Global structure of Gott's two-string spacetime

Curt Cutler
Theoretical Astrophysics 130-33, California Institute of Technology, Pasadena, California 91125
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Gott has recently obtained exact solutions to Einstein's equation representing two infinitely long, straight cosmic strings that gravitationally scatter off each other. A remarkable feature of these solutions is that they contain closed timelike curves when the relative velocity of the strings is sufficiently high. In this paper we elucidate the global structure of Gott's two-string spacetime. In particular, we prove that the closed timelike curves are confined to a certain region of the spacetime, and that the spacetime contains complete spacelike, edgeless, achronal hypersurfaces, from which the causality-violating regions may be said to evolve. We then explicitly determine the boundary of the region containing closed timelike curves.

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I. INTRODUCTION

It is well known that the solution of the Einstein equation representing an infinitely long, infinitesimally thin string is locally flat [1]. It is obtained simply by cutting out a wedge from Minkowski spacetime and then identifying the opposite faces of the wedge (so that each constant-time surface has conical geometry). Gott [2] recently pointed out that an exact solution representing two such strings gravitationally scattering off each other is also easily obtained: take two copies of the single-string spacetime, cut both of them along corresponding flat timelike hypersurfaces, and then paste them together along that hypersurface, but with a relative boost. An intriguing feature of the resulting spacetime is that it contains closed timelike curves (CTC's) when the relative velocity of the strings is sufficiently high.

It was emphasized by Ori [3] (and is implicit in Gott [2]) that the CTC's are not restricted to some "interaction region" where the strings are near each other, but rather CTC's exist arbitrarily far away from both strings. The question then naturally arises whether *all* points in Gott's two-string spacetime lie on some CTC. We prove that the answer to this question is no. Indeed, it follows from the proof that the spacetime contains complete (except for the conical singularity at the strings) spacelike, edgeless, achronal hypersurfaces (where "achronal" means that no two points of the set can be connected by a timelike curve). We are able to determine explicitly the null boundary separating the region that contains CTC's from the region that does not.

The organization of this paper is as follows. In Sec. II we review Gott's two-string spacetime and his demonstration that it contains CTC's. Then in Sec. III we prove that there is a region of the spacetime through which no CTC's pass. In Sec. IV we establish some general properties of the boundary of such a region, which motivates an examination of the null geodesics of the two-string spacetime in Sec. V. This leads directly to an explicit determination of the null boundary of the region containing CTC's in Sec. VI. Having the global structure of the spacetime thus well in hand, we are able to shed some light on the physically interesting question of what can be learned about finite-length curved strings from these infinite-length string spacetimes.

II. THE SPACETIME

Like the single-string spacetime, Gott's two-string spacetimes are locally flat. They are obtained rather directly by simply removing two "wedges" from Minkowski spacetime and identifying appropriately the opposite faces of these wedges. More precisely, consider Minkowski spacetime covered with a Lorentz coordinate system (t_L,x_L,y_L,z_L), so that the metric has the form

$$ds^2=-dt_L^2+dx_L^2+dy_L^2+dz_L^2 \; . \tag{1}$$

(We use the subscript "L" to indicate that this is the "laboratory" frame, with respect to which the strings will have equal but opposite momenta.) It is useful to now define two more Lorentz coordinate systems, (t_1,x_1,y_1,z_1) and (t_2,x_2,y_2,z_2), related to these laboratory coordinates by boosts of velocity $+v$ and $-v$ along the x_L direction:

$$\begin{aligned} t_1&=\frac{(t_L-vx_L)}{\sqrt{1-v^2}}, & t_2&=\frac{(t_L+vx_L)}{\sqrt{1-v^2}} \; , \\ x_1&=\frac{(x_L-vt_L)}{\sqrt{1-v^2}}, & x_2&=\frac{(x_L+vt_L)}{\sqrt{1-v^2}} \; , \\ y_1&=y_L, & y_2&=y_L \; , \\ z_1&=z_L, & z_2&=z_L \; . \end{aligned} \tag{2}$$

These two coordinate systems will define the rest frames of the two strings, respectively. Now, in the region $y_L\equiv y_1\geq d$, remove the wedge whose two faces are given by $\{x_1=(y_1-d)\tan\alpha\}$ and $\{x_1=-(y_1-d)\tan\alpha\}$, and glue together the opposite faces of the wedge by identifying the points $[t_1,(y_1-d)\tan\alpha,y_1,z_1]$ with $[t_1,-(y_1-d)\tan\alpha,y_1,z_1]$. Similarly in the region $y_L\equiv y_2\leq -d$, remove the wedge whose faces are given by $\{x_2=(d+y_2)\tan\alpha\}$ and $\{x_2=-(d+y_2)\tan\alpha\}$ and glue together the opposite faces of this wedge by identifying the points $[t_2,(d+y_2)\tan\alpha,y_2,z_2]$ with $[t_2,-(d+y_2)\tan\alpha,y_2,z_2]$. The resulting spacetime thus contains two parallel strings, each with deficit angle 2α. The strings have speeds (with respect to the laboratory frame) $+v$ and $-v$ along the x_L direction, and their impact parameter is $2d$. Most of the proofs in this paper require $\alpha<\pi/4$, so we will always impose this restriction. (Since the cosmic strings that might arise from grand unified theories have $\alpha\sim10^{-5}$, this is not a serious limitation in practice.)

While the Gott construction does not necessitate having the strings be parallel or of equal deficit angle, we have restricted ourselves to this case for simplicity. In this case, besides the continuous isometry along the z direction, the spacetime possesses the following discrete isometries:

$$\begin{aligned} &D1:(t_L,x_L,y_L,z_L)\to(-t_L,-x_L,y_L,z_L)\ ,\\ &D2:(t_L,x_L,y_L,z_L)\to(t_L,-x_L,-y_L,z_L)\ ,\\ &D3:(t_L,x_L,y_L,z_L)\to(-t_L,x_L,-y_L,z_L)\ , \end{aligned} \tag{3}$$

where $D3$ is just the product of $D1$ and $D2$. (Note that the above discrete isometries are indeed well defined on the "identified" wedge faces of the two-string spacetime; i.e., they map identified points to identified points.)

