathrm{ren}}$ and $\Delta_1^{-1\,\mathrm{fin}}$ in full analogy with $\Sigma_{\mathscr{C}}^{\mathrm{ren}}$ and $\Sigma_{\mathscr{C}}^{\mathrm{fin}}$ (see Sec. IV). $\Delta_1^{-1\,\mathrm{fin}}$ is plotted as a function of p^2 in Fig. 10.

We can now address the problem of a direct determination of the asymptotic large-distance behavior of $B(x-y)$. In momentum space we must look for the first singularity of $\Delta_0^{-1}(p)+\Delta_1^{-1}(p)$; i.e., we must look for a zero of the function $(\Delta_0^{-1}+\Delta_1^{-1})^{-1}$. To first order in the $1/N$ expansion this condition reduces to solving the equation

$$\Delta_0(p)-\Delta_0(p)\Delta_1^{-1}(p)\Delta_0(p)=0 . \quad (8.26)$$

It is almost trivial to show that the necessary condition for $1/N$ expandability of the solution of Eq. (8.26) is the finiteness of $\Sigma(-m_0^2)$, in which case one would simply obtain $p^2\cong-4[m_0^2+1/N\Sigma(-m_0^2)]$, consistent with renormalization group. However this is not the case, and we must resort to an analysis similar to that developed in Sec. V. The expectation that the equation

$$\Delta_0^{-1}(p)=\Delta_1^{-1}(p) \quad (8.27)$$

[equivalent to Eq. (8.26)] be solved by $p^2=-m_B^2$ implies a well-defined singular behavior of $\Delta_1^{-1}(p)$ in the region around $p^2+4m_0^2=0$. Noticing that, in the abovementioned region,

$$\Delta_0^{-1}(p) \underset{p^2+4m_0^2\to0^+}{\sim} \frac{N}{4m_0}\frac{1}{\sqrt{p^2+4m_0^2}} ,$$
$$\Delta_0^{-1}(p) \underset{p^2+4m_0^2\to0^-}{\sim} \pm\frac{iN}{4m_0}\frac{1}{\sqrt{|p^2+4m_0^2|}} , \quad (8.28)$$

we obtain the predictions

$$\Delta_1^{-1\,\mathrm{fin}}(p) \underset{p^2+4m_0^2\to0^+}{\sim} \frac{c_+m_0^2}{(p^2+4m_0^2)^2} , \quad c_+<0 ,$$
$$\Delta_1^{-1\,\mathrm{fin}}(p) \underset{p^2+4m_0^2\to0^-}{\sim} \pm i\frac{c_-m_0^2}{(p^2+4m_0^2)^2} , \quad (8.29)$$
$$c_-=12\pi(-a_1')^{3/2} .$$

Numerically we could only analyze the region $p^2+4m_0^2>0$, where we found consistency with Eq. (8.29) and specifically determined the numerical value $c_+\cong-9.425$. The quantity $(4+p^2/m_0^2)^2\Delta_1^{-1\,\mathrm{fin}}$ is plotted as a function of $\sqrt{4+p^2/m_0^2}$ in Fig. 11.

Altogether, it appears once more that the physical picture of the CP^{N-1} model drawn on the basis of the large-N results is confirmed by the $1/N$ expansion and therefore acceptable for sufficiently high values of N.

For completeness we must mention that, starting from the definition (8.6) and in analogy with Eqs. (5.1) and (5.2)

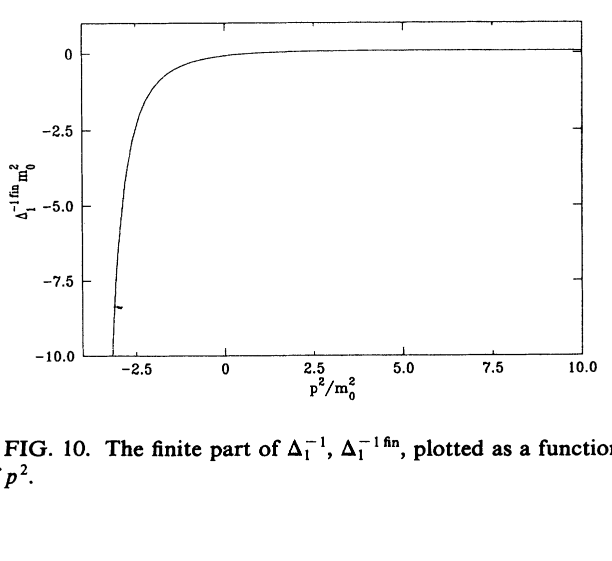

FIG. 10. The finite part of Δ_1^{-1}, $\Delta_1^{-1\,\mathrm{fin}}$, plotted as a function of p^2.

