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Dynamics and gravitational interaction of waves in nonuniform media

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We derive the generally covariant equations describing the propagation of waves with an arbitrary dispersion relation in a nonuniform, nondissipative medium. The back-reaction of the waves on the medium is expressed in terms of the wave energy-momentum tensor. The formalism is based on variations of the Lagrangian of the system with respect to the wave amplitude and phase and the particle orbits. The Lagrangian approach is considered in detail in the context of a cold, unmagnetized plasma. It is shown that the "inertial" mass of a photon in a plasma, namely the plasma frequency, is also its gravitational mass. Extremely precise experiments are needed to measure the gravitational "free fall" of phonons, plasmons, or photons in laboratory media. Finally, we indicate how the formalism can be extended to hot magnetized plasmas.

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I. INTRODUCTION

The theory for the propagation of a wave packet in a nonuniform medium that changes weakly in space and time compared to the wavelength and the period of the wave (WKB theory) is best treated by means of the Lagrangian action principle. This Lagrangian approach was first formulated by Whitham [1] and was subsequently applied by others to various plasma waves [2]. It leads in a natural and unambiguous way to the equations describing the propagation of the wave packet, the evolution of its amplitude, and the correct back-reaction of the wave on the medium.

The Lagrangian treatments so far have been carried out for a Minkowski coordinate system in a flat spacetime [3]. In this work we extend the Lagrangian approach to general coordinate systems in a curved spacetime. This allows one to make use of the principle of general covariance in describing the dynamics of waves in nonuniform media. This generalization has advantages even for a flat spacetime. One can derive the relevant Lagrangian in a local Minkowski frame, and by writing it in a covariant form arrive at the general evolution equations for the waves using a relatively simple procedure. In addition, this method yields the energy-momentum tensor for the wave packet by taking the derivative of the Lagrangian for the entire system (background medium plus wave) with respect to the metric tensor [4]. The resulting fluid equations have an additional term associated with the wave energy-momentum. The wave pressure and energy density are naturally coupled to the fluid equations. Finally, this method allows one to include the effects of gravity on the wave dynamics in a straightforward way.

Rather than treat the most general problem, we illustrate the method for a simple case first. We take for the background a cold plasma consisting of electrons and heavy ions. The ion motion is assumed to be dictated externally. The electrons are forced to follow the ions by a neutralizing electric field, which is negligible if the background is slowly varying compared to the electron plasma frequency. The electrons are then perturbed by a small-amplitude wave packet of a high-frequency electromagnetic wave. The fundamental dynamical variables are the electron position vector (as a function of the initial position) and the small-amplitude vector potential. The electron trajectories include the perturbation associated with the wave packet. The total action for the combined system must be stationary for arbitrary perturbations in the dynamical variables. By properly carrying out these variations we arrive at the mode structure of the waves, their dispersion relation, the ray equations, and the evolution of the wave amplitude. Since the Lagrangian is invariant under coordinate transformations, one can vary the coordinate system and end up with a correction to the fluid equations for the background medium. This correction is associated with the wave energy momentum contribution. After examining waves in cold plasmas in Sec. II, we illustrate in Sec. III the method for treating waves in more complicated media, such as hot magnetized plasmas.

We find some interesting results from this approach. For example, the dispersion relation for a photon wave packet with a wave four-vector $(\omega,\mathbf{k})$ in a homogeneous plasma, namely $\omega^2=|\mathbf{k}|^2+\omega_p^2$, is commonly interpreted as if the photon has an inertial mass equal to the plasma frequency ω_p. Now let us add a gravitational field to the system. It turns out that the photon wave packet falls in the gravitational field as if its gravitational mass was equal to its inertial mass. In fact, under the influence of gravity it moves exactly like a particle whose velocity equals the group velocity of the wave packet. However, this is only true in the absence of refraction. The refractive effects are introduced by gradients in the background electron density as measured in the local Minkowski rest frame of the plasma. The application of the equivalence principle for other types of waves is less straightforward; especially if the waves are related purely to collective

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motions of the medium constituents, and do not exist in vacuum (e.g., acoustic modes). Our discussion may be particularly useful in describing situations in which gravity strongly affects the propagation of waves, e.g., near black holes or neutron stars [5].

II. COVARIANT LAGRANGIAN FORMULATION OF WAVE DYNAMICS IN A COLD PLASMA

A. The ray equations

Consider a cold electron plasma with electromagnetic interactions. The ions are taken as infinitely massive (compared with the electrons), and the time scale for their motion is assumed to be much larger than the electron plasma period. In this case, the electromagnetic fields are negligible in the local Minkowski rest frame of the unperturbed plasma. We analyze this physical system in the general spacetime coordinate system $x^{\alpha}=(t,\mathbf{x})$ with a metric tensor $g_{\mu\nu}(x^{\alpha})$, where $\alpha=0,1,2,3$, and $x^0\equiv t$ is timelike. We use the units $\hbar=c=1$ and a metric signature $(+---)$. Since the plasma is cold, all the possible electron world lines in the unperturbed plasma state form a three-dimensional manifold that fills the entire space and can be labeled by three parameters, $\lambda_1,\lambda_2,\lambda_3$. These could be, for example, the electron positions at some initial time. Because the plasma is cold, there is only one electron trajectory passing through any point in spacetime. The position of each $\boldsymbol{\lambda}$ electron along its trajectory can be labeled by a fourth parameter λ_0. Then, any actual motion of the plasma electrons can be described as a function of λ_{μ}, namely $X^{\mu}=X^{\mu}(\lambda_0,\boldsymbol{\lambda})\equiv X^{\mu}(\lambda)$.

