Two-dimensional gravity as the gauge theory of the Clifford algebra for an even-dimensional generalized Chern-Simons action

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We investigate the two-dimensional version of the Chern-Simons action derived from the recently proposed even-dimensional generalized Chern-Simons action. We show that the two-dimensional topological gravity emerges if we choose the Clifford algebra as a nonstandard gauge symmetry algebra required from the generalized Chern-Simons action. We find a "hidden order parameter" which differentiates the gravity phase and nongravity phase.

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I. INTRODUCTION

Three-dimensional Einstein gravity has been formulated as an ISO(2,1) gauge theory of the standard three-dimensional Chern-Simons action by Witten [1]. It has also been recognized that three-dimensional conformal gravity can be formulated by the SO(3,2) gauge theory of the Chern-Simons action [2]. It is a very natural but nontrivial question if we can extend the three-dimensional treatment to other dimensions. At first thought, it is impossible to extend the three-dimensional Chern-Simons action into other dimensions naively. Recently we have proposed the generalized Chern-Simons actions which are formulated in arbitrary dimensions [3]. In this paper we investigate two-dimensional gravity by the newly proposed even-dimensional generalized Chern-Simons action.

In generalizing the standard Chern-Simons formulation, we have introduced "quaternion algebra" to accommodate even forms, odd forms, fermions, and bosons. In other words, the gauge field is not a standard one-form anymore but includes higher forms whose coefficients of the wedge products may be fermionic and bosonic. As a result the constructed generalized Chern-Simons action has an unusual gauge symmetry which includes anticommutators in the gauge transformation not because of the fermions but because of the fermionic nature of the forms and the quaternion. In this paper we omit introducing fermions while the general formulation to include fermions in the generalized Chern-Simons action has been given in our recent papers [3].

In standard Chern-Simons theory, the relation between the action and the topological quantity such as the Chern character is clear, where the gauge algebra is an SU(N) Lie algebra [4]. In our generalized Chern-Simons formulation, although similar algebraic relations hold as in the standard case, the topological meanings of the new theory are not yet established where the gauge algebra is the Clifford algebra, as we will show in this paper. In the three-dimensional gravity analyses by the standard Chern-Simons action with an ISO(2,1) or SO(3,2) gauge group the topological nature of the action is related to the locally flat nature of the three-dimensional gravity. In other words, the gauge theory formulation of the three-dimensional gravity does not contain dynamical degrees of freedom. It is likely that the two-dimensional gravity formulated by the generalized Chern-Simons action does not contain dynamical degrees of freedom either. In fact we show that some of the equations of motion of the two-dimensional gravity formulated by the generalized Chern-Simons action coincide with a particular gauge-choice version of the two-dimensional topological gravity [5,6]. It is an interesting question how the dynamical degrees of freedom appear after the natural breaking of the full gauge symmetry.

It is generally believed that the vierbein which generates the metric in the realistic gravity, when it is formulated as a gauge theory of some gauge symmetry, contains dynamical degrees of freedom and gets nonzero values and thus the gauge symmetry will be broken. The notion of the broken phase and unbroken phase into the gravity has been introduced by Witten [7]. In this paper we identify the gauge symmetry as the one induced by the Clifford algebra and investigate the classical solutions of the equations of motion of the generalized Chern-Simons action. In identifying two-dimensional gravity, we gauged away the gauge fields to be consistent with the equations of motion. We then find classical solutions which could be interpreted as a nonvanishing zweibein of "broken phase" in certain cases. In our formulation we, however, have a zero-form gauge field which gets a nonvanishing classical value in certain cases and discriminate the gravity phase and the nongravity phase. We claim that there exists a "hidden order parameter" which differentiates the gravity phase and the nongravity phase.

Here we describe how the formulation and the results of this paper could be related to previous works. One of the aims of this paper is to show that the two-dimensional generalized Chern-Simons action with a Clifford algebra leads to a two-dimensional topological gravity, which is contrasted with the three-dimensional case where the three-dimensional standard Chern-Simons action with an ISO(2,1) gauge group has led to the three-dimensional Einstein-Hilbert action [1] while the SO(3,2) gauge group has led to the three-dimensional conformal gravity [2]. The notion of topological gravity was first introduced by Witten, after the formulation of topological field theory [7]. It was recognized later that topological field theory can be reproduced by a particular gauge fixing of the vanishing Lagrangian [5]. This has triggered many investigations of topological field theory and topological gravity from the gauge-fixing point of view [6]. In particular, two-dimensional topological gravity has also been investigated [5,6]. In a certain choice of gauge Clifford algebra, we obtain one version of two-dimensional topological gravity from the two-dimensional generalized Chern-Simons action. Since the important developments of two-dimensional gravity by the matrix model formulation [8,9], it has been recognized that two-dimensional topological gravity is the starting basis of two-dimensional gravity with a matter field [10].

In the meantime topological field theories have been developed from different points of view. In particular, topological meanings and the quantization of the *BF* [the action being the product of a curvature two-form F and $(d-2)$-form B] system have been investigated by many people [11] where higher-order forms were introduced to construct the higher-dimensional topological invariants. The topological invariant of the Abelian version of the *BF* system is related to the linking number and the non-Abelian extension has been intensively studied [11]. Part of our action coincides with the non-Abelian version of the *BF* system. Our original action, however, includes a "quaternion" which leads to a clear difference between the *BF* action and the generalized Chern-Simons action. In other words the anticommutators in the gauge transformation can never be induced by the coupled *BF* system.

In formulating the generalized Chern-Simons action, we introduce higher-order forms. Myers and Periwal have introduced higher-order odd forms to formulate an odd-dimensional version of the Chern-Simons action [12] while we need to introduce both even forms and odd forms and furthermore a "quaternion algebra" to obtain the even-dimensional version of the generalized Chern-Simons action. As a result the gauge transformation includes anticommutators in addition to the standard commutator, which necessitates the Clifford algebra, while the odd-dimensional extension only includes commutators and thus the adjoint representation of the Lie algebra closes the gauge transformation. The introduction of a "quaternion algebra" together with the higher-order forms has led to the new gauge symmetry whose special example can be realized by the Clifford algebra. These are the most unusual aspects of our formulation.

This paper is organized as follows. In Sec. II we derive a pure bosonic version of the even-dimensional generalized Chern-Simons action. In Sec. III we provide the general formulations of the Clifford algebra, which are used in the generalized Chern-Simons formulation as a gauge algebra. We first analyze the case of the simplest algebra in Sec. IV. We then extend the analysis into the next nontrivial algebra where we obtain the two-dimensional topological gravity in Sec. V. Here we investigate several cases of the gauge algebra for Euclidean and Minkowskian gravities. In Sec. VI we investigate the case of a two-dimensional conformal algebra which is extended to include the Poincaré symmetry. In Sec. VII we then try to interpret the classical solutions obtained in the previous sections. In particular the meaning of the hidden order parameter and the classical gravity phase space are discussed. We then provide a conclusion and discussions in the last section.

II. DERIVATION OF TWO-DIMENSIONAL GENERALIZED CHERN-SIMONS ACTION

In this section we summarize the derivation of the pure bosonic version of the even-dimensional generalized Chern-Simons action. The formulation can be straightforwardly extended to include fermions, which have been given in our previous papers [3].

We introduce a gauge field $\mathcal{A}$, an exterior derivative operator Q, and a gauge parameter $\mathcal{V}$ as

$$\begin{aligned} \mathcal{A} &= A\mathbf{j}+\hat{A}\mathbf{k} , \\ Q &= \mathbf{j}d , \\ \mathcal{V} &= \hat{a}\mathbf{1}+a\mathbf{i} , \end{aligned} \tag{2.1}$$

where $d=dx^\mu\partial_\mu$. A and a consist of odd forms while $\hat{A}$ and $\hat{a}$ consist of even forms. The gauge fields and parameters A, $\hat{A}$, $\hat{a}$, and a carry an index of a certain gauge algebra. More explicitly the gauge fields can be expressed as

$$\begin{aligned} \hat{A} &= \frac{T_a}{2}\left[A^{(0)a}+\frac{1}{2}A^{(2)a}_{\mu\nu}dx^\mu\wedge dx^\nu+\cdots \right] , \\ A &= \frac{T_a}{2}\left[A^{(1)a}_{\mu}dx^\mu+\frac{1}{3!}A^{(3)a}_{\mu\nu\rho}dx^\mu\wedge dx^\nu\wedge dx^\rho+\cdots \right] , \end{aligned} \tag{2.2}$$

where $A^{(p)a}_{\mu_1\cdots\mu_p}$ denotes a bosonic p-rank tensor and T_a is a generator of the gauge algebra. The degree of the highest form of $\hat{A}$ or A coincides with the dimension of space-time. The gauge parameters $\hat{a}$ and a have similar forms as the gauge fields. As a base manifold we consider an N-dimensional compact manifold without boundary. $\mathbf{1},\mathbf{i},\mathbf{j},\mathbf{k}$ carry "quaternionic structure" which is defined as

$$\begin{aligned} &\mathbf{1}^2=\mathbf{1}, \quad \mathbf{i}^2=\epsilon_1\mathbf{1}, \quad \mathbf{j}^2=\epsilon_2\mathbf{1}, \quad \mathbf{k}^2=-\epsilon_1\epsilon_2\mathbf{1} , \\ &\mathbf{ij}=-\mathbf{ji}=\mathbf{k}, \quad \mathbf{jk}=-\mathbf{kj}=-\epsilon_2\mathbf{i}, \quad \mathbf{ki}=-\mathbf{ik}=-\epsilon_1\mathbf{j} , \end{aligned} \tag{2.3}$$

where $(\epsilon_1,\epsilon_2)=(-1,-1),(-1,+1),(+1,-1),(+1,+1)$. In the case $(\epsilon_1,\epsilon_2)=(-1,-1)$, $\mathbf{1},\mathbf{i},\mathbf{j},\mathbf{k}$ satisfy the algebra of the quaternion. For the rest of the three cases, $\mathbf{1},\mathbf{i},\mathbf{j},\mathbf{k}$ satisfy the general linear Lie algebra $\mathrm{gl}(2,\mathbb{R})$.

