Klein-Gordon equation is separable on the dyon black-hole metric

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We carry out the separation of variables for the massive complex Klein-Gordon equation in the gravitational and electromagnetic field of a four-parameter (mass, angular momentum, electric and magnetic charges) black hole.

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I. INTRODUCTION

The "no-hair" theorem in general relativity asserts that the metrics of stationary black holes can be described uniquely by three parameters: mass M, charge Q_e (assuming the absence of magnetic charges), and angular momentum per unit mass, a. Therefore, when one writes a matter-field equation on a black-hole background, these parameters become parameters of the field equation.

The Klein-Gordon equation and its separability properties on black-hole metrics have been studied in increasingly complicated contexts. First, Carter [1] pointed out that the Klein-Gordon equation was separable in the Kerr-Newman metric, among others. The actual separated equations for the real massive scalar field on the Kerr metric (M, a) were derived by Brill *et al.* [2], and the separated equations for the complex massive scalar field on the Kerr-Newman metric (M, Q_e, a) are also known [3].

However, black holes could also have magnetic charge, if such exists. Such a black hole would acquire an additional label Q_m for the magnetic charge. The interest in this possibility has grown since magnetic monopoles have been found to be required in various extensions of the standard model of particle physics. For an investigation of the behavior of a scalar field on the dyonic Kerr-Newman black-hole metric (M, Q_e, Q_m, a), and the possible evolution of the black hole by exchanging energy, charge, and angular momentum with the field, one would like to know if the Klein-Gordon equation remains separable.[1] We find that this is the case, and here we present the separated radial and angular equations. We anticipate using this separability in a thought experiment to test the cosmic-censorship conjecture by considering a charged scalar field on a dyonic black hole, i.e., the scalar field analogue of work done in [6].

II. THE SEPARATION

The Klein-Gordon equation in a general spacetime with a background electromagnetic field is

$$(\mathcal{D}_\alpha + ieA_\alpha)(\mathcal{D}^\alpha + ieA^\alpha)\Psi = \mu^2\Psi, \qquad (1)$$

where Ψ is the complex scalar field with mass parameter μ, $\mathcal{D}$ is the (metric-)covariant derivative, and A the four-potential of the electromagnetic field.

A classical black hole with mass M, angular momentum Ma, electric charge Q_e, and magnetic charge Q_m has the metric [2]

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\ &= \frac{a^2\sin^2\theta - \Delta}{\rho^2}dt^2 \\ &\quad + \frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\rho^2}\sin^2\theta\, d\phi^2 \\ &\quad + 2\frac{\Delta - (r^2+a^2)}{\rho^2}a\sin^2\theta dt\, d\phi + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \end{aligned} \qquad (2)$$

and the vector potential

$$A_t = -Q_e\frac{r}{\rho^2} + Q_m\frac{a\cos\theta}{\rho^2}, \qquad (3)$$

$$A_r = A_\theta = 0, \qquad (4)$$

$$A_\phi = Q_e\frac{ar\sin^2\theta}{\rho^2} + Q_m\left(\pm 1 - \cos\theta\frac{r^2+a^2}{\rho^2}\right), \qquad (5)$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta, \qquad (6)$$

$$\Delta = r^2 - 2Mr + a^2 + Q_e^2 + Q_m^2 \qquad (7)$$

and we are using Boyer-Lindquist coordinates.

Of course, the vector potential is unique only up to a gauge transformation, and the magnetic part of A_ϕ contains a string singularity. The two signs in that term correspond to the two gauges we will be using. The upper-sign term puts the string along the negative z axis $(\theta = \pi)$ and will be used when $0 \le \theta \le \pi/2$, the lower-sign term puts it along the positive z axis $(\theta = 0)$ and will be used when $\pi/2 < \theta \le \pi$. Therefore, the wave function is also gauge transformed across the equator, and picks up a factor of $e^{2ieQ_m\phi}$ passing from north to south. This matching of boundary conditions ensures that the problem can be expressed meaningfully without strings of diverging vector potential. Such a wave

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[1] In [4], the separation of variables for all *real, massless single spin* field equations with $s = 0, \frac{1}{2}, 1, 2$ on the seven-parameter metric of [5] has been carried out. However, their coordinates are not of Boyer-Lindquist type.

[2] A seven-parameter class of Petrov type-D solutions of Einstein-Maxwell equations is presented in [5]. The parameters include M, a, Q_e, and Q_m, but the coordinates are not of Boyer-Lindquist type. See also [6] and [7].

function is called a section [8].

We rewrite Eq. (1) as

$$\frac{1}{\sqrt{-g}}\partial_\alpha\left[\sqrt{-g}(\partial^\alpha + ieA^\alpha)\Psi\right] + ieA_\alpha(\partial^\alpha + ieA^\alpha)\Psi = \mu^2\Psi$$

and substitute for the wave function the expansion

$$\Psi = R(r)\Theta(\theta)e^{-i\omega t}e^{i(m\mp eQ_m)\phi}, \tag{8}$$

where the upper sign is used in the upper hemisphere, and the lower sign in the lower. After a few pages of algebra, the equation can be separated into the following angular and radial parts:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left(a^2(\omega^2-\mu^2)\cos^2\theta - \frac{(m-eQ_m\cos\theta)^2}{\sin^2\theta} - 2ameQ_m\cos\theta + \lambda\right)\Theta = 0, \tag{9}$$

$$\frac{d}{dr}\left(\Delta\frac{dR}{dr}\right) + \left(\frac{1}{\Delta}[(r^2+a^2)\omega + eQ_e r - am]^2 - \mu^2 r^2 + 2am\omega - a^2\omega^2 - \lambda\right)R = 0\ . \tag{10}$$

The angular equation (9), together with boundary conditions of regularity at $\theta = 0$ and $\theta = \pi$ constitutes a Sturm-Liouville eigenvalue problem for the separation constant λ for fixed values of a, ω, m, μ and $eQ_m \equiv q$. If we label the eigenvalues by l, we have $\lambda = \lambda_{lmq}(a,\mu,\omega)$. We call the solutions $\Theta_{qlm}(a,\mu,\omega,\theta)$ *monopole spheroidal harmonics* in the spirit of earlier literature [8, 9].

With $\mu = 0$, the equation reduces to the equation (his 4.10) for spin-weighted spheroidal harmonics introduced by Teukolsky [9] and discussed in the literature [10–16], with $-q = -eQ_m$ playing the role of spin (and $a \to -a$). Similiar roles for the same product are known classically [17] and have been pointed out for Abelian monopoles [18–20].

A difference of Eq. (9) from the Teukolsky equation is that m can assume half-integer values. In fact, to make the wave function (8) single valued in ϕ, both $m - eQ_m$ and $m + eQ_m$ have to be integers. That means that m and eQ_m are either both integers or both half-integers. This property of eQ_m is the well-known Dirac quantization condition.

The radial equation (10), on the other hand, is very similiar to the $Q_m = 0$ case, except for the modification in Δ according to Eq. (7). For $Q_m = 0$, it has the same content as the radial equation (their 8) of [3]; and when Q_e is also zero, it further reduces to the $s = 0$ case of Teukolsky's [9] radial equation (his 4.9), as it should. For further work (analyses of perturbations, stability, scattering, etc.) on the dyonic black hole, (10) can be treated the same way [3, 9–12, 15] as its special cases, i.e., brought into a Schrödinger-like form by defining the "tortoise" coordinate r^* via $dr^*/dr = (r^2+a^2)/\Delta$ and substituting $R(r) = U(r)/\sqrt{r^2+a^2}$.

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