The field variables for the Lagrangian of the entire system of the plasma and the electromagnetic wave are the perturbed electron trajectories X'^{μ} and the vector potential A'^{μ} given by

$$\begin{aligned} X'^{\mu}(\lambda) &= X^{\mu}(\lambda)+(\xi^{\mu}(\lambda)\exp\{i\Phi[X'(\lambda)]\}+\text{c.c.})\ , \\ A'^{\mu}(\lambda) &= (a^{\mu}[X'(\lambda)]\exp\{i\Phi[X'(\lambda)]\}+\text{c.c.})\ , \end{aligned} \tag{1}$$

where $X^{\mu}(\lambda)$ represents the undisturbed motion of the electrons, $\xi^{\mu}(\lambda)$ is the amplitude of the perturbed electron motion, $\Phi[x]$ is the phase of the wavelike perturbation, and $a^{\mu}(x)$ is the perturbed vector potential (the unperturbed vector potential $A^{\mu}=0$). By the WKB assumption $|\Phi|\gg 1$. The trajectories of the ion are given by $X^{\mu}(\lambda)$ with no perturbation.

Let $N(\boldsymbol{\lambda})d^3\lambda$ be the number of electrons in the infinitesimal volume $d^3\lambda$ around $\boldsymbol{\lambda}$. Ignoring the ions, the action of the total system is given by the following integral of its Lagrangian density $\mathcal{L}$:

$$\begin{aligned} \mathcal{S} &\equiv \int \mathcal{L}\sqrt{-g}\,d^4x \\ &= -\int N(\boldsymbol{\lambda})\left[m_e \left[\frac{dX'_{\mu}}{d\lambda_0}\frac{dX'^{\mu}}{d\lambda_0}\right]^{1/2} + eA'_{\mu}(\lambda)\frac{dX'^{\mu}}{d\lambda_0}\right] d^4\lambda \\ &\quad -\frac{1}{16\pi}\int F_{\mu\nu}F^{\mu\nu}\sqrt{-g}\,d^4x\ , \end{aligned} \tag{2}$$

where m_e,e are the electron mass and charge $(e<0)$, and $F_{\mu\nu}=(\partial A'_{\nu}/\partial x^{\mu}-\partial A'_{\mu}/\partial x^{\nu})$ is the electromagnetic field tensor. The basic field variables in $\mathcal{L}$ are X'^{μ} and A'_{μ}. The first term in $\mathcal{S}$ sums the total proper length of the electron orbits, $\int N(\boldsymbol{\lambda})d^3\lambda\, d\tau$, where $d\tau\equiv\sqrt{dX'_{\mu}dX'^{\mu}}$ is an element of proper time along an electron orbit. Now consider a local flat coordinate $\bar{x}^{\mu}$ in which the unperturbed plasma is at rest, and in which the electron number density is n. Then, $\int N\,d^3\lambda d\tau=\int N(d\tau/d\lambda_0)d^4\lambda=\int n\,d^4\bar{x}$. The $d^4\lambda$ integral can be transformed to a $d^4\bar{x}$ integral by first transforming from $d^4\lambda$ to d^4x, and then transforming to the $\bar{x}$ coordinates. The Jacobian of the first transformation is $\partial(x^0,\mathbf{x})/\partial(\lambda^0,\boldsymbol{\lambda})$ and that of the second is $\sqrt{-g}$. Thus, we find

$$n(x^{\mu})=N(\boldsymbol{\lambda})\left\{(-g)^{-1/2}\left[\frac{\partial(\lambda_0,\boldsymbol{\lambda})}{\partial(t,\mathbf{x})}\right]\left[\frac{d\tau}{d\lambda_0}\right]\right\}. \tag{3}$$

Let us next introduce the expressions in Eq. (1) for X'^{μ} and A'^{μ} into the action (2) and keep terms to second order in ξ^{μ}, a^{μ}. The result is the sum of a zero-order action $\mathcal{S}_0$ and a second-order action $\mathcal{S}_2$. The first-order terms vanish because $\mathcal{S}$ is stationary for the unperturbed flow. The variation of the action involves phase gradients, defined as

$$k^{\mu}\equiv(\omega,\mathbf{k})\equiv-\frac{\partial\Phi}{\partial x_{\mu}}\ . \tag{4}$$