Note that the opposite faces of the wedges are identified at equal values of time in their respective rest frames. Hence by (2), the coordinate t_L (as well as x_L) is discontinuous on the two-string spacetime. One could, of course, cover the two-string spacetime with a continuous coordinate system; however, for our purposes it is easier to use (t_L,x_L,y_L,z_L) and simply make allowance for the fact that these coordinates are badly behaved. Indeed, using these coordinates it is easy to see how CTC's arise: the value of t_L along a future-directed causal curve increases monotonically where the laboratory coordinates are continuous, but t_L decreases discontinuously when the curve "jumps across" the wedge in the direction opposite to the string's motion. This creates the possibility that a timelike curve can return to its initial spatial position in the laboratory frame at the same value of t_L at which it left.

We conclude this section by briefly repeating Gott's demonstration that the spacetime contains CTC's for $v>\cos\alpha$. For any $w>d\cot\alpha$, consider the points A and B whose coordinates in the (t_1,x_1,y_1,z_1) system are given by $A=(-vw,w,0,0)$ and $B=(vw,-w,0,0)$. Then one easily sees that there exists a geodesic which goes from A to B via the wedge that is at rest in this coordinate system. Straightforward trigonometry shows that the geodesic is timelike if $vw>w\cos\alpha+d\sin\alpha$. Hence for $v>\cos\alpha$, this geodesic is timelike for sufficiently large w. Now, in laboratory coordinates, $A=(0,w\sqrt{1-v^2},0,0)$ and $B=(0,-w\sqrt{1-v^2},0,0)$. Then by acting on the above geodesic with the discrete isometry $D3$, we see there is also a future-directed timelike geodesic which goes from B to A via the wedge that is at rest in the "2" coordinate system. By joining the two curves, we make a CTC. (Note, however, that this CTC is not a geodesic, since its tangent is discontinuous at A and B.) For the remainder of this paper we shall just consider two-string spacetimes having $v>\cos\alpha$.

III. EXISTENCE OF POINTS NOT LYING ON CTC's

It is clear from Gott's example in Sec. II that CTC's exist at arbitrarily large values of $|x_L|$, $|y_L|$, and $|t_L|$. Given this fact, one might wonder whether every point in the spacetime lies on some CTC. We will now prove that this is not the case.

It is useful to first note that, in the construction of the two-string spacetime given in Sec. II, one is of course free to "rotate" the excised wedges in the x_1-y_1 and x_2-y_2 planes, respectively, without changing the resulting spacetime (so long as neither wedge crosses the $y=0$ plane). A different choice for the orientation of the wedges merely changes where the original coordinate systems (the 1, 2, and L systems) are discontinuous on the two-string spacetime. For the explication of the proof below, we find it convenient to introduce new Lorentz coordinates which, while still discontinuous, are discontinuous at different surfaces in the two-string spacetime. To this end, rather than explicitly give the coordinate transformations, it is simpler just to repeat the construction of the two-string spacetime, but stated in terms of coordinates systems $1'$, $2'$, and L'. These "primed" coordinates are related to each other by the same Lorentz transformations as the corresponding unprimed systems (2). Repeating the construction then, starting with Minkowski spacetime, in the region $y_{L'}\equiv y_{1'}\geq d$, we excise the wedge whose two faces are given by $\{x_{1'}=0\}$ and $\{x_{1'}=(y_{1'}-d)\tan2\alpha\}$ and glue together the opposite faces of the wedge by identifying the points $[t_{1'},0,y_{1'},z_{1'}]$ with $[t_{1'},(y_{1'}-d)\sin2\alpha,d+(y_{1'}-d)\cos2\alpha,z_{1'}]$. Similarly, in the region $y_{L'}\equiv y_{2'}\leq -d$, we remove the wedge whose faces are given by $\{x_{2'}=0\}$ and $\{x_{2'}=(y_{2'}+d)\tan2\alpha\}$ and glue together the opposite faces of this wedge by identifying the points $[t_{2'},0,y_{2'},z_{2'}]$ with $[t_{2'},(y_{2'}+d)\sin2\alpha,-d+(y_{2'}+d)\cos2\alpha,z_{2'}]$. To repeat, the resulting two-string spacetime is the same as before. We now proceed with the proof that the two-string spacetime contains points not lying on CTC's.

Choose any $\epsilon>0$ and define two regions in the spacetime: region $\mathrm{I}\equiv\{x_{L'}\geq0,t_{1'}\geq\epsilon\}$ and region $\mathrm{II}\equiv\{x_{L'}\leq0,t_{2'}\geq\epsilon\}$. The boundary of $\mathrm{I}\cup\mathrm{II}$ is just the continuous spacelike surface $\{x_{L'}\geq0,t_{1'}=\epsilon\}\cup\{x_{L'}\leq0,t_{2'}=\epsilon\}$, so clearly no future-directed timelike curve can exit $\mathrm{I}\cup\mathrm{II}$. These spacetime regions are depicted in Fig. 1. Now the coordinate $t_{1'}$ is continuous in region I, and $t_{2'}$ is continuous in region II. (This is possible because only one string enters each region.) Hence, since these functions increase monotonically along future-directed timelike curves (where these functions are continuous), no CTC can remain wholly in region I or wholly in II; any CTC must cross the boundary between I and II. Now, the boundary between regions I and II is just the portion of the $x_{L'}=0$ surface satisfying $t_{L'}\geq\epsilon\sqrt{1-v^2}$. Next note that, on the boundary between I and II, $t_{L'}$ is a monotonic function of