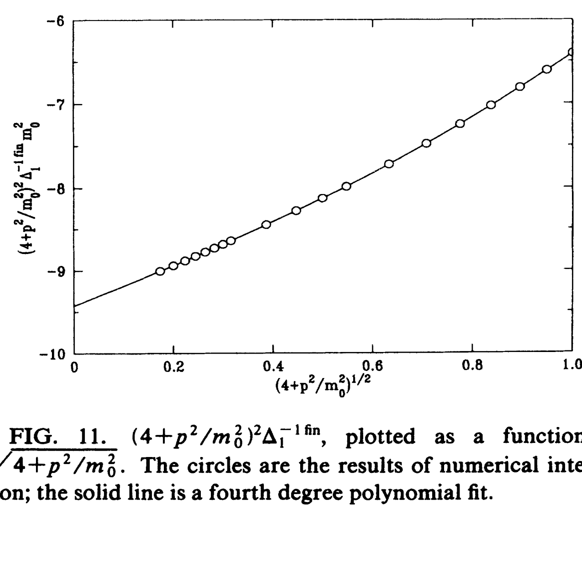

FIG. 11. $(4+p^2/m_0^2)^2\Delta_1^{-1\,\mathrm{fin}}$, plotted as a function of $\sqrt{4+p^2/m_0^2}$. The circles are the results of numerical integration; the solid line is a fourth degree polynomial fit.

it is possible to define a magnetic susceptibility and a moment of the correlation function associated with the P field:

$$\chi_P=\int d^2x\langle \mathrm{tr}P(x)P(0)\rangle\ , \quad (8.30)$$

$$\langle x^2\rangle_P=\frac{\int d^2x(x^2/4)\langle \mathrm{tr}P(x)p(0)\rangle}{\int d^2x\langle \mathrm{tr}P(x)P(0)\rangle}\ , \quad (8.31)$$

where

$$\langle \mathrm{tr}P(x)P(0)\rangle=\frac{N-1}{N}B(x)$$
$$=\frac{4f^2}{N^3}(N^2-1)(\Delta_0^{-1}+\Delta_1^{-1})\ . \quad (8.32)$$

As usual we may evaluate χ_P and $\langle x^2\rangle_P$ starting from the momentum-space representation of $\Delta_0^{-1}+\Delta_1^{-1}$ and considering the behavior of this quantity around $p^2=0$. The divergent regularized terms can be extracted from Eq. (8.22) and the finite parts may be numerically evaluated. We find the following result:

$$\Delta_0^{-1}\underset{p^2\to 0}{\sim}\frac{1}{4\pi m_0^2}\left[1-\frac{p^2}{6m_0^2}\right]+O(p^4)\ , \quad (8.33)$$

$$\Delta_1^{-1\,\mathrm{ren}}\underset{p^2\to 0}{\sim}\frac{1}{4\pi m_0^2 N}\left[-3\ln\ln\frac{M^2}{m_0^2}-\ln\left[\ln\frac{M^2}{m_0^2}-2\right]+c_P+\frac{p^2}{m_0^2}\left[\frac{2}{3}\ln\ln\frac{M^2}{m_0^2}+\frac{2}{3}\ln\left[\ln\frac{M^2}{m_0^2}-2\right]+c_P'\right]\right]+O(p^4)\ , \quad (8.34)$$

and as a consequence

$$\langle x^2\rangle_P=\frac{1}{6m_0^2}\left[1-\frac{1}{N}\ln\ln\frac{M^2}{m_0^2}-\frac{3}{N}\ln\left[\ln\frac{M^2}{m_0^2}-2\right]-\frac{1}{N}(c_P+6c_P')\right]\ , \quad (8.35)$$

where $c_P\cong -0.950\,8$, $c_P'\cong 1.740\,5$.