The resulting action must be stationary with respect to all variations in X^{μ}, A^{μ}, ξ^{μ}, a^{μ}, and Φ. By carrying out a variation of $\mathcal{S}_0$ with respect to X^{μ} and $A^{\mu}\equiv(A_0,\mathbf{A})$ one gets [6] the electron equations of motion and Maxwell's equations, respectively. These are the dynamical equations describing the unperturbed plasma state. The second-order action has terms which vary rapidly and terms which vary slowly with position through Φ [i.e., proportional to $\exp(2i\Phi)$ or $\exp(-2i\Phi)$]. Since $|\Phi|\gg 1$, the rapidly varying terms give zero after being integrated over spacetime. We therefore keep only the slowly varying terms, and assume that the plasma characteristics change weakly on the scale of the perturbation wavelength (the WKB assumption), i.e., $\partial[\ln|(\cdots)|]/\partial x^{\mu}\ll k_{\mu}$. We have simplified the resulting expression for $\mathcal{S}_2$ by choosing the gauge of a_{μ} so that $a_{\mu}U^{\mu}=0$, i.e., $a_0=0$ in the unperturbed plasma rest frame. This yields

$$\begin{aligned} \mathcal{S}_2 &= \int \mathcal{L}_2\sqrt{-g}\,d^4x \\ &= \int d^4x\sqrt{-g}\left[m_e n(|\xi_{\mu}U^{\mu}|^2-\xi_{\mu}\xi^{*\mu})(k_{\alpha}U^{\alpha})^2 \right. \\ &\qquad -ien(a_{\mu}\xi^{*\mu}-a^*_{\mu}\xi^{\mu})(k_{\alpha}U^{\alpha}) \\ &\qquad \left. +\frac{1}{4\pi}|k_{\mu}a^{\mu}|^2-\frac{1}{4\pi}(k_{\mu}k^{\mu}a_{\alpha}a^{*\alpha})^2\right], \end{aligned} \tag{5}$$

where $U^{\mu}\equiv dX^{\mu}/d\tau$, and we have transformed the λ integration to an x integration through $X^{\mu}(\lambda)$. The first term involving $\xi_\mu U^\mu$ vanishes, since the total velocity is a unit four-vector. The above expression still involves Φ through its gradient k_μ.

The averaged action must be stationary for variations with respect to both ξ^μ and a_μ. One can therefore use the perturbation with respect to $\xi^{*\mu}$ to get

$$\xi^\mu = -\frac{iea_\nu g^{\mu\nu}}{m_e(k_\alpha U^\alpha)} , \tag{6}$$

and later vary the resulting Lagrangian, $\mathcal{L}_2$, with respect to a_μ. The above second-order Lagrangian can then be written in the form

$$\mathcal{L}_2 \equiv M^{\mu\nu} a_\mu a_\nu^* \equiv \frac{1}{4\pi}[(\omega_p^2 - k_\alpha k_\beta g^{\alpha\beta}) a_\mu a^{*\mu} + |k_\mu a^\mu|^2] , \tag{7}$$

where $\omega_p \equiv (4\pi n e^2/m_e)^{1/2}$ is the plasma frequency in its rest frame, and $M^{\mu\nu}$ is a 4×4 matrix that can be diagonalized into its principal form.

The Lagrangian $\mathcal{L}_2$ must be stationary under variations of $a^{*\mu}$, subject to the above gauge condition [which we take into account by the Lagrange multiplier term $\Lambda a^{*\mu} U_\mu$, with $\Lambda = (k_\alpha a^\alpha)(k_\beta U^\beta)$]. This leads to two transverse waves with $a_\mu k^\mu = 0$ and $\mathcal{L}_2 = f_T a_\mu a^{*\mu}/4\pi$, and one longitudinal wave with a_μ parallel to $k^\nu(g_{\mu\nu} - U_\mu U_\nu)$ and $\mathcal{L}_2 = f_L a_\mu a^{*\mu}/4\pi$. Here,

$$\mathcal{L}_2 = f a_\mu a^{*\mu}/4\pi , \quad f_T = (\omega_p^2 - k_\alpha k_\beta g^{\alpha\beta}), \quad f_L = (\omega_p^2 - k_\mu U^\mu)^2 . \tag{8}$$

Because of the stationarity of $\mathcal{L}_2$ with respect to variations in a_μ, the wave phase of each mode must evolve subject to the constraint on its derivatives:

$$f(x^\mu, k_\nu, g^{\alpha\beta}) = 0 . \tag{9}$$

This constraint is simply the wave dispersion relation. In a local Minkowski frame it becomes $\omega^2 = \mathbf{k}^2 + \omega_p^2$ for transverse electromagnetic waves, and $\omega^2 = \omega_p^2$ for longitudinal plasma waves.

Next, let us derive the equations for the wave dynamics. By the stationarity of its phase the wavepacket moves at the local group velocity,

$$\frac{dx^i}{dt} = -\left[\frac{\partial\omega}{\partial k_i}\right]_{\mathbf{x},t} = \frac{(\partial f/\partial k_i)_{\mathbf{x},t,\omega}}{(\partial f/\partial\omega)_{\mathbf{x},t,\mathbf{k}}} , \tag{10}$$

where we use the notation $i,j = (1,2,3)$. Combining this result with the identity $dt/dt = 1$ we have

$$\left[\frac{\partial f}{\partial\omega}\right]\frac{dx^\mu}{dt} = \left[\frac{\partial f}{\partial k_\mu}\right]_{x^\alpha=\text{const}} . \tag{11}$$

In addition,

$$\left[\frac{\partial\omega}{\partial x^i}\right]_t = -\frac{\partial^2\Phi}{\partial t\,\partial x^i} = \left[\frac{\partial k_i}{\partial t}\right]_{\mathbf{x}} . \tag{12}$$