The even-dimensional generalized Chern-Simons action is defined by

$$S^{\mathrm{b}}_{\mathrm{e}}=\int \mathrm{Tr}_{\mathbf{k}}\left(\tfrac{1}{2}\mathcal{A}Q\mathcal{A}+\tfrac{1}{3}\mathcal{A}^3\right) , \tag{2.4}$$

where $\mathrm{Tr}_{\mathbf{k}}$ picks up the $\mathbf{k}$th component and takes the trace of the gauge algebra. The action (2.4) is invariant under the gauge transformation

$$\delta\mathcal{A}=Q\mathcal{V}+[\mathcal{A},\mathcal{V}] . \tag{2.5}$$

To prove the gauge invariance of the generalized Chern-Simons action, we need the following settings. We first introduce two types of gauge fields and parameters: $\lambda_1=(\text{even form})\mathbf{1}+(\text{odd form})\mathbf{i}\in\Lambda_1$ and $\lambda_k=(\text{odd}$

form)$\mathbf{j}$+(even form)$\mathbf{k}\in\Lambda_k$. In particular $\mathcal{V}\in\Lambda_1$ and $\mathcal{A},Q\in\Lambda_k$. In case $\mathbf{1},\mathbf{i},\mathbf{j},\mathbf{k}$ satisfy the "quaternion algebra," we can show the following relations.

(i) $\Lambda_1\Lambda_1\sim\Lambda_1,\quad \Lambda_1\Lambda_k\sim\Lambda_k,\quad \Lambda_k\Lambda_k\sim\Lambda_1$,

where the first equation means the following: if $\lambda_1,\lambda_1'\in\Lambda_1$ then $\lambda_1\lambda_1'\in\Lambda_1$ and similarly for the other two equations;

(ii) $\{\vec{Q},\lambda_k\}=Q\lambda_k,[\vec{Q},\lambda_1]=Q\lambda_1$,

where $Q^2=0$ and $Q\in\Lambda_k$;

(iii) $\mathrm{Tr}_{\mathbf{k}}(\lambda_k\lambda_1)=\mathrm{Tr}_{\mathbf{k}}(\lambda_1\lambda_k)$.

The arrow denotes the differential operation on all fields and parameters to the right of the arrow. Using the above relations, it is easy to show the gauge invariance of the action (2.4) under the gauge transformation (2.5). The reason why we take the $\mathbf{k}$th component of the trace for the action comes from relation (iii); i.e., cyclic invariance in the trace is valid only for the $\mathbf{k}$th component.

The equation of motion derived from the action (2.4) is

$$\mathcal{F}=Q\mathcal{A}+\mathcal{A}^2=0\ , \tag{2.6}$$

where $\mathcal{F}$ is a curvature and thus the above equation of motion is nothing but a flat connection condition.

Introducing the defining relations of (2.1) into Eqs. (2.4) and (2.5), we obtain the explicit form of the pure bosonic version of even-dimensional generalized Chern-Simons action:

$$S_{\mathrm{e}}^{\mathrm{b}}=\epsilon_2\int\mathrm{Tr}[\hat{A}(dA+A^2)-\tfrac{1}{3}\epsilon_1\hat{A}^3]\ . \tag{2.7}$$

In the integrand, we pick up the forms whose degrees coincide with dimension N. The above action is invariant under the gauge transformation

$$\begin{aligned}\delta A&=d\hat{a}+[A,\hat{a}]-\epsilon_1\{\hat{A},a\}\ ,\\ \delta\hat{A}&=-da-\{A,a\}+[\hat{A},\hat{a}]\ ,\end{aligned} \tag{2.8}$$

where [,] and { , } are commutator and anticommutator, respectively. The same order of the forms in Eq. (2.8) should be equated. It should be noted that the term $\{\hat{A},a\}$ makes it impossible to close the gauge algebra within the adjoint representation. We need the gauge algebra which is closed under not only commutators but also anticommutators. The equations of motion for A and $\hat{A}$ have the form

$$\begin{aligned}&dA+A^2-\epsilon_1\hat{A}^2=0\ ,\\ &d\hat{A}+[A,\hat{A}]=0\ ,\end{aligned} \tag{2.9}$$

which are derived from (2.6) by using (2.1) or from (2.7) directly.

We now obtain very explicit expressions of two-dimensional action. The gauge fields and parameters are expressed as

$$\begin{aligned}A&=\omega\ ,\\ \hat{A}&=\epsilon_2(\phi+\epsilon_1 B)\ ,\\ \hat{a}&=v+\epsilon_1 b\ ,\\ a&=-\epsilon_1\epsilon_2 u\ ,\end{aligned} \tag{2.10}$$

where ϕ, ω, and B are zero-, one-, and two-form gauge fields while v, u, and b are zero-, one-, and two-form gauge parameters, respectively. By substituting the expressions (2.10) into (2.7), we obtain the explicit form of two-dimensional generalized Chern-Simons action:

$$S_2=\int\mathrm{Tr}[\phi(d\omega+\omega^2)-\phi^2 B]\ , \tag{2.11}$$

which is invariant under the gauge transformations

$$\delta\phi=[\phi,v]\ , \tag{2.12a}$$

$$\delta\omega=dv+[\omega,v]+\{\phi,u\}\ , \tag{2.12b}$$

$$\delta B=du+\{\omega,u\}+[B,v]+[\phi,b]\ . \tag{2.12c}$$

It should be noted that the $\{\phi,u\}$ term is a special term which causes the nonclosure of the gauge algebra within commutators. The equations of motion derived from the action (2.11) are

$$\phi^2=0\ , \tag{2.13a}$$

$$d\phi+[\omega,\phi]=0\ , \tag{2.13b}$$

$$d\omega+\omega^2-\{\phi,B\}=0\ , \tag{2.13c}$$

which are also derived from (2.9). In this paper we study the equations of motion (2.13) and the gauge transformations (2.12) at the classical level. We will find some classical solutions of Eqs. (2.13) modulo the gauge transformations (2.12).

III. REPRESENTATIONS OF ALGEBRA

To carry out the explicit analyses, we need to specify the representations of the gauge algebra which are used in the generalized Chern-Simons theory. As we have pointed out in the preceding section, the gauge symmetry which we are concerned with is an unusual one. The gauge algebra represented by generators T_a has to be closed under both commutators and anticommutators:

$$\begin{aligned}[T_a,T_b]&=f_{ab}{}^{c}T_c\ ,\\ \{T_a,T_b\}&=d_{ab}{}^{c}T_c\ .\end{aligned} \tag{3.1}$$

In other words the algebra has to be closed under multiplications

$$T_aT_b=k_{ab}{}^{c}T_c\ . \tag{3.2}$$

We also require the associativity of T_a; however, we need not require the existence of an inverse element for any T_a. A semigroup algebra is an algebra which satisfies (3.2). Each element of a semigroup algebra is constructed by a linear combination of elements which are the multiplication of an element of a field K and an element of the semigroup so as to satisfy the distributive law. In this paper we consider K as a set of real numbers, and products of γ matrices as the elements of the semigroup.

The simplest examples which satisfy (3.2) are the Clifford algebras. The bases of the Clifford algebra $c(k,n-k)$ are constructed from γ matrices which satisfy $\{\gamma_a,\gamma_b\}=2\eta_{ab}$ with $\eta_{ab}=\mathrm{diag}(1,\ldots,1,-1,\ldots,-1)$. The indices a and b run from 1 to n, where n is an integer. The γ matrices are represented by $2^{[n/2]}\times 2^{[n/2]}$ matrices. The k in $c(k,n-k)$ denotes the number of positive metric with $\eta_{aa}=1$ while $n-k$ in $c(k,n-k)$ denotes the number of negative metric with $\eta_{aa}=-1$.

In the case of $n=$(even integer), the bases of the Clifford algebras $c(k,n-k)$ are, for example,

$$c(k,2-k):\ 1,\gamma_a,\tilde{\gamma}\ , \tag{3.3a}$$

$$c(k,4-k):\ 1,\gamma_a,\gamma_{ab},\tilde{\gamma}\gamma_a,\tilde{\gamma}\ , \tag{3.3b}$$

$$c(k,6-k):\ 1,\gamma_a,\gamma_{ab},\gamma_{abc},\tilde{\gamma}\gamma_{ab},\tilde{\gamma}\gamma_a,\tilde{\gamma}\ , \tag{3.3c}$$

where $\gamma_{ab}=(\gamma_a\gamma_b-\gamma_b\gamma_a)/2, \gamma_{abc}=[\gamma_a\gamma_b\gamma_c+\gamma_b\gamma_c\gamma_a+\gamma_c\gamma_a\gamma_b-(a\leftrightarrow b)]/6$, and $\tilde{\gamma}=\gamma_1\cdots\gamma_n$. The above representations are closed under multiplications explicitly. One can also obtain the special Clifford algebras $sc(k,n-k)$:

$$sc(k,2-k):\ 1,\tilde{\gamma}\ , \tag{3.4a}$$

$$sc(k,4-k):\ 1,\gamma_{ab},\tilde{\gamma}\ , \tag{3.4b}$$

$$sc(k,6-k):\ 1,\gamma_{ab},\tilde{\gamma}\gamma_{ab},\tilde{\gamma}\ , \tag{3.4c}$$

where the special Clifford algebras include the products of only an even number of γ matrices. It should be noted that $sc(k,n-k)$ is isomorphic to $sc(n-k,k)$. These generators are also closed under multiplications. When $\tilde{\gamma}^2=1$, i.e., $\eta_{11}\cdots\eta_{nn}=(-1)^{n(n-1)/2}$, there exist two chiral projection operators $P^{\pm}=(1\pm\tilde{\gamma})/2$ which satisfy $(P^{\pm})^2=P^{\pm}$. Using the projection operator P^+, we can obtain the chiral version of the special Clifford algebra:

$$sc^+(1,1):\ P^+\ , \tag{3.5a}$$

$$sc^+(4,0),sc^+(2,2):\ P^+,P^+\gamma_{ab}\ , \tag{3.5b}$$

$$sc^+(5,1),sc^+(3,3):\ P^+,P^+\gamma_{ab}\ , \tag{3.5c}$$

whose number of elements is half of the special Clifford algebra. Like the special Clifford algebra, $sc^+(k,n-k)$ is isomorphic to $sc^+(n-k,k)$. Similarly $sc^-(k,n-k)$ is obtained by using P^- instead of P^+. $sc^-(k,n-k)$ is always isomorphic to $sc^+(k,n-k)$.