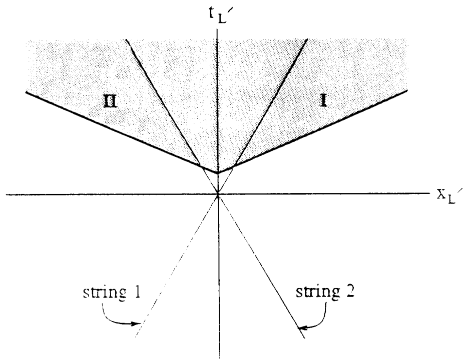

FIG. 1. Spacetime diagram illustrating the proof that there are regions containing no CTC's. Strings 1 and 2 are actually displaced from the $x_{L'}$-$t_{L'}$ plane by $\pm d$ in the $y_{L'}$ direction. Region I∪II, which is devoid of CTC's, is shaded.

$t_{1'}$ and $t_{2'}$ ($t_{1'}=t_{2'}$ on this boundary):

$$t_{1'}=(t_{L'}-vx_{L'})/\sqrt{1-v^2}=t_{L'}/\sqrt{1-v^2}\ ,$$
$$t_{2'}=(t_{L'}+vx_{L'})/\sqrt{1-v^2}=t_{L'}/\sqrt{1-v^2}\ . \tag{4}$$

Hence since $t_{1'}$ increases along every future-directed timelike curve in I, and similarly for $t_{2'}$ in II, whenever a future-directed timelike curve in I∪II intersects the $x_{L'}=0$ boundary, the value of $t_{L'}$ must be larger than at its previous intersection. But then the curve cannot be closed. Hence no point in region I or II can lie on a CTC.

We point out that I∪II is foliated by the surfaces

$$\{x_{L'}\leq 0, t_{2'}=\kappa\}\cup\{x_{L'}\geq 0, t_{1'}=\kappa\} \tag{5}$$

for $\kappa>\epsilon$. These surfaces are continuous but not differentiable at $x_{L'}=0$, but they can certainly be "rounded out" to form smooth spacelike, edgeless, achronal hypersurfaces. (We emphasize that these spacelike hypersurfaces are *not* asymptotically null.) These hypersurfaces are complete except for the conical singularity at the string itself; if the conical singularity is suitably "smoothed out" so that the string has finite thickness, then the hypersurfaces are complete. The existence of CTC's in the spacetime means that the evolution of initial data on these surfaces encounters a Cauchy horizon.

By acting on I∪II with the discrete isometry $D1$, one obtains another, "time-reflected" region devoid of CTC's. By taking the union of I∪II and this time-reflected region, and by taking the limit $\epsilon\rightarrow 0$ in the definition of regions I and II, we also see there is an open neighborhood around each string that intersects no CTC. This latter fact has the interesting implication that none of the CTC's in the spacetime is affected if the infinitesimally thin strings are replaced by strings of sufficiently small but finite thickness. In particular, the boundary of the region containing CTC's, which we find explicitly in Sec. VI, is completely independent of the idealization that the strings are infinitesimally thin.

Also by considering the $\epsilon\rightarrow 0$ limit, we see that the region devoid of CTC's includes an open neighborhood of the origin of laboratory coordinates (0,0,0,0), since any timelike curve passing suitably close to the origin must enter I∪II. (Note that the L and L' coordinates coincide near the origin.)

IV. GENERAL FEATURES OF THE CTC BOUNDARY

We would like to determine precisely which points in the two-string spacetime lie on CTC's. This is of course equivalent to finding the boundary of the region whose point lie on CTC's. In this section we present two rather general results concerning this boundary.

The first result applies in any spacetime: if the boundary of the region containing CTC's is (locally) a smooth hypersurface, then it is a null hypersurface. The proof is simple. First, it is obvious that the boundary cannot be spacelike. We now assume the boundary is timelike, and show that this leads to a contradiction. Consider two nearby points, p and q, on the boundary, connected by a future-directed timelike curve from p to q. We can slightly deform this curve to a future-directed timelike curve whose endpoints p' and q' are interior to the region containing CTC's, while part of the curve lies outside this region. We will then have achieved a contradiction if we can show there is also a future-directed timelike curve from q' to p'. But now consider any curve λ connecting p' and q', but lying entirely interior to the region containing CTC's. Given any point r on λ, there is an open neighborhood of $O\ni r$ such that $r_1,r_2\in O$ implies that the CTC connecting r to itself can be deformed to a future-directed timelike curve from r_1 to r_2. Since λ is compact it can be covered by a finite number of such neighborhoods. We can thus join together the corresponding curves to make a future-directed timelike curve from q' to p', and we are done.

[We remark that it was necessary in the above proof that the points in the interior lie on closed timelike curves, not simply closed causal (i.e., timelike or null) curves. Indeed, it is easy to construct a spacetime containing a region whose points all lie on closed null geodesics, such that the boundary of this region is timelike.]

We next review the concept of a polarized hypersurface, introduced by Kim and Thorne [4] in their analysis of CTC's in wormhole spacetimes. (The term "polarized" originates in the expected divergence at such a hypersurface of the vacuum polarization in quantum field theory.) The Kim and Thorne arguments are rather general, and we expect them to apply in our context. A point p is said to lie on the "Nth polarized hypersurface" if it lies on a null geodesic that leaves p, loops around both strings N times, and then returns to p. Note that the initial and final tangents to the curve at p will generally be different. We will refer to such null geodesics as "self-intersecting." According to Kim and Thorne, one should generally expect the boundary of the region containing CTC's to be just the $N\rightarrow\infty$ limit of the Nth polarized hypersurface. To see this, let q be any point on the boundary, and let q' be any point which is interior to the region containing CTC's and which lies in some convex normal neighborhood of q. Then q' lies on some CTC, which must loop around both strings at least once; say it loops around n times. We now move q' closer to q (along a geodesic connecting them, say) while smoothly deforming the corresponding CTC. At some point, we will not be able to push q' any closer to q while still keeping the n loop timelike. We expect that in this limit the timelike curve convergence to a null geodesic, so that in this limit q' lies on the nth polarized hypersurface. (If the CTC's do converge to a closed curve, they must converge to a null geodesic. We do not know, however, the general conditions under which the CTC's will converge to a closed curve.) Since we can go around this curve twice, the $2n$th polarized hypersurface lies even closer to the CTC boundary. Thus we expect any point on the CTC boundary to be arbitrarily close to self-intersecting null geodesics that loop around both strings an arbitrarily large number of times.