By following the wave packet one therefore obtains the convective derivative of k_i

$$\frac{dk_i}{dt} = \frac{\partial k_i}{\partial t} + \frac{dx^j}{dt}\frac{\partial k_i}{dx^j} = \left[\frac{\partial\omega}{\partial x^i}\right]_t - \left[\frac{\partial\omega}{\partial k_j}\right]_{\mathbf{x},t}\frac{\partial k_i}{\partial x^j} , \tag{13}$$

where ω is considered to be a function of $t,\mathbf{x}$ and also k_i according to the wave dispersion relation $f=0$. The first term on the right-hand side of (13) has contributions both from the explicit dependence of ω on $\mathbf{x}$, and from its dependence on k_i that varies with $\mathbf{x}$. The latter contribution is cancelled by the second term in (13) yielding

$$\frac{dk_i}{dt} = \left[\frac{\partial\omega}{\partial x^i}\right]_{\mathbf{k},t} = -\frac{(\partial f/\partial x^i)_{\mathbf{k},\omega,t}}{(\partial f/\partial\omega)_{\mathbf{x},t,\mathbf{k}}} . \tag{14}$$

Combining this result with

$$\frac{d\omega}{dt} = \frac{\partial\omega}{\partial t} + \left[\frac{\partial\omega}{\partial\mathbf{k}}\right]_{\mathbf{x}}\frac{d\mathbf{k}}{dt} + \left[\frac{\partial\omega}{\partial x^i}\right]_{\mathbf{k}}\frac{dx^i}{dt} = \left[\frac{\partial\omega}{\partial t}\right]_{\mathbf{k},\mathbf{x}} , \tag{15}$$

we have in four-vector notation

$$\left[\frac{\partial f}{\partial\omega}\right]\frac{dk_\mu}{dt} = -\left[\frac{\partial f}{\partial x^\mu}\right]_{k_\alpha=\text{const}} . \tag{16}$$

Here $d[\cdots]/dt \equiv (dx^\mu/dt)(\partial[\cdots]/\partial x^\mu)$, dx^μ/dt being the group velocity of the wave.

The ordinary differential equations (11) and (16) are the complete ray equations describing the dynamics of the wave. These equations are valid for any arbitrary wave with a dispersion relation, whether or not it follows from a classical action principle (e.g., a quantum-mechanical wave function). Their form resembles Hamilton's equations with k_μ being the canonical momentum of the wave position x^μ. The effective Hamiltonian of the wave is given by $\omega = \omega(\mathbf{k},\mathbf{x},t)$ which satisfies $f=0$. The requirement that the wave will satisfy the dispersion relation $f=0$ is equivalent to the assumption that a classical particle is on its mass shell. Thus, a direct analogy exists between the dynamics of waves and classical particles with no reference to quantum mechanics.

Equations (11) and (16) are equivalent to the condition that the wave phase is minimized along its group velocity orbit (Fermat's principle). This can be formulated as an action principle, by imposing the dispersion-relation constraint as a Lagrange multipler in the action,

$$\mathcal{S}_\Phi = \int \left[k_\mu \frac{dx^\mu}{dt} - \Lambda f\right] dt . \tag{17}$$

Keeping $\mathcal{S}_\Phi$ stationary for variations with respect to k_μ and x^μ along a wave orbit (with $df/dt = 0$) can be shown to be equivalent to the above ray equations. $\Lambda = (\partial f/\partial\omega)^{-1}$ enforces the dispersion relation constraint, $f=0$.

In Eqs. (11) and (16) the scalar f is invariant. $\partial f/\partial k_\mu$ is a contravariant four-vector, while from Eq. (11),

$$(d\tau/dt)^2(\partial f/\partial\omega)^2 = (\partial f/\partial k_\mu)(\partial f/\partial k^\mu) ,$$

so that $(\partial f/\partial\omega)(dx^\mu/dt)$ is also a four-vector. Thus, Eq.

(11) is covariant. However, the gradient of f at constant k_μ in Eq. (16) is not covariant since k_μ depends on the coordinate system and its transformation at two different points may be different. In addition, $(\partial f/\partial\omega)(dk_\mu/dt)$ involves an ordinary derivative, *not a covariant* derivative, of the four-vector k_μ. Nevertheless, Eq. (16) as a whole is covariant as it must be by its derivation. It turns out that the same terms (involving the affine connection $\Gamma^\mu_{\alpha\beta}$) must be added to both sides to make each side separately covariant.

The action $\mathcal{S}_2$ must be stationary with respect to variations in the perturbation phase Φ. The Lagrangian $\mathcal{L}_2$ does not involve Φ itself, but only its derivatives k_μ. The Euler-Lagrange equations yield

$$0=\frac{\partial}{\partial t}\left[\frac{\partial(\sqrt{-g}\,\mathcal{L}_2)}{\partial(\partial\Phi/\partial t)}\right]+\frac{\partial}{\partial x^i}\left[\frac{\partial(\sqrt{-g}\,\mathcal{L}_2)}{\partial(\partial\Phi/\partial x^i)}\right]$$
$$=-\frac{\partial}{\partial t}\left[\frac{\partial(\sqrt{-g}\,\mathcal{L}_2)}{\partial\omega}\right]-\frac{\partial}{\partial x^i}\left[\frac{\partial(\sqrt{-g}\,\mathcal{L}_2)}{\partial k_i}\right], \quad (18)$$