In the case of $n=$(odd integer), the bases of the Clifford algebras are

$$c(k,3-k):\ 1,\gamma_a\ , \tag{3.6a}$$

$$c(k,5-k):\ 1,\gamma_a,\gamma_{ab}\ , \tag{3.6b}$$

$$c(k,7-k):\ 1,\gamma_a,\gamma_{ab},\gamma_{abc}\ , \tag{3.6c}$$

where $\gamma_1\cdots\gamma_n$ has to be 1 or -1, i.e., $(-1)^{n(n-1)/2}=-(-1)^k$ in order to be closed under multiplications. The Clifford algebras defined in (3.6) are isomorphic to those in (3.3) as

$$\begin{aligned}
&c(0,3)\cong c(0,2)\ ,\\
&c(2,1)\cong c(2,0)\cong c(1,1)\ ,\\
&c(5,0)\cong c(4,0)\ ,\\
&c(1,4)\cong c(1,3)\cong c(0,4)\ ,\\
&c(3,2)\cong c(3,1)\cong c(2,2)\ ,\\
&c(6,1)\cong c(6,0)\cong c(5,1)\ ,\\
&c(2,5)\cong c(2,4)\cong c(1,5)\ ,\\
&c(4,3)\cong c(4,2)\cong c(3,3)\ ,\\
&c(0,7)\cong c(0,6)\ .
\end{aligned} \tag{3.7}$$

It is easy to check the isomorphisms (3.7) because there is a correspondence between $(\gamma_a,\tilde{\gamma})$ $(a=1,\ldots,n)$ in (3.3) and γ_a $(a=1,\ldots,n+1)$ in (3.6).

We find other isomorphisms:

$$\begin{aligned}
&sc^+(1,1)\cong gl(1,\mathbb{R})\ ,\\
&c(0,3)\cong sc^+(4,0)\ ,\\
&c(2,1)\cong sc^+(2,2)\cong gl(2,\mathbb{R})\ ,\\
&c(5,0)\cong c(1,4)\cong sc^+(5,1)\ ,\\
&c(3,2)\cong sc^+(3,3)\cong gl(4,\mathbb{R})\ ,\\
&c(6,1)\cong c(2,5)\ ,\\
&c(4,3)\cong c(0,7)\cong gl(8,\mathbb{R})\ ,
\end{aligned} \tag{3.8}$$

where $gl(n,\mathbb{R})$ is the Lie algebra of the general linear Lie group $GL(n,\mathbb{R})$. One finds the above isomorphisms from the following correspondences:

$$\begin{aligned}
&\left|\begin{aligned}&c(0,3):\ \gamma_1,\gamma_2,\gamma_3\ ,\\ &sc^+(4,0):\ P\gamma_{14},P\gamma_{24},P\gamma_{34}\ ,\end{aligned}\right.\\
&\left|\begin{aligned}&c(2,1):\ \gamma_1,\gamma_2,\gamma_3\ ,\\ &sc^+(2,2):\ P\gamma_{14},P\gamma_{24},P\gamma_{34}\ ,\end{aligned}\right.\\
&\left|\begin{aligned}&c(5,0):\ \gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5\ ,\\ &c(1,4):\ \gamma_1,\gamma_{12},\gamma_{13},\gamma_{14},\gamma_{15}\ ,\\ &sc^+(5,1):\ P\gamma_{16},P\gamma_{26},P\gamma_{36},P\gamma_{46},P\gamma_{56}\ ,\end{aligned}\right.\\
&\left|\begin{aligned}&c(3,2):\ \gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5\ ,\\ &sc^+(3,3):\ P\gamma_{16},P\gamma_{26},P\gamma_{36},P\gamma_{46},P\gamma_{56}\ ,\end{aligned}\right.\\
&\left|\begin{aligned}&c(6,1):\ \gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5,\gamma_6,\gamma_7\ ,\\ &c(2,5):\ \gamma_1,\gamma_2,\gamma_{345},\gamma_{456},\gamma_{563},\gamma_{634},\gamma_7\ ,\end{aligned}\right.\\
&\left|\begin{aligned}&c(0,7):\ \gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5,\gamma_6,\gamma_7\ ,\\ &c(4,3):\ \gamma_{123},\gamma_{234},\gamma_{341},\gamma_{412},\gamma_5,\gamma_6,\gamma_7\ .\end{aligned}\right.
\end{aligned} \tag{3.9}$$

Those correspondences assure that the above representations in the same category satisfy the same anticommutation relations and the metric structure as the bases of the algebras.

The Clifford algebras defined in (3.4) are isomorphic to gl(n,C) in the case $\tilde{\gamma}^2=-1$:

$$\mathrm{sc}(2,0)\cong \mathrm{gl}(1,\mathbb{C})\ ,$$
$$\mathrm{sc}(3,1)\cong \mathrm{gl}(2,\mathbb{C})\ , \tag{3.10}$$
$$\mathrm{sc}(6,0)\cong \mathrm{sc}(4,2)\cong \mathrm{gl}(4,\mathbb{C})\ .$$

$\tilde{\gamma}$ behaves as the imaginary unit i because its square is -1 and it is commutative with all elements.

The special Clifford algebra sc($k,n-k$) with $\tilde{\gamma}^2=1$ is the direct sum of $\mathrm{sc}^+(k,n-k)$ and $\mathrm{sc}^-(k,n-k)$, i.e., $\mathrm{sc}(k,n-k)\cong \mathrm{sc}^+(k,n-k)\oplus \mathrm{sc}^-(k,n-k)$. General linear algebras gl(n,**R**) and gl(n,C) are represented by $n\times n$ matrices with real elements $\in\mathbf{R}$ and complex elements $\in\mathbb{C}$, respectively. The isomorphism sc(6,0)≅sc(4,2) can be confirmed by the correspondence

$$\left|\begin{array}{l}\mathrm{sc}(6,0):\ \gamma_{16},\gamma_{26},\gamma_{36},\gamma_{46},\gamma_{56}\ ,\\ \mathrm{sc}(4,2):\ \gamma_{14},\gamma_{24},\gamma_{34},\tilde{\gamma}\gamma_{45},\tilde{\gamma}\gamma_{46}\ .\end{array}\right. \tag{3.11}$$

Finally, we should note that the above-mentioned bases of the Clifford algebra are constructed by the direct product of the bases of c(0,3) and c(2,1). The bases of c(0,3) and c(2,1) algebra are constructed by the following 2×2 matrices:

$$\mathrm{c}(0,3):\ \ \mathbf{1}=\begin{bmatrix}1&0\\0&1\end{bmatrix},\ \mathbf{i}=\begin{bmatrix}i&0\\0&-i\end{bmatrix},\ \mathbf{j}=\begin{bmatrix}0&i\\i&0\end{bmatrix},\ \mathbf{k}=\begin{bmatrix}0&1\\-1&0\end{bmatrix}, \tag{3.12a}$$

$$\mathrm{c}(2,1):\ \ \mathbf{1}'=\begin{bmatrix}1&0\\0&1\end{bmatrix},\ \mathbf{i}'=\begin{bmatrix}1&0\\0&-1\end{bmatrix},\ \mathbf{j}'=\begin{bmatrix}0&1\\1&0\end{bmatrix},\ \mathbf{k}'=\begin{bmatrix}0&1\\-1&0\end{bmatrix}. \tag{3.12b}$$

One can then construct the bases of the Clifford algebras by the direct product of c(0,3) and c(2,1) bases, as c(0,3)⊗c(2,1)⊗ · · · ⊗c(2,1) or c(2,1)⊗ · · · ⊗c(2,1):

$$\mathrm{c}(0,2):\ \gamma_1=\mathbf{i},\gamma_2=\mathbf{j}\ , \tag{3.13a}$$

$$\mathrm{c}(0,4):\ \gamma_1=\mathbf{i}\otimes\mathbf{1}',\gamma_2=\mathbf{j}\otimes\mathbf{1}',\gamma_3=\mathbf{k}\otimes\mathbf{i}',\gamma_4=\mathbf{k}\otimes\mathbf{j}'\ , \tag{3.13b}$$

$$\mathrm{c}(2,4):\ \gamma_1=\mathbf{i}\otimes\mathbf{1}'\otimes\mathbf{1}',\gamma_2=\mathbf{j}\otimes\mathbf{1}'\otimes\mathbf{1}',\gamma_3=\mathbf{k}\otimes\mathbf{i}'\otimes\mathbf{1}',\gamma_4=\mathbf{k}\otimes\mathbf{j}'\otimes\mathbf{1}',\gamma_5=\mathbf{k}\otimes\mathbf{k}'\otimes\mathbf{i}',\gamma_6=\mathbf{k}\otimes\mathbf{k}'\otimes\mathbf{j}'\ , \tag{3.13c}$$

and

$$\mathrm{c}(2,0):\ \gamma_1=\mathbf{i}',\gamma_2=\mathbf{j}'\ , \tag{3.14a}$$

$$\mathrm{c}(2,2):\ \gamma_1=\mathbf{i}'\otimes\mathbf{1}',\gamma_2=\mathbf{j}'\otimes\mathbf{1}',\gamma_3=\mathbf{k}'\otimes\mathbf{i}',\gamma_4=\mathbf{k}'\otimes\mathbf{j}'\ , \tag{3.14b}$$

$$\mathrm{c}(4,2):\ \gamma_1=\mathbf{i}'\otimes\mathbf{1}'\otimes\mathbf{1}',\gamma_2=\mathbf{j}'\otimes\mathbf{1}'\otimes\mathbf{1}',\gamma_3=\mathbf{k}'\otimes\mathbf{i}'\otimes\mathbf{1}',\gamma_4=\mathbf{k}'\otimes\mathbf{j}'\otimes\mathbf{1}',\gamma_5=\mathbf{k}'\otimes\mathbf{k}'\otimes\mathbf{i}',\gamma_6=\mathbf{k}'\otimes\mathbf{k}'\otimes\mathbf{j}'\ . \tag{3.14c}$$

It should be noted that the Clifford algebras constructed in (3.14) are manifestly equivalent to gl(n,**R**).