When comparing Eq. (8.35) with Eq. (5.7) we may verify that $\langle x^2\rangle_P$ has the same renormalization-group behavior as $\langle x^2\rangle_{\mathcal{G}}$ and it is therefore a renormalization-group-invariant quantity, as expected. We can also extract from Eq. (8.22) the anomalous dimension of the P field, finding

$$\gamma_P=\frac{1}{N}\frac{f}{2\pi}\left[2-\frac{2}{1-f/\pi}\right]\equiv-\frac{1}{N}\left[\frac{f}{\pi}\right]^2\frac{1}{1-f/\pi}\ . \quad (8.36)$$

The difference between γ_P and 2γ [cf. Eq. (5.9)] is due to the vertex renormalization affecting the renormalization-group properties of the composite operator P; in particular let us notice that, according to Eq. (8.36), γ_P turns out to be $O(f^2)$ in standard perturbation theory. It is pleasant to observe that this result can be shown to reproduce the known anomalous dimension of the field $\mathbf{S}/f$ in the O(3)$\approx$CP^1 case.

IX. CONCLUSIONS AND OUTLOOK

We think we have offered a rather complete overview of the physical predictions that can be obtained from the first nonleading order of the $1/N$ expansion in CP^{N-1} models. We have not discussed the topological properties, because this topic has already received a systematic treatment in the literature, both in the continuum version [1,16] and on the lattice [3,17].

We only review for completeness the basic continuum results: one can define a topological density

$$q(x)=\frac{i}{2\pi}\varepsilon_{\mu\nu}\overline{D_\mu z}D_\nu z=\frac{1}{2\pi}\varepsilon_{\mu\nu}\partial_\mu A_\nu\ , \quad (9.1)$$

and it expectation value in a θ vacuum is

$$\langle q(x)\rangle_\theta\cong i\frac{3m_0^2}{N\pi}\theta+O\left[\frac{1}{N^2}\right]\ ,\quad |\theta|\le\pi\ . \quad (9.2)$$

One can also define a topological susceptibility

$$\chi_t=\int d^2x\langle q(x)q(0)\rangle \quad (9.3)$$

and find the relationship

$$\chi_t\cong\frac{3}{\pi N}m_0^2+O\left[\frac{1}{N^2}\right]\ . \quad (9.4)$$

It is obvious that, like most perturbative results, ours can be in principle improved in many different directions. In our opinion, the most attractive possibility involves the conversion of our results to some lattice version of the models and the numerical exploration of the wide gap separating large-N asymptopia ($N\gtrsim 20$, according to our results) from the solvable $N=2$ case. The qualitative features of the CP$^1\approx$O(3) model still bear a sensible resemblance to the physical picture of the CP^{N-1} models that we have been confirming in our work. Finding a smooth, monotonic dependence of the physical quantities on N would be an encouraging indication about the predictive power of the $1/N$ expansion.

APPENDIX A: EFFECTIVE VERTICES (IN COLLABORATION WITH H. PANAGOPOULOS)

The effective vertices of the $1/N$ expansion are nothing but one-loop integrals over the fundamental field propagators with appropriate couplings to the external lines.

The problem of evaluating the most general one-loop integral in two dimensions has been addressed many times and very general compact formulas have been presented in the literature [18,19,20]. Unfortunately, these vertices are momentum dependent, and the momenta relevant to our problem correspond to exceptional configurations, such that the formal expressions appearing in the literature become formally singular and cannot be used in the computations. We therefore computed from scratch the vertices at exceptional momenta (a procedure simpler than taking the formal limit) and will present here our results.

One basic ingredient is the three-point scalar vertex

$$V_3(p_1,p_2)\equiv\int\frac{d^2q}{(2\pi)^2}\frac{1}{q^2+m_0^2}\frac{1}{(q+p_1)^2+m_0^2}\times\frac{1}{(q+p_2)^2+m_0^2}\,, \tag{A1}$$

a symmetric function of p_1, p_2, and p_1-p_2. We can compute it via the two-dimensional identity

$$\frac{1}{[q^2+m_0^2][(q+p_1)^2+m_0^2][(q+p_2)^2+m_0^2]} = D^{-1}(p_1,p_2)\left\{\frac{p_1^2[p_2(p_2-p_1)]+2(p_1p_2)(p_1q)-2p_1^2(p_2q)}{[q^2+m_0^2][(q+p_1)^2+m_0^2]}+\frac{p_2^2[p_1(p_1-p_2)]+2(p_1p_2)(p_2q)-2p_2^2(p_1q)}{[q^2+m_0^2][(q+p_2)^2+m_0^2]} + \frac{(p_1^2+p_2^2)(p_1p_2)+2p_1^2p_2^2-4(p_1p_2)^2+2(qp_2)[p_1^2-(p_1p_2)]+2(qp_1)[p_2^2-(p_1p_2)]}{[(q+p_1)^2+m_0^2][(q+p_2)^2+m_0^2]}\right\}, \tag{A2}$$