V. NULL GEODESICS IN THE SPACETIME

The argument of the previous section, that the null boundary of the region containing CTC's is the $N\to\infty$ limit of the Nth polarized hypersurface, motivates a close examination of the null geodesics in the spacetime. In this section we show that the condition that a null geodesic loops around the strings a very large number of times puts a strong constraint on either its initial or final direction. As a simple application of this result, we also show that there are no closed null geodesics in the two-string spacetime. [We distinguish between "closed null geodesic" and "self-intersecting null geodesic." In a closed null geodesic, both the position *and* tangent direction of the curve (but not the tangent vector) return to their starting values.]

We first determine how the direction of a null geodesic changes after it has made one loop around both strings. Since we are interested in null geodesics that return to their starting points and since $dz/d\lambda$ is constant along a geodesic, where λ is the curve's affine parameter, we restrict attention to geodesics satisfying $dz/d\lambda=0$. To begin, we define the angle ϕ_1 of the null geodesic projected into the x_1-y_1 plane by

$$\cos\phi_1 \equiv -dx_1/dt_1, \quad \sin\phi_1 \equiv dy_1/dt_1 \ . \tag{6}$$

Similarly we define the angle ϕ_2 by

$$\cos\phi_2 \equiv dx_2/dt_2, \quad \sin\phi_2 \equiv -dy_2/dt_2 \ . \tag{7}$$

Now we see from (2) that the "1"- and "2"-coordinate systems are related by the Lorentz transformation

$$t_1=\frac{(t_2-ux_2)}{\sqrt{1-u^2}}, \quad x_1=\frac{(x_2-ut_2)}{\sqrt{1-u^2}}, \quad y_1=y_2, \quad z_1=z_2 \ , \tag{8}$$

where

$$u\equiv 2v/(1+v^2) \ . \tag{9}$$

Hence the angles ϕ_1 and ϕ_2 are related by

$$\begin{aligned}\cot\phi_1 &\equiv -dx_1/dy_1 \\ &= (1-u^2)^{-1/2}(-dx_2/dy_2+u\,dt_2/dy_2) \\ &= (1-u^2)^{-1/2}(\cot\phi_2-u\csc\phi_2)\end{aligned} \tag{10}$$

or equivalently

$$\begin{aligned}\cot\phi_2 &\equiv -dx_2/dy_2 \\ &= (1-u^2)^{-1/2}(-dx_1/dy_1-u\,dt_1/dy_1) \\ &= (1-u^2)^{-1/2}(\cot\phi_1-u\csc\phi_1) \ .\end{aligned} \tag{11}$$

Now consider a null geodesic that starts from some point on the $y=0$ plane where $x_1>0$, "jumps across" the wedge that is at rest in the "1" system, and returns to the $y=0$ plane. In the course of this half-loop, $\phi_2\to\tilde{\phi}_2$, where

$$\begin{aligned}\cot\tilde{\phi}_2 &= (1-u^2)^{-1/2}(\cot\tilde{\phi}_1-u\csc\tilde{\phi}_1) \\ &= (1-u^2)^{-1/2}[\cot(\phi_1-2\alpha)-u\csc(\phi_1-2\alpha)] \\ &= (1-u^2)^{-1/2}[-\cot(2\alpha-\phi_1)+u\csc(2\alpha-\phi_1)] \ .\end{aligned} \tag{12}$$

Similarly, in a half-loop that starts from some point on the $y=0$ plane where $x_2<0$, "jumps across" the wedge that is at rest in the "2" system, and returns to the $y=0$ plane, the angle $\phi_1\to\tilde{\phi}_1$, where

$$\cot\tilde{\phi}_1=(1-u^2)^{-1/2}[-\cot(2\alpha-\phi_2)+u\csc(2\alpha-\phi_2)] \ . \tag{13}$$

Putting these two half-loop results together, we find that when a null geodesic has spiraled once around *both* strings:

$$\phi_1\to g(g(\phi_1)), \quad \phi_2\to g(g(\phi_2)) \ , \tag{14}$$

where the mapping $g(\phi)$ is defined by

$$g(\phi)=\text{Arccot}\{(1-u^2)^{-1/2}[-\cot(2\alpha-\phi)+u\csc(2\alpha-\phi)]\} \tag{15}$$

(where we take the range of Arccot to be $(0,\pi]$). Iterating, we see that after N loops, $\phi_1\to g^{2N}(\phi_1)$ and $\phi_2\to g^{2N}(\phi_2)$. Clearly if ϕ_1 (or ϕ_2) ever becomes greater than 2α, then the geodesic must cease to "spiral." We now collect the salient features of this map g, which determine its behavior under many iterations. The reader may readily verify that, for $u>[2\cos\alpha/(1+\cos^2\alpha)]$ (i.e., $v>\cos\alpha$), and in the range $0\le\phi\le 2\alpha$, $g(\phi)$ has the following properties:

$$\begin{aligned}&g(0)>0, \quad dg/d\phi>0, \quad d^2g/d\phi^2>0 \ , \\ &g(\alpha)<\alpha, \quad g(2\alpha)=\pi>2\alpha \ .\end{aligned} \tag{16}$$