IV. gl(1,R) AND gl(1,C) MODELS

We consider the simplest example in this section. The generator is composed of only one scalar representation,

$$T_a=\{1\}\ , \tag{4.1}$$

and thus field and parameter components can be expressed simply by

$$\begin{aligned}&\phi=\tfrac{1}{2}\phi_s,\ \ \omega=\tfrac{1}{2}\omega_s,\ \ B=\tfrac{1}{2}B_s\ ,\\ &v=\tfrac{1}{2}v_s,\ \ u=\tfrac{1}{2}u_s,\ \ b=\tfrac{1}{2}b_s\ .\end{aligned} \tag{4.2}$$

It is apparent that the generator (4.1) is closed within commutator and anticommutator by itself. We may say that the scalar representation is commutative but not anticommutative. In other words, the gauge transformation of ω_s includes a $\frac{1}{2}\{\phi_s,u_s\}=\phi_s u_s$ term. Thus the gauge symmetry is not the standard Abelian one but a new type of gauge symmetry. The gauge algebra is isomorphic to the $\mathrm{gl}(1,\mathbf{R})\cong \mathrm{sc}^+(1,1)$ algebra in the present case. In this case the equation of motion (2.13a) leads to

$$\phi_s=0\ , \tag{4.3a}$$

which satisfies (2.13b) automatically. Then Eq. (2.13c) leads to

$$d\omega_s=0\ . \tag{4.3b}$$

Under the condition (4.3a), the gauge transformations (2.12) become

$$\delta\phi_s=0\ , \tag{4.4a}$$

$$\delta\omega_s=dv_s\ , \tag{4.4b}$$

$$\delta B_s=du_s\ . \tag{4.4c}$$

In order to investigate the solution of ω_s and B_s we need to introduce de Rham cohomology. The de Rham cohomology is the set of equivalence classes of closed forms which differ only by exact forms. The p-form de Rham cohomology group $H^{(p)}_{\mathrm{dR}}$ is defined by

$$H_{\rm dR}^{(p)}=Z_{\rm dR}^{(p)}/B_{\rm dR}^{(p)} ,$$
$$Z_{\rm dR}^{(p)}=\{\xi^{(p)};d\xi^{(p)}=0\} , \quad (4.5)$$
$$B_{\rm dR}^{(p)}=\{\xi^{(p)};\xi^{(p)}=d\alpha^{(p-1)}\} ,$$

where $Z_{\rm dR}^{(p)}$ and $B_{\rm dR}^{(p)}$ denote sets of the closed p forms and the exact p forms, respectively. The most important property to define the cohomology is the nilpotency of the derivative operator, i.e., $d^2=0$. Equation (4.3b) means that $\omega_{\rm s}$ belongs to $Z_{\rm dR}^{(1)}$ while Eq. (4.4b) means that $\omega_{\rm s}$ is modulo $B_{\rm dR}^{(1)}$. Therefore $\omega_{\rm s}$ belongs to $H_{\rm dR}^{(1)}$. We also find $B_{\rm s}$ belongs to $H_{\rm dR}^{(2)}$.

If the base manifold is a compact manifold without boundary, de Rham cohomology group is represented by harmonic functions. In order to define harmonic functions we need to introduce the metric. We consider the Euclidean flat metric. The adjoint of the exterior derivative is defined by $d^*=-*d*$, where $*$ is the Hodge star operator. Both d and d^* are nilpotent, i.e., $d^2=d^{*2}=0$. A p-form harmonic function $h_{\rm s}^{(p)}$ is defined so as to satisfy

$$dh_{\rm s}^{(p)}=0,\quad d^*h_{\rm s}^{(p)}=0 . \quad (4.6)$$

Using the Hodge decomposition theorem, we can explicitly show that the solutions of the equations of motion modulo gauge transformations are harmonic functions and thus elements of the cohomology group $H_{\rm dR}^{(p)}$. The gauge fields $\omega_{\rm s}$ and $B_{\rm s}$ are decomposed into

$$\omega_{\rm s}=d\alpha_{\rm s}^{(0)}+d^*\beta_{\rm s}^{(2)}+h_{\rm s}^{(1)} ,$$
$$B_{\rm s}=d\alpha_{\rm s}^{(1)}+h_{\rm s}^{(2)} , \quad (4.7)$$

where $\alpha_{\rm s}^{(0)}$, $\alpha_{\rm s}^{(1)}$, and $\beta_{\rm s}^{(2)}$ denote zero-, one-, two-form functions, respectively, while $h_{\rm s}^{(1)}$ and $h_{\rm s}^{(2)}$ denote one- and two-form harmonic functions defined by (4.6), respectively. Equation (4.3b) leads to $d^*\beta_{\rm s}^{(2)}=0$ while the gauge transformations (4.4b) and (4.4c) make it possible to fix the gauge as $\alpha_{\rm s}^{(0)}=0$ and $\alpha_{\rm s}^{(1)}=0$. Thus we obtain the gauge-fixed solution (4.3a) and harmonic functions:

$$\phi_{\rm s}=0 , \quad (4.8a)$$
$$\omega_{\rm s}=h_{\rm s}^{(1)} , \quad (4.8b)$$
$$B_{\rm s}=h_{\rm s}^{(2)} . \quad (4.8c)$$

In the case that the two-dimensional manifold is a connected Riemann surface with genus g, the dimension of one-form harmonic functions is $2g$ and that of two-form harmonic function is one.

Next we consider the model with gl(1,$\mathbb{C}$) algebra. This algebra is represented by the generators

$$T_a=\{1,i\} . \quad (4.9)$$

The gauge fields and parameters are decomposed as

$$\phi=\tfrac{1}{2}(\phi_s+i\tilde{\phi}_{\rm s}),\quad \omega=\tfrac{1}{2}(\omega_{\rm s}+i\tilde{\omega}_{\rm s}),\quad B=\tfrac{1}{2}(B_{\rm s}+i\tilde{B}_{\rm s}) ,$$
$$v=\tfrac{1}{2}(v_{\rm s}+i\tilde{v}_{\rm s}),\quad u=\tfrac{1}{2}(u_{\rm s}+i\tilde{u}_{\rm s}),\quad b=\tfrac{1}{2}(b_{\rm s}+i\tilde{b}_{\rm s}) . \quad (4.10)$$

Equation (2.13a) leads to

$$\phi_{\rm s}=\tilde{\phi}_{\rm s}=0 , \quad (4.11a)$$

which satisfies (2.13b). Then Eq. (2.13c) leads to

$$d\omega_{\rm s}=d\tilde{\omega}_{\rm s}=0 . \quad (4.11b)$$

The gauge transformations (2.12) are

$$\delta\phi_{\rm s}=\delta\tilde{\phi}_{\rm s}=0 , \quad (4.12a)$$
$$\delta\omega_{\rm s}=dv_{\rm s},\quad \delta\tilde{\omega}_{\rm s}=d\tilde{v}_{\rm s} , \quad (4.12b)$$
$$\delta B_{\rm s}=du_{\rm s},\quad \delta\tilde{B}_{\rm s}=d\tilde{u}_{\rm s} . \quad (4.12c)$$

Therefore the gl(1,$\mathbb{C}$) model is the direct product of two gl(1,$\mathbb{R}$) models at the classical level.

It is important to recognize that the equations of motion over the genus g Riemann surface with the new type of gauge symmetry pick up the important topological information to specify the Riemann surface. The equation of motion of the even-dimensional generalized Chern-Simons action is equivalent to the flat connection condition. It is this new type of gauge symmetry with the flat connection condition that specifies the topological nature of the base manifold even with the simplest symmetry.

V. MODELS OF c(0,3) AND c(2,1) CLIFFORD ALGEBRAS

We next consider implementing two-dimensional gravity in the present framework. In order to investigate two-dimensional gravity from the gauge theory point of view, we need the gauge symmetry which accommodates the zweibein and the spin connection as gauge fields. We show that the gauge symmetry with c(0,3) and c(2,1) algebra can do the job in the two-dimensional generalized Chern-Simons formulation. This is in contrast with the three-dimensional gravity of the standard Chern-Simons action where the ISO(3) and ISO(2,1) Poincaré gauge symmetry or SO(4,1) and SO(3,2) gauge symmetry did the job [1,2]. Just to accommodate the zweibein and the spin connection as gauge fields, we simply need SO(3), SO(2,1), or ISO(1,1) gauge symmetry for two-dimensional gravity. We, however, need the gauge symmetry with the c(0,3) or c(2,1) representation to close the gauge algebra of the two-dimensional generalized Chern-Simons action. In other words, we need one singlet representation in addition to an adjoint representation of SO(3), SO(2,1) to close the algebra (3.1). We will discuss ISO(1,1) Poincaré gauge symmetry in the next section.