where

$$D(p_1,p_2)=p_1^2p_2^2(p_1-p_2)^2+4m_0^2[p_1^2p_2^2-(p_1p_2)^2]\,. \tag{A3}$$

The integration is now straightforward and we obtain

$$V_3(p_1,p_2)=D^{-1}(p_1,p_2)\{p_1^2[p_2(p_2-p_1)]\Delta_{(a)}^{-1}(p_1)+p_2^2[p_1(p_1-p_2)]\Delta_{(a)}^{-1}(p_2)+(p_1-p_2)^2(p_1p_2)\Delta_{(a)}^{-1}(p_1-p_2)\}\,. \tag{A4}$$

Equation (A4) can be shown to agree, after a rotation from Euclidean to Minkowski space, with results appearing in the literature. The exceptional limit of Eq. (A4) corresponds to the case when one of the external momenta vanishes. It is however very easy to see from Eq. (A1) that

$$\lim_{p_2\to 0}V_3(p_1,p_2)=-\frac{1}{2}\frac{\partial}{\partial m_0^2}\Delta_{(a)}^{-1}(p_1)=\frac{1}{p_1^2+4m_0^2}[\Delta_{(a)}^{-1}(p_1)+\Delta_{(a)}^{-1}(0)]\,. \tag{A5}$$

Let us now consider the four-point vertices: the exceptional configurations we are interested in are the cases when the external momenta are equal two by two. Let us define

$$V_4^{(a)}(p_1,p_2)\equiv\int\frac{d^2q}{(2\pi)^2}\frac{1}{[q^2+m_0^2]^2}\frac{1}{(q+p_1)^2+m_0^2}\frac{1}{(q+p_2)^2+m_0^2}\,. \tag{A6}$$

By applying the identity (A2) we are led to an explicitly integrable but cumbersome expression. It takes some algebraic effort to generate the final result

$$\begin{aligned}V_4^{(a)}(p_1,p_2)=&D^{-1}(p_1,p_2)\left\{\frac{[p_2^2-(p_1p_2)]p_1^2}{p_1^2+4m_0^2}[\Delta_{(a)}^{-1}(p_1)+\Delta_{(a)}^{-1}(0)]+\frac{[p_1^2-(p_1p_2)]p_2^2}{p_2^2+4m_0^2}[\Delta_{(a)}^{-1}(p_2)+\Delta_{(a)}^{-1}(0)]\right\}\\&+D^{-2}(p_1,p_2)([(p_1-p_2)^2(p_1p_2)+p_1^2p_2^2-(p_1p_2)^2]\\&\qquad\times\{[p_2^2-(p_1p_2)]p_1^2\Delta_{(a)}^{-1}(p_1)+[p_1^2-(p_1p_2)]p_2^2\Delta_{(a)}^{-1}(p_2)\}\\&\qquad+[(p_1-p_2)^2(p_1p_2)]^2\Delta_{(a)}^{-1}(p_1-p_2))\\&-D^{-2}(p_1,p_2)[p_1^2p_2^2-(p_1p_2)^2]\{(p_1^2+4m_0^2)[(p_1p_2)-p_1^2]\Delta_{(a)}^{-1}(p_1)\\&\qquad+(p_2^2+4m_0^2)[(p_1p_2)-p_2^2]\Delta_{(a)}^{-1}(p_2)\\&\qquad+[(p_1-p_2)^2+4m_0^2](p_1-p_2)^2\Delta_{(a)}^{-1}(p_1-p_2)\}\,.\end{aligned} \tag{A7}$$

We must also evaluate

$$V_4^{(b)}(p_1,p_2)\equiv\int\frac{d^2q}{(2\pi)^2}\frac{1}{q^2+m_0^2}\frac{1}{(q+p_1)^2+m_0^2}\frac{1}{(q+p_2)^2+m_0^2}\frac{1}{(q+p_1+p_2)^2+m_0^2}\ . \tag{A8}$$