where $i=1,2,3$. Since $\mathcal{L}_2=0$ for the waves, $\partial\mathcal{L}_2/\partial k_i=(dx^i/dt)(\partial\mathcal{L}_2/\partial\omega)$. Hence in four-vector form we get the continuity equation

$$\left[\frac{\partial\mathcal{L}_2}{\partial k_\mu}\right]_{;\mu}=\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left[\sqrt{-g}\left[\mathcal{L}_\omega\frac{dx^\mu}{dt}\right]\right]=0\ , \quad (19)$$

where $i=1,2,3$. Since $\mathcal{L}_2=0$ for the waves, $\partial\mathcal{L}_2/\partial k_i=(dx^i/dt)(\partial\mathcal{L}_2/\partial\omega)$. Hence in four-vector form we get the continuity equation

$$\mathcal{L}_\omega=-\{[2k^0a_\mu a^{*\mu}-(a^0k_\mu a^{*\mu}+a^{*0}k_\mu a^\mu)]/4\pi\}\ .$$

By multiplying $\mathcal{L}_\omega$ by the wave group velocity one obtains a conserved current. Thus, the wave action density flows at the group velocity.

The dynamics of the perturbation wave packet as a whole can be obtained after the appropriate averaging of the wave position and momentum over the wave-packet envelope,

$$\langle x^\mu\rangle\equiv\frac{\int x^\mu\mathcal{L}_\omega\sqrt{-g}\,d^3x}{\int\mathcal{L}_\omega\sqrt{-g}\,d^3x}\ ,$$
$$\langle k_\mu\rangle\equiv\frac{\int(\partial\Phi/\partial x^\mu)\mathcal{L}_\omega\sqrt{-g}\,d^3x}{\int\mathcal{L}_\omega\sqrt{-g}\,a^3x}\ . \quad (20)$$

The total photon number in the denominators is constant

$$\frac{d}{dt}\int\mathcal{L}_\omega\sqrt{-g}\,d^3x=\int\frac{\partial}{\partial t}(\sqrt{-g}\,\mathcal{L}_\omega)d^3x$$
$$=\int\frac{\partial}{\partial x^i}(\sqrt{-g}\,\mathcal{L}_{k_i})d^3x=0\ , \quad (21)$$

where $\mathcal{L}_{k_i}\equiv\partial\mathcal{L}_2/\partial k_i$ and we assume that the integrand vanishes at infinity for a finite wave packet. Let us next show that by using Eq. (19) and the condition $(\mathcal{L}_2)_{;\mu}=0$, one gets that $\langle x^\mu\rangle$ and $\langle k_\mu\rangle$ evolve exactly according to the ray equations (11) and (16) with each of the factors and the various terms averaged over $\mathcal{L}_\omega$ (e.g.,

$$\langle\partial f/\partial\omega\rangle=\frac{\int(\partial f/\partial\omega)\mathcal{L}_\omega\sqrt{-g}\,d^3x}{\int\mathcal{L}_\omega\sqrt{-g}\,d^3x}\ ,$$

etc.). From integration by parts (assuming that the surface terms vanish at large distances),

$$\frac{d}{dt}\int\mathbf{x}\mathcal{L}_\omega\sqrt{-g}\,d^3x=-\int\mathbf{x}\cdot\frac{\partial(\sqrt{-g}\,\mathcal{L}_{k_i})}{\partial\mathbf{x}}d^3x$$
$$=\int\frac{(\partial f/\partial k_i)}{(\partial f/\partial\omega)}\mathcal{L}_\omega\sqrt{-g}\,d^3x\ , \quad (22)$$

yielding

$$\frac{d\langle x^\mu\rangle}{dt}=\left\langle\frac{(\partial f/\partial k_\mu)_{x_\alpha}}{(\partial f/\partial\omega)}\right\rangle, \quad (23)$$

where the averages are over $\mathcal{L}_\omega$. In addition

$$\frac{d}{dt}\int k_i\mathcal{L}_\omega\sqrt{-g}\,d^3x$$
$$=-\int\left[\frac{\partial^2\Phi}{\partial x^i\partial t}\mathcal{L}_\omega-\frac{\partial\Phi}{\partial x^i}\frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\,\mathcal{L}_{k_j})}{\partial x^j}\right]\sqrt{-g}\,d^3x\ . \quad (24)$$

Now,

$$0=\int\frac{d(\mathcal{L}_2)}{d\mathbf{x}}\sqrt{-g}\,d^3x$$
$$=\int\left[\mathcal{L}_\omega\frac{\partial\omega}{\partial\mathbf{x}}+\mathcal{L}_{k_i}\frac{\partial k_i}{\partial\mathbf{x}}+\frac{\partial\mathcal{L}_2}{\partial\mathbf{x}}\right]\sqrt{-g}\,d^3x\ . \quad (25)$$

Using Eq. (4), $\partial k_i/\partial x^j=\partial k_j/\partial x^i$. This relation gives together with Eqs. (24) and (25),

$$\frac{d}{dt}\int k_i\mathcal{L}_\omega\sqrt{-g}\,d^3x=-\int\frac{\partial\mathcal{L}_2}{\partial x^i}\sqrt{-g}\,d^3x$$
$$=-\int\frac{\partial f}{\partial x^i}a_\mu a^{*\mu}\sqrt{-g}\,d^3x\ . \quad (26)$$

Consequently

$$\frac{d\langle k_\mu\rangle}{dt}=-\left\langle\frac{(\partial f/\partial x^\mu)_{k_\alpha}}{(\partial f/\partial\omega)}\right\rangle. \quad (27)$$

Although the group velocity is customarily taken to be the propagation velocity of the wave packet, Eqs. (23) and (27) quantify the extent to which this representation is valid if the wave packet is not localized or contains a variety of wave vectors.