As shown in Sec. III, the c(0,3) or c(2,1) algebra includes the generators

$$T_a=\{1,\gamma_a|a=1,2,3\} . \quad (5.1)$$

γ matrices satisfy the properties

$$\{\gamma_a,\gamma_b\}=2\eta_{ab} ,$$
$$[\gamma_a,\gamma_b]=2\epsilon_{abc}\gamma^c , \quad (5.2)$$

where $\epsilon_{123}=1$. The square of $\gamma_1\gamma_2\gamma_3=1$ leads to $\eta_{11}\eta_{22}\eta_{33}=-1$. Therefore the above system is classified into the following two types:

$$\eta_{11}=\eta_{22}=\eta_{33}=-1 \quad \text{for c(0,3) algebra} ,$$
$$\eta_{11}=\eta_{22}=1,\quad \eta_{33}=-1,\quad \text{for c(2,1) algebra} . \quad (5.3)$$

In the case of c(2,1) algebra we have assigned the indefinite metric to the third direction without a loss of generality. c(0,3) and c(2,1) algebras include SO(3) and SO(2,1) symmetry, respectively. We decompose the gauge fields as follows:

$$\phi=\tfrac{1}{2}(\phi_s+\gamma_a\phi^a)\ ,\quad \omega=\tfrac{1}{2}(\omega_s+\gamma_a\omega^a)\ ,\quad B=\tfrac{1}{2}(B_s+\gamma_a B^a)\ . \tag{5.4}$$

Using the generators (5.1), we can rewrite the equation of motion (2.13a) as

$$\phi_s^2+\phi^a\phi_a=0\ ,\quad \phi_s\phi^a=0\ , \tag{5.5}$$

which lead to

$$\phi_s=0\ , \tag{5.6a}$$

$$\phi^a\phi_a=0\ . \tag{5.6b}$$

Other equations (2.13b) and (2.13c) lead to

$$d\phi^a+\epsilon^{abc}\omega_b\phi_c=0\ , \tag{5.6c}$$

$$d\omega_s-\phi^a B_a=0\ , \tag{5.6d}$$

$$d\omega^a+\tfrac{1}{2}\epsilon^{abc}\omega_b\omega_c-\phi^a B_s=0\ . \tag{5.6e}$$

Using the solution of the equation of motion (5.6a), we obtain the explicit forms of the gauge transformation:

$$\delta\phi_s=0\ , \tag{5.7a}$$

$$\delta\phi^a=\epsilon^{abc}\phi_b v_c\ , \tag{5.7b}$$

$$\delta\omega_s=dv_s+\phi^a u_a\ , \tag{5.7c}$$

$$\delta\omega^a=dv^a+\epsilon^{abc}\omega_b v_c+\phi^a u_s\ , \tag{5.7d}$$

$$\delta B_s=du_s\ , \tag{5.7e}$$

$$\delta B^a=du^a+\epsilon^{abc}(\omega_b u_c+B_b v_c+\phi_b b_c)\ . \tag{5.7f}$$

A. c(0,3) model

First we consider a model with c(0,3) algebra. In this case Eq. (5.6b) can be solved:

$$\phi^a=0\ . \tag{5.8a}$$

The remaining equations are

$$d\omega^a+\tfrac{1}{2}\epsilon^{abc}\omega_b\omega_c=0 \tag{5.8b}$$

and

$$d\omega_s=0\ . \tag{5.9}$$

The gauge transformations are

$$\delta\phi^a=0\ , \tag{5.10a}$$

$$\delta\omega^a=dv^a+\epsilon^{abc}\omega_b v_c\ , \tag{5.10b}$$

$$\delta B^a=du^a+\epsilon^{abc}(\omega_b u_c+B_b v_c)\ , \tag{5.10c}$$

and

$$\delta\phi_s=0\ , \tag{5.11a}$$

$$\delta\omega_s=dv_s\ , \tag{5.11b}$$

$$\delta B_s=du_s\ . \tag{5.11c}$$

The singlet components ϕ_s, ω_s, and B_s turn out to satisfy the same equations of motion and gauge transformations and thus pick up the same important topological information of the space-time as the gl$(1,\mathbb{R})$ model in Sec. IV. Therefore, in the rest of this section we concentrate on Eqs. (5.8) with the transformations (5.10). Equation (5.8b) is the flat connection condition of SO(3) gauge symmetry or equivalently the curvature of the gauge symmetry vanishes. The corresponding transformation (5.10b) is the SO(3) gauge transformation.

In this model we can introduce the non-Abelian version of de Rham cohomology as

$$H^{(p)}_{\rm SO}=Z^{(p)}_{\rm SO}/B^{(p)}_{\rm SO}\ ,\quad Z^{(p)}_{\rm SO}=\{\xi^{(p)};D\xi^{(p)}=0\}\ ,\quad B^{(p)}_{\rm SO}=\{\xi^{(p)};\xi^{(p)}=D\alpha^{(p-1)}\}\ , \tag{5.12}$$

where $D=d+\omega$ is the SO(3)-covariant derivative operator and satisfies $D^2=0$, i.e., the vanishing curvature of SO(3) indicated by (5.8b). The B^a modulo the gauge transformation (5.10c) is considered to belong to the two-form cohomology class $H^{(2)}_{\rm SO}$, which is based on similar arguments as B_s in Sec. IV. If the base manifold is a compact manifold without boundary, the cohomology class (5.12) is isomorphic to the set of harmonic functions. Thus B^a can be expressed as $B^a=h^{(2)a}$ after gauge fixing, where $h^{(2)a}$ is a harmonic function which satisfies $D^*h^{(2)}=0$ with a definition $D^*=-*D*$. It now remains to be interpreted for the one-form gauge field ω^a.

To investigate two-dimensional gravity from the gauge theory point of view, we need to assign the zweibein $e^{\bar a}$ $(\bar a=1,2)$ and the spin connection $\omega^{\bar a\bar b}$ to certain gauge fields. For SO(3) gauge symmetry, we take the assignment

$$e^1=\omega^1,\quad e^2=\omega^2\ ,\quad \omega^{12}=-\omega^3\ , \tag{5.13}$$

with the Euclidean metric $\eta_{\bar a\bar b}={\rm diag}(1,1)$. As in the present case we, hereafter, use the different sign assignment of the two-dimensional metric from the original assignment of (5.3) by using the overall sign freedom for the metric. Then Eq. (5.8b) can be decomposed into two equations:

$$de^{\bar a}+\omega^{\bar a}{}_{\bar b}e^{\bar b}=0\ , \tag{5.14a}$$

$$d\omega^{\bar a\bar b}-e^{\bar a}e^{\bar b}=0\ . \tag{5.14b}$$

In order to interpret $e^{\bar a}$ as the zweibein, we need to assume that $e^{\bar a}$ is invertible at any point on the manifold except for some points. The invertibility of $e^{\bar a}$ can be defined as $\det e_\mu{}^{\bar a}\neq 0$ or equivalently $e^1e^2\neq 0$. We recognize that the metric is not defined at the noninvertible points. Equation (5.14a) is the standard torsionless condition. By using the scalar curvature $R=e^{\mu}{}_{\bar a}e^{\nu}{}_{\bar b}R_{\mu\nu}{}^{\bar a\bar b}$ where $R^{\bar a\bar b}=d\omega^{\bar a\bar b}$ and $e^{\mu}{}_{\bar a}e_{\mu}{}^{\bar b}=\delta_{\bar a}{}^{\bar b}$, we can rewrite Eq. (5.14b) as

$$R=+1\ . \tag{5.15}$$

Considering the fact that $(1/2\pi)\int R=2(1-g)=$Euler number, we can identify the manifold in consideration as a sphere ($g=0$). This is a very natural conclusion because the manifold with SO(3) isotropy is a sphere. Note that this fact means one can make the zweibein $e^{\bar a}$ invertible at any point on a sphere; in other words, there exist some points where $e^{\bar a}$ is not invertible unless the base manifold is a sphere. If $e^{\bar a}$ is not invertible, Eqs. (5.14) lead to

$$\omega^a=k^a\omega_0\quad (k^a=\text{any constant parameters})\ , \tag{5.16}$$

where ω_0 is a nonzero one-form and satisfies $d\omega_0=0$. The gauge transformation (5.10b) breaks the relation (5.16) in general. We can thus always make $e^{\bar a}$ invertible locally. However, we cannot remove the points where $e^{\bar a}$ is not invertible from the manifold. The gauge transformation only transfers such points.

B. c(2,1) model

Next we consider a model with c(2,1) algebra. In this case Eq. (5.6b) has a nonzero solution. We first consider the solution with $\phi^a=0$, and study the case of $\phi^a\neq 0$ later. When $\phi^a=0$, we obtain Eqs. (4.3) and (5.8) with the gauge transformations (4.4) and (5.10). We find the same conclusion as that of the c(0,3) model except that there is SO(2,1) gauge symmetry instead of the SO(3) one. For the c(0,3) model we have found Euclidean gravity with the condition (5.15) if the manifold is a sphere; however, for the c(2,1) model we will encounter not only Euclidean gravity but also Minkowskian gravity because SO(2,1) includes both SO(2) and SO(1,1) symmetries. If we assign

$$e^1=\omega^1,\quad e^2=\omega^2\ ,\qquad \omega^{12}=-\omega^3\ , \tag{5.17}$$

and define the Euclidean metric $\eta_{\bar a\bar b}=\mathrm{diag}(1,1)$, we obtain a torsionless condition (5.14a) and

$$R=-1\ . \tag{5.18}$$

Equation (5.18) means one can always make $e^{\bar a}$ invertible at any point on the Riemann surface with $g\geq 2$. It is a well-known fact that $R=-1$ can be satisfied globally at any point on the Riemann surface of $g\geq 2$.