In this case, the computation is made easier by the identity

$$\frac{2(p_1p_2)}{(q^2+m_0^2)[(q+p_1)^2+m_0^2][(q+p_2)^2+m_0^2][(q+p_1+p_2)^2+m_0^2]}$$
$$=\frac{1}{(q^2+m_0^2)[(q+p_1)^2+m_0^2][(q+p_2)^2+m_0^2]}$$
$$-\frac{1}{(q^2+m_0^2][(q+p_1)^2+m_0^2][(q+p_1+p_2)^2+m_0^2]}$$
$$-\frac{1}{(q^2+m_0^2)[(q+p_2)^2+m_0^2][(q+p_1+p_2)^2+m_0^2]}$$
$$+\frac{1}{[(q+p_1)^2+m_0^2][(q+p_2)^2+m_0^2][(q+p_1+p_2)^2+m_0^2]}\ , \tag{A9}$$

leading immediately, also thanks to the symmetries of V_3, to

$$V_4^{(b)}(p_1,p_2)=\frac{1}{(p_1p_2)}[V_3(p_1,p_2)-V_3(p_1,-p_2)]\ . \tag{A10}$$

We insist that, even if Eqs. (A7) and (A10) might in principle be recovered by taking the proper limits of the general expression, in practice we found that the direct derivation of Eq. (A5) from Eq. (A4) is already so involved to discourage us from further pursuing this approach.

For our purposes we also need the mixed four-point scalar-vector vertices in exceptional momentum configurations. We quote here the definitions:

$$V_{\mu\nu}^{(a)}(p,k)=\int\frac{d^2q}{(2\pi)^2}\frac{1}{(q^2+m_0^2)^2}\frac{(2q_\mu+k_\mu)(2q_\nu+k_\nu)}{[(q+p)^2+m_0^2][(q+k)^{2+}m_0^2]}\ , \tag{A11}$$

$$V_{\mu\nu}^{(b)}(p,k)=\int\frac{d^2q}{(2\pi)^2}\frac{1}{q^2+m_0^2}\frac{2q_\mu+k_\mu}{(q+k)^2+m_0^2}\frac{1}{(q+p)^2+m_0^2}\frac{2q_\nu+2p_\nu+k_\nu}{(q+p+k)^2+m_0^2}\ . \tag{A12}$$

Actually we only need the combination of vertices appearing in Eq. (8.19) and this can be shown to be a transverse tensor. Therefore we can limit ourselves to computing

$$\left[\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right]V_{\mu\nu}^{(a)}(p,k)=-(k^2+4m_0^2)V_4^{(a)}(p,k)+2V_3(p,k)+\left[1-\frac{4m_0^2}{k^2}\frac{pk}{p^2}\right]\frac{\Delta_{(\alpha)}^{-1}(0)}{p^2+4m_0^2}$$
$$+\left[(p+k)^2+4m_0^2\left[1+\frac{pk}{p^2}\right]\right]\frac{\Delta_{(\alpha)}^{-1}(p)}{k^2(p^2+4m_0^2)}-\frac{1}{k^2}\Delta_{(\alpha)}^{-1}(p-k) \tag{A13}$$

and

$$\left[\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right]V_{\mu\nu}^{(b)}(p,k)=-(k^2+2p^2+4m_0^2)V_4^{(b)}(p,k)-\frac{2}{k^2}\Delta_{(\alpha)}^{-1}(p)+\frac{1}{k^2}\Delta_{(\alpha)}^{-1}(p-k)$$
$$+\frac{1}{k^2}\Delta_{(\alpha)}^{-1}(p+k)+2V_3(p,k)+2V_3(p,-k)\ . \tag{A14}$$

The derivation of Eq. (8.19) from Eqs. (A13) and (A14) is now straightforward.

APPENDIX B: STRING DIVERGENCES IN COORDINATE SPACE

It is quite instructive to identify the origin of string divergences in coordinate space. We know from Sec. IV that, to $O(1/N)$, string ultraviolet divergences are only originated by the term

$$-\frac{1}{2N}\langle \bar{z}(y)z(x)\rangle_{(0)}\int_y^x dt_\mu \int_y^x dt'_\nu \langle A_\mu(t)A_\nu(t')\rangle_{(0)}$$

$$=-\frac{1}{2N}G_0(x-y)\int_0^1 d\lambda \int_0^1 d\lambda'(y-x)_\mu(y-x)_\nu \Delta^{(\lambda)}_{\mu\nu}((\lambda-\lambda')(y-x))$$

$$=-\frac{1}{2N}G_0(x-y)\int \frac{d^2k}{(2\pi)^2}\frac{\Delta_{(\lambda)}(k)}{k^2}C(k|x-y|)\ , \tag{B1}$$

where we have applied the standard parametrization of straight open lines (4.9) and defined

$$C(z)=\int_0^1 d\lambda \int_0^1 d\lambda' \int \frac{d\theta}{2\pi}e^{i(\lambda-\lambda')z\cos\theta}z^2\sin^2\theta$$

$$=2z\int_0^z J_0(x)dx-2zJ_1(z)+2J_0(z)-2\ . \tag{B2}$$

It is important to notice that the derivation of Eqs. (B1) and (B2) does not really depend on the space dimensionality, and therefore our results equally apply to straight lines in different dimensionalities.