Thus $g(\phi)$ qualitatively has the shape shown in Fig. 2. In particular, $g(\phi)$ has precisely two "fixed points" satisfying $g(\phi)=\phi$. Define ζ to be the smaller of these two fixed points. The condition that ζ is a fixed point,

$$\cot\zeta=(1-u^2)^{-1/2}[-\cot(2\alpha-\zeta)+u\csc(2\alpha-\zeta)] \ , \tag{17}$$

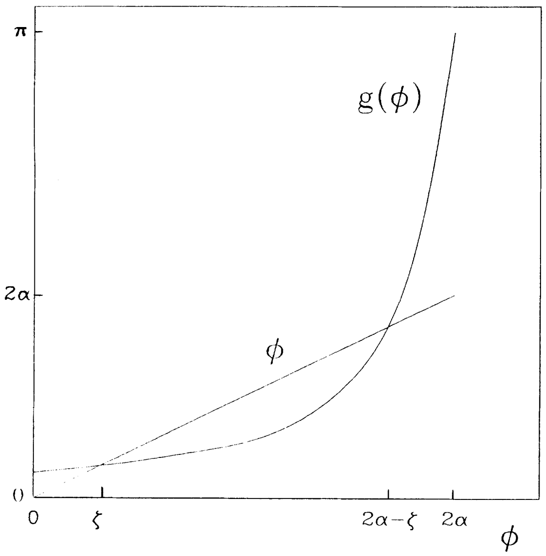

FIG. 2. The map $g(\phi)$ is shown. The angle ζ is a stable fixed point of g, while $2\alpha-\zeta$ is an unstable fixed point.

can be shown to be equivalent to

$$\cot\zeta+\cot(2\alpha-\zeta)=v[\csc\zeta+\csc(2\alpha-\zeta)] , \tag{18}$$

which can be solved to yield

$$\sin\zeta=\frac{\cos\alpha}{v}\left[\sin\alpha-\sqrt{v^2-\cos^2\alpha}\right] . \tag{19}$$

From Eq. (18), it is transparent that the larger fixed-point angle is just $(2\alpha-\zeta)$. We see from Fig. 2 that ζ is a stable fixed point, while $(2\alpha-\zeta)$ is an unstable fixed point. If ϕ is not equal to ζ or $(2\alpha-\zeta)$, then $g^N(\phi)$ is either driven monotonically to ζ as $N\to\infty$, or is monotonically driven past 2α:

$$\begin{aligned}&\phi<\zeta\Longrightarrow g^N(\phi)\to\zeta \quad \text{as } N\to\infty ,\\ &\zeta<\phi<(2\alpha-\zeta)\Longrightarrow g^N(\phi)\to\zeta \quad \text{as } N\to\infty ,\\ &\phi>(2\alpha-\zeta)\Longrightarrow g^N(\phi)>2\alpha \quad \text{for some finite } N .\end{aligned} \tag{20}$$

We note that if we follow null geodesics backwards in time instead of forwards, then the character of the fixed points is reversed: $(2\alpha-\zeta)$ becomes a stable fixed-point angle and ζ an unstable one. Thus the spacetime contains two preferred null directions, **j** and **k**, corresponding to the stable and unstable fixed-point angles, respectively. It is straightforward to determine the components of **j** and **k** in the laboratory frame. First, define the angle ϕ_L of a null geodesic in the laboratory frame by

$$\cos\phi_L\equiv-dx_L/dt_L, \quad \sin\phi_L\equiv dy_L/dt_L . \tag{21}$$

Then we have

$$\begin{aligned}\cot\phi_L&\equiv-dx_L/dy_L\\&=(1-v^2)^{-1/2}(-dx_1/dy_1-v\,dt_1/dy_1)\\&=(1-v^2)^{-1/2}(\cot\phi_1-v\csc\phi_1) .\end{aligned} \tag{22}$$

Define ζ_L to be the stable fixed-point angle as viewed in the laboratory frame, so we have

$$\cot\zeta_L=(1-v^2)^{-1/2}(\cot\zeta-v\csc\zeta) . \tag{23}$$

Then it follows from (18) that the unstable fixed-point angle in the laboratory frame is $(\pi-\zeta_L)$:

$$\cot(\pi-\zeta_L)=(1-v^2)^{-1/2}[\cot(2\alpha-\zeta)-v\csc(2\alpha-\zeta)] . \tag{24}$$

Using (19), ζ_L can be shown to be given by

$$\sin\zeta_L=v^{-1}(1-v^2)^{1/2}\cot\alpha . \tag{25}$$

The coordinate representations of the directions **j** and **k** are constant except for discontinuous "jumps" across the wedges. (Of course the directions are smooth; it is the coordinates that are discontinuous.) The value of dy_L/dt_L changes sign during these jumps, and we use this to indicate what "branch" of the geodesic is intended (e.g., $dy_L/dt_L>0$ on the $y=0$ plane where $x_L>0$, while $dy_L/dt_L<0$ on the $y=0$ plane where $x_L<0$).

$$\mathbf{j}\propto\begin{cases}\mathbf{t}_L-\cos\zeta_L\mathbf{x}_L+\sin\zeta_L\mathbf{y}_L & \text{where } dy_L/dt_L>0 ,\\ \mathbf{t}_L+\cos\zeta_L\mathbf{x}_L-\sin\zeta_L\mathbf{y}_L & \text{where } dy_L/dt_L<0 ,\end{cases} \tag{26}$$

$$\mathbf{k}\propto\begin{cases}\mathbf{t}_L+\cos\zeta_L\mathbf{x}_L+\sin\zeta_L\mathbf{y}_L & \text{where } dy_L/dt_L>0 ,\\ \mathbf{t}_L-\cos\zeta_L\mathbf{x}_L-\sin\zeta_L\mathbf{y}_L & \text{where } dy_L/dt_L<0 ,\end{cases} \tag{27}$$

where $\mathbf{t}_L$, $\mathbf{x}_L$, and $\mathbf{y}_L$ are defined to be the unit vectors in the t_L, x_L, and y_L directions, respectively.