In order to find Minkowskian gravity, we assign

$$e^1=\omega^2,\quad e^2=\omega^3\ ,\qquad \omega^{12}=\omega^1\ , \tag{5.19}$$

and define the Minkowskian metric $\eta_{\bar a\bar b}=\mathrm{diag}(1,-1)$. Then, we find a torsionless condition and

$$R=-1\ . \tag{5.20}$$

If we, however, change the assignment of the zweibein as

$$e^1=\omega^3,\quad e^2=\omega^2\ ,\qquad \omega^{12}=\omega^1\ , \tag{5.21}$$

we find a torsionless condition and

$$R=+1\ . \tag{5.22}$$

This puzzling difference originates from the fact that (5.20) and (5.22) are derived from SO(2,1) and SO(1,2) flat connection conditions, respectively, while SO(2,1) and SO(1,2) are isomorphic. In other words, there is no difference between de Sitter space and anti–de Sitter space in two-dimensional Minkowskian gravity. This means the sign of the cosmological constant is irrelevant in two-dimensional Minkowskian gravity. It should be noted that the different sign assignments of the zweibein do not alter the invertibility. On the other hand it is known that the Minkowskian metric cannot be defined globally on the Riemann surface except for the $g=1$ torus. In the case of $g=1$ torus, nonvanishing constant curvature cannot be defined globally in the Minkowskian metric.

In the rest of this section we consider the solution with $\phi^a\neq 0$ for the c(2,1) model. From Eq. (5.6b) we find $(\phi^3)^2=(\phi^1)^2+(\phi^2)^2\neq 0$ in this case. Using the gauge parameter v^3, we can always let $\phi^2=0$ by the gauge transformation (5.7b). We then obtain

$$\phi^1=\pm\phi^3\neq 0\ . \tag{5.23}$$

Since the residual gauge transformation is the local scaling

$$\delta\phi^3=\mp\phi^3 v^2\ , \tag{5.24}$$

we can always make $\phi^1=\pm\phi^3$ constant, i.e.,

$$d\phi^1=\pm d\phi^3=0\ . \tag{5.25}$$

Taking into account the constraint (5.23), we rewrite the gauge transformations (5.7) which include ϕ^3 and ϕ^1:

$$\delta\omega_s=\pm\phi^3(u^1\mp u^3)+\cdots\ , \tag{5.26a}$$

$$\delta(\omega^1\pm\omega^3)=\pm 2\phi^3 u_s+\cdots\ , \tag{5.26b}$$

$$\delta B^2=\phi^3(b^1\mp b^3)+\cdots\ , \tag{5.26c}$$

$$\delta(B^1\pm B^3)=-2\phi^3 b^2+\cdots\ . \tag{5.26d}$$

From these transformations we can fix the gauge as

$$\omega_s=0,\quad \omega^1\pm\omega^3=0,\quad B^2=0,\quad B^1\pm B^3=0\ , \tag{5.27}$$

using the parameters $u^1\mp u^3$, u_s, $b^1\mp b^3$, b^2, respectively. Under the conditions (5.27), Eqs. (5.6) lead to

$$\omega^2=0,\quad \omega^1\mp\omega^3=0,\quad B_s=0,\quad B^1\mp B^3=0\ . \tag{5.28}$$

Therefore we find that all the components of ω and B vanish.

To summarize, we obtain the following type of the nontrivial classical solution:

$$\phi^1=\pm\phi^3=\text{nonzero constant}\ ,\qquad \text{other fields}=0\ . \tag{5.29}$$

In this solution all the zweibein and the spin connection vanish.

VI. MODELS OF sc(4,0), sc(3,1), AND sc(2,2) SPECIAL CLIFFORD ALGEBRAS

In the analyses of the preceding section, two-dimensional Poincaré symmetry ISO(1,1) has not been realized because the symmetry algebra was too small to accommodate it. We need to extend the symmetry algebra to include ISO(1,1). Since the conformal symmetry is a minimal extension to close the algebra, we consider the two-dimensional conformal gravity. sc(3,1) and sc(2,2) special Clifford algebras include the Lie algebras of SO(3,1) and SO(2,2) groups, respectively, to realize Euclidean and Minkowskian conformal gravity. This is in contrast with the three-dimensional case where ISO(2,1) Poincaré symmetry is extended to SO(3,2) gauge symmetry to investigate the three-dimensional conformal gravity [1,2]. We also consider sc(4,0) special Clifford algebra which includes the Lie algebra of SO(4) symmetry.

The generators of sc(4,0), sc(3,1), and sc(2,2) algebras are represented by the following γ matrices:

$$T_a=\{1,\tilde{\gamma},\gamma_{ab}|a,b=1,2,3,4\}\ , \tag{6.1}$$

where γ_{ab} and $\tilde{\gamma}$ are defined by

$$\begin{aligned}&\gamma_{ab}=\tfrac{1}{2}[\gamma_a,\gamma_b]\ ,\\&\tilde{\gamma}=\gamma_1\gamma_2\gamma_3\gamma_4\ ,\end{aligned} \tag{6.2a}$$

while γ matrices satisfy the standard relation

$$\{\gamma_a,\gamma_b\}=2\eta_{ab}\ . \tag{6.2b}$$

The generators in (6.1) satisfy the following commutation and anticommutation relations:

$$\begin{aligned}&[\gamma_{ab},\gamma_{cd}]=-2(\eta_{ac}\gamma_{bd}+\eta_{bd}\gamma_{ac}-\eta_{bc}\gamma_{ad}-\eta_{ad}\gamma_{bc})\ ,\\&[\gamma_{ab},\tilde{\gamma}]=[\gamma_{ab},1]=[\tilde{\gamma},\tilde{\gamma}]=[\tilde{\gamma},1]=[1,1]=0\ ,\\&\{\gamma_{ab},\gamma_{cd}\}=-2(\eta_{ac}\eta_{bd}-\eta_{bc}\eta_{ad})+2\epsilon_{abcd}\tilde{\gamma}\ ,\\&\{\gamma_{ab},\tilde{\gamma}\}=-\tilde{\eta}\epsilon_{abcd}\gamma^{cd},\quad \{\gamma_{ab},1\}=2\gamma_{ab}\ ,\\&\{\tilde{\gamma},\tilde{\gamma}\}=2\tilde{\eta},\quad \{\tilde{\gamma},1\}=2\tilde{\gamma},\quad \{1,1\}=2\ ,\end{aligned} \tag{6.3}$$

where $\epsilon_{1234}=1$ and $\tilde{\eta}=\eta_{11}\eta_{22}\eta_{33}\eta_{44}$. The above system is classified as

$$\begin{aligned}&\eta_{11}=\eta_{22}=\eta_{33}=\eta_{44}=1 \quad \text{for sc}(4,0)\text{ algebra}\ ,\\&\eta_{11}=\eta_{22}=\eta_{33}=1,\quad \eta_{44}=-1 \quad \text{for sc}(3,1)\text{ algebra}\ ,\\&\eta_{11}=\eta_{22}=1,\quad \eta_{33}=\eta_{44}=-1 \quad \text{for sc}(2,2)\text{ algebra}\ .\end{aligned} \tag{6.4}$$

With respect to the assignment of the indefinite metric, it is sufficient to consider the above three cases without a loss of generality. sc(4,0), sc(3,1), and sc(2,2) algebras include the Lie algebras of SO(4), SO(3,1), and SO(2,2) gauge symmetry, respectively. The gauge fields are decomposed as

$$\begin{aligned}&\phi=\tfrac{1}{2}(\phi_s+\tilde{\gamma}\tilde{\phi}_s+\tfrac{1}{2}\gamma_{ab}\phi^{ab})\ ,\\&\omega=\tfrac{1}{2}(\omega_s+\tilde{\gamma}\tilde{\omega}_s+\tfrac{1}{2}\gamma_{ab}\omega^{ab})\ ,\\&B=\tfrac{1}{2}(B_s+\tilde{\gamma}\tilde{B}_s+\tfrac{1}{2}\gamma_{ab}B^{ab})\ .\end{aligned} \tag{6.5}$$

Under the decomposition (6.5), Eq. (2.13a) leads to

$$\begin{aligned}&\phi_s^2+\tilde{\eta}\tilde{\phi}_s^2-\tfrac{1}{2}\phi^{ab}\phi_{ab}=0\ ,\\&\phi_s\tilde{\phi}_s+\tfrac{1}{8}\epsilon_{abcd}\phi^{ab}\phi^{cd}=0\ ,\\&\phi_s\phi^{ab}-\tfrac{1}{2}\tilde{\eta}\epsilon^{abcd}\tilde{\phi}_s\phi_{cd}=0\ ,\end{aligned} \tag{6.6}$$

which are found to be equivalent to

$$\phi_s=\tilde{\phi}_s=0\ , \tag{6.7a}$$

$$\phi^{ab}\phi_{ab}=0\ , \tag{6.7b}$$

$$\epsilon_{abcd}\phi^{ab}\phi^{cd}=0\ . \tag{6.7c}$$

Other Eqs. (2.13b) and (2.13c) in the present case are

$$d\phi^{ab}+\omega^a{}_c\phi^{cb}-\omega^b{}_c\phi^{ca}=0\ , \tag{6.7d}$$

$$d\omega_s+\tfrac{1}{2}\phi^{ab}B_{ab}=0\ , \tag{6.7e}$$

$$d\tilde{\omega}_s-\tfrac{1}{4}\epsilon_{abcd}\phi^{ab}B^{cd}=0\ , \tag{6.7f}$$

$$d\omega^{ab}+\omega^a{}_c\omega^{cb}-\phi^{ab}B_s+\tfrac{1}{2}\tilde{\eta}\epsilon^{abcd}\phi_{cd}\tilde{B}_s=0\ . \tag{6.7g}$$