Now the crucial observation concerns the asymptotic behavior of $C(z)$ for large z: we can prove that

$$C(z)=2(z-1)-\frac{2}{z}J_1(z)+O\left[\frac{1}{z^2}\right] \tag{B3}$$

and in any case the remainder is an oscillating function. The string divergence in coordinate space is therefore parametrized by

$$-G_0(x-y)\int \frac{d^2k}{(2\pi)^2}\frac{\Delta^{(0)}_{(\lambda)}(k)}{k^2}(k|x-y|-1)\ , \tag{B4}$$

and by recalling that

$$-\frac{1}{N}\int \frac{d^2k}{(2\pi)^2}\frac{\Delta^{(0)}_{(\lambda)}(k)}{k}=\gamma_1\ , \tag{B5}$$

$$\frac{1}{N}\int \frac{d^2k}{(2\pi)^2}\frac{\Delta^{(0)}_{(\lambda)}(k)}{k^2}=\gamma_0\ , \tag{B6}$$

we find the expected divergent behavior

$$G_0(x-y)(\gamma_1|x-y|+\gamma_0)\ . \tag{B7}$$

It is however already clear from Eq. (B3) that, in arbitrary dimensions and in first order of the $1/N$ expansion, the only possible string divergences are a factor growing linearly with the string length (self-mass) and a constant factor (end-point wave-function renormalization).

APPENDIX C: ALTERNATIVE REPRESENTATIONS OF THE FUNCTIONS A AND B AND RELATED INTEGRALS

For the purpose of analytic continuation of $\Sigma^{\rm ren}_{\mathscr{C}}(p)$ to the region $p^2+m_0^2<0$ it may be sometimes useful to possess integral representations depending explicitly only on the variable p^2. This may be achieved by exchanging the integration over the string parameters λ, λ' with the angular integration; the latter can be analytically performed and leaves us with functions of p^2 and k^2. In particular let us notice that

$$\int \frac{d\theta}{2\pi}\left[1+2p^2\sin^2\theta\frac{\partial}{\partial m_0^2}\right]\frac{\partial A}{\partial m_0^2}=\frac{\partial}{\partial m_0^2}\int_{-1}^1 d\lambda(1-|\lambda|)\int \frac{d\theta}{2\pi}\left[1+2p^2\sin^2\theta\frac{\partial}{\partial m_0^2}\right]\frac{1}{[p+\lambda k]^2+m_0^2}$$

$$=\int_{-1}^1 d\lambda(1-|\lambda|)\frac{p^2-m_0^2-\lambda^2k^2}{[(p^2+m_0^2+\lambda^2k^2)^2-4\lambda^2k^2p^2]^{3/2}}\ . \tag{C1}$$

In turn one can show that

$$B=\int_0^1 d\lambda\,\lambda\frac{1}{p^2+\lambda(1-\lambda)k^2+m_0^2}\left[\frac{1}{(p+\lambda k)^2+m_0^2}+\frac{1}{(p-\lambda k)^2+m_0^2}\right] \tag{C2}$$

and

$$\int \frac{d\theta}{2\pi}\left[1+2p^2\sin^2\theta\frac{\partial}{\partial m_0^2}\right]B=\int_0^1 d\lambda\frac{1}{[(p^2+m_0^2+\lambda^2k^2)^2-4\lambda^2k^2p^2]^{1/2}}$$

$$\times\frac{1}{p^2+\lambda(1-\lambda)k^2+m_0^2}\left[\frac{\lambda(k^2+4m_0^2)}{p^2+\lambda(1-\lambda)k^2+m_0^2}-1\right]\ . \tag{C3}$$