We point out that a null geodesic which is spiraling to the future along the **j** or **k** directions is blueshifted or redshifted (with respect to the "1," "2," or "L" frames), respectively, by a constant factor with every successive loop. To see this, consider first that when a null geodesic jumps across the wedge that is at rest in the "1" frame, $dt_1/d\lambda$ is unchanged, where λ is the geodesic's affine parameter. However when the same null geodesic then jumps across the wedge that is at rest in the "2" frame, $dt_1/d\lambda$ changes discontinuously. Thus after one complete loop, $dt_1/d\lambda$ has changed. Since the null geodesic's tangent is always proportional to **j** or **k**, it is just "lengthened or shortened;" i.e., $dt_1/d\lambda$, $dt_2/d\lambda$, and $dt_L/d\lambda$ all change by the same factor after one loop. It is straightforward to work out this factor; the result is

$$\begin{aligned}&dt_L/d\lambda\to\kappa\,dt_L/d\lambda \quad \text{along } \mathbf{j} ,\\ &dt_L/d\lambda\to\kappa^{-1}dt_L/d\lambda \quad \text{along } \mathbf{k} ,\end{aligned} \tag{28}$$

where

$$\kappa=\frac{1-u\cos(2\alpha-\zeta)}{1-u\cos\zeta}>1 . \tag{29}$$

Our analysis of a null geodesic's change in direction angle leads to an easy proof that the Gott two-string spacetime does not contain any closed null geodesics. If such a null geodesic existed, its tangent would have to be proportional to either **j** or **k**, since otherwise its direction angle changes monotonically, and so obviously cannot return to its starting value. Now consider a null geodesic moving along the **j** direction that leaves the $y=0$ plane at

$x_1 = \chi$ (for some $\chi > 0$), is scattered by the string that is at rest in the "1" system, and returns to the $y=0$ plane. We will show that the final value of t_L on this half-loop is greater than the initial value of t_L. The easy way to see this is to first calculate Δt_L in the limit in which the strings' impact parameter $2d$ goes to zero. Let the initial angle, viewed in the "1" frame (6), be the fixed-point angle ζ; then in the $d=0$ case:

$$\Delta t_1 = h[\cot\zeta + \cot(2\alpha - \zeta)] ,$$
$$\Delta x_1 = -h[\csc\zeta + \csc(2\alpha - \zeta)] , \tag{30}$$

where $h \equiv \chi \cos\zeta$. Hence, by (18), in the $d=0$ case,

$$\Delta t_L \equiv \frac{\Delta t_1 + v\Delta x_1}{\sqrt{1-v^2}} = 0 . \tag{31}$$

But clearly if we now increase d while keeping Δx_1 fixed, then Δt_1 increases (i.e., "jumping across" the wedge becomes a less effective "shortcut" as d increases). That is, we have $\Delta t_1 > v|\Delta x_1|$ when $d > 0$, so $\Delta t_L > 0$ for the half-loop.

The same calculation as above shows that $\Delta t_L > 0$ for half-loops through *either* wedge and along *either* the stable or unstable fixed-point directions. Thus $\Delta t_L > 0$ every time a null geodesic with tangent proportional to **j** or **k** spirals once around both strings, and so there cannot be any closed null geodesics. (Note that this in contrast to many well-known examples of spacetimes containing CTC's, such as Taub-NUT (Newman-Unti-Tamburino) and the Morris-Thorne-Yurtsever wormhole spacetimes. Indeed, Hawking [5] has shown that there must exist some closed null geodesic in causality-violating spacetimes that develop from some noncompact initial-data surface, S, such that all the generators of the Cauchy horizon of S enter and subsequently remain within some compact region.)

VI. EXPLICIT DETERMINATION OF THE CTC BOUNDARY

In this section we explicitly determine the boundary of the region containing CTC's. We cannot offer a complete proof; rather we offer a deduction based on some added suppositions (which we believe are completely reasonable and correct). The most important supposition is that the characterization of the boundary as the "$N \to \infty$ limit of the Nth polarized hypersurface," argued for in Sec. IV, is in fact true for the two-string spacetime. We also assume the boundary is a continuous, connected hypersurface, and that the hypersurface is smooth almost everywhere (i.e., except at cusps). Note that it is clear that the boundary must be closed.

A null surface, like a spacelike one, locally divides the spacetime into past and future. We shall say a portion of the CTC boundary is part of the "future boundary" if it is locally to the future of the region containing CTC's; likewise, the "past boundary" is to the past of the region containing CTC's. We want to show that the null tangent to the future boundary must coincide with the unstable null direction **k**, given by (27). The argument is as follows. Consider a point p on the future boundary. Then we can find a sequence of points $p_n \to p$ which are the end points (both initial and final) of self-intersecting null geodesics that loop around both strings at least n times. Let $\mathbf{k}_n$ be the initial, future-directed null tangent to the nth curve at p_n. Then clearly as $n \to \infty$, $\mathbf{k}_n$ approaches the null tangent to the future boundary, since otherwise for sufficiently large n the geodesics would cross the boundary. Next observe that the null geodesic through p that generates the future boundary can never leave the boundary to the future of p, because if it did leave the boundary then, by continuity of solutions of the geodesic equation, the nth self-intersecting null geodesic would also have to leave the region containing CTC's for sufficiently large n. Now consider any null geodesic generator of the future boundary. Since it cannot leave the boundary, the generator must spiral around both strings an infinite number of times. This is because if it ever stops spiralling, it would have to cross into region I∪II (defined in Sec. III).