The explicit forms of the gauge transformations (2.12) are

$$\delta\phi_s=\delta\tilde{\phi}_s=0\ , \tag{6.8a}$$

$$\delta\phi^{ab}=\phi^{ac}v_c{}^b-\phi^{bc}v_c{}^a\ , \tag{6.8b}$$

$$\delta\omega_s=dv_s-\tfrac{1}{2}\phi^{ab}u_{ab}\ , \tag{6.8c}$$

$$\delta\tilde{\omega}_s=d\tilde{v}_s+\tfrac{1}{4}\epsilon_{abcd}\phi^{ab}u^{cd}\ , \tag{6.8d}$$

$$\begin{aligned}\delta\omega^{ab}=&dv^{ab}+\omega^a{}_cv^{cb}-\omega^b{}_cv^{ca}\\&+\phi^{ab}u_s-\tfrac{1}{2}\tilde{\eta}\epsilon^{abcd}\phi_{cd}\tilde{u}_s\ ,\end{aligned} \tag{6.8e}$$

$$\delta B_s=du_s\ , \tag{6.8f}$$

$$\delta\tilde{B}_s=d\tilde{u}_s\ , \tag{6.8g}$$

$$\begin{aligned}\delta B^{ab}=&du^{ab}+\omega^a{}_cu^{cb}-\omega^b{}_cu^{ca}\\&+B^a{}_cv^{cb}-B^b{}_cv^{ca}+\phi^a{}_cb^{cb}-\phi^b{}_cb^{ca}\ .\end{aligned} \tag{6.8h}$$

As is pointed out in Sec. III, there exist chiral projection operators for sc(4,0) and sc(2,2) algebras. The chiral projection operators P_+ and P_- are defined by

$$P_\pm=\frac{1\pm\tilde{\gamma}}{2}\ , \tag{6.9}$$

which satisfy $P_\pm^2=P_\pm$, where $\tilde{\gamma}^2=\tilde{\eta}=1$ for sc(4,0) and sc(2,2) algebras. Using $P_\pm$ we can rewrite the generators as

$$T_{a\pm}=\{P_\pm,P_\pm\gamma_{a4}|a=1,2,3\}\ . \tag{6.10}$$

One should notice that T_{a+} and T_{a-} are decoupled under multiplications. T_{a+} and T_{a-} as generators satisfy the same algebra as (5.1). This comes from the fact that sc(4,0)$\cong$sc$^+$(4,0)$\oplus$sc$^-$(4,0)$\cong$c(0,3)$\oplus$c(0,3) and sc(2,2) $\cong$sc$^+$(2,2)$\oplus$sc$^-$(2,2)$\cong$c(2,1)$\oplus$c(2,1), where sc$^+$(4,0) $\cong$sc$^-$(4,0)$\cong$c(0,3) and sc$^+$(2,2)$\cong$sc$^-$(2,2)$\cong$c(2,1), as is discussed in Sec. III. Therefore sc(4,0) and sc(2,2) algebras in this section are equivalent to the product of two c(0,3) algebras and the product of two c(2,1) algebras, respectively.

We first investigate a model with the sc(4,0) algebra. As discussed above, the sc(4,0) algebra is equivalent to the product of two c(0,3) algebras of Sec. III. Thus we find all the components of ϕ vanish from (5.8) and the analysis of ω and B is the same as the c(0,3) model. Since the sc(4,0) model has an SO(3) one-form gauge field, Euclidean gravity with $R=+1$ exists when the base manifold is a sphere.

Next we study the sc(2,2) model. The sc(2,2) algebra is equivalent to the product of two c(2,1) algebras of Sec. III. Thus the analysis is the same as in the c(2,1) model. If all components of ϕ vanish, we find a two-dimensional Minkowskian conformal gravity which has SO(2,2) gauge symmetry. If the components of ϕ, which correspond to either the generator T_{a+} or T_{a-}, do not vanish and other components of ϕ vanish, we find a nonvanishing SO(2,1) one-form gauge field in one of the c(2,1) sectors. Therefore we obtain two-dimensional Euclidean gravity with $R=-1$ when the base manifold has $g\geq 2$. The arguments on the Minkowskian gravity go parallel to the c(2,1) model. If the components of ϕ which correspond to the generators both T_{a+} and T_{a-} do not vanish, we find all the components of ω and B become zero.

We next investigate the sc(3,1) model. When $\phi^{ab}=0$, we get two-dimensional Euclidean conformal gravity which has SO(3,1) gauge symmetry. ω_s and $\tilde{\omega}_s$ belong to one-form de Rham cohomology class, B_s and $\tilde{B}_s$ belong to two-form de Rham cohomology class, and B^{ab} belongs to the SO(3,1) version of the de Rham cohomology class.

For the solution with $\phi^{ab}\neq 0$ we have to solve the equations carefully. The way of gauge fixing is as follows. First, by using the parameters v^{23} and v^{13} we fix the gauge as $\phi^{13}=\phi^{23}=0$. Next, by the residual gauge parameter v^{12} we fix as $\phi^{24}=0$. From Eq. (6.7b) we find $\phi^{12}\neq 0$ because some components of ϕ^{ab} have to be nonzero. Thus, we find $\phi^{34}=0$ from Eq. (6.7c). Finally, we obtain the nonzero components ϕ^{12} and ϕ^{41} which satisfy $(\phi^{12})^2=(\phi^{41})^2$, i.e.,

$$\phi^{12}=\pm\phi^{41}\neq 0\ . \tag{6.11}$$

Since the residual gauge transformation becomes the local scaling

$$\delta\phi^{12}=\mp\phi^{12}v^{24}\ , \tag{6.12}$$

we can make $\phi^{12}=\pm\phi^{41}$ constant, i.e.,

$$d\phi^{12}=\pm d\phi^{41}=0\ . \tag{6.13}$$

Taking into account the constraint (6.11), we obtain the explicit forms of the gauge transformations (6.8) as

$$\delta\omega_s=-\phi^{12}(u^{12}\mp u^{41})+\cdots\ , \tag{6.14a}$$

$$\delta\tilde{\omega}_s=\phi^{12}(u^{34}\mp u^{23})+\cdots\ , \tag{6.14b}$$

$$\delta(\omega^{12}\pm\omega^{41})=2\phi^{12}u_s+\cdots\ , \tag{6.14c}$$

$$\delta(\omega^{34}\pm\omega^{23})=-2\phi^{12}\tilde{u}_s+\cdots\ , \tag{6.14d}$$

$$\delta B^{24}=\mp\phi^{12}(b^{12}\mp b^{41})+\cdots\ , \tag{6.14e}$$

$$\delta B^{13}=\mp\phi^{12}(b^{34}\mp b^{23})+\cdots\ , \tag{6.14f}$$

$$\delta(B^{12}\pm B^{41})=\mp 2\phi^{12}b^{24}+\cdots\ , \tag{6.14g}$$

$$\delta(B^{34}\pm B^{23})=\mp 2\phi^{12}b^{13}+\cdots\ . \tag{6.14h}$$

From the above transformations, we can fix the gauge as

$$\begin{aligned}&\omega_s=\tilde{\omega}_s=0,\quad \omega^{12}\pm\omega^{41}=\omega^{34}\pm\omega^{23}=0\ ,\\ &B^{24}=B^{13}=0,\quad B^{12}\pm B^{41}=B^{34}\pm B^{23}=0\ ,\end{aligned} \tag{6.15}$$

by using the gauge parameters $u^{12}\mp u^{41}$, $u^{34}\mp u^{23}$, u_s, $\tilde{u}_s$, $b^{12}\mp b^{41}$, $b^{34}\mp b^{23}$, b^{24}, and b^{13}. With the gauge fixing conditions (6.15), Eqs. (6.7) lead to

$$\begin{aligned}&\omega^{24}=\omega^{13}=0,\quad \omega^{12}\mp\omega^{41}=\omega^{34}\mp\omega^{23}=0\ ,\\ &B_s=\tilde{B}_s=0,\quad B^{12}\mp B^{41}=B^{34}\mp B^{23}=0\ .\end{aligned} \tag{6.16}$$

We thus find that all the components of ω and B vanish.

To summarize the present analysis, we have obtained the following type of nontrivial classical solution:

$$\begin{aligned}&\phi^{12}=\pm\phi^{41}=\text{nonzero constant}\ ,\\ &\text{other fields}=0\ .\end{aligned} \tag{6.17}$$

In this solution all the zweibein and the spin connection vanish like the case of the c(2,1) model.

Finally we comment on how we obtain the gravity interpretations from those analyses. In particular two-dimensional Poincaré gravity emerges in a natural way. One should note that SO(4), SO(3,1), and SO(2,2) include the symmetries

$$\begin{aligned}&SO(4)\supset SO(3)\ ,\\ &SO(3,1)\supset ISO(2,0), SO(3), SO(2,1)\ ,\\ &SO(2,2)\supset ISO(1,1), SO(2,1)\ .\end{aligned} \tag{6.18}$$

Here we consider the solution with $\phi^{ab}=0$. When $\phi^{ab}=0$, Eqs. (6.7) lead to

$$d\omega_s=d\tilde{\omega}_s=0\ , \tag{6.19a}$$

$$d\omega^{ab}+\omega^a{}_c\omega^{cb}=0\ . \tag{6.19b}$$

If we reduce the full gauge symmetry by letting $\omega^{14}=\omega^{24}=\omega^{34}=0$ or $\omega^{12}=\omega^{13}=\omega^{14}=0$, we obtain the flat connection condition of SO(3) or SO(2,1) gauge symmetry from (6.19b). The interpretation of such solutions as gravity is similar as in the preceding section. The Poincaré gravity with ISO(2,0) or ISO(1,1) symmetry is obtained by letting $\omega^{12}=\omega^{24}$, $\omega^{13}=\omega^{34}$, and $\omega^{14}=0$. ISO(2,0) and ISO(1,1) Poincaré symmetries lead to Euclidean gravity and Minkowskian gravity, respectively. In the present examples we find the torsionless condition and a vanishing scalar curvature ($R=0$). In the case of Euclidean gravity with $R=0$, one can make the zweibein invertible at any point if and only if the base manifold is a torus ($g=1$). For other base manifolds, we expect to have the regions where the metric does not exist. The zweibein of Minkowskian gravity with $R=0$ can also become invertible at any point on a torus.