APPENDIX D: ASYMPTOTIC BEHAVIOR OF LOOP-LOOP CORRELATIONS

In Sec. VII we derived the integral representation of the loop-loop correlation function $G(r,R)$ for two identical circular loops of radius r at a distance R. Let us define the auxiliary function

$$\chi(r,R)=N\frac{G(r,R)}{(\pi r^2 m_0^2)^2} = -\frac{1}{\pi r^2 m_0^4}\int_0^\infty J_1^2(kr)J_0(kR)\Delta_{(\lambda)}(k)k\,dk\ , \quad (D1)$$

normalized to the areas of the loops in order to admit a finite limit when $r\to 0$. We are interested in the asymptotic behavior of $\chi(r,R)$ when $R\to\infty$, as well as in numerical evaluation for finite r and R. The integral in Eq. (D1) is well defined, but it is not absolutely convergent, and this fact complicates both analytic and numerical manipulations. We can however modify the integration contour in the complex k plane to obtain an integral representation of $\chi(r,R)$ better suited to our needs.

Let us recall the following properties of the Bessel functions:

$$J_0(z)=\tfrac{1}{2}[H_0^{(1)}(z)+H_0^{(2)}(z)]\ , \quad (D2)$$

where

$$H_0^{(1,2)}(z)=J_0(z)\pm iY_0(z) \cong \left[\frac{2}{\pi z}\right]^{1/2} e^{\pm i(z-\pi/4)}\left[1+O\left[\frac{1}{z}\right]\right]\ , \quad (D3)$$

and when z is real $J_0(z)=\mathrm{Re}H_0^{(1)}(z)$. Therefore,

$$\chi(r,R)=-\frac{2}{\pi r^2 m_0^4}\mathrm{Re}\int_0^\infty J_1^2(kr)H_0^{(1)}(kR)\Delta_{(\lambda)}(k)k\,dk\ . \quad (D4)$$

We must now study the singularities of the integrand, that are the singularities of the function

$$\Delta_{(\lambda)}(k)=\frac{2\pi}{\xi\ln\frac{\xi+1}{\xi-1}-2}\ ,\qquad \xi=\left[1+\frac{4m_0^2}{k^2}\right]^{1/2}, \quad (D5)$$

in the complex k plane. The denominator in Eq. (D5) never vanishes for $k\neq 0$; however the square root and the logarithm cause the appearance of branch cuts along the whole imaginary axis, with branch points at $k=\pm 2im_0$. We can therefore rotate the integration contour

$$k\to ix+\varepsilon\ ,\quad x,\varepsilon \text{ real}\ , \quad (D6)$$

and recall the properties

$$J_1(ixr)=iI_1(xr)\ , \quad (D7)$$

$$H_0^{(1)}(ixR)=-\frac{2i}{\pi}K_0(xR)\ . \quad (D8)$$

We then obtain

$$\chi(r,R)=-\frac{4}{\pi^2 r^2 m_0^4}\times\int_0^\infty xI_1^2(xr)K_0(xR)\mathrm{Im}\Delta_{(\lambda)}(ix+\varepsilon)dx\ . \quad (D9)$$

However $\Delta_{(\lambda)}(ix+\varepsilon)$ is real when $0<x<2m_0$, while, when $x>2m_0$,

$$\xi=\left[1-\frac{4m_0^2}{x^2}\right]^{1/2}-i\varepsilon\ ,\qquad \ln\frac{\xi+1}{\xi-1}=\ln\frac{1+\xi}{1-\xi}+i\pi\ . \quad (D10)$$

In conclusion we obtain the representation

$$\chi(r,R)=\frac{8}{r^2 m_0^4}\int_{2m_0}^\infty xI_1^2(xr)K_0(xR)\times\frac{\xi}{(\pi\xi)^2+\left[\xi\ln\frac{1+\xi}{1-\xi}-2\right]^2}dx\ . \quad (D11)$$

Equation (D11) is now ready for numerical evaluation and for an asymptotic expansion at large R, based on the asymptotic expansion of K_0,

$$K_0(z)\cong\left[\frac{\pi}{2z}\right]^{1/2}e^{-z}\left[1+O\left[\frac{1}{z}\right]\right]\ , \quad (D12)$$

and as a consequence, for large R we have

$$\chi(r,R)\cong\frac{\pi}{r^2 m_0^2}I_1^2(2m_0 r)\frac{e^{-2Rm_0}}{R^2 m_0^2} \underset{r\to 0}{\sim}\frac{\pi}{R^2 m_0^2}e^{-2Rm_0}\ . \quad (D13)$$

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