So far we have established that the future boundary is generated by null geodesics that spiral around both strings an infinite number of times. We now show this implies that the tangent to the generator must be proportional to the unstable fixed-point direction **k**, by assuming the opposite and showing this leads to a contradiction.

If the null generator does not coincide with **k**, then since the geodesic spirals to the future an infinite number of times, by (20) its direction angle must be driven asymptotically to the stable fixed-point direction. Therefore after some finite number of loops its laboratory frame direction angle ϕ_L is less than $\pi/2$. Next consider a point where the null generator hits the $y=0$ plane in the region $x_L > 0$, with its direction angle $\phi_L < \pi/2$. Because $\phi_L < \pi/2$, as one passes through the null boundary at this point going "to the right"—that is, increasing x_L while keeping t_L, y_L, and z_L fixed—one passes from the past of the null surface to its future. But it was demonstrated in Sec. III that (in the region $x_L > 0$) by proceeding along the $+x_L$ direction, one crosses from the region that does not have CTC's into the region that does. That is, null planes with null tangent near the stable direction simply tilt the wrong way to be the future boundary of the region containing CTC's. Thus we have achieved a contradiction, and hence the null tangent to the future must everywhere coincide with **k**. Similarly, the null tangent to the past boundary must everywhere coincide with **j**.

In flat spacetime, null hypersurfaces whose null tangents are everywhere parallel are just null planes. Thus, away from the excised wedges, the past and future CTC boundary are portions of null planes. Of course, the boundary is continuous in the physical two-string spacetime, so the intersections of these null planes with the wedge faces "match up" appropriately under the identification described in Sec. II.

A plane in flat spacetime is uniquely determined by knowing its normal and one point through which it passes. Equations (26) and (27) give the normals to the null planes which bound the region containing CTC's; we will show that the symmetries of the spacetime allow the planes to be located exactly. Let the future boundary intersect the x_L axis at $x_L = q > 0$. Then by virtue of the discrete symmetry $D2$, the future boundary also intersects the x_L axis at $x_L = -q$. Similarly, by virtue of $D3$, the past boundary also intersects the x_L axis at $x_L = q$ and $x_L = -q$. Then, by (26), the past boundary is described by

$$-t_L - x_L \cos\zeta_L + y_L \sin\zeta_L = -q \cos\zeta_L \quad \text{where } dy_L/dt_L > 0 \,,$$
$$-t_L + x_L \cos\zeta_L - y_L \sin\zeta_L = -q \cos\zeta_L \quad \text{where } dy_L/dt_L < 0 \,, \tag{32}$$

and by (27) the future boundary is described by

$$-t_L + x_L \cos\zeta_L + y_L \sin\zeta_L = q \cos\zeta_L \quad \text{where } dy_L/dt_L > 0 \,,$$
$$-t_L - x_L \cos\zeta_L - y_L \sin\zeta_L = -q \cos\zeta_L \quad \text{where } dy_L/dt_L < 0 \,. \tag{33}$$

We see that the past and future boundaries meet at a cusp, given by

$$\{x_L = q, -t_L + y_L \sin\zeta_L = 0\} \quad \text{where } dy_L/dt_L > 0 \,,$$
$$\{x_L = -q, t_L + y_L \sin\zeta_L = 0\} \quad \text{where } dy_L/dt_L < 0 \,. \tag{34}$$

We now proceed to calculate q, which will completely determine the boundary. First note that the discrete isometry $D1$ must map the cusp to itself. Next consider the identified faces of the excised wedge that is at rest in the "1" frame. This timelike surface is also mapped to itself by $D1$. Since the intersection of the cusp with this timelike surface is just a single point for each value of z, each point of this intersection must also be invariant under $D1$. Rewriting (3) in terms of the "1" coordinate system, we have

$$D1:(t_1, x_1, y_1, z_1) \to (-t_1, -x_1, y_1, z_1) \,. \tag{35}$$

Hence the only points in the $y_L > 0$ region that are invariant under $D1$ must lie on the $t_1 = 0$ surface. By (2), $t_1 = 0$ implies $t_L = v x_L$ and $x_1 = x_L \sqrt{1 - v^2}$. Now restrict attention to the right (i.e., $x_1 > 0$, where $dy_L/dt_L > 0$) wedge face. There, the intersection of the cusp (34) with the $t_1 = 0$ surface is given by

$$y_1 \sin\zeta_L = vq \,; \tag{36}$$

but the right wedge face also satisfies $x_1 = (y_1 - d)\tan\alpha$, which when combined with $x_1 = x_L \sqrt{1 - v^2}$ and $x_L = q$ yields

$$q\sqrt{1 - v^2} = (y_1 - d)\tan\alpha \,. \tag{37}$$

Combining (25), (36), and (37), we find

$$q = (d \sin\zeta_L)/(v - \sqrt{1 - v^2} \sin\zeta_L \cot\alpha)$$
$$= d \sin\alpha \cos\alpha \sqrt{1 - v^2}/(v^2 - \cos^2\alpha) \,. \tag{38}$$

Notice that in the limit that as $v \to \cos\alpha$, we have $\zeta_L \to 0$ (so the past and future null boundaries degenerate) and $q \to \infty$. That is, the region containing CTC's vanishes in this limit.

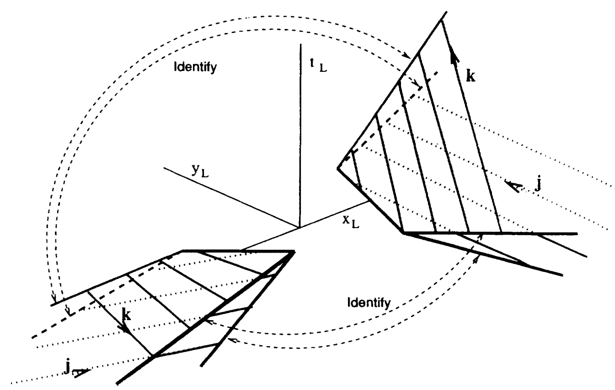

FIG. 3. A sketch showing the general features of the boundary of the region containing CTC's. The past and future boundaries of this region are null planes which meet at a spacelike cusp. CTC's are restricted to the region between these planes. The "identified" edges of the planes represent the intersection of the CTC boundary with the string wedges. (To avoid complicating the figure, we have not drawn in the string wedges.) The null generators **j** and **k** of the past and future CTC boundaries, respectively, are shown inscribed.