B. Energy-momentum tensor of waves

Since spatial and temporal nonuniformities of the background plasma change the energy-momentum of the electromagnetic wave packet, it is clear that the wave packet has a back-reaction effect on the background. This can be interpreted as the effect of a wave pressure, represented by a corresponding energy-momentum tensor for the wave. One way to analyze this effect is to vary the position of the background electrons $X^\mu(\lambda)$. However, the energy-momentum tensor for the wave can be derived more directly by carrying out a variation of the coordinate system $x^\mu \to x'^\mu = x^\mu + \zeta^\mu(x)$ and making use of the fact that the action is covariant and cannot change under this transformation [4]. While carrying out this variation we keep the dynamical variables $X^\mu(\lambda)$ fixed as numerical functions. Under these conditions, one gets a physical variation of the position of the background plasma associated with the coordinate change.

The above coordinate transformation generates a change in the metric, $\delta g_{\mu\nu} = -(\zeta_{\mu;\nu} + \zeta_{\nu;\mu})$. This change induces changes in $\sqrt{-g}$, $d\tau/d\lambda_0$, U^μ, and n. In particular,

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\delta g_{\mu\nu} \tag{28}$$

and

$$\delta\left[\frac{d\tau}{d\lambda_0}\right] = \delta\left[\frac{dX^\mu}{d\lambda_0}\frac{dX^\nu}{d\lambda_0}g_{\mu\nu}\right]^{1/2} = \frac{1}{2}\frac{d\tau}{d\lambda_0}U^\mu U^\nu \delta g_{\mu\nu} , \tag{29}$$

since $dX^\mu/d\lambda_0$ does not vary. In addition,

$$\begin{aligned}\delta n &= \delta\left[[-g]^{-1/2}\frac{\partial(\lambda_0,\boldsymbol{\lambda})}{\partial(t,\mathbf{x})}\frac{d\tau}{d\lambda_0}N(\boldsymbol{\lambda})\right] \\ &= \frac{n}{2}(U^\mu U^\nu - g^{\mu\nu})\delta g_{\mu\nu} ,\end{aligned} \tag{30}$$

where the Jacobian and $N(\boldsymbol{\lambda})$ do not vary. Finally

$$\delta U^\mu = \delta\left[\frac{dX^\mu}{d\lambda_0}\frac{d\lambda_0}{d\tau}\right] = -\frac{1}{2}U^\mu U^\alpha U^\beta \delta g_{\alpha\beta} . \tag{31}$$

Thus, the variations of all the terms in the Lagrangian of the action

$$\mathcal{S} \equiv \mathcal{S}_0 + \mathcal{S}_2 = \int (\mathcal{L}_0 + \mathcal{L}_2)\sqrt{-g}\, d^4x$$

are proportional to $\delta g_{\mu\nu} = -(\zeta_{\mu;\nu} + \zeta_{\nu;\mu})$. Therefore, one can define [4] an energy-momentum tensor $T_{\mu\nu}$

$$\frac{1}{2}\sqrt{-g}\, T_{\mu\nu} \equiv \frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial g^{\mu\nu}} , \tag{32}$$

so that [4]

$$\begin{aligned}\delta\mathcal{S} &= -\frac{1}{2}\int T^{\mu\nu}\delta g_{\mu\nu}\sqrt{-g}\, d^4x \\ &= -\int T^{\mu\nu}{}_{;\mu}\zeta_\nu\sqrt{-g}\, d^4x = 0 .\end{aligned} \tag{33}$$

Since ζ_μ is arbitrary this result yields the energy-momentum conservation equation for the plasma and the wave

$$T^{\mu\nu}{}_{;\mu} = (T^{\mu\nu}_{\text{medium}} + T^{\mu\nu}_{\text{wave}})_{;\mu} = 0 . \tag{34}$$

In a self-consistent derivation of the metric from Einstein's equations, the term $T^{\mu\nu}_{\text{wave}}$ must be taken into account in addition to the medium energy-momentum tensor.

Let us use Eq. (32) to explicitly find these energy-momentum tensors. With no background fields we find for a cold electron-proton plasma

$$\mathcal{L}_0 = -n(m_e + m_p) , \tag{35}$$

where m_p is the proton mass. Therefore

$$\begin{aligned}\frac{1}{2}\sqrt{-g}\, T^{\mu\nu}_{\text{medium}} &= \frac{\partial(\sqrt{-g}\, n[m_e + m_p])}{\partial g_{\mu\nu}} \\ &= \frac{1}{2}\sqrt{-g}\, n(m_e + m_p)U^\mu U^\nu ,\end{aligned} \tag{36}$$

or

$$T^{\mu\nu}_{\text{medium}} = n(m_e + m_p)U^\mu U^\nu . \tag{37}$$

In the absence of waves the covariant divergence of this tensor vanishes, giving the covariant fluid equations for the plasma.