VII. HIDDEN ORDER PARAMETER AND TWO-DIMENSIONAL TOPOLOGICAL GRAVITY

In identifying the two-dimensional gravity, we have gauged away the gauge fields as many as possible to be consistent with the equations of motion. We have then tried to find the simplest possible equations of motion. It turns out that most of the components of the zero-form gauge field could be either gauged away or set to zero by equations of motion except two of the components which could possess a nonvanishing constant value. In other words, there is a constant "hidden parameter" in the zero-form sector, which could not be gauged away and could be interpreted as a solution of equations of motion. We have found the equations of motion of the two-dimensional gravity, the torsion free condition, and the constant curvature condition, in case the "hidden parameter" vanishes, i.e., all the zero-form components vanish. On the other hand the zweibein and the spin connection could be gauged away in case the "hidden parameter" does not vanish. We could then call this "hidden parameter" the "hidden order parameter" to discriminate the gravity phase and the nongravity phase. It should be noted that these equations of motion could be identified as a particular version of the equations of motion of the two-dimensional topological gravity which has been intensively investigated [5,6,10].

In considering quantum gravity, we need to identify the classical phase space which can be identified as the space of all solutions of the classical equations, modulo gauge transformations [13]. Although we are not concerned with quantum gravity in this paper, it is important to recognize that the classical solutions of the equations of motion of the two-dimensional generalized Chern-Simons action with a particular choice of the Clifford algebra define the classical phase space of two-dimensional gravity. The remaining equations of motion together with the classical solutions are representatives of those that belong to the same equivalence class modulo gauge transformations.

Among the classical solutions and a part of equations of motion given in the previous sections, we summarize the results of three models where we omit describing the singlet and two-form components:

(i) c(0,3) model with $\eta_{\bar{a}\bar{b}}=\mathrm{diag}(1,1)$

$$\begin{aligned}&\phi^{a}=0\ ,\\ &T^{\bar{a}}=de^{\bar{a}}+\omega^{\bar{a}}{}_{\bar{b}}e^{\bar{b}}=0,\quad R=+1\ .\end{aligned} \tag{7.1}$$

(ii) c(2,1) model

(A) $\eta_{\bar{a}\bar{b}}=\mathrm{diag}(1,1)$

$$\phi^{a}=0,\quad T^{\bar{a}}=0,\quad R=-1\ . \tag{7.2a}$$

(B) $\eta_{\bar{a}\bar{b}}=\mathrm{diag}(1,-1)$

$$\phi^{a}=0,\quad T^{\bar{a}}=0,\quad R=\pm 1\ . \tag{7.2b}$$

(C) $\phi^{1}=\pm\phi^{3}=\langle\phi\rangle=$ nonzero constant ,

$$\text{other fields}=0\ . \tag{7.2c}$$

(iii) sc(3,1) model

(A) $\phi^{ab}=0$,

two-dimensional Euclidean conformal gravity with SO(3,1) [$\supset$ISO(2,0),SO(3),SO(2,1)] gauge symmetry with symmetry reductions ,

$$T^{\bar{a}}=0,\quad R=\begin{cases}0 & \text{for ISO}(2,0),\\ +1 & \text{for SO}(3),\\ -1 & \text{for SO}(2,1)\end{cases} ,\tag{7.3a}$$

with $\eta_{\bar{a}\bar{b}}=\mathrm{diag}(1,1)$.

(B) $\phi^{12}=\pm\phi^{41}=\langle\phi\rangle=$ nonzero constant ,

$$\text{other fields}=0\ . \tag{7.3b}$$

As pointed out in the preceding section, we should get rid of the noninvertible zweibein, which are of "measure zero" but change the topology of field space (and of the space of gauge transformations) and permit the occurrence of global anomalies [1].

Concerning the constraints of the scalar curvature in the above examples, the base manifold could be chosen without metric singularities and global anomalies to be compatible with the invertibility as follows: (i) sphere, (ii A) Riemann surface with $g\geq 2$, (ii B) nonexisting, (iii A) torus ($g=1$) for ISO(2,0), sphere for SO(3), and Riemann surface with $g\geq 2$ for SO(2,1). In other words, if the base manifold is chosen otherwise as mentioned above we expect to have metric singularities and/or global anomalies.

As we can see in the above examples, the classical phase space corresponding to the two-dimensional gravity is obtained only when the zero-form gauge fields vanish. In the case of nonvanishing zero-form gauge fields, the classical phase space has nothing to do with the gravity. In other words, the zweibein and the spin connection would be gauged away. In cases (ii) and (iii), there are two types of solutions which are characterized by the "hidden parameter" $\langle\phi\rangle$. $\langle\phi\rangle\neq 0$ specifies the nongravity phase space while $\langle\phi\rangle=0$ specifies the phase space of the gravity. In this sense the "hidden order parameter" specifies the classical solutions of the two-dimensional gravity. Even if $\langle\phi\rangle=0$, nonsingular classical solutions of the gravity may not exist unless the base manifold is correctly chosen.

When we investigate the gravity by the gauge theory point of view, there is the following folklore stressed by Witten [7]: The perturbative classical vacuum corresponding to the vanishing vielbein and spin connection may play an important role in considering quantum gravity and is related to the unbroken symmetry. On the other hand in the realistic gravity, nonvanishing vielbein leading to the nonvanishing metric corresponds to the broken symmetry. We may expect that there exists some order parameter to differentiate the two different phases just like the vacuum expectation value of the scalar field in the standard gauge theory.

It is interesting to recognize that the hidden parameter $\langle \phi \rangle$ plays a similar role as the vacuum expectation value of the scalar field in the Nambu-Goldstone mechanism. One important difference from the order parameter of unbroken and broken symmetry is that the hidden parameter $\langle \phi \rangle$ does not specify the breakdown of the gauge symmetry but the different equivalence classes of the classical solutions (or classical phase space) of the gravity modulo gauge transformations. In our formulation of gravity, the gauge symmetry in consideration is *unbroken* in the case of $\mathcal{A}=\omega\mathbf{j}+\epsilon_2(\phi+\epsilon_1 B)\mathbf{k}=0$, and *broken* in the case of $\mathcal{A}\neq 0$, respectively. The gauge symmetry is, of course, broken in the case $\langle \phi \rangle \neq 0$.

VIII. CONCLUSION AND DISCUSSIONS

We have given the derivation of pure bosonic version of the even-dimensional generalized Chern-Simons action, which has the unusual gauge symmetry including the anticommutator. We have shown that the gauge symmetry can be realized by Clifford-algebra-valued gauge fields and parameters. It has been shown that the two-dimensional generalized Chern-Simons action with a particular choice of the Clifford algebra leads to two-dimensional topological gravity, which should be contrasted to the three-dimensional case where the three-dimensional standard Chern-Simons action with an ISO(2,1) gauge group and SO(3,2) gauge group led to the three-dimensional version of the Einstein-Hilbert action and the conformal gravity, respectively [1,2]. We have investigated the classical solutions for various cases of the Clifford algebra and found the intimate relations between the value of the scalar curvature and the possible base manifold to define nonsingular two-dimensional gravity. In a special case of the classical solutions, we found the nongravity solution which includes a nonvanishing zero-form even after any possible gauging away. We claim to interpret this nonvanishing zero-form value as a "hidden order parameter" to differentiate the gravitational and nongravitational classical phase spaces, or gravity phase and nongravity phase.

We have identified the gravity theory by reducing the equations of motion to the simplest form by using gauge freedom. In this sense, two-dimensional gravity is reproduced in the special gauge choice. Following from the arguments that the classical solutions modulo gauge transformation define the classical phase space, we claim that the equations of motion of two-dimensional gravity obtained in this way define the classical phase space of the gravity as an equivalence class of the gauge symmetry.

The constant value of the scalar curvature as one of the equations of motion has specified the possible base manifold compatible with the invertibility of the zweibein on the Riemann surface. In the case that the indefinite metric is assigned to the Riemann surface, a sign ambiguity appears for the constant scalar curvature because of the peculiarity of the isomorphism between SO(2,1) and SO(1,2). Furthermore a theorem says that the Minkowskian metric cannot be defined globally on the Riemann surface except for the $g=1$ torus. It is interesting to note that the Minkowskian gravity can be defined on the $g=1$ torus without noninvertible points when the cosmological constant vanishes.

We have found a new formulation to treat even-dimensional gravity theories and applied to two-dimensional gravity at the classical level. It is a quite natural and important question to ask if we can carry out similar analyses in four dimensions as in the two-dimensional case. In fact in a separate paper [14], we have shown that we can obtain four-dimensional conformal gravity from a four-dimensional generalized Chern-Simons action. As pointed out above, the classical gravity theory in four dimensions is also defined as the equivalence class of the equations of motion modulo gauge transformation. The next natural question is how we quantize these gravity theories. Since we have formulated two-dimensional gravity as a gauge theory from the generalized Chern-Simons formulation, we expect the quantization can be carried out as in the three-dimensional gravity of the standard Chern-Simons action [1,2].

In the standard three-dimensional Chern-Simons action, the relations between the action and the topological index with the help of the Chern character is clear. In the generalized Chern-Simons action, the algebraic relations hold quite parallel with the standard relations. We, however, do not yet have a clear understanding of the topological meaning of the generalized Chern-Simons theory where the gauge algebra is the Clifford algebra unlike the standard SU(N) Lie algebra. We believe that it is a mathematically very interesting subject to clear the topological meaning of the generalized Chern-Simons theory. As was pointed out in our previous paper, there is already an interesting relation between the generalized Chern-Simons theory and the topological particle field theory which may provide new insight into the relations between the algebra and fermions [3].

Although the topological meaning of the generalized Chern-Simons theory is not yet established, it has a clear connection with topological two-dimensional gravity. It is general folklore that the topological gravity does not contain the dynamical degrees of freedom. It is then an interesting question how the dynamical degrees appear with a natural breaking of the full gauge symmetry of the Clifford algebra. We believe that the analyses of the breaking pattern of the gauge symmetry and finding dynamical degrees of freedom would help us to understand the Einstein gravity which certainly includes dynamical degrees of freedom in four dimensions.

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