In summary, a rather elegant picture has emerged of the boundary of the region containing CTC's, which is depicted in Fig. 3. (Note that we have not attempted to show the location of the string wedges in Fig. 3; however, the "identified" edges of the CTC boundary are intended to represent where these planes intersect the string wedges.) Locally the past and future CTC boundaries are null planes, but globally they have topology $S^1 \times R^2$. The past and future boundaries are generated by null geodesics which emerge orthogonally from a central cusp and spiral around both strings an infinite number of times. Since these null geodesic congruences have zero expansion, the cusp is a "marginally trapped surface." Note also from (34) that the cusp is foliated by closed spatial geodesics (one spatial geodesic for each value of z). A cross section of Fig. 3, showing its intersection with the $y_L = 0$ plane, is depicted in Fig. 4.

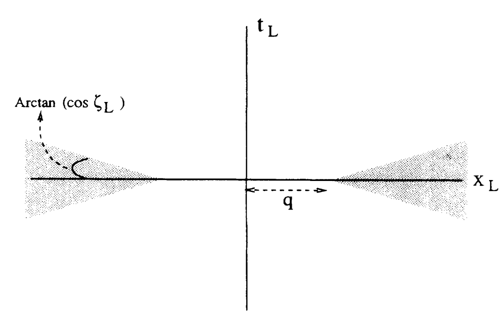

FIG. 4. Represents the x_L-t_L cross section ($y_L = z_L = 0$) of the CTC boundary depicted in Fig. 3. The region containing CTC's is shaded.

It is also clear what the self-intersecting null geodesics corresponding to the *N*th polarized hypersurface "look like" for large *N*. For initial points very close to the future CTC boundary, the null geodesic spirals very close to the boundary—its direction angle just slightly less than the unstable fixed-point direction angle—for most of the *N* loops. But eventually the geodesic's direction angle is driven sufficiently far from the fixed-point value that t_L can decrease over the course of a loop, and in several more spirals the curve returns to its starting point.

Finally, perhaps the most interesting question that arises from the Gott two-string solution is what happens when nearly straight sections of closed loops of string pass each other at high relative velocity. Just as one expects the metric very near a *single* loop of string to be well approximated by the infinite-length, straight string metric, so one might naively expect the metric very near scattering loops of string to be well approximated by the Gott solutions, so long as the impact parameter was very small compared to the radii of curvature of the strings (so the strings could be approximated as straight on this length scale). On the other hand, Tipler [6] has proved (roughly) that if a spacetime (1) is asymptotically flat and contains CTC's, (2) contains a complete, edgeless achronal hypersurface, whose domain of dependence terminates at the CTC boundary, and (3) satisfies the weak energy condition and the generic condition, then the spacetime must be singular, in the sense of being null geodesically incomplete. So we cannot use finite loops of string to "create" CTC's in an asymptotically flat spacetime without also creating a singularity. Having determined the boundary of the causality-violating region in the two-string spacetime, we can now see why our naive expectations might fail—why the causality violation that arises in the infinite-length string case could perhaps not arise in string spacetimes that are asymptotically flat. We have seen that points on the past CTC boundary lie on null geodesics that spiral out infinitely far. Hence if p is some point on the past CTC boundary, and Σ is some complete, edgeless, achronal surface to the past of p, then the intersection of the causal past of p with Σ, $J^-(p)\cap\Sigma$, is not contained within any compact region of Σ. Equivalently, given a sequence of points p_n approaching p from below, $J^-(p_n)\cap\Sigma$ extends outside any compact region of Σ for sufficiently large n, so as you approach the causality-violating region in Gott's two-string spacetime, you "can tell" that the spacetime is not asymptotically flat.

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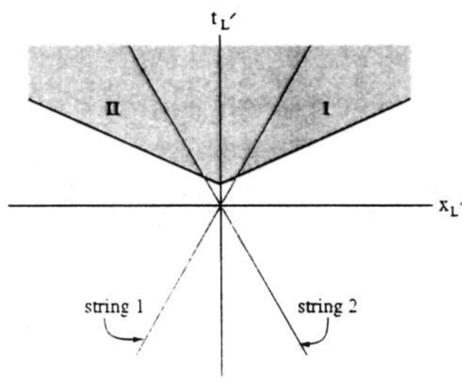

FIG. 1. Spacetime diagram illustrating the proof that there are regions containing no CTC's. Strings 1 and 2 are actually displaced from the $x_{L'}$-$t_{L'}$ plane by $\pm d$ in the $y_{L'}$ direction. Region $\mathrm{I} \cup \mathrm{II}$, which is devoid of CTC's, is shaded.

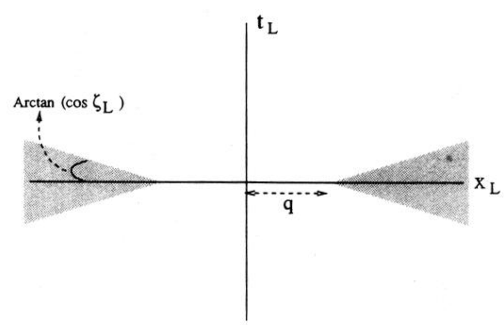

FIG. 4. Represents the x_L-t_L cross section ($y_L = z_L = 0$) of the CTC boundary depicted in Fig. 3. The region containing CTC's is shaded.