For the wave Lagrangian we have $\mathcal{L}_2 = f a_\mu a^{*\mu}/4\pi$, with f_T and f_L given by Eq. (8). Along the wave orbit $f = 0$, and therefore

$$T^{\mu\nu}_{\text{wave}} = -\frac{a_\alpha a^{*\alpha}}{2\pi}\left[\frac{\partial f}{\partial g_{\mu\nu}}\right] . \tag{38}$$

For transverse waves one gets

$$T_T^{\mu\nu} = -[2k^\mu k^\nu + \omega_p^2(U^\mu U^\nu - g^{\mu\nu})]a_\alpha a^{*\alpha}/4\pi \tag{39}$$

and for longitudinal waves

$$T_L^{\mu\nu} = -\omega_p^2(3U^\mu U^\nu - g^{\mu\nu})a_\alpha a^{*\alpha}/4\pi . \tag{40}$$

Note that $a_\alpha a^{*\alpha} < 0$ so T^{00}_{wave} is positive as required. In the rest frame we have $T_T^{00} = -\omega^2 a_\alpha a^{*\alpha}/2\pi$ and $T_L^{00} = -\omega_p^2 a_\alpha a^{*\alpha}/2\pi$. In this frame the longitudinal mode has no momentum density or energy flow.

C. Gravitational effects on wave dynamics

An external gravitational field affects the wave dynamics through the metric $g^{\mu\nu}$ introduced in f. For example, consider a medium at rest, with a Newtonian potential $|V| \ll 1$ and $g^{00} = 1 - 2V$. For transverse photons, Eqs. (7)–(16) yield $k^\nu k_{\mu;\nu} = \frac{1}{2}\omega_p^2 \partial \ln(n)/\partial x^\mu$, giving in the limit of $|d\mathbf{x}/dt| \ll 1$

$$\frac{d\mathbf{k}}{dt} = -\omega_p \nabla V - \frac{\omega_p}{2n}\nabla n , \tag{41}$$

and $d\mathbf{x}/dt = \mathbf{k}/\omega$. In a homogeneous medium the refractive (density-gradient) term vanishes and the photon couples to gravity exactly like a particle with a mass ω_p, a momentum $\mathbf{k}$, and an energy ω. Therefore, the plasma frequency plays the role of a gravitational mass for the photon. It is possible to bind a photon gravitationally in a homogeneous plasma if its group velocity,

$$|d\mathbf{x}/dt|=[1+(\omega_p/|\mathbf{k}|)^2]^{-1/2},$$

is smaller than the escape velocity from the gravitational field. The gravitational binding can be viewed as a result of a redshift in ω down to its minimal allowed value, ω_p. In general, the gravitational effects become more important as the group velocity of the wave decreases or the magnitude of the gravitational field increases. The gravitational effects we have found may strongly affect wave trapping in magnetospheres or accretion flows near compact objects, such as neutron stars or black holes [5] where $|V|\sim 1$. However, for $|V|\ll 1$ the refractive effects originating from the second term in Eq. (41) usually dominate, if the temperature of the medium T is nonrelativistic. At equilibrium, the medium becomes inhomogeneous due to the gravitational potential, $n\propto\exp(-m_pV/T)$ (where m_p is the ion mass), leading to a typical ratio of $(T/m_p)\sim 10^{-9}(T/\mathrm{eV})$ between the first and second terms in Eq. (41).

The polarization four-vector $e_\mu\equiv a_\mu/\sqrt{-a_\mu a^{*\mu}}$ of transverse electromagnetic waves ($k_\mu e^\mu=0$) can also be changed by gravity. In the gauge $e_\mu U^\mu=0$, Eqs. (11) and (16) yield the evolution equation

$$\frac{de_\mu}{dt}=\frac{e^\sigma}{k_\alpha k^\alpha-(k_\beta U^\beta)^2}\left[(k_\gamma U^\gamma k_\mu-k_\delta k^\delta U_\mu)\frac{dU_\sigma}{dt}+(k_\gamma U^\gamma U_\mu-k_\mu)\frac{dk_\sigma}{dt}\right], \tag{42}$$

where the plasma hydrodynamic four-velocity satisfies $U_\mu U^\mu=1$. Near the turning point of a gravitationally bound photon the polarization can be reversed. The photon momentum is reversed there while its polarization direction does not change, and therefore its net helicity flips.

III. ELECTROMAGNETIC WAVES IN HOT MAGNETIZED PLASMAS

For the sake of simplicity we considered only cold plasmas with no background fields in Sec. II. We have shown how one can derive all the essential properties of the electromagnetic waves in these plasmas from the consideration of their Lagrangian. The general method of treating the WKB theory of waves by the least action principle can be used to describe more complicated systems, once the second-order action $\mathcal{S}_2$ is calculated. One can arrive at this action by first deriving the Lagrangian density in the local Minkowski rest frame of the medium, and then generalizing it to a covariant form integrated over space-time. In this section we will outline this method for a specific example, namely a hot plasma in a constant magnetic field. In order to avoid extended derivations, we will only illustrate the least-action approach in this plasma without deriving all its complicated dielectric properties in detail.

First, it is easy to see that a hot plasma in a magnetic field admits a second-order Lagrangian. This Lagrangian can be derived by replacing the four-function of the four variables $X^\mu(\lambda)$ that were employed for the cold-plasma trajectories by four-functions of eight parameters $\{\lambda_i\}_{i=0}^7$ that give a complete description of all the orbits (e.g., $\{\lambda_1,\dots,\lambda_7\}$ may characterize different orbits and λ_0 the position along an orbit). The total Lagrangian of the system can be expressed in terms of these orbits. However, the procedure outlined in Sec. II leads to a considerable complication, especially when one substitutes the perturbed electron orbits in terms of the perturbed electromagnetic fields of the wave. This complication can be avoided by considering the Lagrangian in the locally flat rest frame of the plasma where the background electric field vanishes and the background magnetic field $\mathbf{B}_0$ is constant. In this frame, the effective Lagrangian density of the plasma can be written as

$$\mathcal{L}_2=\varepsilon_{\rm kin}-j^\mu a_\mu+(|\mathbf{E}|^2-|\mathbf{B}|^2)/8\pi, \tag{43}$$

where $\mathbf{E},\mathbf{B}$ are the perturbed electric and magnetic fields of the wave, derived from the vector potential a^μ; $\varepsilon_{\rm kin}$ is the perturbed kinetic energy of the electrons and ions; and $j^\mu=(j^0,\mathbf{j})$ is the perturbed four-current in the plasma. In proceeding further, it is necessary to assume that there are no resonant particles, since these particles cannot satisfy the WKB approximation. Under this assumption, the waves are characterized by a dielectric tensor

$$\epsilon^{ij}(i,j=1,2,3), \quad 4\pi\mathbf{j}^i=-i\omega(\epsilon^{ij}-I^{ij})\mathbf{E}_j, \tag{44}$$

where I is the unit 3×3 matrix. The dielectric tensor is a function of $\omega,\mathbf{k}$, and the properties of the plasma, such as the magnetic field $\mathbf{B}_0$ and the unperturbed electron distribution function. The total energy of the wave can be shown to be [7]

$$\varepsilon_{\rm kin}+\frac{(|\mathbf{E}|^2+|\mathbf{B}|^2)}{8\pi}=\frac{1}{8\pi}\left[\frac{\partial}{\partial\omega}(\omega\epsilon^{ij})\mathbf{E}_i^*\mathbf{E}_j+|\mathbf{B}|^2\right]. \tag{45}$$

Finally, the second term on the right-hand side of Eq. (43) can also be expressed in terms of ϵ^{ij} to give

$$\mathcal{L}_2=\frac{1}{8\pi}\left\{|\mathbf{E}|^2-|\mathbf{B}|^2+\frac{1}{\omega^2}\frac{\partial}{\partial\omega}[(\omega^3[\epsilon^{ij}-I^{ij}])\mathbf{E}_i^*\mathbf{E}_j]\right\}. \tag{46}$$

In this result, ϵ^{ij} involves the standard [8] integration over momentum space of the derivatives of the electron distribution function with respect to the electron momentum parallel or perpendicular to $\mathbf{B}_0$. Once the resulting expression is written in a covariant form, it may be employed in the wave action equation (19), and used to derive the wave energy-momentum tensor. We leave these extensive details to another investigation.

IV. CONCLUSIONS

The results derived in this work can be easily generalized to any type of waves in nondissipative media (that can be described by a Lagrangian), with their corresponding dispersion relation $f(x^\mu, k_\nu, g^{\alpha\beta})=0$. Generally, if the dispersion relation of a wave is known in terms of ω and $\mathbf{k}$ in the local rest frame of the medium, one can transform it to the covariant form $f=0$ in a general frame, by substituting $\omega \equiv k_\mu U^\mu$ and $\mathbf{k}^i = k_\mu (g^{\mu i} - U^\mu U^i)$. For a hot (or a degenerate) collisionless medium, $\mathcal{S}$ should contain the particle distribution function integrated over particle trajectories in the 8-dimensional phase-space. By including higher-order terms in the Lagrangian, one can also analyze nonlinear wave-wave interactions.

An extremely precise experiment is required to measure the gravitational effect of the earth on the dynamics of collective excitations in the laboratory. Similarly to gravitational lensing of photons, one may look for deflections in the trajectories of excitation wave packets in solids. For modes that can propagate with a small group velocity (nonrelativistic "particles"), such as phonons or plasmons, the gravitational effects are much more pronounced than for vacuum photons [9]. However, these effects may be easily dominated by noise from refraction due to very small density or temperature inhomogeneities in the solid. Wave damping processes (that were ignored in this work) should also be minimized. In principle, it is possible to look for gravitational effects on phonons in superfluid helium at low temperatures, on waves in media that are accelerated by an external force other than gravity (e.g., a centrifugal force), or in wave guides.

The formalism of this paper describes the behavior of a large class of waves, which appear in a variety of applications in astrophysics and plasma physics. In particular, it provides a simple approach for calculating gravitational lensing phenomena in a background medium. Besides photons, it can describe the propagation of other excitations, such as acoustic modes, or of particles, such as neutrinos [10], which have a dispersion relation in nondissipative media. Additional applications include, for example, turbulent heating and self-focusing phenomena in fusion plasmas or astrophysical jets [11], effects on the propagation of cosmic rays through the galactic magnetic field [12], the structure of collisionless shocks [13], and wave phenomena near compact objects [5].

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